EDUCA

Ejercicio Propuesto 5.1

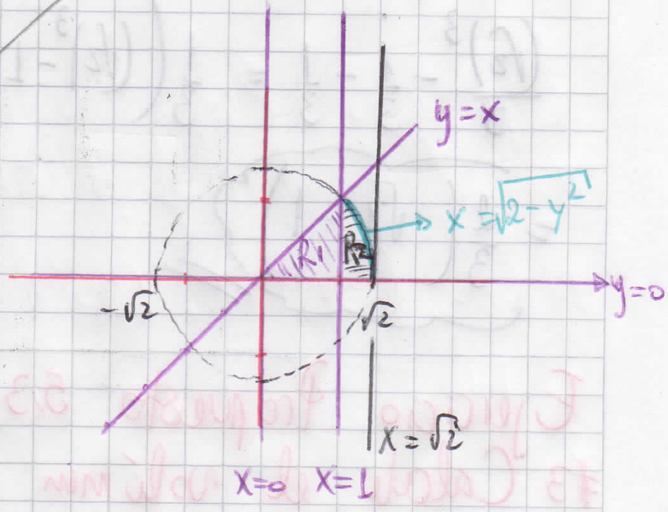
#10 Invertir el orden de integración y evaluar

$$\underbrace{\int_0^1\int_0^x y\,dy\,dx}_{R_1} + \underbrace{\int_1^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} y\,dy\,dx}_{R_2}$$

$y^2 = 2 - x^2$

$x^2 + y^2 = 2$

$$R_1 \begin{cases} y = x \\ y = 0 \\ x = 1 \\ x = 0 \end{cases} \qquad R_2 \begin{cases} y = \sqrt{2-x^2} \\ y = 0 \\ x = \sqrt{2} \\ x = 1 \end{cases}$$



El nuevo orden es:

$$\int_0^1\int_y^{\sqrt{2-y^2}} y\,dx\,dy = \int_0^1 y\,x\Big|_y^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 y\left(\sqrt{2-y^2} - y\right) dy = \int_0^1 y\left(\sqrt{2-y^2}\right) dy - \int_0^1 y^2\,dy$$

$$\int_0^1 y\left(\sqrt{2-y^2}\right) dy =$$

$u^2 = 2 - y^2$

$2u\,du = -2y\,dy$

$$\boxed{\frac{-u\,du}{y} = dy}$$

$y = 1 \rightarrow u = 1$

$y = 0 \rightarrow u = \sqrt{2}$

$$-\int_0^1 \cancel{y}\,u\,\frac{u\,du}{\cancel{y}} = -\int_0^1 u^2\,du = -\frac{u^3}{3}\Big|_{\sqrt{2}}^1 = -\left[\frac{1}{3} - \frac{(\sqrt{2})^3}{3}\right]$$

$$\boxed{= \frac{(\sqrt{2})^3}{3} - \frac{1}{3}}$$

$$(\sqrt{2})^3 = 2^{3/2} = \sqrt{8} = 2\sqrt{2}$$

$$-\int_0^1 y^2\,dy \quad -\frac{y^3}{3}\Big|_0^1 = -\frac{1}{3}$$

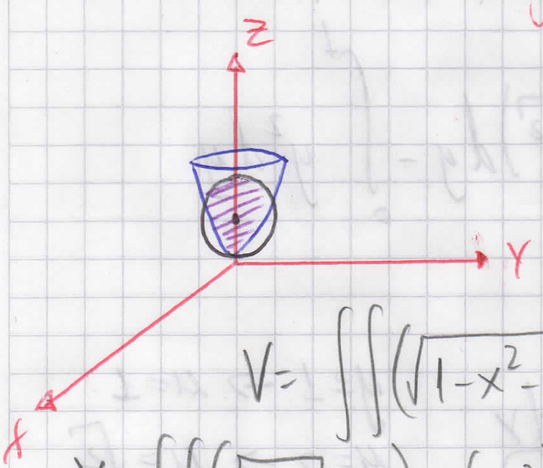
Sumando todo:

$$\frac{(\sqrt{2})^3}{3} - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}\left((\sqrt{2})^3 - 1 - 1\right) = \frac{1}{3}\left((\sqrt{2})^3 - 2\right) = \frac{2\sqrt{2}}{3} - \frac{2}{3}$$

$$= \frac{2}{3}\left(\sqrt{2} - 1\right)$$

Ejercicio Propuesto 5.3

#3 Calcule el volúmen del Sólido interior de la esfera $x^2+y^2+z^2 = 2z$ y arriba del paraboloide $x^2+y^2 = z$



$$x^2+y^2+z^2-2z=0$$
$$x^2+y^2+z^2-2z+1=1$$
$$x^2+y^2+(z-1)^2=1$$
$$z=\sqrt{1-x^2-y^2}+1$$

$$V = \iint \left(\sqrt{1-x^2-y^2}+1\right) - \left(x^2+y^2\right)$$

$$V = \iint \left(\sqrt{1-r^2}+1\right) - \left(r^2\right) r\,dr\,d\theta$$

$$z = 1-(z-1)^2 \rightarrow z = 1-(z^2-2z+1) \rightarrow z^2 - z = 0$$

$$z(z-1) = 0 \quad z=0 \lor z=1$$

$$V = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,r + r - r^3\,dr\,d\theta$$

(2)

$$V = \iint \sqrt{1-r^2}\,r\,dr\,d\theta + \iint r\,dr\,d\theta - \iint r^3\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\, r\, dr\, d\theta \qquad u^2 = 1-r^2$$
$$\cancel{2}u\,du = -\cancel{2}r\,dr$$

$$-\iint u\cancel{r}\,\frac{u\,du}{\cancel{r}}\,d\theta = -\iint u^2\,du\,d\theta = -\frac{u^3}{3} = -\left(\frac{1}{3}\left(\sqrt{1-r^2}\right)^3\right)\Big|_0^1$$

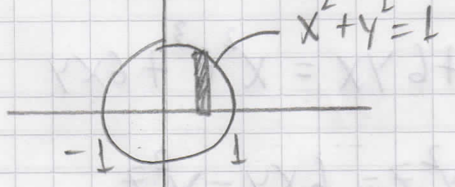
$$= \frac{1}{3}\int_0^{2\pi} d\theta = \frac{1}{3}\,\theta\Big|_0^{2\pi} = \boxed{\frac{1}{3}(2\pi)}$$

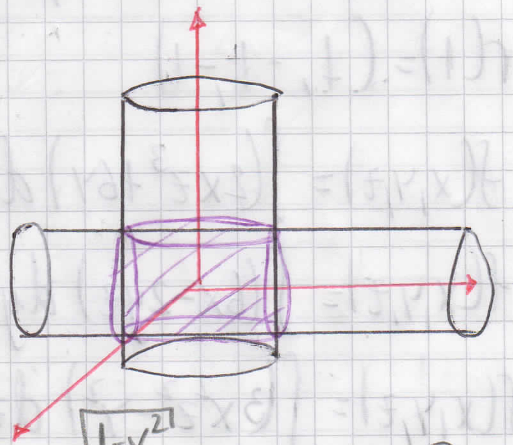
$$\int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \frac{r^2}{2}\Big|_0^1 = \frac{1}{2}\int_0^{2\pi} d\theta = \frac{1}{2}\,\theta\Big|_0^{2\pi} = \boxed{\pi}$$

$$-\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta \quad \frac{r^4}{4}\Big|_0^1 = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{1}{4}\,\theta\Big|_0^{2\pi} = \boxed{-\frac{\pi}{2}}$$

$$\frac{2\pi}{3} + \pi - \frac{\pi}{2} = \frac{7\pi}{6}$$

#5 Calcule el volúmen del Sólido intersección de $x^2+y^2=1$ y $y^2+z^2=1$





$V_{total} = 8V$

$$V = \int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx = \int_0^1 \sqrt{1-x^2}\; y\Big|_0^{\sqrt{1-x^2}} dx$$

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$$\int_0^1 \left(\sqrt{1-x^2}\right)^2 dx = \int_0^1 (1-x^2)\,dx = x - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$V_{total} = 3\left(\frac{2}{3}\right) = \frac{16}{3}$$

Ejercicios Propuestos 7.2

#3 Calcular $\int_C F \cdot dr$ siendo C la trayectoria

$r(t) = (t-1)^3+1,\ \cos^5(\pi t),\ -\cos^8(\pi t) > t \in [1,2]$

y $F(x,y,z) = 2xz^3+6y,\ 6x-2yz,\ 3x^2z^2-y^2$

$$\text{Rotacional} = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ (2xz^3+6y) & (6x-2yz) & (3x^2z^2-y^2) \end{vmatrix}$$

$i\,(-2y)-(-2y)=0 \quad -j(6xz^2-6xz^2)=0 \quad k(6-6)=0$

$\therefore$ Campo Conservativo

$r(1) = (1,-1,-1) \qquad r(2) = (2,1,-1)$

$f(x,y,z) = \int (2xz^3+6y)\,dx = \frac{2x^2}{2}z^3 + 6yx = x^2z^3+6xy$

$f(x,y,z) = \int (6x-2yz)\,dy = 6xy - \frac{2y^2}{2}z = 6xy - y^2z$

$f(x,y,z) = \int (3x^2z^2-y^2)\,dz = \frac{3x^2z^3}{3} - y^2z = x^2z^3 - y^2z$

$$\boxed{f(x,y,z) = x^2z^3+6xy-y^2z}$$

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$f(2,1,-1) = 9 \qquad f(1,-1,-1) = -6$

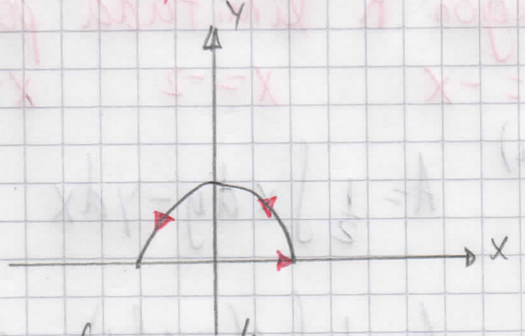
$$f_{final} - f_{inicial} = 9+6 = 15$$

Ejercicios Propuestos 7.3

#6 Una partícula empieza en el punto $(-2,0)$ se mueve a lo largo del eje x hacia $(2,0)$ y luego a lo largo de la semicircunferencia $y=\sqrt{4-x^2}$ hacia el punto inicial. Encontrar el trabajo sobre esta partícula por el campo de fuerzas $F(x,y) = (x, x^3+3xy^2)$

$y^2 = 4-x^2$

$x^2+y^2 = 4$



Probamos si es o no Conservativo:

$$rot = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ x & x^3+3xy^2 & 0 \end{vmatrix}$$

$i(-(x^3+3xy^2)dz) = 0$

$-j(-x\,dz) = 0$

$k((x^3+3xy^2)dx - x\,dy) = 3x^2+3y^2$

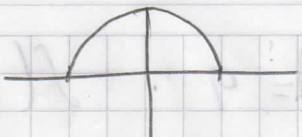
$(0, 0, 3x^2+3y^2)$ Campo NO Conservativo

Teorema de Green:

$$W = \iint (N_x - M_y)\,dA = \iint [(x^3+3xy^2)dx - x\,dy]\,dA$$

$$\iint (3x^2+3y^2)\,dA = 3\iint x^2+y^2\,dA$$

Hacemos Cambio de Variable a Polares.



$$= 3\int_0^{\pi}\int_0^{2} r^2\, r\, dr\, d\theta$$

$$x^2+y^2=4$$

$$r^2(\cos\theta + \operatorname{sen}\theta) = 4$$

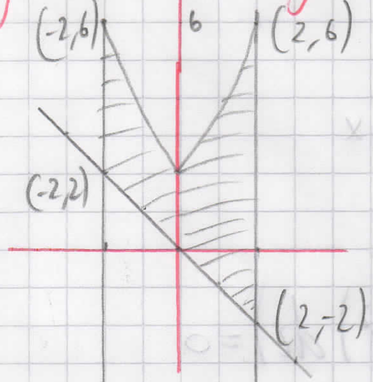
$$r^2=4 \rightarrow r=2$$

$$= 3\int_0^{\pi}\int_0^{2} r^3\, dr\, d\theta = 3\int_0^{\pi} \frac{r^4}{4}\Big|_0^2 d\theta$$

$$= \frac{3}{4}(16)\int_0^{\pi} d\theta = 12\,\theta\Big|_0^{\pi} = 12\pi$$

#9 Empleando una integral de línea encuentre el área de la región R limitada por las gráficas

$y = x^2+2$ $\quad y=-x \quad x=-2 \quad x=2$



$$A = \frac{1}{2}\int x\,dy - y\,dx \qquad \underline{y=x^2+2} \quad \underline{dy=2x\,dx}$$

$$A = \frac{1}{2}\int x(2x\,dx) - x^2 - 2 = \frac{1}{2}\int_2^{-2}(x^2-2)\,dx$$

$$A = \frac{1}{2}\left[\frac{x^3}{3} - 2x\right]_2^{-2} = \boxed{\frac{4}{3}}$$

$$A = \frac{1}{2}\int x\,dy - y\,\cancel{dx}^{\,0} \qquad \underline{x=-2} \quad \underline{dx=0}$$

$$A = \frac{1}{2}\int_6^2 -2\,dy = -\int_6^2 dy = -y\Big|_6^2 = -(2-6) = \boxed{4}$$

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$$A = \frac{1}{2}\int x\,dy - y\,dx \qquad \underline{y=-x} \quad \underline{dy=-dx}$$

$$A = \frac{1}{2}\int x(-dx) - (-x)\,dx = \frac{1}{2}\int -x\,dx + x\,dx = \frac{1}{2}\int_{-2}^{2} dx$$

$$\frac{1}{2}\left[x\Big|_{-2}^{2}\right] = \boxed{2}$$

$$A = \frac{1}{2}\int x\,dy - \cancel{y\,dx}^{\,0}$$

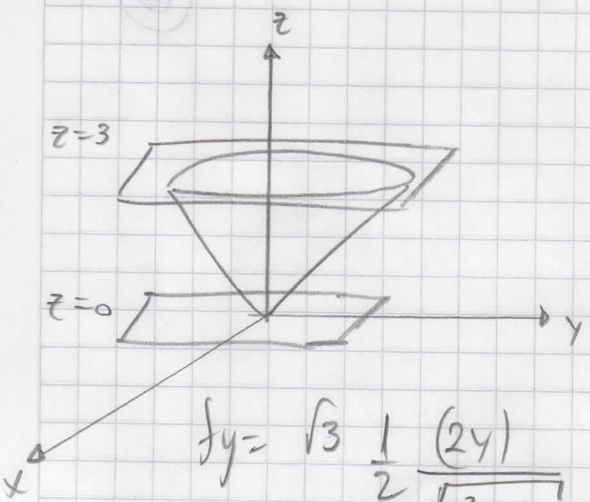
$x = 2$ $dx = 0$

$$A = \frac{1}{2}\int 2\,dy = \int_{-2}^{6} dy = y\Big|_{-2}^{6} = \boxed{8}$$

Sumando: $\frac{4}{3} + 4 + 2 + 8 = \frac{4}{3} + 14 = \frac{4+42}{3} = \frac{46}{3}$

Ejercicios Propuestos 7.4

#1 Evaluar $\iint_S (x^2+y^2)\,ds$, siendo S la superficie del cono $z^2 = 3(x^2+y^2)$ entre $z=0$ y $z=3$.



$$\iint_S (x^2+y^2)\sqrt{1+f_x^2+f_y^2}\;dy\,dx$$

$$f_x = \sqrt{3}\;\frac{1}{2}\frac{(2x)}{\sqrt{x^2+y^2}} = \left(\frac{\sqrt{3}\,x}{\sqrt{x^2+y^2}}\right)^2 = \frac{3x^2}{x^2+y^2}$$

$$f_y = \sqrt{3}\;\frac{1}{2}\frac{(2y)}{\sqrt{x^2+y^2}} = \left(\frac{\sqrt{3}\,y}{\sqrt{x^2+y^2}}\right)^2 = \frac{3y^2}{x^2+y^2}$$

$$\iint_S x^2+y^2\sqrt{1+\frac{3x^2}{x^2+y^2}+\frac{3y^2}{x^2+y^2}}\;dy\,dx$$

$$\iint_S x^2+y^2\sqrt{\frac{x^2+y^2+3x^2+3y^2}{x^2+y^2}}\;dy\,dx$$

EDUCA

$$\iint_S (x^2+y^2)\sqrt{\frac{4x^2+4y^2}{x^2+y^2}}\,dy\,dx = 2\iint x^2+y^2\,\frac{\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}}$$

$$= 2\iint_S x^2+y^2\,dy\,dx = 2\iint r^2\,r\,dr\,d\theta = 2\int_0^{2\pi}\int_0^{\sqrt{3}} r^3\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\frac{r^4}{4}\Big|_0^{\sqrt{3}}\,d\theta = 2\int_0^{2\pi}\frac{(\sqrt{3})^4}{4}\,d\theta = \frac{1}{2}(\sqrt{3})^4\,\theta\Big|_0^{2\pi} = \frac{1}{2}(\sqrt{3})^4(2\pi)$$

$z=3 \quad (3)^2 = 3(x^2+y^2)$

$x^2+y^2=3$

$$= 9\pi \;//$$

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