ESPOL

ECUACIONES DIFERENCIALES (1ER PARCIAL)

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[ERICK CONDE]

ECUACIONES DIFERENCIALES SEPARABLES

1) $\boldsymbol{dy(xy-2x+4y-8)-dx(xy+3x-y-3)=0}$

$dy[x(y-2)+4(y-2)]-dx[x(y+3)-(y+3)]=0$

$dy(y-2)(x+4)-dx(y+3)(x-1)=0$

$dy(y-2)(x+4)=dx(y+3)(x-1)$

$dy\frac{(y-2)}{(y+3)}=dx\frac{(x-1)}{(x+4)}$

$\int\frac{y-2}{y+3}dy=\int\frac{x-1}{x+4}dx$

$\int\frac{y-2+5-5}{y+3}dy=\int\frac{x-1+5-5}{x+4}dx$

$\int\left(\frac{y+3}{y+3}-\frac{5}{y+3}\right)dy=\int\left(\frac{x+4}{x+4}-\frac{5}{x+4}\right)dx$

$\boxed{\boldsymbol{y-5\,ln|y+3|=x-5\,ln|x+4|+c}}$

2) $\boldsymbol{\frac{dy}{dx}=y-x-1+(x-y+2)^{-1}}$

$\frac{dy}{dx}=-(x-y)-1+\frac{1}{(x-y)+2}$

$t=x-y \quad \Rightarrow \quad \frac{dt}{dx}=1-\frac{dy}{dx}$

$1-\frac{dt}{dx}=-t-1+\frac{1}{t+2} \quad \Rightarrow \quad \frac{dt}{dx}=t+2-\frac{1}{t+2} \quad \Rightarrow \quad \frac{dt}{dx}=\frac{(t+2)^2-1}{t+2}$

$\frac{dt}{dx}=\frac{t^2+4t+4-1}{t+2} \quad \Rightarrow \quad \frac{dt}{dx}=\frac{t^2+4t+3}{t+2} \quad \Rightarrow \quad \frac{dt}{dx}=\frac{(t+3)(t+1)}{(t+2)} \quad \Rightarrow \quad \int\frac{(t+2)}{(t+3)(t+1)}dt=\int dx$

$\frac{(t+2)}{(t+3)(t+1)}=\frac{A}{(t+3)}+\frac{B}{(t+1)}$

$t+2=A(t+1)+B(t+3) \quad \Rightarrow \quad t+2=(A+B)t+(A+3B)$

$1=A+B$

$2=A+3B$

Resolviendo el sistema A = ½, B = ½

$\int\left(\frac{A}{t+3}+\frac{B}{t+1}\right)dt=\int dx \quad \Rightarrow \quad A\,ln|t+3|+B\,ln|t+1|=x+c$

$\boxed{\boldsymbol{\frac{1}{2}ln|(x-y)+3|+\frac{1}{2}ln|(x-y)+1|=x+c}}$

3) $\boldsymbol{\frac{dy}{dx}+x=\frac{x}{y}}$

$$\frac{dy}{dx}=\frac{x}{y}-x \quad \Rightarrow \quad \frac{dy}{dx}=\frac{x-xy}{y}$$

$$\frac{dy}{dx}=\frac{x(1-y)}{y}$$

$$\frac{y}{(1-y)}dy=x\,dx$$

$$\int\frac{y}{(1-y)}dy=\int x\,dx \quad \Rightarrow \quad \int\left(-1+\frac{1}{1-y}\right)dy=\int x\,dx$$

$$\boxed{-y-ln|1-y|=\frac{x^2}{2}+c}$$

4) $\boldsymbol{(x+y)(y'+1)=y'}$

$$t=(x+y) \quad \Rightarrow \quad \frac{dt}{dx}=1+\frac{dy}{dx}$$

$$t\left(\frac{dt}{dx}-1+1\right)=\frac{dt}{dx}-1$$

$$t\frac{dt}{dx}=\frac{dt}{dx}-1 \quad \Rightarrow \quad t\frac{dt}{dx}-\frac{dt}{dx}=-1$$

$$\frac{dt}{dx}(t-1)=-1 \quad \Rightarrow \quad dt(t-1)=-dx$$

$$\int(t-1)dt=\int -dx \quad \Rightarrow \quad \frac{t^2}{2}-t=-x+c$$

$$\boxed{\frac{(x+y)^2}{2}-(x+y)=-x+c}$$

5) $\boldsymbol{\frac{dy}{dx}=\frac{2x+3y-1}{4x+6y+3}}$

$$\frac{dy}{dx}=\frac{(2x+3y)-1}{2(2x+3y)+3}$$

$$t=2x+3y \quad \Rightarrow \quad \frac{dt}{dx}=2+3\frac{dy}{dx}$$

$$\frac{1}{3}\left(\frac{dt}{dx}-2\right)=\frac{t-1}{2t+3}$$

$$\frac{dt}{dx}-2=\frac{3(t-1)}{2t+3}$$

$$\frac{dt}{dx}=\frac{3t-3+4t+6}{2t+3} \quad \Rightarrow \quad \frac{dt}{dx}=\frac{7t+3}{2t+3}$$

$$\frac{2t+3}{7t+3}dt = dx \quad \Rightarrow \quad \int \frac{2t+3}{7t+3}dt = \int dx$$

$$2\int \frac{t}{7t+3}dt + 3\int \frac{dt}{7t+3} = \int dx$$

$$u = 7t+3 \quad \Rightarrow \quad t = \frac{u-3}{7}$$

$$du = 7dt \;\Rightarrow dt = \frac{du}{7}$$

$$2\int \frac{(u-3)}{49u}du + 3\int \frac{du}{7u} = \int dx$$

$$\frac{2}{49}[u - 3ln|u|] + \frac{3}{7}ln|u| = x + c$$

$$\frac{2}{49}[7t+3-3ln|7t+3|] + \frac{3}{7}ln|7t+3| = x + c$$

$$\boxed{\boldsymbol{\frac{2}{49}[7(2x+3y)+3-3ln|7(2x+3y)+3|] + \frac{3}{7}ln|7(2x+3y)+3| = x + c}}$$

6) $\boldsymbol{\frac{dy}{dx} = \frac{y^3}{1-2xy^2} \quad ; \quad y(0) = 1}$

Para este ejercicio lo mejor es "invertir la ecuación"

$$\frac{dx}{dy} = \frac{1-2xy^2}{y^3} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{y^3} - \frac{2x}{y} \quad \Rightarrow \quad \frac{dx}{dy} + \frac{2x}{y} = \frac{1}{y^3} \;\; ; \;\; esta\ ecuación\ tiene\ la\ forma\ x' + p(y)x = g(y)$$

$$\mu(y) = e^{\int p(y)dy} \quad \Rightarrow \quad \mu(y) = e^{\int \frac{2}{y}dy} \quad \Rightarrow \quad \mu(y) = e^{2\,ln|y|}$$

$$\mu(y) = y^2$$

$$y^2\frac{dx}{dy} + 2xy = \frac{1}{y}$$

$$\frac{d}{dy}(xy^2) = \frac{1}{y} \quad \Rightarrow \quad d(xy^2) = \frac{1}{y}dy$$

$$\int d(xy^2) = \int \frac{1}{y}dy \quad \Rightarrow \quad xy^2 = ln|y| + c \quad \Rightarrow \quad (0)(1)^2 = ln|1| + c \quad \Rightarrow \quad c = 0$$

$$\boxed{\boldsymbol{x = \frac{1}{y^2}ln|y|}}$$

MÉTODO DEL FACTOR INTEGRANTE

1) $\left(y - Sen^2 x \right) dx + Sen\ x\ dy = 0$

$Sen\ x\ dy = (y - Sen^2 x)dx$

$\frac{dy}{dx} = \frac{Sen^2 x - y}{Sen\ x}$

$\frac{dy}{dx} + \frac{y}{Sen\ x} = Sen\ x$

$\mu(x) = e^{\int p(x)dx} \quad \Rightarrow \quad \mu(x) = e^{\int \frac{1}{Sen\ x}dx} \quad \Rightarrow \quad \mu(x) = e^{ln|Csc\ x - Cot\ x\ |}$

$\mu(x) = Csc\ x - Cot\ x$

$(Csc\ x - Cot\ x)\ y' + \frac{Csc\ x - Cot\ x}{Sen\ x}\ y = (Csc\ x - Cot\ x)Sen\ x$

$\frac{d}{dx}[(Csc\ x - Cot\ x)\ y] = (Csc\ x - Cot\ x)Sen\ x$

$\frac{d}{dx}[(Csc\ x - Cot\ x)\ y] = \left(\frac{1}{Sen\ x} - \frac{Cos\ x}{Sen\ x}\right) Sen\ x$

$d\ [(Csc\ x - Cot\ x)\ y] = \left(\frac{1 - Cos\ x}{Sen\ x}\right) Sen\ x\ dx$

$\int d\ [(Csc\ x - Cot\ x)\ y] = \int (1 - Cos\ x)dx$

$(Csc\ x - Cot\ x)\ y = x - Sen\ x + c$

$$\boxed{\boldsymbol{y = \frac{x - Sen\ x + c}{Csc\ x - Cot\ x}}}$$

2) $y' + \frac{y}{x} = 3\ Cos(2x)$

$\mu(x) = e^{\int p(x)dx} \quad \Rightarrow \quad \mu(x) = e^{\int \frac{1}{x}dx} \quad \Rightarrow \quad \mu(x) = e^{ln|x|}$

$\mu(x) = x$

$y'x + y = 3x\ Cos(2x)$

$\frac{d}{dx}(xy) = 3x\ Cos(2x) \quad \Rightarrow \quad d(xy) = 3x\ Cos(2x)\ dx$

$\int d(xy) = \int 3x\ Cos(2x)\ dx$

$u = x \quad \Rightarrow \quad du = dx$

$dv = Cos(2x)dx \quad \Rightarrow \quad v = \frac{Sen(2x)}{2}$

$xy = \frac{3}{2}x\ Sen(2x) - \frac{3}{2}\int Sen(2x)dx \quad \Rightarrow \quad \boxed{\boldsymbol{xy = \frac{3}{2}x\ Sen(2x) + \frac{3}{4}Cos(2x) + c}}$

3) $\boldsymbol{(x^2+1)\frac{dy}{dx}+3x^3y=6xe^{-\frac{3}{2}x^2}}$

$$\frac{dy}{dx}+\frac{3x^3y}{x^2+1}=\frac{6xe^{-\frac{3}{2}x^2}}{x^2+1}$$

$$\mu(x)=e^{\int p(x)dx} \quad \Rightarrow \quad \mu(x)=e^{\int \frac{3x^3}{x^2+1}dx} \quad \Rightarrow \quad \mu(x)=e^{\int 3\left(x-\frac{x}{x^2+1}\right)dx}$$

$$\mu(x)=e^{3\left(\frac{x^2}{2}-\frac{1}{2}ln|x^2+1|\right)} \quad \Rightarrow \quad \mu(x)=e^{3x^2/2}(x^2+1)^{-3/2}$$

$$\left[\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}\right]\frac{dy}{dx}+\left[\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}\right]\frac{3x^3y}{x^2+1}=\left[\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}\right]\frac{6xe^{-\frac{3}{2}x^2}}{x^2+1}$$

$$\frac{d}{dx}\left[y\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}\right]=\frac{6x}{(x^2+1)^{5/2}} \quad \Rightarrow \quad d\left[y\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}\right]=\frac{6x}{(x^2+1)^{5/2}}\,dx$$

$$\int d\left[y\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}\right]=\int\frac{6x}{(x^2+1)^{5/2}}\,dx$$

$$u=x^2+1 \quad \Rightarrow \quad du=2x\,dx$$

$$y\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}=\int\frac{3}{u^{5/2}}du \quad \Rightarrow \quad \boxed{\boldsymbol{y\frac{e^{3x^2/2}}{(x^2+1)^{3/2}}=-\frac{2}{(x^2+1)^{3/2}}+c}}$$

4) $\boldsymbol{\frac{dy}{dx}=Sen(x+y)}$

$$t=x+y \quad \Rightarrow \quad \frac{dt}{dx}=1+\frac{dy}{dx}$$

$$\frac{dt}{dx}-1=Sen(t) \quad \Rightarrow \quad \frac{dt}{dx}=Sen(t)+1 \quad \Rightarrow \quad \frac{dt}{Sen(t)+1}=dx$$

$$\int\frac{dt}{Sen(t)+1}=\int dx$$

$$\int\frac{\frac{2}{1+m^2}dm}{\frac{2m}{1+m^2}+1}=\int dx \quad \Rightarrow \quad \int\frac{2}{m^2+2m+1}dm=\int dx$$

$$\int\frac{2}{(m+1)^2}dm=\int dx \quad \Rightarrow \quad -\frac{2}{m+1}=x+c \quad \Rightarrow \quad -\frac{2}{Tan\left(\frac{t}{2}\right)+1}=x+c$$

$$\boxed{\boldsymbol{-\frac{2}{Tan\left(\frac{x+y}{2}\right)+1}=x+c}}$$

5) $\boldsymbol{y' + 2y = \begin{cases} 1 \;;\; 0 \le x \le 1 \\ 0 \;;\; x > 1 \end{cases} \quad ; \quad g(0) = 0}$

<u>$0 \le x \le 1$</u>

$y' + 2y = 1$

$\mu(x) = e^{\int p(x)dx} \quad \Rightarrow \quad \mu(x) = e^{\int 2dx} \quad \Rightarrow \quad \mu(x) = e^{2x}$

$y' e^{2x} + 2\, e^{2x} y = e^{2x}$

$\frac{d}{dx}(ye^{2x}) = e^{2x} \quad \Rightarrow \quad d(ye^{2x}) = e^{2x}\, dx$

$\int d(ye^{2x}) = \int e^{2x}\, dx \quad \Rightarrow \quad ye^{2x} = \frac{e^{2x}}{2} + c \quad \Rightarrow \quad (0)e^{2(0)} = \frac{e^{2(0)}}{2} + c \quad \Rightarrow \quad c = -\frac{1}{2}$

$y = \frac{1}{2} - \frac{1}{2}e^{-2x}$

<u>$x > 1$</u>

$y' + 2y = 0$

$y' e^{2x} + 2\, e^{2x} y = 0$

$\frac{d}{dx}(ye^{2x}) = 0 \quad \Rightarrow \quad d(ye^{2x}) = 0$

$\int d(ye^{2x}) = \int 0\, dx \quad \Rightarrow \quad ye^{2x} = c$

$y = ce^{-2x}$

$y(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2x} \;;\; 0 \le x \le 1 \\ ce^{-2x} \;;\; x > 1 \end{cases}$

$\lim_{x \to 1^+} ce^{-2x} = \lim_{x \to 1^-} \frac{1}{2} - \frac{1}{2}e^{-2x}$

$ce^{-2} = \frac{1}{2} - \frac{1}{2}e^{-2} \quad \Rightarrow \quad c = \frac{1}{2}e^{2} - \frac{1}{2}$

$$\boxed{\boldsymbol{y(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2x} \;;\; 0 \le x \le 1 \\ \left(\frac{1}{2}e^{2} - \frac{1}{2}\right)e^{-2x} \;;\; x > 1 \end{cases}}}$$

ECUACIONES DIFERENCIALES DE BERNOULLI

1) $2y^2dx + (3xy - x^2)dy = 0$

$(3xy - x^2)dy = -2y^2dx$

$$\frac{dy}{dx} = \frac{-2y^2}{3xy - x^2} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{3xy - x^2}{-2y^2}$$

$$\frac{dx}{dy} = \frac{x^2}{2y^2} - \frac{3x}{2y} \quad \Rightarrow \quad \frac{dx}{dy} + \frac{3}{2y}x = \frac{1}{2y^2}x^2$$

$$\frac{1}{x^2}\frac{dx}{dy} + \frac{3}{2y}\frac{x}{x^2} = \frac{1}{2y^2} \quad \Rightarrow \quad \frac{1}{x^2}\frac{dx}{dy} + \frac{3}{2y}x^{-1} = \frac{1}{2y^2}$$

$$t = x^{-1} \quad \Rightarrow \quad \frac{dt}{dy} = -\frac{1}{x^2}\frac{dx}{dy} \quad \Rightarrow \quad \frac{1}{x^2}\frac{dx}{dy} = -\frac{dt}{dy}$$

$$\frac{dt}{dy} - \frac{3}{2y}t = -\frac{1}{2y^2}$$

$$\mu(y) = e^{\int p(y)dy} \quad \Rightarrow \quad \mu(y) = e^{\int -\frac{3}{2y}dy} \quad \Rightarrow \quad \mu(y) = e^{-\frac{3}{2}ln|y|}$$

$$\mu(y) = y^{-\frac{3}{2}}$$

$$\left(y^{-\frac{3}{2}}\right)\frac{dt}{dy} - \left(y^{-\frac{3}{2}}\right)\frac{3}{2y}t = -\left(y^{-\frac{3}{2}}\right)\frac{1}{2y^2}$$

$$\frac{d}{dy}\left(y^{-\frac{3}{2}}t\right) = -\frac{1}{2}y^{-\frac{7}{2}}$$

$$\int d\left(y^{-\frac{3}{2}}t\right) = \int -\frac{1}{2}y^{-\frac{7}{2}}\,dy$$

$$y^{-\frac{3}{2}}t = -\frac{1}{2}\left(-\frac{2}{5}\right)y^{-\frac{5}{2}} + c$$

$$y^{-\frac{3}{2}}x^{-1} = -\frac{1}{2}\left(-\frac{2}{5}\right)y^{-\frac{5}{2}} + c$$

$$\boxed{\boldsymbol{x = \frac{y^{-\frac{3}{2}}}{\frac{1}{5}y^{-\frac{5}{2}} + c}}}$$

2) $\boldsymbol{y' + \frac{1}{x}y = x^3y^3}$

$\frac{1}{y^3}y' + \frac{1}{x}\frac{y}{y^3} = x^3 \quad \Rightarrow \quad \frac{1}{y^3}y' + \frac{1}{x}y^{-2} = x^3$

$t = y^{-2} \quad \Rightarrow \quad \frac{dt}{dx} = -2\frac{1}{y^3}\frac{dy}{dx} \quad \Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dt}{dx}$

$-\frac{1}{2}\frac{dt}{dx} + \frac{1}{x}t = x^3 \quad \Rightarrow \quad \frac{dt}{dx} - \frac{2}{x}t = -2x^3$

$\mu(x) = e^{\int p(x)dx} \quad \Rightarrow \quad \mu(x) = e^{\int -\frac{2}{x}dx} \quad \Rightarrow \quad \mu(x) = e^{-2\,ln|x|}$

$\mu(x) = x^{-2}$

$(x^{-2})\frac{dt}{dx} - (x^{-2})\frac{2}{x}t = -2x^3(x^{-2}) \quad \Rightarrow \quad \frac{d}{dx}(x^{-2}t) = -2x$

$\int d\,(x^{-2}t) = \int -2x\,dx \quad \Rightarrow \quad x^{-2}t = -x^{-2} + c$

$$\boxed{\boldsymbol{y = \pm\sqrt{\frac{1}{x^2\left(c - \frac{1}{x^2}\right)}}}}$$

3) $\boldsymbol{\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3}$

$\frac{1}{y^3}\frac{dy}{dx} - 5\frac{y}{y^3} = -\frac{5}{2}x \quad \Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} - \frac{5}{y^2} = -\frac{5}{2}x$

$t = y^{-2} \quad \Rightarrow \quad \frac{dt}{dx} = -2\frac{1}{y^3}\frac{dy}{dx} \quad \Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dt}{dx}$

$-\frac{1}{2}\frac{dt}{dx} - 5t = -\frac{5}{2}x \quad \Rightarrow \quad \frac{dt}{dx} + 10t = 5x$

$\mu(x) = e^{\int p(x)dx} \quad \Rightarrow \quad \mu(x) = e^{\int 10dx} \quad \Rightarrow \quad \mu(x) = e^{10x}$

$(e^{10x})\frac{dt}{dx} + (e^{10x})10t = 5xe^{10x} \quad \Rightarrow \quad \frac{d}{dx}(te^{10x}) = 5xe^{10x}$

$\int d\,(te^{10x}) = \int 5xe^{10x}\,dx$

$u = x \quad \Rightarrow \quad du = dx$

$dv = e^{10x}dx \quad \Rightarrow \quad v = \frac{e^{10x}}{10}$

$te^{10x} = 5\left(\frac{xe^{10x}}{10} - \frac{1}{10}\int e^{10x}dx\right) \quad \Rightarrow \quad te^{10x} = \frac{xe^{10x}}{2} - \frac{e^{10x}}{20} + c \quad \Rightarrow \quad \boxed{\boldsymbol{y^{-2}e^{10x} = \frac{xe^{10x}}{2} - \frac{e^{10x}}{20} + c}}$

4) $\boldsymbol{x\,dy - [y + xy^3(1 + lnx)]dx = 0}$

$x\,dy = [y + xy^3(1 + lnx)]dx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + y^3(1 + lnx)$

$\frac{dy}{dx} - \frac{y}{x} = y^3(1 + lnx) \quad \Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} - \frac{1}{xy^2} = (1 + lnx)$

$t = y^{-2} \quad \Rightarrow \quad \frac{dt}{dx} = -2\frac{1}{y^3}\frac{dy}{dx} \quad \Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dt}{dx}$

$-\frac{1}{2}\frac{dt}{dx} - \frac{t}{x} = (1 + lnx) \quad \Rightarrow \quad \frac{dt}{dx} + \frac{2t}{x} = -2(1 + lnx)$

$\mu(x) = e^{\int p(x)dx} \quad \Rightarrow \quad \mu(x) = e^{\int \frac{2}{x}dx} \quad \Rightarrow \quad \mu(x) = e^{2\,ln|x|}$

$\mu(x) = x^2$

$(x^2)\frac{dt}{dx} + (x^2)\frac{2t}{x} = -2(x^2)(1 + lnx) \quad \Rightarrow \quad \frac{d}{dx}(x^2t) = -2x^2(1 + lnx)$

$\int d\,(x^2t) = -2\int (x^2 + x^2lnx)dx$

$u = lnx \quad \Rightarrow \quad du = \frac{dx}{x}$

$dv = x^2dx \quad \Rightarrow \quad v = \frac{x^3}{3}$

$x^2t = -2\left[\frac{x^3}{3} + \left(\frac{x^3}{3}lnx - \frac{1}{3}\int x^2dx\right)\right] \quad \Rightarrow \quad x^2t = -\frac{2}{3}x^3 - \frac{2}{3}x^3lnx + \frac{1}{9}x^3 + c$

$$\boxed{\boldsymbol{x^2y^{-2} = \frac{1}{9}x^3 - \frac{2}{3}x^3 - \frac{2}{3}x^3lnx + c}}$$

ECUACIONES DIFERENCIALES EXACTAS

1) $(4x^2 - 2xy)dx + (5y - x^2)dy = 0$

$$\frac{\partial M}{\partial y} = -2x \qquad \frac{\partial N}{\partial x} = -2x$$

$$\therefore Es\ exacta$$

$$\frac{\partial \mathbb{Q}}{\partial x} = 4x^2 - 2xy \quad ; \quad \frac{\partial \mathbb{Q}}{\partial y} = 5y - x^2$$

$$\int \partial \mathbb{Q} = \int (4x^2 - 2xy)dx$$

$$\mathbb{Q} = \frac{4}{3}x^3 - x^2y + h(y)$$

$$\frac{d}{dy}\left[\frac{4}{3}x^3 - x^2y + h(y)\right] = 5y - x^2 \quad \Rightarrow \quad -x^2 + h'(y) = 5y - x^2$$

$$h'(y) = 5y \quad \Rightarrow \quad h(y) = \int 5y\, dy$$

$$h(y) = \frac{5}{2}y^2 + c$$

$$\boxed{\frac{4}{3}x^3 - x^2y + \frac{5}{2}y^2 + c = 0}$$

2) $2(x-1)y^3dx + \left[3(x-1)^2y^2 + 2y\right]dy = 0$

$$\frac{\partial M}{\partial y} = 3y^2[2(x-1)] \qquad \frac{\partial N}{\partial x} = 3y^2[2(x-1)]$$

$$\therefore Es\ exacta$$

$$\frac{\partial \mathbb{Q}}{\partial x} = 2(x-1)y^3 \quad ; \quad \frac{\partial \mathbb{Q}}{\partial y} = 3(x-1)^2y^2 + 2y$$

$$\int \partial \mathbb{Q} = \int [3(x-1)^2y^2 + 2y]dy$$

$$\mathbb{Q} = (x-1)^2y^3 + y^2 + h(x)$$

$$\frac{d}{dx}[(x-1)^2y^3 + y^2 + h(x)] = 2(x-1)y^3 \quad \Rightarrow \quad 2(x-1)y^3 + h'(x) = 2(x-1)y^3$$

$$h'(x) = 0 \quad \Rightarrow \quad h(x) = c$$

$$\boxed{(x-1)^2y^3 + y^2 + c = 0}$$

3) $(\mathbf{2x + ye^{xy}})\mathbf{dx} + \mathbf{xe^{xy}dy = 0}$

$$\frac{\partial M}{\partial y} = yxe^{xy} \qquad \frac{\partial N}{\partial x} = xye^{xy}$$

$$\therefore Es\ exacta$$

$\frac{\partial \mathbb{Q}}{\partial x} = 2x + ye^{xy}$; $\frac{\partial \mathbb{Q}}{\partial y} = xe^{xy}$

$\int \partial \mathbb{Q} = \int xe^{xy}\,dy$

$u = xy \Rightarrow \frac{du}{dy} = x \Rightarrow du = x\,dy$

$\mathbb{Q} = \int e^u\,du$

$\mathbb{Q} = e^{xy} + h(x)$

$\frac{d}{dx}[e^{xy} + h(x)] = 2x + ye^{xy} \Rightarrow ye^{xy} + h'(x) = 2x + ye^{xy}$

$h'(x) = 2x \Rightarrow h(x) = \int 2x\ dx$

$h(x) = x^2 + c$

$\boxed{\mathbf{e^{xy} + x^2 + c = 0}}$

4) $\left[\mathbf{4x^3y - \frac{e^{xy}}{x} + y\,ln(x) + x\sqrt[3]{x-4}}\right]\mathbf{dx} + \left[\mathbf{x^4 - \frac{e^{xy}}{y} + x\,ln(x) - x}\right]\mathbf{dy = 0}$

$$\frac{\partial M}{\partial y} = 4x^3 - \frac{1}{x}e^{xy}x + ln(x) \qquad \frac{\partial N}{\partial x} = 4x^3 - \frac{1}{y}e^{xy}y + l\,n(x) + 1 - 1$$

$$\frac{\partial M}{\partial y} = 4x^3 - e^{xy} + ln(x) \qquad \frac{\partial N}{\partial x} = 4x^3 - e^{xy} + ln(x)$$

$$\therefore Es\ exacta$$

$\frac{\partial \mathbb{Q}}{\partial x} = 4x^3y - \frac{e^{xy}}{x} + y\,ln(x) + x\sqrt[3]{x-4}$; $\frac{\partial \mathbb{Q}}{\partial y} = x^4 - \frac{e^{xy}}{y} + x\,ln(x) - x$

$\int \partial \mathbb{Q} = \int \left[x^4 - \frac{e^{xy}}{y} + x\,ln(x) - x\right]dy$

$\mathbb{Q} = x^4y - \int \frac{e^{xy}}{y}dy + xy\,l\,n(x) - xy$

$$Para\ resolver \int \frac{e^{xy}}{y} dy\ es\ necesario\ utilizar\ series$$

$$\frac{e^{xy}}{y} = \frac{1}{y}\sum_{n=0}^{+\infty} \frac{(xy)^n}{n!}$$

$$\frac{e^{xy}}{y} = \sum_{n=0}^{+\infty} \frac{(x)^n (y)^{n-1}}{n!}$$

$$\int \frac{e^{xy}}{y} dy = \int \left[\frac{1}{y} + \sum_{n=1}^{+\infty} \frac{(x)^n (y)^{n-1}}{n!}\right] dy$$

$$\int \frac{e^{xy}}{y} dy = ln|y| + \sum_{n=1}^{+\infty} \frac{(x)^n (y)^n}{(n)n!}$$

$$\omega = x^4 y - ln|y| - \sum_{n=1}^{+\infty} \frac{(x)^n (y)^n}{(n)n!} + xy\, l\,n(x) - xy + h(x)$$

$$\frac{d}{dx}\left[x^4 y - ln|y| - \sum_{n=1}^{+\infty} \frac{(x)^n (y)^n}{(n)n!} + xy\, l\,n(x) - xy + h(x)\right] = 4x^3 y - \frac{e^{xy}}{x} + y\, ln(x) + x\sqrt[3]{x-4}$$

$$4x^3 y - \sum_{n=1}^{+\infty} \frac{n(x)^{n-1} (y)^n}{(n)n!} + y\,[l\,n(x) + 1] - y + h'(x) = 4x^3 y - \frac{e^{xy}}{x} + y\, ln(x) + x\sqrt[3]{x-4}$$

$$4x^3 y - \sum_{n=1}^{+\infty} \frac{(x)^{n-1} (y)^n}{n!} + y\, l\,n(x) + y - y + h'(x) = 4x^3 y - \frac{e^{xy}}{x} + y\, ln(x) + x\sqrt[3]{x-4}$$

$$4x^3 y - \frac{e^{xy}}{x} + y\, l\,n(x) + h'(x) = 4x^3 y - \frac{e^{xy}}{x} + y\, ln(x) + x\sqrt[3]{x-4}$$

$$h'(x) = x\sqrt[3]{x-4} \quad \Rightarrow \quad h(x) = \int x\sqrt[3]{x-4}\; dx$$

$$u = x - 4 \implies x = u + 4$$

$$du = dx$$

$$h(x) = \int (u+4)u^{\frac{1}{3}}\; du \quad \Rightarrow \quad h(x) = \int \left(u^{\frac{4}{3}} + 4u^{\frac{1}{3}}\right) du$$

$$h(x) = \frac{3}{7}u^{\frac{7}{3}} + 3u^{\frac{4}{3}} + c \quad \Rightarrow \quad h(x) = \frac{3}{7}(x-4)^{\frac{7}{3}} + 3(x-4)^{\frac{4}{3}} + c$$

$$\boxed{\mathbf{x^4 y - ln|y| - \sum_{n=1}^{+\infty} \frac{(x)^n (y)^n}{(n)n!} + xy\, l\,n(x) - xy + \frac{3}{7}(x-4)^{\frac{7}{3}} + 3(x-4)^{\frac{4}{3}} + c = 0}}$$

ECUACIONES DIFERENCIALES EXACTAS CON FACTOR INTEGRANTE

1) $\mathbf{4xy^2 + \left(3x^2y - 1\right)y' = 0}$

$$4xy^2 + (3x^2y - 1)\frac{dy}{dx} = 0 \quad \Rightarrow \quad 4xy^2dx + (3x^2y - 1)dy = 0$$

$$\frac{\partial M}{\partial y} = 8xy \qquad \frac{\partial N}{\partial x} = 6xy$$

$$\therefore No\ es\ exacta$$

$$\mu(y) = e^{\int \frac{6xy - 8xy}{4xy^2}dy} = e^{\int -\frac{2xy}{4xy^2}dy} \quad \Rightarrow \quad \mu(y) = e^{\int -\frac{dy}{2y}}$$

$$\mu(y) = e^{-\frac{1}{2}ln|y|} \quad \Rightarrow \quad \mu(y) = e^{ln|y|^{-\frac{1}{2}}} \quad \Rightarrow \quad \mu(y) = y^{-\frac{1}{2}}$$

$$4xy^2\left(y^{-\frac{1}{2}}\right)dx + (3x^2y - 1)\left(y^{-\frac{1}{2}}\right)dy = 0$$

$$4xy^{\frac{3}{2}}dx + \left(3x^2y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 4x\left(\frac{3}{2}\right)y^{\frac{1}{2}} \qquad \frac{\partial N}{\partial x} = 6xy^{\frac{1}{2}}$$

$$\frac{\partial M}{\partial y} = 6xy^{\frac{1}{2}} \qquad \frac{\partial N}{\partial x} = 6xy^{\frac{1}{2}}$$

$$\therefore Es\ exacta$$

$$\frac{\partial \mathbb{Q}}{\partial x} = 4xy^{\frac{3}{2}} \quad ; \quad \frac{\partial \mathbb{Q}}{\partial y} = 3x^2y^{\frac{1}{2}} - y^{-\frac{1}{2}}$$

$$\int \partial \mathbb{Q} = \int 4xy^{\frac{3}{2}}\,dx$$

$$\mathbb{Q} = 2x^2y^{\frac{3}{2}} + h(y)$$

$$\frac{d}{dy}\left[2x^2y^{\frac{3}{2}} + h(y)\right] = 3x^2y^{\frac{1}{2}} - y^{-\frac{1}{2}} \quad \Rightarrow \quad 2x^2\left(\frac{3}{2}\right)y^{\frac{1}{2}} + h'(y) = 3x^2y^{\frac{1}{2}} - y^{-\frac{1}{2}}$$

$$3xy^{\frac{1}{2}} + h'(y) = 3x^2y^{\frac{1}{2}} - y^{-\frac{1}{2}}$$

$$h'(y) = -y^{-\frac{1}{2}} \quad \Rightarrow \quad h(y) = \int -y^{-\frac{1}{2}}\,dy$$

$$h(y) = -\frac{1}{2}y^{\frac{1}{2}} + c$$

$$\boxed{\mathbf{2x^2y^{\frac{3}{2}} - \frac{1}{2}y^{\frac{1}{2}} + c = 0}}$$

2) $[\boldsymbol{x}+\boldsymbol{Sen}(\boldsymbol{x})+\boldsymbol{Sen}(\boldsymbol{y})]\boldsymbol{dx}+\boldsymbol{Cos}(\boldsymbol{y})\boldsymbol{dy}=\mathbf{0}$

$$\frac{\partial M}{\partial y}=Cos(y) \qquad \frac{\partial N}{\partial x}=0$$

$$\therefore No\ es\ exacta$$

$$\mu(x)=e^{\int\frac{Cos(y)}{Cos(y)}dx}=e^{\int dx} \quad \Rightarrow \quad \mu(y)=e^{x}$$

$$[e^{x}x+e^{x}Sen(x)+e^{x}Sen(y)]dx+e^{x}Cos(y)dy=0$$

$$\frac{\partial M}{\partial y}=e^{x}Cos(y) \qquad \frac{\partial N}{\partial X}=e^{x}Cos(y)$$

$$\therefore Es\ exacta$$

$$\frac{\partial \mathbb{Q}}{\partial x}=e^{x}x+e^{x}Sen(x)+e^{x}Sen(y) \quad ; \quad \frac{\partial \mathbb{Q}}{\partial y}=e^{x}Cos(y)$$

$$\int \partial \mathbb{Q}=\int e^{x}Cos(y)dy$$

$$\mathbb{Q}=e^{x}Sen(y)+h(x)$$

$$\frac{d}{dx}[e^{x}Sen(y)+h(x)]=e^{x}x+e^{x}Sen(x)+e^{x}Sen(y) \quad \Rightarrow \quad e^{x}Sen(y)+h^{'}(x)=e^{x}x+e^{x}Sen(x)+e^{x}Sen(y)$$

$$h^{'}(x)=e^{x}x+e^{x}Sen(x) \quad \Rightarrow \quad h(x)=\int[e^{x}x+e^{x}Sen(x)]dx$$

$$u=x \quad \Rightarrow \quad du=dx \quad ; \quad dv=e^{x}dx \quad \Rightarrow \quad v=e^{x}$$

$$\int e^{x}x=xe^{x}-\int e^{x}dx \quad \Rightarrow \quad \int e^{x}x=xe^{x}-e^{x}$$

$$\int e^{x}Sen(x)dx$$

$$t=e^{x} \quad \Rightarrow \quad du=e^{x}dx \quad ; \quad dv=Sen(x)dx \quad \Rightarrow \quad v=-Cos(x)$$

$$\int e^{x}Sen(x)dx=-e^{x}Cos(x)+\int e^{x}Cos(x)dx$$

$$t=e^{x} \quad \Rightarrow \quad du=e^{x}dx \quad ; \quad dv=Cos(x)dx \quad \Rightarrow \quad v=Sen(x)$$

$$\int e^{x}Sen(x)dx=-e^{x}Cos(x)+\left[e^{x}Sen(x)-\int e^{x}Sen(x)dx\right] \quad \Rightarrow \quad \int e^{x}Sen(x)dx=\frac{e^{x}}{2}[Sen(x)-Cos(x)]$$

$$h(x)=xe^{x}-e^{x}+\frac{e^{x}}{2}[Sen(x)-Cos(x)]+c$$

$$\boxed{\boldsymbol{e^{x}Sen(y)+xe^{x}-e^{x}+\frac{e^{x}}{2}[Sen(x)-Cos(x)]+c=0}}$$

3) $\boldsymbol{y\,dx + \left(2xy - e^{-2y}\right)dy = 0}$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2y$$

$$\therefore No\ es\ exacta$$

$\mu(y) = e^{\int \frac{2y-1}{y} dy} = e^{2y - ln|y|} \quad \Rightarrow \quad \mu(y) = e^{2y}y^{-1}$

$e^{2y}y^{-1}y\,dx + e^{2y}y^{-1}(2xy - e^{-2y})dy = 0$

$$\frac{\partial M}{\partial y} = e^{2y}(2) \qquad \frac{\partial N}{\partial x} = 2e^{2y}$$

$$\therefore Es\ exacta$$

$\frac{\partial \mathbb{Q}}{\partial x} = e^{2y} \quad ; \quad \frac{\partial \mathbb{Q}}{\partial y} = 2xe^{2y} - y^{-1}$

$\int \partial \mathbb{Q} = \int e^{2y}dx$

$\mathbb{Q} = xe^{2y} + h(y)$

$\frac{d}{dy}[xe^{2y} + h(y)] = 2xe^{2y} - y^{-1} \quad \Rightarrow \quad 2xe^{2y} + h'(y) = 2xe^{2y} - y^{-1}$

$h'(y) = -y^{-1} \quad \Rightarrow \quad h(y) = -ln|y| + c$

$\boxed{\boldsymbol{xe^{2y} - ln|y| + c = 0}}$

ECUACIONES DIFERENCIALES HOMOGÉNEAS

1) $x(\ln x - \ln y)dy - y\,dx = 0$

$$x\,ln\left(\frac{x}{y}\right)dy - y\,dx = 0$$

$$x\,ln\left(\frac{x}{y}\right)dy = y\,dx \quad \Rightarrow \quad \frac{x}{y}ln\left(\frac{x}{y}\right) = \frac{dx}{dy}$$

$$t = \frac{x}{y} \quad \Rightarrow \quad x = ty \quad \Rightarrow \quad \frac{dx}{dy} = t + y\frac{dt}{dy}$$

$$t\ln(t) = t + y\frac{dt}{dy} \quad \Rightarrow \quad t\ln(t) - t = y\frac{dt}{dy}$$

$$\frac{dy}{y} = \frac{dt}{t\,(\ln(t) - 1)} \quad \Rightarrow \quad \int \frac{y}{dy} = \int \frac{dt}{t\,(\ln(t) - 1)}$$

$$\boxed{\boldsymbol{ln|y| + c = ln\left|ln\left(\frac{x}{y}\right) - 1\right|}}$$

2) $\dfrac{dy}{dx} = \dfrac{y^2 + x\sqrt{x^2 + y^2}}{xy}$

$$\frac{dy}{dx} = \frac{y^2}{xy} + \frac{x\sqrt{x^2 + y^2}}{xy}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\left(\frac{x}{y}\right)^2 + 1}$$

$$t = \frac{y}{x} \quad \Rightarrow \quad y = tx \quad \Rightarrow \quad \frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = t + \sqrt{\frac{1}{t^2} + 1}$$

$$x\frac{dt}{dx} = t + \sqrt{\frac{1}{t^2} + 1} - t \quad \Rightarrow \quad x\frac{dt}{dx} = \sqrt{\frac{1 + t^2}{t^2}}$$

$$\frac{t}{\sqrt{1 + t^2}}dt = \frac{dx}{x} \quad \Rightarrow \quad \int \frac{t}{\sqrt{1 + t^2}}dt = \int \frac{dx}{x}$$

$$u = 1 + t^2 \quad \Rightarrow \quad du = 2t\,dt$$

$$\frac{1}{2}\int \frac{du}{\sqrt{u}} = \int \frac{dx}{x} \quad \Rightarrow \quad \sqrt{u} = ln|x| + c \quad \Rightarrow \quad \sqrt{1 + t^2} = ln|x| + c$$

$$\boxed{\boldsymbol{\sqrt{1 + \left(\frac{x}{y}\right)^2} = ln|x| + c}}$$

$$\mathbf{3)}\ \boldsymbol{\frac{dy}{dx} = \frac{(y^2+3xy+6x^2)x}{(y-x)(y^2+6xy+13x^2)} + \frac{y}{x}}$$

$$\frac{dy}{dx} = \frac{x^2\left[\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) + 6\right]x}{x\left(\frac{y}{x}-1\right)x^2\left[\left(\frac{y}{x}\right)^2 + 6\left(\frac{y}{x}\right) + 13\right]} + \frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\left[\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) + 6\right]}{\left(\frac{y}{x}-1\right)\left[\left(\frac{y}{x}\right)^2 + 6\left(\frac{y}{x}\right) + 13\right]} + \frac{y}{x}$$

$$t = \frac{y}{x} \quad \Rightarrow \quad y = tx \quad \Rightarrow \quad \frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = \frac{t^2+3t+6}{(t-1)(t^2+6t+13)} + t \quad \Rightarrow \quad x\frac{dt}{dx} = \frac{t^2+3t+6}{(t-1)(t^2+6t+13)}$$

$$\frac{(t-1)(t^2+6t+13)}{t^2+3t+6}dt = \frac{dx}{x} \quad \Rightarrow \quad \frac{t^3+5t^2+7t-13}{t^2+3t+6}dt = \frac{dx}{x}$$

$$\int \frac{t^3+5t^2+7t-13}{t^2+3t+6}dt = \int \frac{dx}{x}$$

Resolviendo la división entre polinomios de la 1era integral, nos queda lo siguiente:

$$\int \left(t + 2 - \frac{5t+25}{t^2+3t+6}\right) = \int \frac{dx}{x}$$

$$\frac{5t+25}{t^2+3t+6} = \frac{A(2t+3)+B}{t^2+3t+6}$$

$$5t+25 = 2At + (3A+B)$$

$$5 = 2A$$

$$25 = 3A + B$$

Resolviendo el sistema A = 5/2, B = 35/2

$$\int (t+2)dt - \int \left[\frac{A(2t+3)}{t^2+3t+6} + \frac{B}{t^2+3t+6}\right]dt = \int \frac{dx}{x}$$

$$\int (t+2)dt - \int \left[\frac{A(2t+3)}{t^2+3t+6} + \frac{B}{\left(t^2+3t+\frac{9}{4}\right)+6-\frac{9}{4}}\right]dt = \int \frac{dx}{x}$$

$$\int (t+2)dt - \int \left[\frac{A(2t+3)}{t^2+3t+6} + \frac{B}{\left(t+\frac{3}{2}\right)^2+\frac{15}{4}}\right]dt = \int \frac{dx}{x}$$

$$\frac{t^2}{2} - \left[\frac{5}{2}ln|t^2+3t+6| + \frac{35}{2}\left(\frac{2}{\sqrt{15}}\right)Arctan\left[\frac{2\left(t+\frac{3}{2}\right)}{\sqrt{15}}\right]\right] = ln|x| + c$$

$$\boxed{\boldsymbol{\frac{1}{2}\left(\frac{y}{x}\right)^2 - \frac{5}{2}ln\left|\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) + 6\right| - \frac{35}{\sqrt{15}}Arctan\left[\frac{2\left(\frac{y}{x}+\frac{3}{2}\right)}{\sqrt{15}}\right] = ln|x| + c}}$$

4) $\boldsymbol{y' = \dfrac{2xy}{3x^2 - y^2}}$

$$y' = \frac{\frac{2xy}{x^2}}{\frac{3x^2}{x^2} - \frac{y^2}{x^2}} \quad \Rightarrow \quad y' = \frac{2\left(\frac{y}{x}\right)}{3 - \left(\frac{y}{x}\right)^2}$$

$$t = \frac{y}{x} \quad \Rightarrow \quad y = tx \quad \Rightarrow \quad \frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = \frac{2t}{3 - t^2}$$

$$x\frac{dt}{dx} = \frac{2t}{3 - t^2} - t$$

$$x\frac{dt}{dx} = \frac{2t - 3t + t^3}{3 - t^2}$$

$$\int \frac{3 - t^2}{t^3 - t}dt = \int \frac{dx}{x}$$

$$\int \frac{3 - t^2}{t(t^2 - 1)}dt = \int \frac{dx}{x}$$

$$\frac{3 - t^2}{t(t^2 - 1)} = \frac{A}{t} + \frac{B(2t) + C}{(t^2 - 1)}$$

$$3 - t^2 = A(t^2 - 1) + B(2t^2) + Ct$$

$$3 - t^2 = (A + 2B)t^2 + Ct - A$$

$$3 = -A$$

$$-1 = A + 2B$$

$$C = 0$$

Resolviendo el sistema A = -3, B = ½ , C = 0

$$\int \frac{A}{t}dt + \int \frac{2Bt}{t^2 - 1}dt + \int \frac{C}{t^2 - 1}dt = \int \frac{dx}{x}$$

$$-3ln|t| + ln|t^2 - 1| = ln|x| + c$$

$$\boxed{\boldsymbol{-3ln\left|\frac{y}{x}\right| + ln\left|\left(\frac{y}{x}\right)^2 - 1\right| = ln|x| + c}}$$

ECUACIONES DIFERENCIALES DE COEFICIENTES LINEALES

1) $\frac{dy}{dx} = \frac{2y - x + 5}{2x - y - 4}$

$$y = Y + h \quad \Rightarrow \quad dy = dY \quad ; \quad x = X + k \quad \Rightarrow \quad dx = dX$$

$$\frac{dY}{dX} = \frac{2(Y+h) - (X+k) + 5}{2(X+k) - (Y+h) - 4} = \frac{2Y + 2h - X - k + 5}{2X + 2k - Y - h - 4}$$

$$\frac{dY}{dX} = \frac{2Y - X + (2h - k + 5)}{2X - Y + (2k - h - 4)}$$

$$\begin{cases} 2h - k + 5 = 0 \\ 2k - h - 4 = 0 \end{cases} \quad ; \quad \textit{Resolviendo el sistema} \quad k = 1\,, h = -2$$

$$\frac{dY}{dX} = \frac{2Y - X}{2X - Y} \quad \Rightarrow \quad \frac{dY}{dX} = \frac{2\left(\frac{Y}{X}\right) - 1}{2 - \left(\frac{Y}{X}\right)}$$

$$t = \frac{Y}{X} \quad \Rightarrow \quad Y = tX \quad \Rightarrow \quad \frac{dY}{dX} = t + X\frac{dt}{dX}$$

$$t + X\frac{dt}{dX} = \frac{2t - 1}{2 - t} \quad \Rightarrow \quad X\frac{dt}{dX} = \frac{2t - 1 - 2t + t^2}{2 - t}$$

$$X\frac{dt}{dX} = \frac{t^2 - 1}{2 - t} \quad \Rightarrow \quad \frac{2 - t}{(t+1)(t-1)}dt = \frac{dX}{X}$$

$$\int \frac{2 - t}{(t+1)(t-1)}dt = \int \frac{dX}{X}$$

$$\frac{2 - t}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1}$$

$$2 - t = A(t-1) + B(t+1) \quad \Rightarrow \quad 2 - t = (A+B)t + (B - A)$$

$$-1 = A + B$$

$$2 = B - A$$

Resolviendo el sistema A = -3/2, B = ½

$$\int \left(\frac{A}{t+1} + \frac{B}{t-1}\right)dt = \int \frac{dX}{X} \quad \Rightarrow \quad A\,ln|t+1| + B\,ln|t-1| = ln|X| + c$$

$$-\frac{3}{2}ln\left|\frac{Y}{X} + 1\right| + \frac{1}{2}ln\left|\frac{Y}{X} - 1\right| = ln|X| + c \quad \Rightarrow \quad -\frac{3}{2}ln\left|\frac{y-h}{x-k} + 1\right| + \frac{1}{2}ln\left|\frac{y-h}{x-k} - 1\right| = ln|x-k| + c$$

$$\boxed{-\frac{3}{2}ln\left|\frac{y+2}{x-1} + 1\right| + \frac{1}{2}ln\left|\frac{y+2}{x-1} - 1\right| = ln|x-1| + c}$$

2) $(-3x + y + 6)dx + (x + y + 2)dy = 0$

$$\frac{dy}{dx} = \frac{3x - y - 6}{x + y + 2}$$

$$y = Y + h \quad \Rightarrow \quad dy = dY \quad ; \quad x = X + k \quad \Rightarrow \quad dx = dX$$

$$\frac{dY}{dX} = \frac{3(X + k) - (Y + h) - 6}{(X + k) + (Y + h) + 2} = \frac{3X + 3k - Y - h - 6}{X + k + Y + h + 2}$$

$$\frac{dY}{dX} = \frac{3X - Y + (3h - k - 6)}{X + Y + (k + h + 2)}$$

$$\begin{cases} 3h - k - 6 = 0 \\ k + h + 2 = 0 \end{cases} \quad ; \quad \textit{Resolviendo el sistema} \quad k = 1\,, h = -3$$

$$\frac{dY}{dX} = \frac{3X - Y}{X + Y} \quad \Rightarrow \quad \frac{dY}{dX} = \frac{3 - \left(\frac{Y}{X}\right)}{1 + \left(\frac{Y}{X}\right)}$$

$$t = \frac{Y}{X} \quad \Rightarrow \quad Y = tX \quad \Rightarrow \quad \frac{dY}{dX} = t + X\frac{dt}{dX}$$

$$t + X\frac{dt}{dX} = \frac{3 - t}{1 + t} \quad \Rightarrow \quad X\frac{dt}{dX} = \frac{3 - t - t - t^2}{1 + t}$$

$$X\frac{dt}{dX} = \frac{-(t^2 + 2t - 3)}{1 + t} \quad \Rightarrow \quad -\frac{t + 1}{(t + 3)(t - 1)}dt = \frac{dX}{X}$$

$$-\int \frac{t + 1}{(t + 3)(t - 1)}dt = \int \frac{dX}{X}$$

$$\frac{t + 1}{(t + 3)(t - 1)} = \frac{A}{t + 3} + \frac{B}{t - 1}$$

$$t + 1 = A(t - 1) + B(t + 3) \quad \Rightarrow \quad t + 1 = (A + B)t + (3B - A)$$

$$1 = A + B$$

$$1 = 3B - A$$

Resolviendo el sistema A = ½ , B = ½

$$-\int \left(\frac{A}{t + 3} + \frac{B}{t - 1}\right)dt = \int \frac{dX}{X} \quad \Rightarrow \quad -A\,ln|t + 3| - B\,ln|t - 1| = ln|X| + c$$

$$-\frac{1}{2}ln\left|\frac{Y}{X} + 3\right| - \frac{1}{2}ln\left|\frac{Y}{X} - 1\right| = ln|X| + c \quad \Rightarrow \quad -\frac{1}{2}ln\left|\frac{y - h}{x - k} + 3\right| - \frac{1}{2}ln\left|\frac{y - h}{x - k} - 1\right| = ln|x - k| + c$$

$$\boxed{-\frac{1}{2}\boldsymbol{ln}\left|\frac{y + 3}{x - 1} + 1\right| - \frac{1}{2}\boldsymbol{ln}\left|\frac{y + 3}{x - 1} - 1\right| = \boldsymbol{ln}|x - 1| + c}$$

3) $\boldsymbol{\frac{dy}{dx} = -\frac{4x+3y+5}{2x+y+7}}$

$$y = Y + h \quad \Rightarrow \quad dy = dY \quad ; \quad x = X + k \quad \Rightarrow \quad dx = dX$$

$$\frac{dy}{dx} = -\frac{4(X+k)+3(Y+h)+5}{2(X+k)+(Y+h)+7} = -\frac{4X+k+3Y+3h+5}{2X+2k+Y+h+7}$$

$$\frac{dY}{dX} = -\frac{4X+3Y+(4h+3k+5)}{2X+Y+(2k+y+7)}$$

$$\begin{cases} 4h+3k+5=0 \\ 2k+y+7=0 \end{cases} \quad ; \quad \textit{Resolviendo el sistema} \;\; k=-8\,, h=9$$

$$\frac{dY}{dX} = -\frac{4X+3Y}{2X+Y} \quad \Rightarrow \quad \frac{dY}{dX} = -\frac{4+3\left(\frac{Y}{X}\right)}{2+\left(\frac{Y}{X}\right)}$$

$$t = \frac{Y}{X} \quad \Rightarrow \quad Y = tX \quad \Rightarrow \quad \frac{dY}{dX} = t + X\frac{dt}{dX}$$

$$t + X\frac{dt}{dX} = -\frac{4+3t}{2+t} \quad \Rightarrow \quad X\frac{dt}{dX} = -\frac{4+3t-2t-t^2}{2+t}$$

$$X\frac{dt}{dX} = -\frac{-t^2+t-4}{2+t} \quad \Rightarrow \quad \frac{2+t}{t^2-t+4}dt = \frac{dX}{X}$$

$$\int \frac{2+t}{t^2-t+4}dt = \int \frac{dX}{X}$$

$$\frac{2+t}{t^2-t+4} = \frac{A(2t-1)+B}{t^2-t+4}$$

$$2+t = 2At + (B-A)$$

$$1 = 2A$$

$$1 = B - A$$

Resolviendo el sistema A = ½ , B = 5/2

$$\int \left[\frac{A(2t-1)}{t^2-t+4} + \frac{B}{\left(t^2-t+\frac{1}{4}\right)+4-\frac{1}{4}}\right]dt = \int \frac{dX}{X} \quad \Rightarrow \quad \int \left[\frac{A(2t-1)}{t^2-t+4} + \frac{B}{\left(t-\frac{1}{2}\right)^2-\frac{17}{4}}\right]dt = \int \frac{dX}{X}$$

$$\frac{1}{2}ln|t^2-t+4| + \frac{5}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2-\frac{17}{4}}dt = ln|x-k| + c$$

$$\int \frac{1}{\left(t-\frac{1}{2}\right)^2-\frac{17}{4}}dt$$

$$u = t - \frac{1}{2} \quad \Rightarrow \quad du = dt$$

$$\int \frac{1}{u^2-\frac{17}{4}}dt \quad ; \quad Sea\ A = \frac{17}{4}$$

$$\int \frac{1}{u^2 - A}du \quad \Rightarrow \quad \int \frac{du}{(u+\sqrt{A})(u-\sqrt{A})}$$

$$\frac{1}{(u+\sqrt{A})(u-\sqrt{A})} = \frac{C}{u+\sqrt{A}} + \frac{D}{u-\sqrt{A}}$$

$$1 = C(u-\sqrt{A}) + D(u+\sqrt{A}) \quad \Rightarrow \quad 1 = (C+D)u + (D\sqrt{A} - C\sqrt{A})$$

Resolviendo el sistema $C = -\frac{1}{2\sqrt{A}}$, $D = \frac{1}{2\sqrt{A}}$

$$\int \left(\frac{C}{u+\sqrt{A}} + \frac{D}{u-\sqrt{A}}\right)du \quad \Rightarrow \quad C\ ln|u+\sqrt{A}| + D\ ln|u-\sqrt{A}| \quad \Rightarrow \quad -\frac{1}{2\sqrt{A}}\ ln|u+\sqrt{A}| + \frac{1}{2\sqrt{A}}\ ln|u-\sqrt{A}|$$

$$-\frac{1}{\sqrt{17}}\ ln\left|t-\frac{1}{2}+\frac{2}{\sqrt{17}}\right| + \frac{1}{\sqrt{17}}\ ln\left|t-\frac{1}{2}-\frac{2}{\sqrt{17}}\right|$$

Retornando a la ecuación

$$\frac{1}{2}ln|t^2-t+4| + \frac{5}{2}\left(-\frac{1}{\sqrt{17}}\ ln\left|t-\frac{1}{2}+\frac{2}{\sqrt{17}}\right| + \frac{1}{\sqrt{17}}\ ln\left|t-\frac{1}{2}-\frac{2}{\sqrt{17}}\right|\right) = ln|x-k| + c$$

$$\frac{1}{2}ln\left|\left(\frac{Y}{X}\right)^2 - \frac{Y}{X} + 4\right| + \frac{5}{2}\left(-\frac{1}{\sqrt{17}}\ ln\left|\frac{Y}{X}-\frac{1}{2}+\frac{2}{\sqrt{17}}\right| + \frac{1}{\sqrt{17}}\ ln\left|\frac{Y}{X}-\frac{1}{2}-\frac{2}{\sqrt{17}}\right|\right) = ln|x-k| + c$$

$$\frac{1}{2}ln\left|\left(\frac{y-h}{x-k}\right)^2 - \frac{y-h}{x-k} + 4\right| + \frac{5}{2\sqrt{17}}\left(ln\left|\frac{y-h}{x-k}-\frac{1}{2}-\frac{2}{\sqrt{17}}\right| - ln\left|\frac{y-h}{x-k}-\frac{1}{2}+\frac{2}{\sqrt{17}}\right|\right) = ln|x-k| + c$$

$$\boxed{\frac{1}{2}ln\left|\left(\frac{y-9}{x+8}\right)^2 - \frac{y+9}{x+8} + 4\right| + \frac{5}{2\sqrt{17}}\left(ln\left|\frac{y-9}{x+8}-\frac{1}{2}-\frac{2}{\sqrt{17}}\right| - ln\left|\frac{y-9}{x+8}-\frac{1}{2}+\frac{2}{\sqrt{17}}\right|\right) = ln|x+8| + c}$$

APLICACIONES

LEY DE ENFRIAMIENTO DE NEWTON

1) Un asado de 4 libras inicialmente a 50°F se coloca en un horno a 375°F a las 5 P.M después de 75 minutos se observa que la temperatura del asado es 125°F ¿Cuándo estará el asado a 150°F?

$$\frac{dT}{dt} = k(A - T)\,;\;\; donde\ A\ es\ la\ temperatura\ ambiente$$

$$\frac{dT}{dt} = k(375 - T) \quad \Rightarrow \quad \int \frac{dT}{375 - T} = K\int dt \quad \Rightarrow \quad -ln|375 - T| = kt + c$$

No tomamos en cuenta el valor absoluto, debido a que la temperatura del asado nunca superará la temperatura del horno, por lo tanto siempre será un valor positivo

$$e^{ln|375-T|} = e^{-kt-c} \quad \Rightarrow \quad 375 - T = e^{-kt}\underbrace{e^{-c}}_{Constante} \quad \Rightarrow \quad 375 - T = e^{-kt}A$$

$$T(t) = 375 - Ae^{-kt} \quad ; \quad Ecuación\ de\ la\ temperatura\ del\ asado\ para\ cualquier\ tiempo\ "t"$$

Según las condiciones que nos da el problema para:

$$T(0) = 50°F \quad y \quad T\left(1\frac{1}{4}\right) = 125°F$$

Reemplazando nuestra primera condición:

$$50 = 375 - Ae^{-k(0)} \quad \Rightarrow \quad 50 = 375 - Ae^{0} \quad \Rightarrow \quad A = 325$$

Reemplazando la segunda condición:

$$125 = 375 - 325(e^{-k})^{5/4} \quad \Rightarrow \quad (e^{-k})^{5/4} = \frac{375 - 125}{325} \quad \Rightarrow \quad e^{-k} = \sqrt[5]{\left(\frac{250}{325}\right)^4}$$

Entonces reemplazando todas las constantes en nuestra ecuación, nos queda:

$$T(t) = 375 - A(e^{-k})^t \quad \Rightarrow \quad T(t) = 375 - 325\left[\sqrt[5]{\left(\frac{250}{325}\right)^4}\right]^t$$

Como el problema nos pide el tiempo en que el asado estará a 150°F, solo nos queda reemplazar, es decir T(t) = 150°F

$$150 = 375 - 325\left[\sqrt[5]{\left(\frac{250}{325}\right)^4}\right]^t \quad \Rightarrow \quad \left[\sqrt[5]{\left(\frac{250}{325}\right)^4}\right]^t = \frac{375 - 150}{325}$$

$$ln\left[\sqrt[5]{\left(\frac{250}{325}\right)^4}\right]^t = ln\left(\frac{9}{13}\right) \quad \Rightarrow \quad t = {ln\left(\frac{9}{13}\right)} \Big/ {ln\left[\sqrt[5]{\left(\frac{250}{325}\right)^4}\right]}$$

$$\boxed{t \approx 1.75\ horas}$$

2) Justo antes del mediodía se encuentra el cuerpo de una víctima de un presunto homicidio dentro de un cuarto que se conserva a una temperatura constante de 70°F. A las 12 del día la temperatura del cuerpo es de 80°F y a la 1 P.M es de 75°F. Considere que la temperatura del cuerpo al morir es de 98.6°F y que este se ha enfriado de acuerdo con la ley de enfriamiento de Newton. Determine la hora del homicidio.

$$\frac{dT}{dt} = k(A - T) \quad \Rightarrow \quad \int \frac{dT}{A - T} = k \int dt \quad \Rightarrow \quad -ln|A - T| = kt + c$$

$$e^{ln|A-T|} = e^{-kt-c} \quad \Rightarrow \quad A - T = Be^{-kt}$$

$$T(t) = A - Be^{-kt}$$

El problema nos dice que el cuerpo está en un cuarto donde existe una temperatura constante, entonces A = 70

$$T(t) = 70 - Be^{-kt}$$

Si consideramos las 12 del día como t = 0, entonces T(0) = 80

$$80 = 70 - Be^{-k(0)} \quad \Rightarrow \quad 80 = 70 - B \quad \Rightarrow \quad B = -10$$

Después, nos dice que transcurrido una hora la temperatura del cuerpo es de 75°F, entonces T(1) = 75

$$75 = 70 + 10e^{-k} \quad \Rightarrow \quad e^{-k} = \frac{1}{2}$$

Reemplazando todas estas constantes en nuestra ecuación:

$$T(t) = 70 + 10\left(\frac{1}{2}\right)^t$$

La última condición, nos dice que la temperatura del cuerpo al morir es de 98.6°F, es decir T(t) = 98.6

$$98.6 = 70 + 10\left(\frac{1}{2}\right)^t \quad \Rightarrow \quad \left(\frac{1}{2}\right)^t = 2.86 \quad \Rightarrow \quad ln\left(\frac{1}{2}\right)^t = ln(2.86) \quad \Rightarrow \quad t = \frac{ln(2.86)}{ln(0.5)}$$

$$t \approx -1.5 \; horas$$

Pero la hora del asesinato sería 12 – 1.5 horas, es decir la hora del asesinato fue aproximadamente a las 10 y 30

PROBLEMAS DE MEZCLAS

1) Un tanque contiene originalmente 400 lit. de agua limpia. Entonces se vierte en el tanque agua que contiene 0.05 kg de sal por litro a una velocidad de 8 lit. por minuto y se deja que la mezcla salga bien homogenizada con la misma rapidez ¿Cuál es la cantidad de sal en el recipiente luego de 10 min?

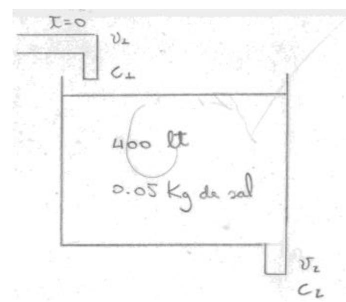
Tasa de acumulación = Tasa de entrada – Tasa de salida

Sea x(t) las libras de sal en el instante t

$\frac{dx}{dt}$ = Tasa de acumulación = Tasa de entrada del soluto – Tasa de salida

$$\frac{dx}{dt} = v_1c_1 - v_2c_2 \;\; ; \;\; donde\ c_1 es\ la\ concentración\ que\ entra\ y\ c_2\ es\ la\ concentración\ de\ salida$$

El problema no dice con que rapidez sale la mezcla, por lo tanto asumimos que la velocidad con que entra la mezcla es la misma que sale.



$$\frac{dx}{dt} = v_1c_1 - v_2c_2 \quad \Rightarrow \quad \frac{dx}{dt} = 8(0.05) - 8\frac{x}{400} \quad \Rightarrow \quad \frac{dx}{dt} = 0.4 - 0.02x$$

$$\frac{dx}{0.4 - 0.02x} = dt \quad \Rightarrow \quad \int \frac{dx}{0.4 - 0.02x} = \int dt$$

$$-\frac{1}{0.02} ln|0.4 - 0.02x| = t + c \quad \Rightarrow \quad ln|0.4 - 0.02x| = -0.02t - 0.02c$$

$$e^{ln|0.4-0.02x|} = e^{(-0.02t-0.02c)} \quad \Rightarrow \quad e^{ln|0.4-0.02x|} = e^{-0.02t}e^{-0.02c}$$

$$0.4 - 0.02x = Be^{-0.02t} \quad \Rightarrow \quad 0.02x = 0.4 - Be^{-0.02t}$$

La ecuación para determinar la cantidad de sal en el tanque en cualquier tiempo "t" es:

$$x(t) = \frac{0.4}{0.02} - \frac{B}{0.02}e^{-0.02t} \quad \Rightarrow \quad x(t) = 20 - ke^{-0.02t}$$

Sabemos que en t = 0, x = 0 , reemplazando esa condición resulta:

$$0 = 20 - k \quad \Rightarrow \quad k = 20$$

Finalmente la ecuación es:

$$x(t) = 20 - 20e^{-0.02t}$$

El problema nos pide la cantidad de sal después de 10 min

$$x(10) = 20 - 20e^{-0.02(10)} \quad \Rightarrow \quad x(10) = 20 - 20e^{-0.2}$$

La cantidad de sal después de 10 min es:

$$\boxed{\boldsymbol{x(10) \approx 3.63\ kg}}$$

Después de 10 min se detiene el proceso y se vierte agua limpia al tanque con la misma rapidez de entrada. ¿Cuál es la cantidad de sal a los 20 min de toda la operación?

$$\frac{dx}{dt} = v_1c_1 - v_2c_2$$

$$\frac{dx}{dt} = v_1(0) - 8\frac{x}{400}$$

$$\frac{dx}{dt} = -0.02x \quad \Rightarrow \quad \frac{dx}{x} = -0.02\,dt$$

$$\int \frac{dx}{x} = -0.02\int dt \quad \Rightarrow \quad ln|x| = -0.02t + c$$

$$e^{ln|x|} = e^{(-0.02t+c)}$$

$$x(t) = Ae^{-0.02t}$$

Pero para t = 0 , x = 3.63

$$3.63 = A$$

Entonces, nuestra ecuación nos queda:

$$x(t) = 3.63e^{-0.02t}$$

Finalmente, la cantidad de sal a los 20 min es:

$$x(20) = 3.63e^{-0.02(20)}$$

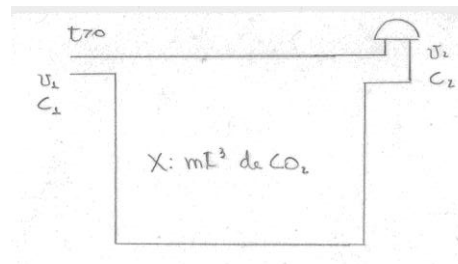
$$\boxed{\boldsymbol{x(20) \approx 2.43\ kg}}$$

2) El aire de un teatro de dimensiones 12x8x4 mt contiene 0.12% de su solución de CO_2. Se desea renovar en 10 minutos el aire, de modo que llegue a contener solamente el 0.06% de CO_2. Calcular el número de mt^3 por minuto que deben renovarse, suponiendo que el aire exterior contiene 0.04% de CO_2.

Nos dice que determinemos la velocidad con que debe renovarse el aire de manera que llegue a contener el 0.06% de CO_2

El problema no menciona cual es la velocidad de salida de la mezcla, por lo tanto asumiremos que la velocidad de entrada es la misma que la de salida, entonces $v_1 = v_2 = v$

Sea "x" la concentración de CO_2 *para cualquier tiempo "t", entonces:*



$$\frac{dx}{dt} = v_1 c_1 - v_2 c_2 \quad \Rightarrow \quad \frac{dx}{dt} = vc_1 - vc_2$$

$$\frac{dx}{dt} = 0.04\%\, v - v\frac{x}{12*8*4} \quad \Rightarrow \quad \frac{dx}{dt} = 4*10^{-4}v - \frac{vx}{384}$$

Para mayor comodidad, A = 384 y B = $4*10^{-4}$*, entonces:*

$$\frac{dx}{dt} = Bv - \frac{vx}{A} \quad \Rightarrow \quad \frac{dx}{dt} + \frac{v}{A}x = Bv$$

Resolviendo la ecuación diferencial:

$$\mu(t) = e^{\int p(t)dt} \quad \Rightarrow \quad \mu(t) = e^{\int \frac{v}{A}dt} \quad \Rightarrow \quad \mu(t) = e^{\frac{v}{A}t}$$

$$\left(e^{\frac{v}{A}t}\right)\frac{dx}{dt} + \left(e^{\frac{v}{A}t}\right)\frac{v}{A}x = \left(e^{\frac{v}{A}t}\right)Bv$$

$$\frac{d}{dt}\left(e^{\frac{v}{A}t}x\right) = \left(e^{\frac{v}{A}t}\right)Bv$$

$$d\left(e^{\frac{v}{A}t}x\right) = \left(e^{\frac{v}{A}t}\right)Bv\,dt$$

$$\int d\left(e^{\frac{v}{A}t}x\right) = Bv\int e^{\frac{v}{A}t}dt$$

La ecuación en forma implícita de la concentración de CO_2 *es:*

$$e^{\frac{v}{A}t}x = Bv\left(\frac{A}{v}\right)\left[e^{\frac{v}{A}t} + c\right]$$

$$e^{\frac{v}{A}t}x = BA\left(e^{\frac{v}{A}t} + c\right)$$

Pero en t = 0 , x =0.12%A , entonces x = 0.0012A = 0.4608, por lo tanto vamos a llamar k = 0.4608, entonces:

$$e^0 k = BA\left(e^{\frac{v}{A}(0)} + c\right)$$

$$c = \frac{k}{BA} - 1$$

Entonces:

$$e^{\frac{v}{A}t}x = BA\left(e^{\frac{v}{A}t} + \frac{k}{BA} - 1\right)$$

Pero se desea que en 10 minutos el aire llegue a contener el 0.06% de CO_2 *entonces en t = 0 , x = 6**$10^{-4}A =$ *0.2304, vamos a llamar z = 0.2304, entonces:*

$$e^{\frac{v}{A}(10)}z = BA\left[e^{\frac{v}{A}(10)} + \frac{k}{BA} - 1\right]$$

$$e^{\frac{v}{A}(10)}z = BAe^{\frac{v}{A}(10)} + k - BA$$

$$e^{\frac{v}{A}(10)}z - BAe^{\frac{v}{A}(10)} = k - BA$$

$$e^{\frac{v}{A}(10)}(z - BA) = k - BA$$

$$e^{\frac{v}{A}(10)} = \frac{k - BA}{z - BA}$$

$$ln\left[e^{\frac{v}{A}(10)}\right] = ln\left(\frac{k - BA}{z - BA}\right)$$

$$10\frac{v}{A} = ln\left(\frac{k - BA}{z - BA}\right)$$

Finalmente despejando v encontraremos la velocidad con que debe renovarse el aire del teatro

$$v = \frac{A}{10}ln\left(\frac{k - BA}{z - BA}\right)$$

Reemplazando, tenemos que:

$$v = \frac{384}{10}ln\left[\frac{0.4608 - (4 * 10^{-4})(384)}{0.2304 - (4 * 10^{-4})(384)}\right]$$

$$\boxed{\boldsymbol{v = 53.23\ mt^3 de\ aire/minuto}}$$

3) Un tanque contiene inicialmente agua pura. Salmuera que contiene 2 lib de sal/gal entra al tanque a una velocidad de 4 gal/min. Asumiendo la mezcla uniforme, la salmuera sale a una velocidad de 3 gal/min. Si la concentración alcanza el 90% de su valor máximo en 30 min. Encuentre la ecuación que determine la cantidad de agua que había inicialmente en el tanque.

Sea Q la cantidad de H_2O inicialmente en el tanque.

$$\frac{dx}{dt} = v_1c_1 - v_2c_2 \quad \Rightarrow \quad \frac{dx}{dt} = v_1c_1 - v_2\frac{x}{Q + (v_1 - v_2)t}$$

$$\frac{dx}{dt} = 4(2) - 3\frac{x}{Q + (4 - 3)t} \quad \Rightarrow \quad \frac{dx}{dt} = 8 - \frac{3x}{Q + t}$$

$$\frac{dx}{dt} + \frac{3}{Q + t}x = 8$$

$$\mu(t) = e^{\int p(t)dt} \quad \Rightarrow \quad \mu(t) = e^{\int \frac{3}{Q+t}dt} \quad \Rightarrow \quad \mu(t) = e^{3\,ln(Q+t)} \quad \Rightarrow \quad \mu(t) = (Q + t)^3$$

$$\frac{d}{dt}[(Q + t)^3x] = 8(Q + t)^3 \quad \Rightarrow \quad d[(Q + t)^3x] = 8(Q + t)^3dt$$

$$\int d[(Q + t)^3x] = 8\int (Q + t)^3dt$$

$$(Q + t)^3x = 2(Q + t)^4 + c$$

El problema nos dice que inicialmente el tanque contiene agua pura entonces en t = 0 , x = 0

$$0 = 2(Q)^4 + c \quad \Rightarrow \quad c = -2Q^4$$

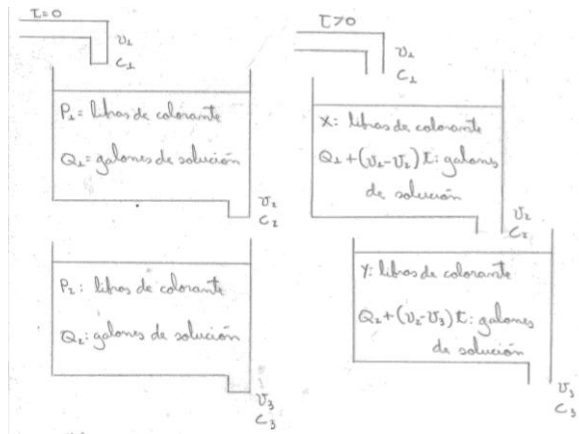
Entonces, nuestra ecuación en forma implícita es:

$$(Q + t)^3x = 2(Q + t)^4 - 2Q^4$$

Nos dice el problema que en 30 min la concentración alcanza el 90% de su valor máximo, es decir que en t = 30 , x = 0.9Q, por lo tanto la ecuación que nos pide, será expresada en forma implícita:

$$\boxed{\boldsymbol{(Q + t)^3(0.9Q) = 2(Q + 30)^4 - 2Q^4}}$$

4) Un colorante sólido disuelto en un líquido no voltil, entra a un tanque a una velocidad v_1 galones de solución por minuto y con concentración c_1 libras de colorante/galón de solución. La solución bien homogenizada sale del tanque a una velocidad v_2 galones de solución/min, y entra en un segundo tanque del cual sale posteriormente a una velocidad de v_3 galones de solución/min. Inicialmente el primer tanque tenía P_1 libras de colorante disueltas en Q_1 galones de solución y el segundo tanque P_2 libras de colorante disueltas en Q_2 galones de solución. Encontrar dos ecuaciones que determinen las libras de colorante presentes en cada tanque en cualquier tiempo t.



Sea:

x = libras de colorante en el primer tanque en el instante t

y = libras de colorante en el segundo tanque en el instante t

E.D para el primer tanque:

$$\frac{dx}{dt} = v_1c_1 - v_2c_2 \quad \Rightarrow \quad \frac{dx}{dt} = v_1c_1 - v_2\frac{x}{Q_1 + (v_1 - v_2)t}$$

$$\frac{dx}{dt} + \frac{v_2}{Q_1 + (v_1 - v_2)t}x = v_1c_1$$

$$\mu(t) = e^{\int p(t)dt} \quad \Rightarrow \quad \mu(t) = e^{\int \frac{v_2}{Q_1+(v_1-v_2)t}dt} \quad \Rightarrow \quad \mu(t) = e^{\frac{v_2}{v_1-v_2}ln[Q_1+(v_1-v_2)t]}$$

$$\mu(t) = [Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}$$

$$[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}\frac{dx}{dt} + [Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}\frac{v_2}{Q_1 + (v_1 - v_2)t}x = v_1c_1[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}$$

$$\frac{d}{dt}\left[[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}\, x\right] = v_1c_1[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}$$

$$\int d\left[[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}\, x\right] = \int v_1c_1[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}\, dt$$

$$[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}\, x = \frac{v_1c_1}{v_1 - v_2}\left(\frac{1}{\frac{v_2}{v_1 - v_2} + 1}\right)[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}+1} + c$$

$$[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}\, x = \frac{v_1c_1}{v_1 - v_2}\left(\frac{v_1 - v_2}{v_2 + v_1 - v_2}\right)[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}+1} + c$$

$$[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}\, x = c_1[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}+1} + c$$

$$x(t) = \frac{c_1[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}+1}}{[Q_1 + (v_1 - v_2)t]^{\frac{v_2}{v_1-v_2}}} + c[Q_1 + (v_1 - v_2)t]^{-\frac{v_2}{v_1-v_2}}$$

$$x(t) = c_1[Q_1 + (v_1 - v_2)t] + c[Q_1 + (v_1 - v_2)t]^{-\frac{v_2}{v_1-v_2}}$$

Pero sabemos que en t = 0 , x = P_1 , entonces:

$$P_1 = c_1Q_1 + cQ_1^{-\frac{v_2}{v_1-v_2}} \quad \Rightarrow \quad c = (P_1 - c_1Q_1)Q_1^{\frac{v_2}{v_1-v_2}}$$

Finalmente la ecuación para nuestro primer tanque es:

$$\boxed{\boldsymbol{x(t) = c_1[Q_1 + (v_1 - v_2)t] + (P_1 - c_1Q_1)Q_1^{\frac{v_2}{v_1-v_2}}[Q_1 + (v_1 - v_2)t]^{-\frac{v_2}{v_1-v_2}}}}$$

Ahora, vamos a encontrar la ecuación para el segundo tanque:

E.D para el segundo tanque:

$$\frac{dy}{dt} = v_2c_2 - v_3c_3 \quad \Rightarrow \quad \frac{dy}{dt} = v_2\frac{x}{Q_1 + (v_1 - v_2)t} - v_3\frac{y}{Q_2 + (v_2 - v_3)t}$$

$$\frac{dy}{dt} + \frac{v_3}{Q_2 + (v_2 - v_3)t}y = \frac{v_2}{Q_1 + (v_1 - v_2)t}x$$

$$\mu(t) = e^{\int p(t)dt} \quad \Rightarrow \quad \mu(t) = e^{\int\frac{v_3}{Q_2+(v_2-v_3)t}dt} \quad \Rightarrow \quad \mu(t) = e^{\frac{v_3}{v_2-v_3}ln[Q_2+(v_2-v_3)t]}$$

$$\mu(t) = [Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}$$

$$[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}\frac{dy}{dt} + [Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}\frac{v_3}{Q_2 + (v_2 - v_3)t}y = \frac{v_2}{Q_1 + (v_1 - v_2)t}x\,[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}$$

$$\frac{d}{dt}\left[[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}\,y\right] = \frac{v_2}{Q_1 + (v_1 - v_2)t}x\,[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}$$

$$\int d\left[[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}\,y\right] = \frac{v_2}{Q_1 + (v_1 - v_2)t}x\int[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}\,dt$$

$$[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}\,y = \left[\frac{v_2}{Q_1 + (v_1 - v_2)t}x\right]\left(\frac{1}{v_2 - v_3}\right)\left(\frac{1}{\frac{v_3}{v_2 - v_3} + 1}\right)[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}+1} + c$$

$$[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}\,y = \left[\frac{v_2}{Q_1 + (v_1 - v_2)t}x\right]\left(\frac{1}{v_2 - v_3}\right)\left(\frac{v_2 - v_3}{v_3 + v_2 - v_3}\right)[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}+1} + c$$

$$[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}\,y = \frac{x}{Q_1 + (v_1 - v_2)t}[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}+1} + c$$

$$y(t) = \left[\frac{x}{Q_1 + (v_1 - v_2)t}\right]\frac{[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}+1}}{[Q_2 + (v_2 - v_3)t]^{\frac{v_3}{v_2-v_3}}} + c[Q_2 + (v_2 - v_3)t]^{-\frac{v_3}{v_2-v_3}}$$

$$y(t) = \frac{x}{Q_1 + (v_1 - v_2)t}[Q_2 + (v_2 - v_3)t] + c[Q_2 + (v_2 - v_3)t]^{-\frac{v_3}{v_2-v_3}}$$

Pero sabemos que en $t = 0$, $y = P_2$, entonces:

$$P_2 = \frac{x}{Q_1} Q_2 + c Q_2^{-\frac{v_3}{v_2 - v_3}} \quad \Rightarrow \quad c = \left(P_2 - x\frac{Q_2}{Q_1}\right) Q_2^{\frac{v_3}{v_2 - v_3}}$$

Reemplazando en la ecuación:

$$y(t) = \frac{x}{Q_1 + (v_1 - v_2)t}[Q_2 + (v_2 - v_3)t] + \left(P_2 - x\frac{Q_2}{Q_1}\right) Q_2^{\frac{v_3}{v_2 - v_3}} [Q_2 + (v_2 - v_3)t]^{-\frac{v_3}{v_2 - v_3}}$$

$$y(t) = x\frac{Q_2 + (v_2 - v_3)t}{Q_1 + (v_1 - v_2)t} + \left(P_2 - x\frac{Q_2}{Q_1}\right) Q_2^{\frac{v_3}{v_2 - v_3}} [Q_2 + (v_2 - v_3)t]^{-\frac{v_3}{v_2 - v_3}}$$

Reemplazando "x", obtendremos la ecuación para el segundo tanque:

$$\boldsymbol{y(t) = \left[c_1\ [Q_1 + (v_1 - v_2)t] + (P_1 - c_1 Q_1)\ Q_1^{\frac{v_2}{v_1 - v_2}}\ [Q_1 + (v_1 - v_2)t]^{-\frac{v_2}{v_1 - v_2}}\right]\ \frac{Q_2 + (v_2 - v_3)t}{Q_1 + (v_1 - v_2)t}\ +}$$
$$\boldsymbol{\left[P_2 - \left[c_1[Q_1 + (v_1 - v_2)t] + (P_1 - c_1 Q_1) Q_1^{\frac{v_2}{v_1 - v_2}}[Q_1 + (v_1 - v_2)t]^{-\frac{v_2}{v_1 - v_2}}\right] - \frac{Q_2}{Q_1}\right]\ Q_2^{\frac{v_3}{v_2 - v_3}}[Q_2 + (v_2 - v_3)t]^{-\frac{v_3}{v_2 - v_3}}}$$

ANTIGÜEDAD DE UN FÓSIL

Se ha encontrado que un hueso fosilizado contiene 1/1000 de la cantidad original de carbono 14. Determine la edad del fósil, sabiendo que el tiempo de vida media del carbono 14 es 5600 años.

Sea Q la cantidad de carbono en cualquier tiempo "t":

$$\frac{dQ}{dt} = kQ \quad \Rightarrow \quad \frac{dQ}{Q} = k\,dt$$

$$\int \frac{dQ}{Q} = k \int dt \quad \Rightarrow \quad ln|Q| = kt + c$$

$$e^{\,ln|Q|} = e^{(kt+c)} \quad \Rightarrow \quad e^{\,ln|Q|} = e^{kt}e^{c}$$

Entonces nuestra ecuación es:

$$Q(t) = Ae^{kt}$$

Sabemos que en t = 0 la cantidad de carbono presente en el hueso es Q_0

$$Q_0 = Ae^{0} \quad \Rightarrow \quad A = Q_0$$

Reemplazando la constante en la ecuación:

$$Q(t) = Q_0 e^{kt}$$

Una de las condiciones nos dice que la edad media del carbono es 5600 años, entonces en t = 5600 , Q = $\frac{1}{2}Q_0$

$$\frac{1}{2}Q_0 = Q_0 e^{5600\,k} \quad \Rightarrow \quad e^{5600\,k} = \frac{1}{2}$$

$$ln(e^{5600\,k}) = ln\left(\frac{1}{2}\right) \quad \Rightarrow \quad k = \frac{ln(1/2)}{5600} \quad \Rightarrow \quad k = -1.24 * 10^{-4}$$

Entonces:

$$Q(t) = Q_0 e^{-1.24*10^{-4}t}$$

Nos dice que se ha encontrado un hueso cuya cantidad de carbono es 1/1000 de la cantidad original, es decir Q(t) = $\frac{Q_0}{1000}$

$$\frac{Q_0}{1000} = Q_0 e^{-1.24*10^{-4}t} \quad \Rightarrow \quad \frac{1}{1000} = e^{-1.24*10^{-4}t}$$

$$ln\left(\frac{1}{1000}\right) = ln\left(e^{-1.24*10^{-4}t}\right) \quad \Rightarrow \quad -1.24 * 10^{-4}t = ln\left(\frac{1}{1000}\right)$$

Despejando "t" encontraremos la edad del hueso fosilizado

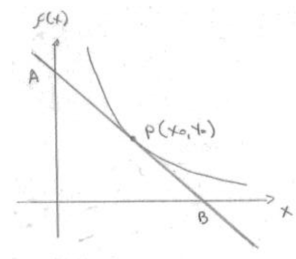
$$t = -\frac{ln(1/1000)}{1.24 * 10^{-4}} \quad \Rightarrow \quad \boxed{\boldsymbol{t \approx 55707.7 años}}$$

APLICACIONES GEOMÉTRICAS

1) Encontrar las curvas para las cuales la tangente en un punto P (x,y) y tiene intercepto sobre los ejes "x" y "y" cuya suma es 2(x+y)

Sabemos que la ecuación de una recta es: $(y - y_0) = m(x - x_0)$; $pero\ \ m = \frac{dy}{dx}$

Entonces:



$$(y - y_0) = \frac{dy}{dx}(x - x_0)$$

Encontrando el intercepto en "x" , y = 0

$$-y_0 = \frac{dy}{dx}(x - x_0) \quad \Rightarrow \quad -y_0\frac{dx}{dy} = x - x_0 \quad \Rightarrow \quad x = x_0 - y_0\frac{dx}{dy}$$

Encontrando el intercepto en "y" , x = 0

$$(y - y_0) = -x_0\frac{dy}{dx} \quad \Rightarrow \quad y = y_0 - x_0\frac{dy}{dx}$$

Pero queremos la recta tangente en un punto (x,y), es decir en (x_0, y_0), y si llamamos el intercepto en y como A y el intercepto en x como B, entonces:

$$B = x_0 - y_0\frac{dx_0}{dy_0} \quad ; \quad A = y_0 - x_0\frac{dy_0}{dx_0}$$

La condición dice que la suma de los interceptos es 2(x+y), entonces:

$$A + B = 2(x_0 + y_0)$$

$$x_0 - y_0\frac{dx_0}{dy_0} + y_0 - x_0\frac{dy_0}{dx_0} = 2(x_0 + y_0) \quad \Rightarrow \quad (x_0 + y_0) - x_0\frac{dy_0}{dx_0} - y_0\frac{dx_0}{dy_0} = 2(x_0 + y_0)$$

$$x_0\frac{dy_0}{dx_0} + y_0\frac{dx_0}{dy_0} = -(x_0 + y_0) \quad \Rightarrow \quad x_0 + x_0\frac{dy_0}{dx_0} + y_0 + y_0\frac{dx_0}{dy_0} = 0$$

$$x_0\left(1 + \frac{dy_0}{dx_0}\right) + y_0\left(1 + \frac{dx_0}{dy_0}\right) = 0 \quad \Rightarrow \quad x_0\left(\frac{dx_0 + dy_0}{dx_0}\right) + y_0\left(\frac{dx_0 + dy_0}{dy_0}\right) = 0 \quad \Rightarrow \quad \left(\frac{x_0}{dx_0} + \frac{y_0}{dy_0}\right)(dx_0 + dy_0) = 0$$

Entonces:

$$\frac{x_0}{dx_0} + \frac{y_0}{dy_0} = 0 \quad ; \quad dx_0 + dy_0 = 0$$

Resolviendo cada ecuación:

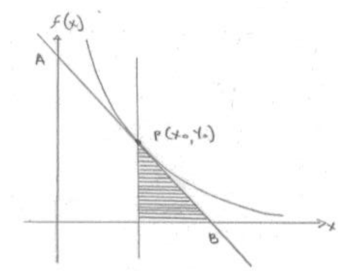
$$dy_0 = -dx_0 \quad \Rightarrow \quad \int dy_0 = -\int dx_0 \quad \Rightarrow \quad y = -x + c \ \ (ecuación\ de\ recta)$$

$$\frac{y_0}{dy_0} = -\frac{x_0}{dx_0} \quad \Rightarrow \quad \int \frac{dy_0}{y_0} = -\int \frac{dx_0}{x_0} \quad \Rightarrow \quad ln(y) = -ln(x) + c \quad \Rightarrow \quad y = x^{-1}A \ \ (ecuación\ de\ una\ curva)$$

La curva a encontrar es: $\boxed{\boldsymbol{yx = A}}$

2) Hallar la ecuación de todas las curvas del plano xy que tienen la propiedad de que el triángulo formado por la tangente a la curva, el eje "x" y la recta vertical que pasa por el punto de tangencia siempre tiene un área igual a la suma de los cuadrados de las coordenadas del punto de tangencia.

$(y - y_0) = \frac{dy}{dx}(x - x_0)$



Encontrando el intercepto en "x" , y = 0

$$-y_0 = \frac{dy}{dx}(x - x_0) \quad \Rightarrow \quad -y_0 \frac{dx}{dy} = x - x_0 \quad \Rightarrow \quad B = x_0 - y_0 \frac{dx_0}{dy_0}$$

Encontrando el intercepto en "y" , x = 0

$$(y - y_0) = -x_0 \frac{dy}{dx} \quad \Rightarrow \quad A = y_0 - x_0 \frac{dy_0}{dx_0}$$

El área del triángulo formado según el problema es:

$$Área = {x_0}^2 + {y_0}^2 \quad \Rightarrow \quad \frac{(B - x_0)y_0}{2} = {x_0}^2 + {y_0}^2$$

$$\frac{\left(x_0 - y_0 \frac{dx_0}{dy_0} - x_0\right)y_0}{2} = {x_0}^2 + {y_0}^2 \quad \Rightarrow \quad -{y_0}^2 \frac{dx_0}{dy_0} = 2({x_0}^2 + {y_0}^2) \quad \Rightarrow \quad \frac{dx_0}{dy_0} = -2\left(\frac{x_0}{y_0}\right)^2 - 2$$

Entonces:

$$t = \frac{x_0}{y_0} \quad \Rightarrow \quad x_0 = y_0 t \quad \Rightarrow \quad \frac{dx_0}{dy_0} = t + y_0 \frac{dt}{dy_0}$$

Reemplazando:

$$t + y_0 \frac{dt}{dy_0} = -2t^2 - 2 \quad \Rightarrow \quad y_0 \frac{dt}{dy_0} = -2t^2 - 2 - t \quad \Rightarrow \quad \int \frac{dt}{2t^2 + t + 2} = -\int \frac{dy_0}{y_0}$$

$$\int \frac{dt}{2\left(t^2 + \frac{1}{2}t + 1\right)} = -ln(y) + c \quad \Rightarrow \quad \frac{1}{2}\int \frac{dt}{\left(t^2 + \frac{1}{2}t + \frac{1}{16}\right) + 1 - \frac{1}{16}} = -ln(y) + c$$

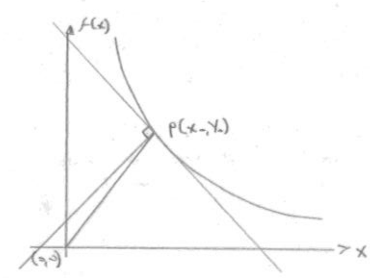
$$\frac{1}{2}\int \frac{dt}{\left(t + \frac{1}{4}\right)^2 + \frac{15}{16}} = -ln(y) + c \quad \Rightarrow \quad \frac{1}{2}\left(\frac{4}{\sqrt{15}}\right) Arctan\left[\frac{4\left(t + \frac{1}{4}\right)}{\sqrt{15}}\right] = -ln(y) + c$$

La ecuación en forma implícita de la curva es:

$$\boxed{\frac{\mathbf{2}}{\sqrt{\mathbf{15}}} \boldsymbol{Arctan}\left[\frac{\mathbf{4}\left(\frac{\boldsymbol{x}}{\boldsymbol{y}} + \frac{\mathbf{1}}{\mathbf{4}}\right)}{\sqrt{\mathbf{15}}}\right] = -\boldsymbol{ln}(\boldsymbol{y}) + \boldsymbol{c}}$$

3) Hallar la ecuación de todas las curvas que tienen la propiedad de que la distancia de cualquier punto al origen, es igual a la longitud del segmento de normal entre el punto y el intercepto con el eje x.

Tenemos que encontrar la ecuación de una recta normal en dicho punto, pero sabemos que



$m_{tangente} * m_{normal} = -1$,

Entonces:

$$m_{tangente} * m_{normal} = -1 \quad \Rightarrow \quad \frac{dy_0}{dx_0} * m_{normal} = -1 \quad \Rightarrow \quad m_{normal} = -\frac{dx_0}{dy_0}$$

Ahora la ecuación de la recta norma es:

$$(y - y_0) = m_{normal}\,(x - x_0)$$

Encontrando el intercepto en "x" , y = 0

$$-y_0 = -\frac{dx_0}{dy_0}(x - x_0) \quad \Rightarrow \quad y_0\frac{dy_0}{dx_0} = x - x_0 \quad \Rightarrow \quad x = x_0 + y_0\frac{dy_0}{dx_0}$$

Planteando la condición:

$d_1 = d_2$; siendo d_1 la distancia entre el punto y el intercepto con "x" y d_2 la distancia entre el punto y el origen.

Entonces:

$$\sqrt{(x_0 - x)^2 + (y_0 - 0)^2} = \sqrt{(x_0 - 0)^2 + (y_0 - 0)^2}$$

$$(x_0 - x)^2 + {y_0}^2 = {x_0}^2 + {y_0}^2 \quad \Rightarrow \quad \left(x_0 - x_0 - y_0\frac{dy_0}{dx_0}\right)^2 + {y_0}^2 = {x_0}^2 + {y_0}^2$$

$$\left(y_0\frac{dy_0}{dx_0}\right)^2 + {y_0}^2 = {x_0}^2 + {y_0}^2 \quad \Rightarrow \quad {y_0}^2\left(\frac{dy_0}{dx_0}\right)^2 = {x_0}^2 \quad \Rightarrow \quad \left(\frac{dy_0}{dx_0}\right)^2 = \left(\frac{x_0}{y_0}\right)^2$$

$$\int y\,dy = \int x\,dx \quad \Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + c$$

Finalmente la ecuación de todas las curvas es:

$$\boldsymbol{y = \pm\sqrt{2\left(\frac{x^2}{2} + c\right)}}$$

CRECIMIENTO LOGÍSTICO

1) Encuentre el modelo de la ecuación diferencial si P(t) representa el número de individuos en una población con cierta enfermedad contagiosa incurable y esta se extiende por encuentros casuales entonces la razón de cambio del número de personas infectadas con respecto al tiempo, es proporcional al número de personas infectadas en ese momento y al número de personas no infectadas en ese momento.

$$\frac{dP}{dt} = K \overbrace{P}^{infectados} \underbrace{(M-P)}_{no\ infectados}$$

Suponga es ese modelo en el tiempo t = 0 , 10000 personas en una ciudad con una población de 100000 han oído un cierto rumor. Después de una semana el número de personas de aquellas que la escucharon se incrementa a 20000 ¿Cuándo escucharan el 80% de la población el rumor?

$$\frac{dP}{dt} = KP(100000 - P)$$

$$\int \frac{dP}{P(100000-P)} = K\int dt$$

Resolviendo por fracciones parciales, tenemos:

$$\frac{1}{P(100000-P)} = \frac{A}{P} + \frac{B}{100000-P}$$

$$1 = A(100000-P) + BP \quad \Rightarrow \quad 1 = (B-A)P + 100000\,A$$

$$1 = 10000000\,A$$

$$0 = (B-A)P$$

Resolviendo el sistema A = 1/1000000 , B = 1/1000000

Entonces:

$$\int \left(\frac{A}{P} + \frac{B}{100000-P}\right) dP = K\int dt$$

$$\frac{1}{1000000}[ln|P| - ln|100000-P|] = Kt + c$$

$Sabemos\ que\ en\ \ t = 0\ , P = 10000$

$$\frac{1}{1000000}[ln|10000| - ln|100000-10000|] = c$$

$$c = \frac{1}{1000000} ln\left(\frac{1}{9}\right)$$

Además nos dice que después de una semana el número de personas infectadas es 20000, entonces en t = 1 , P = 20000

$$\frac{1}{1000000}[ln|20000| - ln|100000 - 20000|] = K + \frac{1}{1000000} ln\left(\frac{1}{9}\right)$$

$$K = \frac{1}{1000000} ln\left(\frac{1}{4}\right) - \frac{1}{1000000} ln\left(\frac{1}{9}\right)$$

$$K = \frac{1}{1000000} ln\left(\frac{9}{4}\right)$$

Entonces nuestra ecuación nos queda lo siguiente:

$$\frac{1}{1000000} ln\left(\frac{P}{100000 - P}\right) = \frac{1}{1000000} ln\left(\frac{9}{4}\right)\ t + \frac{1}{1000000} ln\left(\frac{1}{9}\right)$$

Pero el problema nos pide el tiempo en que el 80% de la población total escuchan el rumor, es decir P(t) = 80000

$$\frac{1}{1000000} ln\left(\frac{80000}{100000 - 80000}\right) = \frac{1}{1000000} ln\left(\frac{9}{4}\right)\ t + \frac{1}{1000000} ln\left(\frac{1}{9}\right)$$

$$ln(4) = ln\left(\frac{9}{4}\right)\ t + ln\left(\frac{1}{9}\right)$$

$$ln(4) - ln\left(\frac{1}{9}\right) = ln\left(\frac{9}{4}\right)\ t$$

$$ln(36) = ln\left(\frac{9}{4}\right)\ t$$

Entonces:

$$t = \frac{ln(36)}{ln\left(\frac{9}{4}\right)} \quad \Rightarrow \quad \boxed{\boldsymbol{t \approx 4\ semanas}}$$

2) Suponga que en una comunidad cuenta con 15000 personas que son susceptibles de adquirir una enfermedad contagiosa, en el tiempo t = 0, el número de personas que han desarrollado la enfermedad es 5000 y este se incrementa a una tasa proporcional al producto del número de aquellas que han adquirido la enfermedad y de aquellas que no. (Considere que la tasa inicial de incremento es de 500 sujetos por día). ¿Cuánto tiempo pasara para que otras personas desarrollen la enfermedad?

Planteando el modelo:

$$\frac{dN}{dt} = KN\,(15000 - N)$$

$$\int \frac{dN}{N(15000 - N)} = K \int dt$$

Entonces:

$$\frac{1}{15000} ln\left(\frac{N}{15000 - N}\right) = Kt + c$$

Pero en t = 0 , N = 5000

$$\frac{1}{15000} ln\left(\frac{5000}{15000 - 5000}\right) = c$$

$$c = \frac{1}{15000} ln\left(\frac{1}{2}\right)$$

Además nos dice que consideremos que la tasa inicial de incremento es 500 personas por día, es decir $\frac{dN}{dt} = 500$

$$\frac{dN}{dt} = K(5000)(15000 - 5000)$$

$$500 = K(5000)(15000 - 5000)$$

$$K = \frac{1}{100000}$$

Finalmente nos dice que, cuanto tiempo pasará para que otras 5000 personas se infecte, es decir el número total de infectados será de 10000, entonces:

$$\frac{1}{15000} ln\left(\frac{10000}{15000 - 10000}\right) = \frac{1}{100000} t + \frac{1}{15000} ln\left(\frac{1}{2}\right)$$

$$\frac{1}{15000} ln(2) - \frac{1}{15000} ln\left(\frac{1}{2}\right) = \frac{1}{100000} t$$

$$\frac{1}{15000} ln(4) = \frac{1}{100000} t$$

Entonces:

$$t = \frac{20}{3} ln(4) \quad \Rightarrow \quad \boxed{\boldsymbol{t \approx 9\ días}}$$

APLICACIONES A LA FÍSICA

1) Una pelota se proyecta hacia arriba desde el piso con una velocidad v_0 y la resistencia del aire es proporcional al cuadrado de la velocidad. Demuestre que la altura máxima que alcanza la pelota es $y_{máx} = \frac{1}{2\rho} ln\left(1 + \rho \frac{{v_0}^2}{g}\right)$, donde $\rho = \frac{k}{m}$

$$\sum F = ma \quad \Rightarrow \quad \sum F = m\frac{dv}{dt}$$

$$-mg - kv^2 = m\frac{dv}{dt} \quad \Rightarrow \quad -mg - \rho m v^2 = m\frac{dv}{dt}$$

$$-g - \rho v^2 = \frac{dv}{dt} \quad \Rightarrow \quad \frac{dv}{g + \rho v^2} = -dt$$

$$\int \frac{dv}{g + \rho v^2} = -\int dt \quad \Rightarrow \quad \frac{1}{g}\int \frac{dv}{\frac{g}{\rho} + \frac{\rho v^2}{\rho}} = -\int dt$$

$$\frac{1}{\rho}\int \frac{dv}{\frac{g}{\rho} + v^2} = -\int dt \quad \Rightarrow \quad \frac{1}{\rho}\left(\frac{1}{\sqrt{\frac{g}{\rho}}}\right) Arctan\left(\frac{v}{\sqrt{\frac{g}{\rho}}}\right) = -t + c$$

$$\frac{1}{\rho}\left(\sqrt{\frac{\rho}{g}}\right) Arctan\left(\sqrt{\frac{\rho}{g}}\,v\right) = -t + c \quad \Rightarrow \quad \frac{1}{\sqrt{\rho g}} Arctan\left(\sqrt{\frac{\rho}{g}}\,v\right) = -t + c$$

$$Arctan\left(\sqrt{\frac{\rho}{g}}\,v\right) = -t\sqrt{\rho g} + \overbrace{c\sqrt{\rho g}}^{constante} \quad \Rightarrow \quad \sqrt{\frac{\rho}{g}}\,v = Tan\left(-t\sqrt{\rho g} + b\right)$$

La ecuación que describe la velocidad de la pelota en cualquier tiempo "t" es:

$$\boxed{v(t) = \sqrt{\frac{g}{\rho}}\,Tan\left(-t\sqrt{\rho g} + b\right)}$$

El problema nos dice que en $t = 0$ la velocidad de la pelota es v_0, es decir $v(0) = v_0$

$$v_0 = \sqrt{\frac{g}{\rho}}\,Tan(b) \quad \Rightarrow \quad b = Tan^{-1}\left(\sqrt{\frac{\rho}{g}}\,v_0\right)$$

Pero sabemos que:

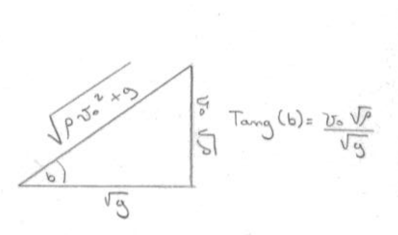
$$v(t) = \frac{dx}{dt} \quad \Rightarrow \quad dx = v(t)dt \quad \Rightarrow \quad x(t) = \int v(t)dt \quad ; \quad \text{pero en este caso} \quad y(t) = \int v(t)dt$$

$$y(t) = \int \sqrt{\frac{g}{\rho}}\,Tan\left(-t\sqrt{\rho g} + b\right)dt \quad \Rightarrow \quad y(t) = \sqrt{\frac{g}{\rho}}\int \frac{Sen\left(-t\sqrt{\rho g} + b\right)}{Cos\left(-t\sqrt{\rho g} + b\right)}dt$$

$$u = Cos\left(-t\sqrt{\rho g} + b\right) \quad \Rightarrow \quad du = -Sen\left(-t\sqrt{\rho g} + b\right)\left(-\sqrt{\rho g}\right) \quad \Rightarrow \quad du = \left(\sqrt{\rho g}\right)Sen\left(-t\sqrt{\rho g} + b\right)$$

$$y(t) = \sqrt{\frac{g}{\rho}}\left(\frac{1}{\sqrt{\rho g}}\right)\int\frac{du}{u} \quad \Rightarrow \quad y(t) = \frac{1}{\rho}ln\left|Cos\left(-t\sqrt{\rho g}+b\right)\right| + a$$

Sabemos que en $t = 0$, $y = 0$



$$0 = \frac{1}{\rho}ln|Cos(b)| + a \quad \Rightarrow \quad a = -\frac{1}{\rho}ln|Cos(b)|$$

$$a = -\frac{1}{\rho}ln\left|\frac{\sqrt{g}}{\sqrt{\rho {v_0}^2 + g}}\right|$$

La ecuación que describe la posición de la pelota en cualquier tiempo "t" es:

$$y(t) = \frac{1}{\rho}ln\left|Cos\left(-t\sqrt{\rho g}+b\right)\right| - \frac{1}{\rho}ln|Cos(b)|$$

$$\boxed{y(t) = \frac{1}{\rho}ln\left|Cos\left(-t\sqrt{\rho g}+b\right)\right| - \frac{1}{\rho}ln\left|\frac{\sqrt{g}}{\sqrt{\rho {v_0}^2 + g}}\right|}$$

El problema nos dice que determinemos la altura máxima que alcanza la pelota, entonces en el punto más alto la velocidad de la pelota es cero

$$0 = \sqrt{\frac{g}{\rho}}Tan\left(-t_{máx}\sqrt{\rho g}+b\right)$$

Como el resultado de esa ecuación es cero, entonces el ángulo tiene que ser ***π*** *ó* ***0*** *es decir:*

$$0 = \sqrt{\frac{g}{\rho}}Tan\overbrace{\left(-t_{máx}\sqrt{\rho g}+b\right)}^{\pi \text{ ó } 0}$$

Entonces:

$$y_{máx} = \frac{1}{\rho}ln\left|Cos\left(-t_{máx}\sqrt{\rho g}+b\right)\right| - \frac{1}{\rho}ln\left|\frac{\sqrt{g}}{\sqrt{\rho {v_0}^2 + g}}\right|$$

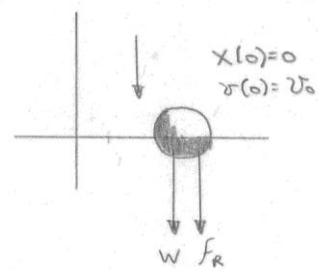
$$y_{máx} = \frac{1}{\rho}ln\left|\frac{Cos\overbrace{\left(-t_{máx}\sqrt{\rho g}+b\right)}^{0}}{\frac{\sqrt{g}}{\sqrt{\rho {v_0}^2 + g}}}\right| \quad \Rightarrow \quad y_{máx} = \frac{1}{\rho}ln\left|\frac{Cos(0)}{\frac{\sqrt{g}}{\sqrt{\rho {v_0}^2 + g}}}\right| \quad \Rightarrow \quad y_{máx} = \frac{1}{\rho}ln\left|\frac{\sqrt{\rho {v_0}^2 + g}}{\sqrt{g}}\right|$$

Pero $\frac{\sqrt{\rho {v_0}^2 + g}}{\sqrt{g}}$ *siempre nos dará como resultado un número positivo, por lo tanto el valor absoluto lo podemos despreciar.*

$$y_{máx} = \frac{1}{\rho}ln\sqrt{\frac{\rho {v_0}^2 + g}{g}} \quad \Rightarrow \quad y_{máx} = \frac{1}{\rho}ln\left(\frac{\rho {v_0}^2}{g}+1\right)^{1/2} \quad \Rightarrow \quad y_{máx} = \frac{1}{\rho}\left(\frac{1}{2}\right)ln\left(\frac{\rho {v_0}^2}{g}+1\right) \;\; ; \;\; y\ finalmente$$

$$\boxed{\boldsymbol{y_{máx} = \frac{1}{2\rho}ln\left(\frac{\rho {v_0}^2}{g}+1\right)}}$$

2) Un cuerpo con masa "m" se lanza verticalmente hacia abajo con una velocidad inicial v_0 en un medio que ofrece una resistencia proporcional a la raíz cuadrada de la magnitud de la velocidad. Encuentre la relación que hay entre la velocidad y el tiempo "t". Determine la velocidad límite.



$$\sum F = ma \quad \Rightarrow \quad \sum F = m\frac{dv}{dt}$$

$$mg - k\sqrt{v} = m\frac{dv}{dt}$$

$$\frac{dv}{mg - k\sqrt{v}} = \frac{dt}{m}$$

$$\int \frac{dv}{mg - k\sqrt{v}} = \int \frac{dt}{m}$$

$$v = u^2 \quad \Rightarrow \quad dv = 2u\,du$$

$$\int \frac{dt}{m} = 2\int \frac{u}{mg - ku}du$$

Realizando la división entre polinomios, nos queda:

$$\int \frac{dt}{m} = 2\int \left(-\frac{1}{k} + \frac{\frac{mg}{k}}{mg - ku}\right)du$$

$$\int \frac{dt}{m} = 2\left(-\int \frac{1}{k}du + \frac{mg}{k}\int \frac{du}{mg - ku}\right)$$

$$\frac{1}{m}t = -\frac{2}{k}u - \frac{2mg}{k^2}ln|mg - ku| + c$$

La ecuación en forma implícita que describe la posición del cuerpo en cualquier tiempo "t" es:

$$\frac{1}{m}t = -\frac{2}{k}\sqrt{v} - \frac{2mg}{k^2}ln|mg - k\sqrt{v}| + c$$

Pero sabemos que en $t = 0$, $v = v_0$

$$0 = -\frac{2}{k}\sqrt{v_0} - \frac{2mg}{k^2}ln|mg - k\sqrt{v_0}| + c$$

$$c = \frac{2}{k}\sqrt{v_0} + \frac{2mg}{k^2}ln|mg - k\sqrt{v_0}|$$

Entonces nuestra ecuación finalmente nos queda:

$$\frac{1}{m}t = -\frac{2}{k}\sqrt{v} - \frac{2mg}{k^2}ln|mg - k\sqrt{v}| + \frac{2}{k}\sqrt{v_0} + \frac{2mg}{k^2}ln|mg - k\sqrt{v_0}|$$

La relación que existe entre la velocidad y el tiempo, es que, la velocidad límite es cuando "t" tiende al infinito.

$$\infty = \frac{2}{k}\left(\sqrt{v_0} - \sqrt{v}\right) + \frac{2mg}{k^2} ln\left|\frac{mg - k\sqrt{v_0}}{mg - k\sqrt{v}}\right|$$

Si el resultado de esa ecuación es ∞ , entonces:

$$\infty = \overbrace{\left[\frac{2}{k}\left(\sqrt{v_0} - \sqrt{v_{lím}}\right)\right]}^{algo\ \ grande} + \frac{2mg}{k^2}\left[\overbrace{ln\left|\frac{mg - k\sqrt{v_0}}{mg - k\sqrt{v_{lím}}}\right|}^{\infty}\right]$$

Para que:

$$ln\left|\frac{mg - k\sqrt{v_0}}{mg - k\sqrt{v_{lím}}}\right| = \infty \ \ ; \ \ entonces \ \ \ mg - k\sqrt{v_{lím}} = 0$$

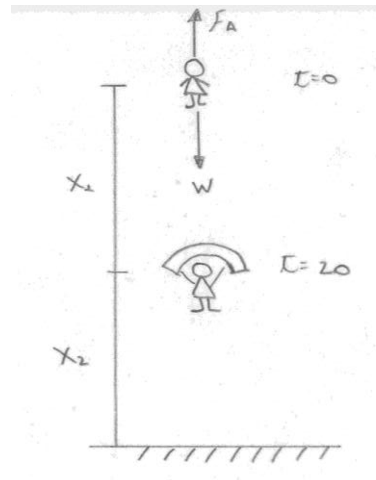
Por lo tanto, la velocidad límite es:

$$mg - k\sqrt{v_{lím}} = 0$$

$$mg = k\sqrt{v_{lím}}$$

$$\boxed{\boldsymbol{v_{lím} = \left(\frac{mg}{k}\right)^2}}$$

3) Una mujer que se lanza en paracaídas desde un avión a una altitud de 10000 pies cae libremente por 20 seg y entonces abre el paracaídas. Si se conoce que la resistencia debido al aire es proporcional a la velocidad (la constante de proporcionalidad es 0.15 sin paracaídas y 1.5 con paracaídas). Encontrar la ecuación en forma implícita del tiempo que demorar la mujer en llegar al piso



Para el 1er tramo tenemos

$$\sum F = ma \quad \Rightarrow \quad \sum F = m\frac{dv}{dt}$$

$$mg - kv = m\frac{dv}{dt} \quad \Rightarrow \quad mg - 0.15v = m\frac{dv}{dt}$$

Pero sabemos que:

$$v(t) = \frac{dx}{dt} \quad \Rightarrow \quad dx = v(t)dt \quad \Rightarrow \quad x(t) = \int v(t)dt$$

Entonces:

$$\frac{dv}{dt} + \frac{0.15}{m}dv = g$$

Resolviendo la ecuación diferencial:

$$\mu(t) = e^{\int p(t)dt} \quad \Rightarrow \quad \mu(t) = e^{\int \frac{0.15}{m}dt} \quad \Rightarrow \quad \mu(x) = e^{\frac{0.15}{m}t}$$

$$\left(e^{\frac{0.15}{m}t}\right)\frac{dv}{dt} + \left(e^{\frac{0.15}{m}t}\right)\frac{0.15}{m}dv = \left(e^{\frac{0.15}{m}t}\right)g$$

$$\frac{d}{dt}\left(e^{\frac{0.15}{m}t}\, v\right) = \left(e^{\frac{0.15}{m}t}\right)g \quad \Rightarrow \quad \int d\left(e^{\frac{0.15}{m}t}\, v\right) = \int \left(e^{\frac{0.15}{m}t}\right)g\, dt \quad \Rightarrow \quad e^{\frac{0.15}{m}t}\, v = \frac{gm}{0.15}e^{\frac{0.15}{m}t} + c$$

Por lo tanto, la ecuación que describe la velocidad para cualquier tiempo "t" es:

$$v_1(t) = \frac{gm}{0.15} + ce^{-\frac{0.15}{m}t}$$

Pero en $t = 0$, $v = 0$

$$0 = \frac{gm}{0.15} + c \quad \Rightarrow \quad c = -\frac{gm}{0.15}$$

Finalmente nuestra ecuación de velocidad, nos queda:

$$v_1(t) = \frac{gm}{0.15} - \frac{gm}{0.15}e^{-\frac{0.15}{m}t}\ ; \quad si\ llamamos\ k = \frac{gm}{0.15}\ ; \quad entonces$$

$$v_1(t) = \frac{gm}{0.15} - \frac{gm}{0.15}e^{-\frac{0.15}{gm}gt} \quad \Rightarrow \quad v_1(t) = k - ke^{-\frac{gt}{k}}$$

$$\boxed{v_1(t) = k - ke^{-\frac{10t}{k}}}$$

La velocidad de la mujer hasta que abra el paracaídas, es decir después de 20 seg es:

$$v_1(20) = k - ke^{-\frac{200}{k}}$$

Sabemos que:

$$x(t) = \int v(t)dt$$

$$x_1(t) = \int \left(k - ke^{-\frac{10t}{k}}\right)dt \quad \Rightarrow \quad x_1(t) = kt - \frac{k^2}{10}e^{-\frac{10t}{k}} + a$$

Pero en $t = 0$ *,* $x = 0$

$$0 = \frac{k^2}{10} + a \quad \Rightarrow \quad a = -\frac{k^2}{10}$$

La ecuación que describe la posición de la mujer durante el 1er tramo es:

$$\boxed{x_1(t) = kt - \frac{k^2}{10}e^{-\frac{10t}{k}} - \frac{k^2}{10}}$$

La posición de la mujer en $t = 20$ *seg es:*

$$x_1(20) = 20k - \frac{k^2}{10}e^{-\frac{200}{k}} - \frac{k^2}{10}$$

<u>Para el 2do tramo, tenemos que:</u>

$$mg - kv = m\frac{dv}{dt} \quad \Rightarrow \quad mg - 1.5v = m\frac{dv}{dt}$$

Realizando todo el procedimiento anterior para encontrar la ecuación de la velocidad, nos queda lo siguiente:

$$v_2(t) = \frac{gm}{1.5} + ze^{-\frac{1.5}{gm}gt}$$

Realizando un artificio para que esta ecuación nos quede en término de "k"

$$v_2(t) = \frac{\frac{gm}{10}}{\frac{1.5}{10}} + ze^{-\frac{\frac{1.5}{10}}{\frac{gm}{10}}gt} \quad \Rightarrow \quad v_2(t) = \frac{1}{10}\left(\frac{gm}{0.15}\right) + ze^{-\frac{10}{k}gt}$$

$$v_2(t) = \frac{k}{10} + ze^{-\frac{100}{k}t}$$

Pero sabemos que la velocidad final del 1er tramo es la velocidad inicial del 2do tramo, entonces en $t = 0$ *, la velocidad es* $k - ke^{-\frac{200}{k}}$ *, por lo tanto:*

$$\frac{k}{10} + ze^{-\frac{100}{k}(0)} = k - ke^{-\frac{200}{k}} \quad \Rightarrow \quad \frac{k}{10} + z = k - ke^{-\frac{200}{k}} \quad \Rightarrow \quad z = \frac{9}{10}k - ke^{-\frac{200}{k}}$$

Entonces la ecuación que describe la velocidad para cualquier tiempo "t" durante el 2do tramo es:

$$\boxed{v_2(t)=\frac{k}{10}+\left(\frac{9}{10}k-ke^{-\frac{200}{k}}\right)e^{-\frac{100}{k}t}}$$

Encontrando la ecuación que describe la posición:

$$x(t)=\int v(t)dt$$

$$x_2(t)=\int\left[\frac{k}{10}+\left(\frac{9}{10}k-ke^{-\frac{200}{k}}\right)e^{-\frac{100}{k}t}\right]dt \quad\Rightarrow\quad x_2(t)=\frac{k}{10}\int dt+\left(\frac{9}{10}k-ke^{-\frac{200}{k}}\right)\int e^{-\frac{100}{k}t}dt$$

$$x_2(t)=\frac{k}{10}t-\left(\frac{9}{10}k-ke^{-\frac{200}{k}}\right)e^{-\frac{100}{k}t}+b$$

Si consideramos un nuevo sistema de referencia para el 2do tramo, entonces en $t=0$, $x=0$

$$0=-\left(\frac{9}{10}k-ke^{-\frac{200}{k}}\right)+b \quad\Rightarrow\quad b=\frac{9}{10}k-ke^{-\frac{200}{k}}$$

Finalmente la ecuación que describe la posición de la mujer durante el 2do tramo es:

$$\boxed{x_2(t)=\frac{k}{10}t-\left(\frac{9}{10}k-ke^{-\frac{200}{k}}\right)e^{-\frac{100}{k}t}+\frac{9}{10}k-ke^{-\frac{200}{k}}}$$

Sabemos que el tiempo que demora en recorrer $\boldsymbol{x_1}$ *y* $\boldsymbol{x_2}$ *, es el tiempo que tomará en llegar al piso, por lo tanto:*

$$x_1+x_2=10000 \quad\Rightarrow\quad x_2=10000-x_1$$

Entonces nuestra ecuación final, nos quedará:

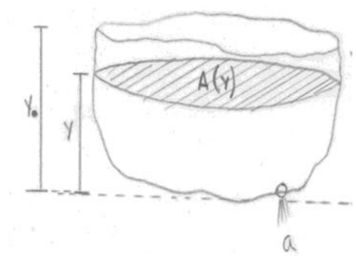
$$\frac{k}{10}t-\left(\frac{9}{10}k-ke^{-\frac{200}{k}}\right)e^{-\frac{100}{k}t}+\frac{9}{10}k-ke^{-\frac{200}{k}}=10000-20k-\frac{k^2}{10}e^{-\frac{200}{k}}-\frac{k^2}{10}$$

$$\frac{\left(\frac{gm}{0.15}\right)}{10}t-\left(\frac{gm}{0.15}\right)\left(\frac{9}{10}-e^{-\frac{200(0.15)}{gm}}\right)e^{-\frac{100(0.15)}{gm}t}+\left(\frac{9}{10}-e^{-\frac{200(0.15)}{gm}}\right)=10000-20\left(\frac{gm}{0.15}\right)-\frac{\left(\frac{gm}{0.15}\right)^2}{10}e^{-\frac{200(0.15)}{gm}}-\frac{\left(\frac{gm}{0.15}\right)^2}{10}$$

$$\boxed{\boldsymbol{\frac{3}{2}gmt-\left(\frac{gm}{0.15}\right)\left(\frac{9}{10}-e^{-\frac{30}{gm}}\right)e^{-\frac{15}{gm}t}+\left(\frac{9}{10}-e^{-\frac{30}{gm}}\right)=10000-\frac{400}{30}gm-\frac{9}{40}(gm)^2e^{-\frac{30}{gm}}-\frac{9}{40}(gm)^2}}$$

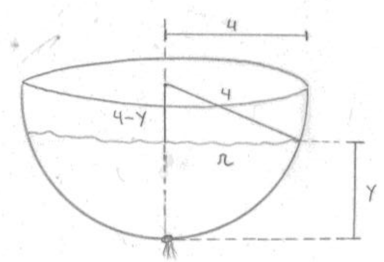
LEY DE TORRICELLI

Esta ley nos dice como cambia la altura con respecto al tiempo, es decir:



$$A(y)\frac{dy}{dt} = -K\sqrt{y} \quad ; \quad donde \;\; K = a\sqrt{2g} \;\; y \; a \; es \; el \; área \; del \; orificio \; de \; salida$$

Un tanque semiesférico tiene un radio de 4 pies, en t = 0 está lleno de agua, en ese momento se hace un orificio circular con un diámetro de una pulgada en el fondo del tanque ¿Cuánto tiempo tomará a toda el agua salir?



$$A(y)\frac{dy}{dt} = -K\sqrt{y} \quad \Rightarrow \quad \pi r^2\frac{dy}{dt} = -K\sqrt{y}$$

Pero como podemos observar:

$$(4)^2 = (4-y)^2 + r^2 \quad \Rightarrow \quad 16 = 16 - 8y + y^2 + r^2 \quad \Rightarrow \quad r^2 = 8y - y^2$$

Reemplazando:

$$\pi(8y - y^2)\frac{dy}{dt} = -K\sqrt{y} \quad \Rightarrow \quad \pi\int\frac{8y - y^2}{\sqrt{y}}dy = -K\int dt \quad \Rightarrow \quad \pi\int\left(8y^{1/2} - y^{3/2}\right)dy = -K\int dt$$

$$\pi\left(\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right) = -Kt + c$$

Pero en t = 0 , y = 4

$$\pi\left[\frac{16}{3}(4)^{3/2} - \frac{2}{5}(4)^{5/2}\right] = c \quad \Rightarrow \quad c = \frac{448}{15}\pi$$

Entonces:

$$\pi\left(\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right) = -Kt + \frac{448}{15}\pi$$

Nos pide que determinemos el tiempo en que demora en vaciarse el tanque, es decir y(t) = 0

$$0 = -Kt + \frac{448}{15}\pi \quad \Rightarrow \quad t = \frac{448}{15K}\pi \quad \Rightarrow \quad t = \frac{448\pi}{15}\left[\frac{1}{\pi}\left(\frac{2}{d}\right)^2 \frac{1}{\sqrt{2g}}\right]$$

Hay que recordar que hay que convertir a pies

$$t = \frac{1792}{15\,(0.8333)^2\,\sqrt{2(32.15)}} \approx 2162.64\ seg$$

$$\boxed{\boldsymbol{t \approx 36.044\ minutos}}$$

REACCIONES QUÍMICAS

Cuando se combinan dos sustancia A y B, se forma un compuesto C. L a reacción entre ambas es tal que, por cada gramo de A se usan 4 gramos de B. Se observa que a los 10 minutos se han formado 30 gramos del producto C. Calcule la cantidad de C en función del tiempo si la velocidad de la reacción es proporcional a las cantidades A y B que quedan y al principio hay 50 gramos de A y 32 gramos de B. ¿Qué cantidad de compuesto C hay a los 15 minutos? Interprete la solución cuando t → ∞

Sean x(t) los gramos del compuesto C presentes en un tiempo t.

Si por ejemplo, hay 2 gramos del producto C, hemos debido de usar, digamos "a" gramos de A y "b" gramos de B, de tal modo que a+b = 2 y b = 4a ; por consiguiente , reemplazando nos queda que a = 2/5 = 2(1/5)g de la sustancia A y b = 8/5 =2(4/5)g de B. En general, para obtener x gramos de C debemos emplear.

$$\frac{x}{5} g\ de\ A \quad y \quad \frac{4}{5} x\ g\ de\ B$$

Entonces las cantidades de A y B que quedan en cualquier momento son:

$$50 - \frac{x}{5} \quad y \quad 32 - \frac{4}{5}x$$

Respectivamente, sabemos que la rapidez de formación del compuesto C está definida por:

$$\frac{dx}{dt} \propto \left(50 - \frac{x}{5}\right)\left(32 - \frac{4}{5}x\right)$$

Para simplificar las operaciones algebraicas, sacaremos a 1/5 como factor común del primer término, 4/5 del segundo término e introduciremos la constante de proporcionalidad.

$$\frac{dx}{dt} = k(250 - x)(40 - x)$$

Separamos variables y por fracciones parciales llegamos a:

$$-\frac{1/210}{250 - x}dx + \frac{1/210}{40 - x}dx = k\ dt$$

Integrando:

$$\int \left(-\frac{1/210}{250 - x} + \frac{1/210}{40 - x}\right)dx = k \int dt$$

$$\frac{1}{210} ln\left|\frac{250 - x}{40 - x}\right| = kt + c$$

$$ln\left|\frac{250 - x}{40 - x}\right| = 210\ kt + a$$

Despejando "x":

$$e^{ln\left|\frac{250-x}{40-x}\right|} = e^{(210\,kt+a)} \quad \Rightarrow \quad \frac{250-x}{40-x} = be^{210\,kt}$$

Pero sabemos que en t = 0 , x = 0

$$b = \frac{25}{4}$$

Y también que en t = 10 , x = 30

$$\frac{250-30}{40-30} = \left(\frac{25}{4}\right)e^{210\,k(10)}$$

$$\frac{88}{25} = e^{210\,k(10)}$$

$$ln\left(\frac{88}{25}\right) = ln\left(e^{210\,k(10)}\right)$$

$$210\,k = \frac{1}{10}ln\left(\frac{88}{25}\right)$$

El problema nos pide la cantidad de sustancia a los 15 minutos, pero primero despejemos "x":

$$\frac{250-x}{40-x} = \left(\frac{25}{4}\right)e^{\frac{1}{10}ln\left(\frac{88}{25}\right)t}$$

$$250-x = \left(\frac{25}{4}\right)(40-x)e^{At}$$

$$1000-4x = 1000e^{At} - 25xe^{At}$$

$$x(t) = 1000\frac{(1-e^{At})}{(4-25e^{At})}$$

$$x(15) = 1000\frac{\left[1-e^{\frac{1}{10}ln\left(\frac{88}{25}\right)(15)}\right]}{\left[4-25e^{\frac{1}{10}ln\left(\frac{88}{25}\right)(15)}\right]}$$

$$\boxed{\boldsymbol{x(15) = 34.78\ g}}$$

<u>*Interpretando cuando $t \to \infty$*</u>

$$x(t) = 1000\frac{(1-e^{At})}{(4-25e^{At})} \quad ; \quad sacando\ factor\ común\ e^{At}, entonces \quad x(t) = 1000\frac{\left(\frac{1}{e^{At}}-1\right)}{\left(\frac{4}{e^{At}}-25\right)}$$

$x(\infty) = 1000\frac{\left(\frac{1}{\infty}-1\right)}{\left(\frac{4}{\infty}-25\right)}$; $esto\ quiere\ decir\ que\ cuando\ t\ \to \infty\ se\ forman$ $\boxed{\boldsymbol{40\ g\ de\ C}}$ *y quedan:*

$$\frac{40}{5} = \boxed{\boldsymbol{8\ g\ de\ A}} \quad y \quad \frac{4}{5}(40) = \boxed{\boldsymbol{32\ g\ de\ B}}$$

ECUACIONES DIFERENCIALES DE ORDEN SUPERIOR

ECUACIONES DIFERENCIALES EN LAS QUE FALTA LA VARIABLE DEPENDIENTE "y"

1) $\mathbf{2x^2y'' + (y')^3 = 2xy'}$

$$t = y' \quad \Rightarrow \quad t = \frac{dy}{dx}$$

$$\frac{dt}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \Rightarrow \quad \frac{dt}{dx} = \frac{d^2y}{dx^2}$$

$$2x^2\frac{dt}{dx} + t^3 = 2xt \quad \Rightarrow \quad 2x^2\frac{dt}{dx} - 2xt = -t^3 \quad \Rightarrow \quad \frac{dt}{dx} - \frac{t}{x} = -\frac{t^3}{2x^2}$$

$$\frac{1}{t^3}\frac{dt}{dx} - \frac{t^{-2}}{x} = -\frac{1}{2x^2}$$

$$u = t^{-2} \quad \Rightarrow \quad \frac{du}{dx} = -\frac{2}{t^3}\frac{dt}{dx}$$

$$-\frac{1}{2}\frac{du}{dx} - \frac{u}{x} = -\frac{1}{2x^2} \quad \Rightarrow \quad \frac{du}{dx} + \frac{2u}{x} = \frac{1}{x^2}$$

$$\mu(x) = e^{\int p(x)dx} \quad \Rightarrow \quad \mu(x) = e^{\int \frac{2}{x}dx} \quad \Rightarrow \quad \mu(y) = e^{2\,ln|x|}$$

$$\mu(x) = x^2$$

$$(x^2)\frac{du}{dx} + (x^2)\frac{2u}{x} = (x^2)\frac{1}{x^2}$$

$$\frac{d}{dx}(x^2\,u) = 1$$

$$\int d\,(x^2\,u) = \int dx$$

$$x^2\,u = x + c \quad \Rightarrow \quad x^2\,t^{-2} = x + c \quad \Rightarrow \quad x^2\,(y')^{-2} = x + c$$

$$y' = \pm\frac{x}{\sqrt{x+c}} \quad \Rightarrow \quad dy = \pm\frac{x}{\sqrt{x+c}}dx \quad \Rightarrow \quad \int dy = \pm\int\frac{x}{\sqrt{x+c}}dx$$

$$m = x + c \quad \Rightarrow \quad dm = dx$$

$$\int dy = \pm\int\frac{m-c}{\sqrt{m}}dx \quad \Rightarrow \quad \int dy = \pm\int\left(\sqrt{m} - \frac{c}{\sqrt{m}}\right)dx$$

$$y = \pm\left(\frac{2}{3}m^{\frac{3}{2}} - cm^{\frac{1}{2}} + d\right) \quad \Rightarrow \quad \boxed{\mathbf{y = \pm\left[\frac{2}{3}(x+c)^{\frac{3}{2}} - c(x+c)^{\frac{1}{2}} + d\right]}}$$

$\mathbf{2)}\ \boldsymbol{y'' = (x + y')^2}$

$t = x + y' \quad \Rightarrow \quad \dfrac{dt}{dx} = 1 + y''$

$\dfrac{dt}{dx} - 1 = t^2$

$\Rightarrow \quad \dfrac{dt}{dx} = t^2 + 1 \quad \Rightarrow \quad \dfrac{dt}{t^2 + 1} = dx$

$\displaystyle\int \frac{dt}{t^2 + 1} = \int dx$

$Arctan(t^2) = x + c$

$Arctan(x + y') = x + c \quad \Rightarrow \quad x + y' = Tan(x + c\,)$

$y' = Tan(x + c\,) - x \quad \Rightarrow \quad dy = [Tan(x + c\,) - x]dx$

$\displaystyle\int dy = \int \frac{Sen(x + c\,)}{Cos(x + c\,)} dx - \int x\, dx$

$u = Cos(x + c\,) \quad \Rightarrow \quad du = -Sen(x + c\,)$

$\displaystyle\int dy = -\int \frac{1}{u} dx - \int x\, dx \quad \Rightarrow \quad y = -ln|u| - \frac{x^2}{2} + d$

$$\boxed{\boldsymbol{y = -ln|Cos(x + c\,)| - \frac{x^2}{2} + d}}$$

ECUACIONES DIFERENCIALES EN LAS QUE FALTA LA VARIABLE INDEPENDIENTE "x"

1) $\mathbf{2y^2y'' + 2y(y')^2 = 1}$

$t = \frac{dy}{dx}$

$\frac{dt}{dx} = \frac{d^2y}{dx^2} \quad \Rightarrow \quad \frac{dt}{dx}\left(\frac{dy}{dy}\right) = \frac{d^2y}{dx^2} \quad \Rightarrow \quad \frac{dt}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \quad \Rightarrow \quad \frac{dt}{dy}(t) = \frac{d^2y}{dx^2}$

$2y^2t\frac{dt}{dy} + 2yt^2 = 1$

$\frac{dt}{dy} + \frac{2yt^2}{2y^2t} = \frac{1}{2y^2t}$

$t\frac{dt}{dy} + \frac{t^2}{y} = \frac{1}{2y^2}$

$u = t^2 \quad \Rightarrow \quad \frac{du}{dy} = 2t\,\frac{dt}{dy}$

$\frac{1}{2}\frac{du}{dy} + \frac{u}{y} = \frac{1}{2y^2} \quad \Rightarrow \quad \frac{du}{dy} + \frac{2u}{y} = \frac{1}{y^2}$

$\mu(y) = e^{\int p(y)dy} \quad \Rightarrow \quad \mu(y) = e^{\int \frac{2}{y}dy} \quad \Rightarrow \quad \mu(y) = e^{2\,ln|y|}$

$\mu(y) = y^2$

$(y^2)\frac{du}{dy} + (y^2)\frac{2u}{y} = (y^2)\frac{1}{y^2}$

$\frac{d}{dy}(y^2\,u) = 1 \quad \Rightarrow \quad \int d\,(y^2\,u) = \int dy$

$y^2t^2 = y + c \quad \Rightarrow \quad y^2\left(\frac{dy}{dx}\right)^2 = y + c$

$\frac{dy}{dx} = \pm\frac{\sqrt{y+c}}{y} \quad \Rightarrow \quad \int \frac{y}{\sqrt{y+c}}dy = \pm\int dx$

$$\boxed{\left[\frac{2}{3}(y+c)^{\frac{3}{2}} - c(y+c)^{\frac{1}{2}}\right] = \pm(x+d)}$$

2) $\boldsymbol{yy'' + (y')^2 = yy'}$

$$t = \frac{dy}{dx}$$

$$\frac{dt}{dx} = \frac{d^2y}{dx^2} \quad \Rightarrow \quad \frac{dt}{dx}\left(\frac{dy}{dy}\right) = \frac{d^2y}{dx^2} \quad \Rightarrow \quad \frac{dt}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \quad \Rightarrow \quad \frac{dt}{dy}(t) = \frac{d^2y}{dx^2}$$

$$y\frac{dt}{dy}(t) + t^2 = yt$$

$$\frac{dt}{dy} + \frac{1}{y}t = 1$$

$$\mu(y) = e^{\int p(y)dy} \quad \Rightarrow \quad \mu(y) = e^{\int \frac{1}{y}dy} \quad \Rightarrow \quad \mu(y) = e^{ln|y|}$$

$$\mu(y) = y$$

$$y\frac{dt}{dy} + t = y$$

$$\frac{d}{dy}(yt) = y \quad \Rightarrow \quad \int d(yt) = \int y\,dy$$

$$yt = y^2 + c$$

$$y\frac{dy}{dx} = \frac{y^2}{2} + c \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y^2 + 2c}{2y}$$

$$\int \frac{2y}{y^2 + 2c} = \int dx$$

$$\boxed{\boldsymbol{ln|y^2 + 2c| = x + d}}$$

FÓRMULA DE ABEL

1) $\boldsymbol{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$; $\boldsymbol{Si\ f(x) = x^{-\frac{1}{2}}Sen(x)\ es\ una\ soluci\acute{o}n\ de\ la\ ecuaci\acute{o}n}$

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0$$

$$y_2 = v(x)y_1$$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{1}{x}dx}}{\left(x^{-\frac{1}{2}}Sen(x)\right)^2}dx$$

$$v(x) = \int \frac{e^{-ln|x|}}{x^{-1}Sen^2(x)}dx \quad \Rightarrow \quad v(x) = \int \frac{x^{-1}}{x^{-1}Sen^2(x)}dx$$

$$v(x) = \int Csc^2(x)dx \quad \Rightarrow \quad v(x) = -Cot(x)$$

$$y_2 = -Cot(x)x^{-\frac{1}{2}}Sen(x) = -x^{-\frac{1}{2}}Cos(x)$$

$$y(x) = C_1y_1 + C_2y_2 \quad \Rightarrow \quad \boxed{\boldsymbol{y(x) = C_1\left[x^{-\frac{1}{2}}Sen(x)\right] - C_2\left[x^{-\frac{1}{2}}Cos(x)\right]}}$$

2) $\boldsymbol{x\frac{d^2y}{dx^2} - (1-5x)\frac{dy}{dx} - 5y = 0}$; $\boldsymbol{Si\ f(x) = e^{-5x}\ es\ una\ soluci\acute{o}n\ de\ la\ ecuaci\acute{o}n}$

$$y'' - \frac{(1-5x)}{x}y' - \frac{5y}{x} = 0$$

$$y_2 = v(x)y_1$$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int -\frac{(1-5x)}{x}dx}}{(e^{-5x})^2}dx$$

$$v(x) = \int \frac{e^{ln|x|-5x}}{e^{-10x}}dx \quad \Rightarrow \quad v(x) = \int \frac{xe^{-5x}}{e^{-10x}}dx$$

$$v(x) = \int xe^{5x}dx$$

$$u = x \quad \Rightarrow \quad du = dx$$

$$dv = e^{5x}dx \quad \Rightarrow \quad v = \frac{1}{5}e^{5x}$$

$$v(x) = \frac{x}{5}e^{5x} - \frac{1}{5}\int e^{5x}dx \quad \Rightarrow \quad v(x) = \frac{x}{5}e^{5x} - \frac{e^{5x}}{25}$$

$$y_2 = \left(\frac{x}{5}e^{5x} - \frac{e^{5x}}{25}\right)e^{-5x} \quad \Rightarrow \quad y_2 = \left(\frac{x}{5} - \frac{1}{25}\right)$$

$$y(x) = C_1y_1 + C_2y_2 \quad \Rightarrow \quad \boxed{\boldsymbol{y(x) = C_1e^{-5x} - C_2\left(\frac{x}{5} - \frac{1}{25}\right)}}$$

ECUACIONES DIFERENCIALES DE COEFICIENTES CONSTANTES

1) $\boldsymbol{y'' - 6y' + 9y = 0}$

$y(x) = e^{rx}$

$y'(x) = re^{rx}$

$y''(x) = r^2e^{rx}$

$r^2e^{rx} - 6re^{rx} + 9e^{rx} = 0$

$e^{rx}(r^2 - 6r + 9) = 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r_1 = r_2 = 3$

$y(x) = C_1e^{r_1x} + C_2xe^{r_1x}$

$$\boxed{\boldsymbol{y(x) = C_1e^{3x} + C_2xe^{3x}}}$$

2) $\boldsymbol{2y'' - 3y' + 5y = 0}$

$y(x) = e^{rx}$

$y'(x) = re^{rx}$

$y''(x) = r^2e^{rx}$

$2r^2e^{rx} - 3re^{rx} + 5e^{rx} = 0$

$e^{rx}(2r^2 - 3r + 5) = 0$

$2r^2 - 3r + 5 = 0$

$$r_{1,2} = \frac{3 \pm \sqrt{9 - 4(2)(5)}}{2(2)} = \frac{3 \pm \sqrt{-31}}{4}$$

$$r_{1,2} = \frac{3}{4} \pm \frac{\sqrt{31}}{4}i$$

$y(x) = e^{\lambda x}[C_1\,Cos(\mu x) + C_2\,Sen(\mu x)]$

$$\boxed{\boldsymbol{y(x) = e^{\frac{3}{4}x}\left[C_1\,Cos\left(\frac{\sqrt{31}}{4}x\right) + C_2\,Sen\left(\frac{\sqrt{31}}{4}x\right)\right]}}$$

ECUACIÓN DIFERENCIAL DE EULER – CAUCHY

1) $\boldsymbol{x^2y'' + 2xy' - y = 0}$

$x = e^z \quad \Rightarrow \quad z = ln(x) \quad \Rightarrow \quad \frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dx}\left(\frac{dz}{dz}\right) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dz}\left(\frac{dz}{dx}\right) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dz}\left(\frac{1}{x}\right) \quad \Rightarrow \quad \boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dz}{dz} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dz}\left[\frac{dy}{dz}\left(\frac{dz}{dx}\right)\right]\frac{dz}{dx}$

$\frac{d^2y}{dx^2} = \frac{d}{dz}\left[\frac{dy}{dz}\left(\frac{1}{x}\right)\right]\frac{dz}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dz}\left[\frac{dy}{dz}e^{-z}\right]\frac{dz}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{d^2y}{dz^2}e^{-z} - e^{-z}\frac{dy}{dz}\right)\frac{1}{x}$

$\frac{d^2y}{dx^2} = e^{-z}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\frac{1}{x} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\frac{1}{x} \quad \Rightarrow \quad \boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}}$

$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 2\frac{dy}{dz} - y = 0$

$y'' - y' + 2y' - y = 0$

$y'' + y' - y = 0$

$y(z) = e^{rz}$

$y'(z) = re^{rz}$

$y''(z) = r^2e^{rz}$

$r^2e^{rz} + re^{rz} - e^{rz} = 0$

$e^{rz}(r^2 + r - 1) = 0 \quad \Rightarrow \quad r^2 + r - 1 = 0$

$r_{1,2} = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$

$y(z) = C_1e^{\left(\frac{1+\sqrt{5}}{2}\right)z} + C_2e^{\left(\frac{1-\sqrt{5}}{2}\right)z}$

$y(x) = C_1e^{\left(\frac{1+\sqrt{5}}{2}\right)ln(x)} + C_2e^{\left(\frac{1-\sqrt{5}}{2}\right)ln(x)}$

$\boxed{\boldsymbol{y(x) = C_1x^{\left(\frac{1+\sqrt{5}}{2}\right)} + C_2x^{\left(\frac{1-\sqrt{5}}{2}\right)}}}$

2) $\boldsymbol{x^2y'' - xy' + 2y = 0}$

Estableciéndolo como una fórmula

$$x\frac{dy}{dx} = \frac{dy}{dz} \quad \Rightarrow \quad x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \quad \Rightarrow \quad x^2\frac{d^2y}{dx^2} = D^2y - Dy \quad \Rightarrow \quad x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$D(D-1)y - Dy + 2y = 0$$

$$D^2y - Dy - Dy + 2y = 0$$

$$D^2y - 2Dy + 2y = 0$$

$$y'' - 2y' + 2y = 0$$

$$y(z) = e^{rz}$$

$$y'(z) = re^{rz}$$

$$y''(z) = r^2e^{rz}$$

$$r^2e^{rz} - 2re^{rz} + 2e^{rz} = 0$$

$$e^{rz}(r^2 - 2r + 2) = 0$$

$$r^2 - 2r + 2 = 0$$

$$r_{1,2} = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} \quad \Rightarrow \quad r_{1,2} = 1 \pm i$$

$$y(z) = e^z[C_1\,Cos(z) + C_2\,Sen(z)]$$

$$y(x) = e^{ln(x)}[C_1\,Cos(lnx) + C_2\,Sen(lnx)]$$

$$\boxed{\boldsymbol{y(x) = x[C_1\,Cos(lnx) + C_2\,Sen(lnx)]}}$$

MÉTODO DE COEFICIENTES INDETERMINADOS

1) $\boldsymbol{y'' - 2y' + y = xe^{x} + 4}$

Encontrando la solución complementaria

$y'' - 2y' + y = 0$

$y(x) = e^{rx} \quad \Rightarrow \quad y'(x) = re^{rx} \quad \Rightarrow \quad y''(x) = r^2 e^{rx}$

$r^2 e^{rx} - 2re^{rx} + e^{rx} = 0 \quad \Rightarrow \quad e^{rx}(r^2 - 2r + 1) = 0 \quad \Rightarrow \quad r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 1$

$\therefore\ y_c = C_1 e^{x} + C_2 x e^{x} \quad \Rightarrow \quad C.F.S = \{e^{x}, xe^{x}\}$

Encontrando la solución particular

$y_{p_1} = x^s e^{x}(Ax + B) \quad ; \quad s = 2$

$y_{p_1} = x^2 e^{x}(Ax + B)$

$y_{p_2} = x^s C \quad ; \quad s = 0$

$y_{p_2} = C$

$y_p = y_{p_1} + y_{p_2}$

$y_p = Ax^3 e^{x} + Bx^2 e^{x} + C$

$y'_p = (3Ax^2 e^{x} + Ax^3 e^{x}) + (2Bxe^{x} + Bx^2 e^{x})$

$y''_p = (6Axe^{x} + 3Ax^2 e^{x} + 3Ax^2 e^{x} + Ax^3 e^{x}) + (2Be^{x} + 2Bxe^{x} + 2Bxe^{x} + Bx^2 e^{x})$

$y_p + y'_p + y''_p = xe^{x} + 4$

$$Ax^3 e^{x} + Bx^2 e^{x} + C - 2(3Ax^2 e^{x} + Ax^3 e^{x} + 2Bxe^{x} + Bx^2 e^{x}) + 6Axe^{x} + 3Ax^2 e^{x} + 3Ax^2 e^{x} + Ax^3 e^{x} + 2Be^{x} + 2Bxe^{x} + 2Bxe^{x} + Bx^2 e^{x} = xe^{x} + 4$$

$6Axe^{x} + 2Be^{x} + C = xe^{x} + 4$

$6A = 1 \quad \Rightarrow \quad A = \frac{1}{6}$

$B = 0$

$C = 4$

$y(x) = y_c + y_p$

$$\boxed{\boldsymbol{y(x) = C_1 e^{x} + C_2 x e^{x} + \frac{1}{6} x^3 e^{x} + 4}}$$

2) $x^2y'' + 3xy' + 4y = Cos(4\ lnx)$

Encontrando la solución complementaria

$x = e^z \quad \Rightarrow \quad z = ln(x)$

$D(D-1)y + 3Dy + 4y = 0$

$D^2y - Dy + 3Dy + 4y = 0$

$D^2y + 2Dy + 4y = 0 \quad \Rightarrow \quad y'' + 2y' + 4y = 0$

$y(x) = e^{rz} \quad \Rightarrow \quad y'(x) = re^{rz} \quad \Rightarrow \quad y''(x) = r^2e^{rz}$

$r^2e^{rz} + 2re^{rz} + 4e^{rz} = 0 \quad \Rightarrow \quad e^{rz}(r^2 + 2r + 4) = 0 \quad \Rightarrow \quad r^2 + 2r + 1 = 0$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4(1)(4)}}{2(1)} = \frac{-2 \pm \sqrt{-12}}{2} \quad \Rightarrow \quad r_{1,2} = -1 \pm \sqrt{3}i$$

$y_c(z) = e^{-z}\left[C_1\ Cos\left(\sqrt{3}z\right) + C_2\ Sen\left(\sqrt{3}z\right)\right]$

$\therefore\ C.F.S = \left\{e^{-z}Cos\left(\sqrt{3}z\right), e^{-z}Sen\left(\sqrt{3}z\right)\right\}$

Reemplazando $x = e^z$ *en la ecuación para encontrar la solución particular*

$y'' + 2y' + 4y = Cos(4z)$

$y_p = z^s[Ax^0Cos(4z) + Bx^0Sen(4z)] \quad ; \quad s = 0$

$y_p = A\ Cos(4z) + B\ Sen(4z)$

$y'_p = -4A\ Sen(4z) + 4B\ Cos(4z)$

$y''_p = -16A\ Cos(4z) - 16B\ Sen(4z)$

Reemplazando:

$-16A\ Cos(4z) - 16B\ Sen(4z) + 2[-4A\ Sen(4z) + 4B\ Cos(4z)] + 4[A\ Cos(4z) + B\ Sen(4z)] = Cos(4z)$

$-16A\ Cos(4z) - 16B\ Sen(4z) - 8A\ Sen(4z) + 8B\ Cos(4z) + 4A\ Cos(4z) + 4B\ Sen(4z) = Cos(4z)$

$(8B - 12A)Cos(4z) + (-12B - 8A)Sen(4z) = Cos(4z)$

$1 = 8B - 12A$

$0 = -12B - 8A$

Resolviendo el sistema A = -3/40 , B = 1/20

$$y_p = -\frac{3}{40}\ Cos(4z) + \frac{1}{20}\ Sen(4z)$$

$$y(z) = y_c + y_p \quad \Rightarrow \quad y(z) = e^{-z}\left[C_1\ Cos\left(\sqrt{3}z\right) + C_2\ Sen\left(\sqrt{3}z\right)\right] - \frac{3}{40}\ Cos(4z) + \frac{1}{20}\ Sen(4z)$$

$$\boxed{\boldsymbol{y(x) = x^{-1}\left[C_1\ Cos\left(\sqrt{3}ln(x)\right) + C_2\ Sen\left(\sqrt{3}ln(x)\right)\right] - \frac{3}{40}\ Cos(4ln(x)) + \frac{1}{20}\ Sen(4ln(x))}}$$

VARIACIÓN DE PARÁMETRO

1) $\boldsymbol{y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}}$

Encontrando la solución complementaria

$y'' + 4y' + 4y = 0$

$y(x) = e^{rx} \quad \Rightarrow \quad y'(x) = re^{rx} \quad \Rightarrow \quad y''(x) = r^2e^{rx}$

$r^2e^{rx} + 4re^{rx} + 4e^{rx} = 0 \quad \Rightarrow \quad e^{rx}(r^2 + 4r + 4) = 0 \quad \Rightarrow \quad r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = -2$

$\therefore\ y_c = C_1e^{-2x} + C_2xe^{-2x} \quad \Rightarrow \quad C.F.S = \{e^{-2x}, xe^{-2x}\}$

Encontrando la solución particular

$y_p = u_1y_1 + u_2y_2$

$u_1 = \int \frac{-y_2\, g(x)}{w(y_1, y_2)}dx \quad ; \quad u_2 = \int \frac{y_1\, g(x)}{w(y_1, y_2)}dx$

Determinando el wronskiano

$w(e^{-2x}, xe^{-2x}) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix}$

$w(e^{-2x}, xe^{-2x}) = e^{-4x} - 2xe^{-4x} + 2xe^{-4x}$

$w(e^{-2x}, xe^{-2x}) = e^{-4x}$

$u_1 = \int \frac{-xe^{-2x}\ \frac{e^{-2x}}{x^2}}{e^{-4x}}dx \quad \Rightarrow \quad u_1 = \int -\frac{1}{x}dx \quad \Rightarrow \quad u_1 = -ln(x)$

$u_2 = \int \frac{e^{-2x}\ \frac{e^{-2x}}{x^2}}{e^{-4x}}dx \quad \Rightarrow \quad u_2 = \int \frac{1}{x^2}dx \quad \Rightarrow \quad u_2 = -\frac{1}{x}$

$y_p = -ln(x)e^{-2x} - \frac{1}{x}xe^{-2x}$

$\boxed{\boldsymbol{y(x) = C_1e^{-2x} + C_2xe^{-2x} - ln(x)e^{-2x} - e^{-2x}}}$

2) $\boldsymbol{(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}}$; ***Si*** $\boldsymbol{f(x) = e^x}$ ***es una solución de la ecuación homogénea***

$$y'' + \frac{x}{1-x}y' - \frac{y}{1-x} = -2(x-1)e^{-x}$$

$$y_2 = v(x)y_1$$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{x}{1-x}dx}}{e^{2x}}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int\left(-1+\frac{1}{1-x}\right)dx}}{e^{2x}}dx$$

$$v(x) = \int \frac{e^{-[-x-ln(x-1)]}}{e^{2x}}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{x}(x-1)}{e^{2x}}dx \quad \Rightarrow \quad v(x) = \int e^{-x}(x-1)\,dx$$

$$u = x-1 \quad \Rightarrow \quad du = dx$$

$$dv = e^{-x}dx \quad \Rightarrow \quad v = -e^{-x}$$

$$v(x) = -e^{-x}(x-1) + \int e^{-x}dx \quad \Rightarrow \quad v(x) = e^{-x}(1-x) - e^{-x}$$

$$y_2 = -xe^{-x}e^{x} \quad \Rightarrow \quad y_2 = -x$$

$$C.F.S = \{e^x, x\}$$

Encontrando la solución particular

$$y_p = u_1y_1 + u_2y_2$$

$$u_1 = \int \frac{-y_2\, g(x)}{w(y_1,y_2)}dx \quad ; \quad u_2 = \int \frac{y_1\, g(x)}{w(y_1,y_2)}dx$$

Determinando el wronskiano

$$w(e^x, x) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix}$$

$$w(e^x, x) = e^x - xe^x$$

$$u_1 = \int \frac{2x(x-1)e^{-x}}{e^x - xe^x}dx \quad \Rightarrow \quad u_1 = 2\int \frac{x(x-1)e^{-x}}{-(x-1)e^x}dx \quad \Rightarrow \quad u_1 = -2\int xe^{-2x}dx$$

$$u = x \quad \Rightarrow \quad du = dx \quad ; \quad dv = e^{-2x}dx \quad \Rightarrow \quad v = -\frac{1}{2}e^{-2x}$$

$$u_1 = -2\left(-\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx\right) \quad \Rightarrow \quad u_1 = xe^{-2x} + \frac{1}{2}e^{-2x}$$

$$u_2 = \int \frac{-2e^x(x-1)e^{-x}}{e^x - xe^x}dx \quad \Rightarrow \quad u_2 = -2\int \frac{e^x(x-1)e^{-x}}{-(x-1)e^x}dx \quad \Rightarrow \quad u_2 = 2\int e^{-x}dx$$

$$u_2 = -2e^{-x}$$

$$y_p = \left(xe^{-2x} + \frac{1}{2}e^{-2x}\right)e^x - 2xe^{-x}$$

$$\boxed{\boldsymbol{y(x) = C_1e^x + C_2x - xe^{-x} + \frac{1}{2}e^{-x}}}$$

3) $\boldsymbol{x^2y'' - 2xy' + 2y = x - ln(x)}$

$x = e^z$

$D(D-1)y - 2Dy + 2y = e^z - ln(e^z) \quad \Rightarrow \quad D^2y - Dy - 2Dy + 2y = e^z - z$

$D^2y - 3Dy + 2y = e^z - z \quad \Rightarrow \quad y'' - 3y' + 2y = e^z - z$

$y(z) = e^{rz} \quad \Rightarrow \quad y'(z) = re^{rz} \quad \Rightarrow \quad y''(z) = r^2e^{rz}$

$r^2e^{rz} - 3re^{rz} + 2e^{rz} = 0 \quad \Rightarrow \quad e^{rz}(r^2 - 3r + 2) = 0 \quad \Rightarrow \quad r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0 \quad \Rightarrow \quad r_1 = 2 \;\; ; \;\; r_2 = 1$

$y_c = C_1e^{2z} + C_2e^z \quad \Rightarrow \quad C.F.S = \{e^{2z}, e^z\}$

Encontrando la solución particular

$y_p = u_1y_1 + u_2y_2$

$$u_1 = \int \frac{-y_2\, g(x)}{w(y_1, y_2)}dx \quad ; \quad u_2 = \int \frac{y_1\, g(x)}{w(y_1, y_2)}dx$$

Determinando el wronskiano

$$w(e^{2z}, e^z) = \begin{vmatrix} e^{2z} & e^z \\ 2e^{2z} & e^z \end{vmatrix}$$

$w(e^{2z}, e^z) = e^{3z} - 2e^{3z} \quad \Rightarrow \quad w(e^{2z}, e^z) = -e^{3z}$

$$u_1 = \int \frac{-e^z(e^z - z)}{-e^{3z}}dz \quad \Rightarrow \quad u_1 = \int (e^{-z} - ze^{-2z})dz$$

$u = z \quad \Rightarrow \quad du = dz$

$$dv = e^{-2z}dz \quad \Rightarrow \quad v = -\frac{1}{2}e^{-2z}$$

$$u_1 = -e^{-z} - \left(-\frac{1}{2}ze^{-2z} - \frac{1}{4}e^{-2z}\right) \quad \Rightarrow \quad u_1 = \frac{1}{2}ze^{-2z} + \frac{1}{4}e^{-2z} - e^{-z}$$

$$u_2 = \int \frac{e^{2z}(e^z - z)}{-e^{3z}}dz \quad \Rightarrow \quad u_2 = \int (-1 + ze^{-z})\,dz \quad \Rightarrow \quad u_2 = -z + (-ze^{-z} - e^{-z})$$

$$y_p = \left(\frac{1}{2}ze^{-2z} + \frac{1}{4}e^{-2z} - e^{-z}\right)e^{2z} - (z + ze^{-z} + e^{-z})e^z$$

$$y(x) = C_1e^{2z} + C_2e^z + \frac{1}{2}z + \frac{1}{4} - e^z - ze^z - z - 1$$

$$y(x) = C_1e^{2z} + \underbrace{(C_2 - 1)}_{constante}e^z - \frac{1}{2}z - \frac{3}{4} - ze^z \quad \Rightarrow \quad y(x) = C_1e^{2\,ln(x)} + C_2e^{ln(x)} - \frac{1}{2}ln(x) - \frac{3}{4} - ln(x)e^{ln(x)}$$

$$\boxed{\boldsymbol{y(x) = C_1x^2 + C_2x - -\frac{1}{2}ln(x) - \frac{3}{4} - xln(x)}}$$

ECUACIONES DIFERENCIALES DE ORDEN SUPERIOR

$1)\ y''' - y = 0$

$y(x) = e^{rx}$

$y'(x) = re^{rx}$

$y''(x) = r^2e^{rx}$

$y'''(x) = r^3e^{rx}$

Reemplazando:

$r^3e^{rx} - e^{rx} = 0$

$e^{rx}(r^3 - 1) = 0$

$r^3 - 1 = 0 \quad \Rightarrow \quad (r-1)(r^2 + r + 1) = 0$

$r_1 = 1 \quad ; \quad r_{2,3} = \dfrac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

$$\boxed{\boldsymbol{y(x) = C_1e^x + e^{-\frac{1}{2}x}\left[C_2\,Cos\left(\frac{\sqrt{3}}{2}x\right) + C_3\,Sen\left(\frac{\sqrt{3}}{2}x\right)\right]}}$$

$2)\ y^{VI} + y = 0$

De forma directa, tenemos que:

$e^{rx}(r^6 + 1) = 0$

$r^6 + 1 = 0$

El problema se reduce a extraer las raíces sextas de -1, entonces:

$z^6 = -1$

Recordando que todas las raíces n-ésimas de $z = re^{i\theta}$ *vienes dadas por la expresión:*

$z_k = |z|^{1/n}e^{i\left(\frac{\theta + 2k\pi}{n}\right)}\ para\ \ k = 0\,,1\,,2\,, \ldots \ldots .\,n-1$

Entonces:

$z_k = |1|^{1/6}\,e^{i\left(\frac{\pi + 2k\pi}{6}\right)}\ \ para\ k = 0, 1, 2, 3, 4, 5$

$z_0 = e^{i\left(\frac{\pi}{6}\right)} = Cos\left(\dfrac{\pi}{6}\right) + i\,Sen\left(\dfrac{\pi}{6}\right) \quad \Rightarrow \quad z_0 = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

$z_1 = e^{i\left(\frac{\pi}{2}\right)} = Cos\left(\dfrac{\pi}{2}\right) + i\,Sen\left(\dfrac{\pi}{2}\right) \quad \Rightarrow \quad z_1 = i$

$z_2 = e^{i\left(\frac{5\pi}{6}\right)} = Cos\left(\dfrac{5\pi}{6}\right) + i\,Sen\left(\dfrac{5\pi}{6}\right) \quad \Rightarrow \quad z_2 = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

Las tres raíces restantes, son la conjugada de las tres primeras, entonces:

$z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$z_4 = -i$

$z_5 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

Entonces:

$$\boldsymbol{y(x) = e^{\frac{\sqrt{3}}{2}x}\left[C_1\, Cos\left(\frac{1}{2}x\right) + C_2\, Sen\left(\frac{1}{2}x\right)\right] + e^{0x}[C_3\, Cos(x) + C_4\, Sen(x)] + e^{-\frac{\sqrt{3}}{2}x}\left[C_5\, Cos\left(\frac{1}{2}x\right) + C_6\, Sen\left(\frac{1}{2}x\right)\right]}$$

3) $\boldsymbol{y^{VI} - 3y^{IV} + 3y^{II} - y = 0}$

$e^{rx}(r^6 - 3r^4 + 3r^2 - 1) = 0$

$(r^6 - 3r^4 + 3r^2 - 1) = 0$

Sabemos que:

$(a - b)^3 = a^2 - 3a^2b + 3ab^2 - b^3$

Entonces:

$(r^2 - 1)^3 = 0$

$r^2 - 1 = 0$

$r = \pm 1$

Recordemos que son 6 raíces, entonces:

$r_1 = r_2 = r_3 = 1 \quad y \quad r_4 = r_5 = r_6 - 1$

$$\boldsymbol{y(x) = C_1e^x + C_2xe^x + C_3x^2e^x + C_4e^{-x} + C_5xe^{-x} + C_6x^2e^{-x}}$$

4) $\boldsymbol{x^3y''' + 6x^2y'' + 7xy' = 0}$

$$x = e^z \quad \Rightarrow \quad z = ln(x) \quad \Rightarrow \quad \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dx}\left(\frac{dz}{dz}\right) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dz}\left(\frac{dz}{dx}\right) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dz}\left(\frac{1}{x}\right) \quad \Rightarrow \quad \boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dz}{dz} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dz}\left[\frac{dy}{dz}\left(\frac{dz}{dx}\right)\right]\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left[\frac{dy}{dz}\left(\frac{1}{x}\right)\right]\frac{dz}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dz}\left[\frac{dy}{dz}e^{-z}\right]\frac{dz}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{d^2y}{dz^2}e^{-z} - e^{-z}\frac{dy}{dz}\right)\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = e^{-z}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\frac{1}{x} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\frac{1}{x} \quad \Rightarrow \quad \boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \quad \Rightarrow \quad \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)\frac{dz}{dz} \quad \Rightarrow \quad \frac{d^3y}{dx^3} = \frac{d}{dz}\left(\frac{d^2y}{dx^2}\right)\frac{dz}{dx}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dz}\left[\left(\frac{d^2y}{dz^2}e^{-z} - e^{-z}\frac{dy}{dz}\right)\frac{1}{x}\right]\frac{dz}{dx} \quad \Rightarrow \quad \frac{d^3y}{dx^3} = \frac{d}{dz}\left[\left(\frac{d^2y}{dz^2}e^{-z} - e^{-z}\frac{dy}{dz}\right)e^{-z}\right]\frac{dz}{dx}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dz}\left(\frac{d^2y}{dz^2}e^{-2z} - e^{-2z}\frac{dy}{dz}\right)\frac{1}{x} \quad \Rightarrow \quad \frac{d^3y}{dx^3} = \left(\frac{d^3y}{dz^3}e^{-2z} - 2\frac{d^2y}{dz^2}e^{-2z} - \frac{d^2y}{dz^2}e^{-2z} + 2\frac{dy}{dz}e^{-2z}\right)\frac{1}{x}$$

$$\frac{d^3y}{dx^3} = e^{-2z}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)e^{-z} \quad \Rightarrow \quad \frac{d^3y}{dx^3} = \frac{1}{x^2}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)\frac{1}{x} \quad \Rightarrow \quad \boxed{x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}}$$

Reemplazando

$$\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + 6\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 7\left(\frac{dy}{dz}\right) = 0$$

$$y''' - 3y'' + 2y' + 6y'' - 6y' + 7y' = 0 \quad \Rightarrow \quad y''' + 3y'' + 3y' = 0$$

$$e^{rz}(r^3 + 3r^2 + 3r) = 0 \quad \Rightarrow \quad (r^3 + 3r^2 + 3r) = 0 \quad \Rightarrow \quad r(r^2 + 3r + 3) = 0$$

$$r_1 = 0 \quad ; \quad r_{2,3} = \frac{-3 \pm \sqrt{9 - 12}}{2} = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$$

$$y(z) = C_1e^{0z} + e^{-\frac{3}{2}z}\left[C_2\,Cos\left(\frac{\sqrt{3}}{2}z\right) + C_3\,Sen\left(\frac{\sqrt{3}}{2}z\right)\right]$$

$$\boxed{\boldsymbol{y(x) = C_1 + x^{-\frac{3}{2}}\left[C_2\,Cos\left(\frac{\sqrt{3}}{2}lnx\right) + C_3\,Sen\left(\frac{\sqrt{3}}{2}lnx\right)\right]}}$$

5) $\boldsymbol{x^4y^{IV} + 5x^3y^{III} + 7x^2y^{II} + 8xy^{I} = \ln x - (\ln x)^2}$

Encontrando la solución complementaria

$x = e^z$

$D(D-1)(D-2)(D-3)y + 5D(D-1)(D-2)y + 7D(D-1)y + 8Dy = z - z^2$

$D^4y - 6D^3y + 11D^2y - 6Dy + 5D^3y - 15D^2y + 10Dy + 7D^2y - 7Dy + 8Dy = z - z^2$

$y^{IV} - y^{III} - 3y^{II} + 5y^{I} = z - z^2$

$y = e^{rz}$

$e^{rz}(r^4 - r^3 - 3r^2 + 5r) = 0$

$r(r^3 - r^2 - 3r + 5) = 0 \quad \Rightarrow \quad r(r+1)(r^2 - 2r + 5) = 0$

$r_1 = 0 \quad ; \quad r_2 = -1 \quad ; \quad r_{3,4} = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$y_c = C_1e^{0z} + C_2e^{-z} + e^z[C_3\, Cos(2z) + C_4\, Sen(2z)]$

$y_c = C_1 + C_2e^{-z} + e^z[C_3\, Cos(2z) + C_4\, Sen(2z)]$

$\therefore C.F.S = \{1, e^{-z}, e^zCos(2z), e^zSen(2z)\}$

Encontrando la solución particular

$y_p = z^s(Az^2 + Bz + C) \quad ; \quad s = 1$

$y_p = Az^3 + Bz^2 + Cz$

$y'_p = 3Az^2 + 2Bz + C$

$y''_p = 6Az + 2B$

$y'''_p = 6A$

$y''''_p = 0$

Reemplazando:

$-6A + 3(6Az + 2B) + 5(3Az^2 + 2Bz + C) = z - z^2$

$-6A + 18Az + 6B + 15Az^2 + 10Bz + 5C = z - z^2$

$15Az^2 + (18A + 10B)z + (-6A + 6B + 5C) = z - z^2$

$-1 = 15A$

$1 = 18A + 10B$

$0 = -6A + 6B + 5C$

Resolviendo el sistema; A = -1/15, B = 11/50, C = -43/125

$$y_p = -\frac{1}{15}z^3 + \frac{11}{50}z^2 - \frac{43}{125}z$$

Entonces:

$$y(z) = C_1 + C_2e^{-z} + e^z[C_3\,Cos(2z) + C_4\,Sen(2z)] - \frac{1}{15}z^3 + \frac{11}{50}z^2 - \frac{43}{125}z$$

$$\boxed{\boldsymbol{y(x) = C_1 + C_2x^{-1} + x[C_3\,Cos(2lnx) + C_4\,Sen(2lnx)] - \frac{1}{15}(lnx)^3 + \frac{11}{50}(lnx)^2 - \frac{43}{125}(lnx)}}$$

6) $\boldsymbol{y''' + y' = Sec(x)Tan(x) \quad , \quad 0 < x < \pi/2}$

Encontrando la solución complementaria

$$e^{rx}(r^3 + r) = 0 \quad \Rightarrow \quad r(r^2 + 1) = 0$$

$$r_1 = 0 \quad ; \quad r^2 + 1 = 0 \quad \Rightarrow \quad r_{2,3} = \pm i$$

$$y_c = C_1e^{0x} + e^{0x}[C_2\,Cos(x) + C_3Sen(x)] \quad \Rightarrow \quad y_c = C_1 + [C_2\,Cos(x) + C_3Sen(x)]$$

$$\therefore C.F.S = \{1\,, Sen(x)\,, Cos(x)\}$$

Encontrando la solución particular

$$y_p = u_1y_1 + u_2y_2 + u_3y_3$$

Determinando el wronskiano

$$w(y_1, y_2, y_3) = \begin{vmatrix} 1 & Sen(x) & Cos(x) \\ 0 & Cos(x) & -Sen(x) \\ 0 & -Sen(x) & -Cos(x) \end{vmatrix} = \begin{vmatrix} Cos(x) & -Sen(x) \\ -Sen(x) & -Cos(x) \end{vmatrix} = -Cos^2(x) - Sen^2(x) = -1$$

Determinando u_1

$$u_1 = \int \frac{\begin{vmatrix} 0 & Sen(x) & Cos(x) \\ 0 & Cos(x) & -Sen(x) \\ Sec(x)Tan(x) & -Sen(x) & -Cos(x) \end{vmatrix}}{-1}dx = -\int Sec(x)Tan(x)\begin{vmatrix} Sen(x) & Cos(x) \\ Cos(x) & -Sen(x) \end{vmatrix}dx$$

$$u_1 = -\int Sec(x)Tan(x)[-Sen^2(x) - Cos^2(x)]dx = \int Sec(x)Tan(x)dx = \int \frac{1}{Cos(x)}\frac{Sen(x)}{Cos(x)}dx = \int \frac{Sen(x)}{Cos^2(x)}dx$$

$$u = Cos(x) \quad \Rightarrow \quad du = -Sen(x)$$

$$u_1 = -\int \frac{1}{u^2}du \quad \Rightarrow \quad u_1 = \frac{1}{u} \quad \Rightarrow \quad \boldsymbol{u_1 = Sec(x)}$$

Determinando u_2

$$u_2 = \int \frac{\begin{vmatrix} 1 & 0 & Cos(x) \\ 0 & 0 & -Sen(x) \\ 0 & Sec(x)Tan(x) & -Cos(x) \end{vmatrix}}{-1} dx = -\int 1 \begin{vmatrix} 0 & -Sen(x) \\ Sec(x)Tan(x) & -Cos(x) \end{vmatrix} dx$$

$$u_2 = -\int Sen(x)Sec(x)Tan(x)dx = -\int Sen(x)\frac{1}{Cos(x)}\frac{Sen(x)}{Cos(x)}dx = -\int \frac{Sen^2(x)}{Cos^2(x)}dx$$

$$u_2 = -\int \frac{1 - Cos^2(x)}{Cos^2(x)}dx = -\int Sec^2(x)dx + \int dx \quad \Rightarrow \quad \boldsymbol{u_2 = -Tan(x) + x}$$

Determinando u_3

$$u_3 = \int \frac{\begin{vmatrix} 1 & Sen(x) & 0 \\ 0 & Cos(x) & 0 \\ 0 & -Sen(x) & Sec(x)Tan(x) \end{vmatrix}}{-1} dx = -\int 1 \begin{vmatrix} Cos(x) & 0 \\ -Sen(x) & Sec(x)Tan(x) \end{vmatrix} dx$$

$$u_3 = -\int Cos(x)Sec(x)Tan(x)dx = -\int Cos(x)\frac{1}{Cos(x)}\frac{Sen(x)}{Cos(x)}dx = -\int \frac{Sen(x)}{Cos(x)}dx$$

$$u_3 = -\int Tan(x)dx \quad \Rightarrow \quad \boldsymbol{u_3 = ln|Cos(x)|}$$

Entonces:

$$\boxed{\boldsymbol{y(x) = C_1 + [C_2\, Cos(x) + C_3 Sen(x)] + Sec(x) + [x - Tan(x)]Sen(x) + ln|Cos(x)|Cos(x)}}$$

RESOLUCIÓN DE ECUACIONES DIFERENCIALES POR SERIES DE POTENCIAS

1) $(\boldsymbol{x}-\boldsymbol{1})\boldsymbol{y}'' + \boldsymbol{y}' = \boldsymbol{0}$; $\boldsymbol{alrededor\ de\ x_0 = 0}$

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{+\infty} a_n(n)x^{n-1}$$

$$y''(x) = \sum_{n=2}^{+\infty} a_n(n)(n-1)x^{n-2}$$

Reemplazando

$$(x-1)\sum_{n=2}^{+\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{+\infty} a_n(n)x^{n-1} = 0$$

$$x\sum_{n=2}^{+\infty} a_n(n)(n-1)x^{n-2} - \sum_{n=2}^{+\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{+\infty} a_n(n)x^{n-1} = 0$$

$$\sum_{n=2}^{+\infty} a_n(n)(n-1)x^{n-1} - \sum_{n=2}^{+\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{+\infty} a_n(n)x^{n-1} = 0$$

$M = n-1 \qquad K = n-2 \qquad S = n-1$

$$\sum_{M=1}^{+\infty} a_{M+1}(M+1)(M)x^M - \sum_{K=0}^{+\infty} a_{K+2}(K+2)(K+1)x^K + \sum_{S=0}^{+\infty} a_{S+1}(S+1)x^S = 0$$

$$-a_2(2)(1)x^0 + a_1(1)x^0 + \sum_{n=1}^{+\infty}[a_{n+1}(n+1)(n) - a_{n+2}(n+2)(n+1) + a_{n+1}(n+1)]x^n = 0$$

$$-2a_2 + a_1 = 0 \quad \Rightarrow \quad \boldsymbol{a_2 = \frac{a_1}{2}}$$

$$a_{n+1}(n+1)(n) - a_{n+2}(n+2)(n+1) + a_{n+1}(n+1) = 0$$

$$a_{n+2} = \frac{(n+1)(n) + (n+1)}{(n+2)(n+1)}a_{n+1} \quad \Rightarrow \quad \boldsymbol{a_{n+2} = \frac{n+1}{n+2}a_{n+1}} \ ; \ \forall\, \boldsymbol{n \geq 1}$$

Dando valores a "n":

n = 1

$$a_3 = \frac{2}{3}a_2 \quad \Rightarrow \quad a_3 = \frac{2}{3}\left(\frac{a_1}{2}\right) \quad \Rightarrow \quad a_3 = \frac{a_1}{3}$$

n =2

$$a_4 = \frac{3}{4}a_3 \quad \Rightarrow \quad a_4 = \frac{2}{4}\left(\frac{a_1}{3}\right) \quad \Rightarrow \quad a_4 = \frac{a_1}{4}$$

Podemos observar un patrón:

n = 3

$$a_5 = \frac{a_1}{5}$$

n =4

$$a_6 = \frac{a_1}{6}$$

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Entonces:

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \ldots \ldots \ldots \ldots.$$

$$y(x) = a_0 + a_1 x + \frac{a_1}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_1}{4}x^4 + \frac{a_1}{5}x^5 + \ldots \ldots \ldots \ldots.$$

$$y(x) = a_0(1) + a_1\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots \ldots \ldots \ldots.\right)$$

Sabemos que:

$$\frac{1}{1-x}=\sum_{n=0}^{+\infty} x^n$$

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+\ \dots\dots\dots.$$

Integrando tenemos:

$$\int \frac{1}{1-x}=\int (1+x+x^2+x^3+x^4+x^5+\ \dots\dots\dots.)dx$$

$$-ln|1-x|=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\ \dots\dots\dots$$

Entonces, la solución homogénea es:

$$\boxed{\boldsymbol{y(x)=a_0(1)+a_1\ ln|1-x|}}$$

2) $\boldsymbol{(x^2-1)y'' + 4xy' + 2y = 0 \quad ; \quad alrededor\ de\ x_0 = 0}$

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{+\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2}$$

$$(x^2-1)\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + 4x\sum_{n=1}^{+\infty} a_n (n) x^{n-1} + 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

$$x^2\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + 4x\sum_{n=1}^{+\infty} a_n (n) x^{n-1} + 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

$$\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n} - \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + 4\sum_{n=1}^{+\infty} a_n (n) x^{n} + 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

$$K = n-2$$

$$\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n} - \sum_{K=0}^{+\infty} a_{K+2}(K+2)(K+1) x^{K} + 4\sum_{n=1}^{+\infty} a_n (n) x^{n} + 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

$$-a_2(2)(1) - a_3(3)(2)x + 4a_1(1)x + 2a_0 + 2a_1x + \sum_{n=2}^{+\infty}[a_n(n)(n-1) - a_{n+2}(n+2)(n+1) + 4a_n(n) + 2a_n]x^n = 0$$

$$-2a_2 - 6a_3x + 4a_1x + 2a_0 + 2a_1x + \sum_{n=2}^{+\infty}[a_n(n)(n-1) - a_{n+2}(n+2)(n+1) + 4a_n(n) + 2a_n]x^n = 0$$

$$(2a_0 - 2a_2) + (-6a_3 + 4a_1 + 2a_1)x + \sum_{n=2}^{+\infty}[a_n(n)(n-1) - a_{n+2}(n+2)(n+1) + 4a_n(n) + 2a_n]x^n = 0$$

$2a_0 - 2a_2 = 0 \quad \Rightarrow \quad \boldsymbol{a_0 = a_2}$

$-6a_3 + 4a_1 + 2a_1 = 0 \quad \Rightarrow \quad \boldsymbol{a_1 = a_3}$

$a_n(n)(n-1) - a_{n+2}(n+2)(n+1) + 4a_n(n) + 2a_n = 0$

$a_{n+2} = \dfrac{n(n-1)+4n+2}{(n+2)(n+1)}a_n \quad \Rightarrow \quad \boldsymbol{a_{n+2} = a_n \ ; \ \forall\, n \geq 2}$

Dando valores a "n":

$n = 2$

$a_4 = a_2 \quad \Rightarrow \quad a_4 = a_0$

$n = 3$

$a_5 = a_3 \quad \Rightarrow \quad a_4 = a_1$

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Entonces:

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = a_0 + a_1 x + a_0 x^2 + a_2 x^3 + a_0 x^4 + \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = a_0(1 + x^2 + x^4 + x^6 + \ldots\ldots\ldots) + a_1 x(1 + x^2 + x^4 + x^6 + \ldots\ldots\ldots)$$

Sabemos que:

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$$

$$\frac{1}{1-x^2} = \sum_{n=0}^{+\infty} x^{2n}$$

$$\frac{1}{1-x^2} = x + x^2 + x^4 + x^6 + \ldots\ldots\ldots$$

Entonces:

$$y(x) = a_0\left(\frac{1}{1-x^2}\right) + a_1 x\left(\frac{1}{1-x^2}\right)$$

$$\boxed{\boldsymbol{y(x) = a_0\left(\frac{1}{1-x^2}\right) + a_1\left(\frac{x}{1-x^2}\right)}}$$

3) $\boldsymbol{(x^2+1)y'' + 2xy' - 2y = 0 \quad ; \quad alrededor\ de\ x_0 = 0}$

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{+\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2}$$

$$(x^2+1)\sum_{n=2}^{+\infty} a_n (n)(n-1)x^{n-2} + 2x\sum_{n=1}^{+\infty} a_n(n)x^{n-1} - 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

$$x^2\sum_{n=2}^{+\infty} a_n (n)(n-1)x^{n-2} + \sum_{n=2}^{+\infty} a_n (n)(n-1)x^{n-2} + 2x\sum_{n=1}^{+\infty} a_n(n)x^{n-1} - 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

$$\sum_{n=2}^{+\infty} a_n (n)(n-1)x^{n} + \sum_{n=2}^{+\infty} a_n (n)(n-1)x^{n-2} + 2\sum_{n=1}^{+\infty} a_n(n)x^{n} - 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

$$K = n-2$$

$$\sum_{n=2}^{+\infty} a_n (n)(n-1)x^{n} + \sum_{K=0}^{+\infty} a_{K+2}(K+2)(K+1)x^{K} + 2\sum_{n=1}^{+\infty} a_n(n)x^{n} - 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

$$a_2(2)(1) + a_3(3)(2)x + 2a_1(1)x - 2a_0 - 2a_1x + \sum_{n=2}^{+\infty}[a_n(n)(n-1) + a_{n+2}(n+2)(n+1) + 2a_n(n) - 2a_n]x^n = 0$$

$$2a_2 + 6a_3x + 2a_1x - 2a_0 - 2a_1x + \sum_{n=2}^{+\infty}[a_n(n)(n-1) + a_{n+2}(n+2)(n+1) + 2a_n(n) - 2a_n]x^n = 0$$

$$2a_2 - 2a_0 = 0 \quad \Rightarrow \quad \boldsymbol{a_2 = a_0}$$

$$6a_3 = 0 \quad \Rightarrow \quad \boldsymbol{a_3 = 0}$$

$$a_n(n)(n-1) + a_{n+2}(n+2)(n+1) + 2a_n(n) - 2a_n = 0$$

$$a_{n+2} = \frac{2-2n-(n)(n-1)}{(n+2)(n+1)}a_n \quad \Rightarrow \quad \boldsymbol{a_{n+2} = -\frac{n-1}{n+1}a_n \quad ; \quad \forall\, n \geq 2}$$

Dando valores a "n":

n = 2

$a_4 = -\frac{1}{3}a_2 \quad \Rightarrow \quad a_4 = -\frac{1}{3}a_0$

n = 3

$a_5 = -\frac{2}{4}a_3 \quad \Rightarrow \quad a_5 = 0$

n = 4

$a_6 = -\frac{3}{5}a_4 \quad \Rightarrow \quad a_4 = \frac{1}{5}a_0$

n = 5

$a_7 = 0$

n = 6

$a_8 = -\frac{5}{7}a_2 \quad \Rightarrow \quad a_4 = \frac{1}{7}a_0$

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Entonces:

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ \ldots \ldots \ldots \ldots \ldots$$

$$y(x) = a_0 + a_1 x + a_0 x^2 + 0 - \frac{a_0}{3}x^4 + 0 + \frac{a_0}{5}x^6 + 0 - \frac{a_0}{7}x^8 \ \ldots \ldots \ldots \ldots \ldots$$

$$y(x) = a_0\left(1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} \ \ldots \ldots \ldots \ldots \ldots .\right) + a_1(x)$$

Sabemos que:

$$\frac{d}{dx}[Arctan(x)] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[Arctan(x)] = \sum_{n=0}^{+\infty}(-x^2)^n$$

$$\frac{d}{dx}[Arctan(x)] = \sum_{n=0}^{+\infty}(-1)^n x^{2n}$$

$$\frac{d}{dx}[Arctan(x)] = 1 - x^2 + x^3 - x^4 + x^5 - \ \dots\dots\dots\dots$$

Integrando tenemos:

$$\int d[Arctan(x)] = \int (1 - x^2 + x^3 - x^4 + x^5 - \ \dots\dots\dots\dots)dx$$

$$Arctan(x) = x - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^6}{6} - \ \dots\dots\dots\dots$$

Multiplicando por "x":

$$xArctan(x) = x^2 - \frac{x^4}{3} + \frac{x^5}{4} - \frac{x^6}{5} + \frac{x^7}{6} - \ \dots\dots\dots\dots$$

Entonces:

$$\boxed{\boldsymbol{y(x) = a_0[1 + x\,Arctan(x)] + a_1(x)}}$$

4) $\boldsymbol{y'' + 2xy' - 2y = x}$

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{+\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2}$$

$$\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + 2x \sum_{n=1}^{+\infty} a_n (n) x^{n-1} - 2 \sum_{n=0}^{+\infty} a_n x^n = x$$

$$\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + 2 \sum_{n=1}^{+\infty} a_n (n) x^{n} - 2 \sum_{n=0}^{+\infty} a_n x^n = x$$

$$K = n - 2$$

$$\sum_{K=0}^{+\infty} a_{K+2} (K+2)(K+1) x^{K} + 2 \sum_{n=1}^{+\infty} a_n (n) x^{n} - 2 \sum_{n=0}^{+\infty} a_n x^n = x$$

$$a_2(2)(1) - 2a_0 + \sum_{n=0}^{+\infty} [a_{n+2}(n+2)(n+1) + 2a_n(n) - 2a_n] x^n = x$$

$$2a_2 - 2a_0 = 0 \quad \Rightarrow \quad \boldsymbol{a_2 = a_0}$$

$$[a_3(3)(2) + 2a_1(1) - 2a_1]x = x \quad \Rightarrow \quad 6a_3 - 2a_1 - 2a_1 = 1 \quad \Rightarrow \quad \boldsymbol{a_3 = \frac{1}{6} + \frac{2}{3} a_1}$$

$$a_{n+2}(n+2)(n+1) + 2a_n(n) - 2a_n = 0$$

$$a_{n+2} = \frac{2(n+1)}{(n+2)(n+1)} a_n \quad \Rightarrow \quad \boldsymbol{a_{n+2} = \frac{2}{n+2} a_n} \ ; \ \ \forall\, \boldsymbol{n \geq 2}$$

Dando valores a "n":

n = 2

$$a_4 = \frac{2}{4}a_2 \quad \Rightarrow \quad a_4 = \frac{1}{2}a_0$$

n = 3

$$a_5 = \frac{2}{5}a_3 \quad \Rightarrow \quad a_5 = \frac{2}{5}\left(\frac{1}{6} + \frac{2}{3}a_1\right) \quad \Rightarrow \quad a_5 = \frac{1}{3*5} + \frac{2*2}{3*5}a_1$$

n = 4

$$a_6 = \frac{2}{6}a_4 \quad \Rightarrow \quad a_6 = \frac{2}{6}\left(\frac{1}{2}a_0\right) \quad \Rightarrow \quad a_6 = \frac{1}{6}a_0$$

n = 5

$$a_7 = \frac{2}{7}a_5 \quad \Rightarrow \quad a_7 = \frac{2}{7}\left(\frac{2}{5*6} + \frac{2*2}{3*5}a_1\right) \quad \Rightarrow \quad a_7 = \frac{2}{3*5*7} + \frac{2*2*2}{3*5*7}a_1$$

n = 6

$$a_8 = \frac{2}{8}a_6 \quad \Rightarrow \quad a_8 = \frac{2}{8}\left(\frac{1}{6}a_0\right) \quad \Rightarrow \quad a_8 = \frac{1}{24}a_0$$

n = 7

$$a_9 = \frac{2}{9}a_7 \quad \Rightarrow \quad a_9 = \frac{2}{9}\left(\frac{2*2}{5*6*7} + \frac{2*2*2}{3*5*7}a_1\right) \quad \Rightarrow \quad a_9 = \frac{2*2}{3*5*7*9} + \frac{2*2*2*2}{3*5*7*9}a_1$$

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Entonces:

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = a_0 + a_1 x + a_0 x^2 + \left(\frac{1}{6} + \frac{2}{3}a_1\right)x^3 + \frac{a_0}{2}x^4 + \left(\frac{2}{5*6} + \frac{2*2}{3*5}a_1\right)x^5 + \frac{a_0}{6}x^6 + \left(\frac{2*2}{5*6*7} + \frac{2*2*2}{3*5*7}a_1\right)x^7 + \ \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = a_0 + a_1 x + a_0 x^2 + \frac{x^3}{6} + \frac{2}{3}a_1 x^3 + \frac{a_0}{2}x^4 + \frac{2x^5}{5*6} + \frac{2*2}{3*5}a_1 x^5 + \frac{a_0}{6}x^6 + \frac{2*2}{5*6*7}x^7 + \frac{2*2*2}{3*5*7}a_1 x^7 + \ \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = a_0\left(1 + x^2 + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + \ \dots\dots\right) + a_1\left(x + \frac{2}{3}x^3 + \frac{2*2}{3*5}x^5 + \frac{2*2*2}{3*5*7}x^7 + \frac{2*2*2*2}{3*5*7*9}x^9 \ \dots\dots\right) +$$

$$\left(\frac{1}{2*3}x^3 + \frac{1}{3*5}x^5 + \frac{2}{3*5*7}x^7 + \frac{2*2}{3*5*7*9} \ \dots\dots\right)$$

Haciendo artificios para dar forma de sumatoria a la sucesión:

$$y(x) = a_0\left(1 + x^2 + \frac{x^6}{2*3} + \frac{x^8}{2*3*4} + \frac{x^{10}}{2*3*4*5} + \ \dots\dots\right) + a_1\left(x + \frac{\mathbf{2}*2}{\mathbf{2}*3}x^3 + \frac{\mathbf{2}*\mathbf{4}*2*2}{\mathbf{2}*\mathbf{4}*3*5}x^5 + \frac{\mathbf{2}*\mathbf{4}*\mathbf{6}*2*2*2}{\mathbf{2}*\mathbf{4}*\mathbf{6}*3*5*7}x^7 + \ \dots\dots\right) +$$

$$\left(\frac{\mathbf{2}}{\mathbf{2}*3}x^3 + \frac{\mathbf{2}*\mathbf{4}}{3*5*\mathbf{2}*\mathbf{4}}x^5 + \frac{2*\mathbf{2}*\mathbf{4}*\mathbf{6}}{3*5*7*\mathbf{2}*\mathbf{4}*\mathbf{6}}x^7 + \frac{2*2*\mathbf{2}*\mathbf{4}*\mathbf{6}*\mathbf{8}}{3*5*7*9*\mathbf{2}*\mathbf{4}*\mathbf{6}*\mathbf{8}} \ \dots\dots\right)$$

$$y(x) = a_0\left(1 + x^2 + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \ \dots\dots\right) + a_1\left(x + \frac{(2*2)}{2*3}x^3 + \frac{(2*2*2*2)*2}{2*3*4*5}x^5 + \frac{(2*2*2*2*2*2)*2*3}{2*3*4*5*6*7}x^7 + \ \dots\dots\right) +$$

$$\left(\frac{1}{2*3}x^3 + \frac{(2*2)*2}{2*3*4*5}x^5 + \frac{(2*2*2*2)*2*3}{2*3*4*5*6*7}x^7 + \frac{(2*2*2*2*2*2)*2*3*4}{2*3*4*5*6*7*8*9}x^9 \ \dots\dots\right)$$

$$y(x) = a_0\sum_{n=0}^{+\infty}\frac{(x^2)^n}{n!} + a_1\sum_{n=0}^{+\infty}\frac{2^{2n}n!}{(2n+1)!}x^{2n+1} + \sum_{n=0}^{+\infty}\frac{2^{2n}n!}{(2n+3)!}x^{2n+3}$$

$$\boxed{\boldsymbol{y(x) = a_0 e^{x^2} + a_1\sum_{n=0}^{+\infty}\frac{2^{2n}n!}{(2n+1)!}x^{2n+1} + \sum_{n=0}^{+\infty}\frac{2^{2n}n!}{(2n+3)!}x^{2n+3}}}$$

5) $\boldsymbol{(x^2+1)y'' + xy' - y = 0}$

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{+\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2}$$

$$(x^2+1)\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + x\sum_{n=1}^{+\infty} a_n (n) x^{n-1} - \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$x^2\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + x\sum_{n=1}^{+\infty} a_n (n) x^{n-1} - \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n} + \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=1}^{+\infty} a_n (n) x^{n} - \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$K = n-2$$

$$\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n} + \sum_{K=0}^{+\infty} a_{K+2} (K+2)(K+1) x^{K} + \sum_{n=1}^{+\infty} a_n (n) x^{n} - \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$a_2(2)(1) + a_3(3)(2)x + a_1(1)x - a_0 - a_1 x + \sum_{n=2}^{+\infty}[a_n(n)(n-1) + a_{n+2}(n+2)(n+1) + a_n(n) - a_n]x^n = 0$$

$$(2a_2 - a_0) + 6a_3 + \sum_{n=2}^{+\infty}[a_n(n)(n-1) + a_{n+2}(n+2)(n+1) + a_n(n) - a_n]x^n = 0$$

$$2a_2 - a_0 = 0 \quad \Rightarrow \quad \boldsymbol{a_2 = \frac{1}{2}a_0}$$

$$6a_3 = 0 \quad \Rightarrow \quad \boldsymbol{a_3 = 0}$$

$$a_n(n)(n-1) + a_{n+2}(n+2)(n+1) + a_n(n) - a_n = 0$$

$$a_{n+2} = \frac{1-n-n(n-1)}{(n+2)(n+1)}a_n \quad \Rightarrow \quad \boldsymbol{a_{n+2} = \frac{1-n}{n+2}a_n \ ; \ \forall\, n \geq 2}$$

Dando valores a "n":

n = 2

$$a_4 = -\frac{1}{4}a_2 \quad \Rightarrow \quad a_4 = -\frac{1}{4}\left(\frac{1}{2}a_0\right) \quad \Rightarrow \quad a_4 = -\frac{1}{2*4}a_0$$

n = 3

$$a_5 = -\frac{2}{5}a_3 \quad \Rightarrow \quad a_5 = -\frac{2}{5}(0) \quad \Rightarrow \quad a_5 = 0$$

n = 4

$$a_6 = -\frac{3}{6}a_4 \quad \Rightarrow \quad a_6 = -\frac{3}{6}\left(-\frac{1}{2*4}a_0\right) \quad \Rightarrow \quad a_6 = \frac{3}{2*4*6}a_0$$

n = 5

$$a_7 = -\frac{4}{7}a_5 \quad \Rightarrow \quad a_7 = 0$$

n = 6

$$a_8 = -\frac{5}{8}a_6 \quad \Rightarrow \quad a_4 = -\frac{5}{8}\left(\frac{3}{2*4*6}a_0\right) \quad \Rightarrow \quad a_4 = -\frac{3*5}{2*4*6*8}a_0$$

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Entonces:

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = a_0 + a_1x + \frac{1}{2}a_0x^2 + 0 - \frac{1}{2*4}a_0x^4 + 0 + \frac{3}{2*4*6}a_0x^6 - \frac{3*5}{2*4*6*8}a_0x^8 \ \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = a_1(x) + a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{(2*2)*2}x^4 + \frac{3}{(2*2*2)*2*3}x^6 - \frac{3*5}{(2*2*2*2)*2*3*4}x^8 \ \ldots\ldots\ldots\right)$$

$$\boxed{\boldsymbol{y(x) = a_1(x) + a_0\left[1 + \frac{1}{2}x^2 + \sum_{n=2}^{+\infty}(-1)^{n-1}\frac{1*3*5****(2n-3)}{2^{2n}n!}x^{2n}\right]}}$$

6) $\boldsymbol{y'' + (Cos\ x)y = 0}$

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{+\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2}$$

Sabemos que:

$$Cos\ x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n+1)}$$

$$Cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \ldots\ldots\ldots\ldots$$

Entonces:

$$\sum_{n=2}^{+\infty} a_n (n)(n-1) x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^3}{4!} - \frac{x^4}{6!} + \ \ldots\ldots\ldots\ldots\right) \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$K = n - 2$$

$$\sum_{K=0}^{+\infty} a_{K+2} (K+2)(K+1) x^{K} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \ldots\ldots\ldots\ldots\right) \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$\sum_{n=0}^{+\infty} a_{n+2} (n+2)(n+1) x^{n} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \ldots\ldots\ldots\ldots\right) \sum_{n=0}^{+\infty} a_n x^n = 0$$

Resolviendo la sumatoria:

$$(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ \ldots\ldots\ldots) + \left(1 - \frac{x^2}{2} + \frac{x^3}{24} - \frac{x^6}{720} + \ \ldots\ldots\ldots\right)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ \ldots\ldots\ldots) = 0$$

$$2a_2 + a_0 + (6a_3 + a_1)x + \left(12a_4 + a_2 - \frac{a_0}{2}\right)x^2 + \left(20a_5 + a_3 - \frac{a_1}{2}\right)x^3 + \ \ldots\ldots\ldots\ldots$$

La última expresión tiene que ser igual a cero, de modo que se debe cumplir que:

$$2a_2 + a_0 = 0 \quad \Rightarrow \quad a_2 = -\frac{a_0}{2}$$

$$6a_3 + a_1 = 0 \quad \Rightarrow \quad a_3 = -\frac{a_1}{6}$$

$$12a_4 + a_2 - \frac{a_0}{2} = 0 \quad \Rightarrow \quad 12a_4 = \frac{a_0}{2} - a_2 \quad \Rightarrow \quad 12a_4 = \frac{a_0}{2} + \frac{a_0}{2} \quad \Rightarrow \quad a_4 = \frac{a_0}{12}$$

$$20a_5 + a_3 - \frac{a_1}{2} = 0 \quad \Rightarrow \quad 20a_5 = \frac{a_1}{2} - a_3 \quad \Rightarrow \quad 20a_5 = \frac{a_1}{2} + \frac{a_1}{6} \quad \Rightarrow \quad a_5 = \frac{a_1}{30}$$

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Entonces:

$$\boxed{\boldsymbol{y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \ \ldots\ldots\ldots\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \ \ldots\ldots\ldots\right)}}$$

ESPOL

ECUACIONES DIFERENCIALES (2DO PARCIAL)

- **RESOLUCIÓN DE ECUACIONES DIFERENCIALES ALREDEDOR DE PUNTOS SINGULARES**
- **TRANSFORMADA DE LAPLACE**
- **RESOLUCIÓN DE ECUACIONES DIFERENCIALES MEDIANTE TRANSFORMADA DE LAPLACE**
- **TRANSFORMADA INVERSA DE LAPLACE**
- **RESOLUCIÓN DE SISTEMAS DE ECUACIONES DIFERENCIALES**
- **APLICACIONES DE LAS ECUACIONES DIFERENCIALES DE SEGUNDO ORDEN**
- **SERIES DE FOURIER**
- **ECUACIONES EN DERIVADA PARCIALES**

[ERICK CONDE]

RESOLUCIÓN DE ECUACIONES DIFERENCIALES ALREDEDOR DE PUNTOS SINGULARES

MÉTODO DE FROBENIUS

1) $\boldsymbol{xy'' - y' + 4x^3y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{-1}{x} = -1$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{4x^3}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2} - \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4x^3\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4\sum_{n=0}^{+\infty} a_n\, x^{n+r+3} = 0$$

Multiplicando por "x" a toda la expresión:

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4x\sum_{n=0}^{+\infty} a_n\, x^{n+r+3} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} - \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + 4\sum_{n=0}^{+\infty} a_n\, x^{n+r+4} = 0$$

$$M = n+4$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} - \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + 4\sum_{M=4}^{+\infty} a_{M-4}\, x^{M+r} = 0$$

Generando términos hasta n=4

$$a_0(r)(r-1)x^r + a_1(r+1)(r)x^{r+1} + a_2(r+2)(R+1)x^{r+2} + a_3(r+3)(r+2)x^{r+3} -$$

$$a_0(r)x^r - a_1(r+1)x^{r+1} - a_2(r+2)x^{r+2} - a_3(r+3)x^{r+3} + \sum_{n=4}^{+\infty}[(n+r)(n+r-1)a_n - (n+r)a_n + 4a_{n-4}]x^{n+r} = 0$$

$a_0x^r[r(r-1)-r]=0$

$r(r-1-1)=0 \quad \Longrightarrow \quad r_1=0 \quad , \quad r_2=2 \ \Longrightarrow \boldsymbol{a_0 \neq 0}$

$a_1x^{r+1}[r(r+1)-(r+1)]=0$

$a_1x^{r+1}[2(2+1)-(2+1)]=0 \Longrightarrow a_1x^{r+1}(3)=0 \Longrightarrow \ \boldsymbol{a_1=0}$

$a_2x^{r+2}[(r+1)(r+2)-(r+2)]=0$

$a_2x^{r+2}[(2+1)(2+2)-(2+2)]=0 \Longrightarrow a_2x^{r+2}(8)=0 \Longrightarrow \boldsymbol{a_2=0}$

$a_3x^{r+3}[(r+3)(r+2)-(r+3)]=0$

$a_3x^{r+3}[(2+3)(2+2)-(2+3)]=0 \Longrightarrow a_2x^{r+2}(15)=0 \Longrightarrow \boldsymbol{a_3=0}$

$[(n+r)(n+r-1)a_n-(n+r)a_n+4a_{n-4}]x^{n+r}=0$

$(n+r)(n+r-1)a_n-(n+r)a_n+4a_{n-4}=0 \quad \Rightarrow \quad \boldsymbol{a_n(r)=\frac{4a_{n-4}}{(n+r)(2-n-r)}} \ ; \ \forall \boldsymbol{n} \geq \boldsymbol{4}$

<u>*Para r = 2*</u>

$a_n=-\frac{4a_{n-4}}{(n+2)(n)} \ ; \ \forall n \geq 4$

$n=4 \quad \Longrightarrow \quad a_4=-\frac{4a_0}{4*6}$

$n=5 \quad \Longrightarrow \quad a_5=-\frac{4a_1}{5*7} \quad \Longrightarrow \quad a_5=0$

$n=6 \quad \Longrightarrow \quad a_6=-\frac{4a_2}{6*8} \quad \Longrightarrow \quad a_6=0$

$n=7 \quad \Longrightarrow \quad a_7=-\frac{4a_3}{7*9} \quad \Longrightarrow \quad a_7=0$

$n=8 \quad \Longrightarrow \quad a_8=-\frac{4a_4}{8*10} \quad \Longrightarrow \quad a_8=\frac{4*4a_0}{4*6*8*10}$

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$n=12 \quad \Longrightarrow \quad a_{12}=-\frac{4a_8}{12*14} \quad \Longrightarrow \quad a_{12}=\frac{4*4*4a_0}{4*6*8*10*12*14}$
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Entonces:

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$
$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+2}$$

$$y_1(x) = a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + a_4x^6 + a_5x^7 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + a_4x^6 + a_5x^7 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0x^2 - \frac{4a_0}{4*6}x^6 + \frac{4*4a_0}{4*6*8*10}x^{10} - \frac{4*4*4a_0}{4*6*8*10*12*14}x^{14} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0\left(x^2 - \frac{4}{3!*2^2}x^6 + \frac{4^2}{5!*2^4}x^{10} - \frac{4^3}{7!*2^6}x^{14} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\right)$$

$$y_1(x) = a_0\sum_{n=0}^{+\infty}\frac{x^{4n+2}*4^n*(-1)^n}{(2n+1)!*2^{2n}}$$

$$y_1(x) = a_0\sum_{n=0}^{+\infty}\frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!}$$

$$y_1(x) = a_0\, Sen(x^2)$$

$$y_2(x) = v(x)y_1(x)$$

Encontrando v(x)

$$v(x) = \int\frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad\Rightarrow\quad v(x) = \int\frac{e^{-\int -\frac{1}{x}dx}}{Sen(x^2)^2}dx$$

$$v(x) = \int\frac{e^{\int\frac{1}{x}dx}}{Sen(x^2)^2}dx \quad\Rightarrow\quad v(x) = \int\frac{e^{ln|x|}}{Sen(x^2)^2}dx$$

$$v(x) = \int\frac{x}{Sen(x^2)^2}dx$$

Integrando por cambio de variable:

$$u = x^2 \qquad du = 2x\,dx$$

$$v(x) = \int\frac{1}{2}\frac{du}{Sen(u)^2} \quad\Rightarrow\quad v(x) = -\frac{1}{2}Cot(u) \quad\Rightarrow\quad v(x) = -\frac{1}{2}Cot(x^2)$$

$$y_2(x) = -\frac{1}{2}Cot(x^2)\,Sen(x^2) = -\frac{1}{2}Cos(x^2)$$

$$\boxed{\boldsymbol{y_1(x) = Sen(x^2)}}$$

$$\boxed{\boldsymbol{y_2(x) = -\frac{1}{2}Cos(x^2)}}$$

2) $\mathbf{2xy'' + (1+x)y' + y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\;\Rightarrow\;\; \lim_{x\to 0} x\,\frac{(1+x)}{2x} = \frac{1}{2}$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\;\Rightarrow\;\; \lim_{x\to 0} x^2\,\frac{1}{2x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$2x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-2} + (1+x)\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + \sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$2\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + \sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

Multiplicando por "x":

$$2x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} + x\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + x\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$2\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r+1} + \sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$M = n+1$$

$$2\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + \sum_{M=1}^{+\infty}(M-1+r)a_n\, x^{M+r} + \sum_{M=1}^{+\infty} a_{M-1}\, x^{M+r} = 0$$

Generando términos hasta n=1

$$2a_0(r)(r-1)x^r + a_0(r)x^r + \sum_{n=1}^{+\infty}[2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-1}(n-1+r) + a_{n-1}]x^{n+r} = 0$$

$a_0x^r[2r(r-1)+r] = 0 \;\;\Longrightarrow\;\; r(2r-2+1) = 0 \;\;\Longrightarrow\;\; r_1 = 0 \;\;,\;\; r_2 = {}^1\!/_2 \;\Longrightarrow \boldsymbol{a_0 \neq 0}$

$[2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-1}(n-1+r) + a_{n-1}]x^{n+r} = 0$

$2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-1}(n-1+r) + a_{n-1} = 0$

$$\boldsymbol{a_n(r) = -\frac{a_{n-1}(n-1+r+1)}{(n+r)[2(n+r-1)+1]}}\ ;\ \forall \boldsymbol{n \geq 1}$$

$a_n(r) = -\frac{a_{n-1}}{(2n+2r-1)}$; $\forall n \geq 1$

$Para \quad r = {}^1\!/_2$

$a_n = -\frac{a_{n-1}}{2n}$; $\forall n \geq 1$

$n = 1 \quad \Rightarrow \quad a_1 = -\frac{a_0}{2}$

$n = 2 \quad \Rightarrow \quad a_2 = -\frac{a_1}{4} \quad \Rightarrow \quad a_2 = \frac{a_0}{2*4}$

$n = 3 \quad \Rightarrow \quad a_3 = -\frac{a_2}{6} \quad \Rightarrow \quad a_3 = -\frac{a_0}{2*4*6}$

$n = 4 \quad \Rightarrow \quad a_7 = -\frac{a_3}{8} \quad \Rightarrow \quad a_7 = \frac{a_0}{2*4*6*8}$

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Entonces:

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+\frac{1}{2}}$$

$$y_1(x) = a_0 x^{1/2} + a_1 x^{3/2} + a_2 x^{5/2} + a_3 x^{7/2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0 x^{1/2} - \frac{a_0}{2} x^{3/2} + \frac{a_0}{2*4} x^{5/2} - \frac{a_0}{2*4*6} x^{7/2} + \frac{a_0}{2*4*6*8} x^{9/2} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0 \left(x^{1/2} - \frac{1}{2} x^{3/2} + \frac{1}{2!*2^2} x^{5/2} - \frac{1}{3!*2^3} x^{7/2} + \frac{1}{4!*2^4} x^{9/2} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \right)$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{(-1)^n x^{\frac{2n+1}{2}}}{n!*2^n}$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{(-1)^n x^n * x^{\frac{1}{2}}}{n!*2^n}$$

$$y_1(x) = a_0 \sqrt{x} \sum_{n=0}^{+\infty} \frac{\left(-\frac{x}{2}\right)^n}{n!}$$

$$y_1(x) = a_0\, \sqrt{x}\, e^{-x/2}$$

$y_2(x) = v(x)y_1(x)$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{1+x}{2x}dx}}{\left(\sqrt{x}\, e^{-x/2}\right)^2}dx$$

$$v(x) = \int \frac{e^{-\frac{1}{2}(ln|x|+x)}}{xe^{-x}}dx \quad \Rightarrow \quad v(x) = \int \frac{x^{-1/2}\, e^{-x/2}}{xe^{-x}}dx$$

$$v(x) = \int x^{-3/2}\, e^{x/2}\, dx$$

$Para\ resolver \int x^{-3/2}\, e^{x/2}\, dx\ \ es\ necesario\ utilizar\ series$

$$e^{x/2} = \sum_{n=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^n}{n!}$$

$$\frac{e^{x/2}}{x^{3/2}} = \sum_{n=0}^{+\infty} \frac{\left(\frac{1}{2}\right)^n x^{n-\frac{3}{2}}}{n!}$$

$$\int \frac{e^{x/2}}{x^{3/2}}dx = \int \left[\sum_{n=0}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-3}{2}}}{n!}\right]dx$$

$$\int \frac{e^{x/2}}{x^{3/2}}dx = \int \left[\frac{1}{x^{3/2}} + \frac{1}{2x^{1/2}} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-3}{2}}}{n!}\right]dx$$

$$\int \frac{e^{x/2}}{x^{3/2}}dx = -\frac{2}{x^{3/2}} + \frac{x^{1/2}}{2} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-1}{2}}}{\left(\frac{2n-1}{2}\right)n!}$$

$$v(x) = -\frac{2}{x^{3/2}} + \frac{x^{1/2}}{2} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-1}{2}}}{\left(\frac{2n-1}{2}\right)n!}$$

$$y_2(x) = \sqrt{x}\, e^{-x/2}\left[-\frac{2}{x^{3/2}} + \frac{x^{1/2}}{2} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-1}{2}}}{\left(\frac{2n-1}{2}\right)n!}\right]$$

$$\boxed{\boldsymbol{y_1(x) = \sqrt{x}\, e^{-x/2}}}$$

$$\boxed{\boldsymbol{y_2(x) = \sqrt{x}\, e^{-x/2}\left[-\frac{2}{x^{3/2}} + \frac{x^{1/2}}{2} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-1}{2}}}{\left(\frac{2n-1}{2}\right)n!}\right]}}$$

3) $\boldsymbol{xy'' + (3-x)y' - y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{(3-x)}{x} = 3$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{-1}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2} + (3-x)\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} - \sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

Multiplicado por “x”

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 3x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} - x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} - \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r+1} - \sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$M = n+1$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} - \sum_{M=1}^{+\infty} (M-1+r)a_n\, x^{M+r} - \sum_{M=1}^{+\infty} a_{M-1}\, x^{M+r} = 0$$

Generando términos hasta n=1

$$a_0(r)(r-1)x^r + 3a_0(r)x^r + \sum_{n=1}^{+\infty} [(n+r)(n+r-1)a_n + 3(n+r)a_n - a_{n-1}(n-1+r) - a_{n-1}]x^{n+r} = 0$$

$$a_0x^r[r(r-1)+3r] = 0 \;\Rightarrow\; r(r-1+3) = 0 \;\Rightarrow\; r_1 = 0 \;\;,\;\; r_2 = -2 \;\Rightarrow \boldsymbol{a_0 \neq 0}$$

$$[(n+r)(n+r-1)a_n + 3(n+r)a_n - a_{n-1}(n-1+r) - a_{n-1}]x^{n+r} = 0$$

$$a_n(r) = \frac{a_{n-1}(n-1+r+1)}{(n+r)[(n+r-1)+3]}\;;\;\forall n \geq 1$$

$$\boldsymbol{a_n(r) = \frac{a_{n-1}}{(n+r+2)}\;;\;\forall n \geq 1}$$

$\underline{Para\ \ r = 0}$

$a_n = \dfrac{a_{n-1}}{n+2}$; $\forall n \geq 1$

$n = 1 \quad \Longrightarrow \quad a_1 = \frac{a_0}{3}$

$n = 2 \quad \Longrightarrow \quad a_2 = \frac{a_1}{4} \quad \Longrightarrow \quad a_2 = \frac{a_0}{3*4}$

$n = 3 \quad \Longrightarrow \quad a_3 = \frac{a_2}{5} \quad \Longrightarrow \quad a_3 = \frac{a_0}{3*4*5}$

$n = 4 \quad \Longrightarrow \quad a_7 = \frac{a_3}{6} \quad \Longrightarrow \quad a_7 = \frac{a_0}{3*4*5*6}$

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Entonces:

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n}$$

$$y_1(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ \dots\dots\dots\dots\dots\dots\dots\dots.$$

$$y_1(x) = a_0 + \frac{a_0}{3}\,x + \frac{a_0}{3*4}\,x^2 + \frac{a_0}{3*4*5}x^3 + \frac{a_0}{3*4*5*6}\,x^4 + \ \dots\dots\dots\dots\dots\dots\dots\dots.$$

$$y_1(x) = a_0\left(1 + \frac{x}{3} + \frac{x^2}{3*4} + \frac{x^3}{3*4*5} + \frac{x^4}{3*4*5*6} + \ \dots\dots\dots\dots\dots\dots\dots\dots.\right)$$

$$y_1(x) = 2a_0\left(\frac{1}{2} + \frac{x}{2*3} + \frac{x^2}{2*3*4} + \frac{x^3}{2*3*4*5} + \frac{x^4}{2*3*4*5*6} + \ \dots\dots\dots\dots\dots\dots\dots\dots.\right)$$

$$y_1(x) = 2a_0 \sum_{n=0}^{+\infty} \frac{x^n}{(n+2)!}$$

$Como\ no\ sabemos\ a\ que\ converge\ la\ sumatoria\ probemos\ con\ r_2 = -2, siempre\ y\ cuando\ a_n(r) = \dfrac{a_{n-1}}{(n+r+2)}\ exista, \forall n \geq 1$

$\underline{Para\ \ r = -2}$

$a_n = \dfrac{a_{n-1}}{n}$; $\forall n \geq 1$

$n = 1 \quad \Longrightarrow \quad a_1 = a_0$

$n = 2 \quad \Longrightarrow \quad a_2 = \frac{a_1}{4} \quad \Longrightarrow \quad a_2 = \frac{a_0}{2}$

$n = 3 \quad \Longrightarrow \quad a_3 = \frac{a_2}{5} \quad \Longrightarrow \quad a_3 = \frac{a_0}{2*3}$

$n = 4 \quad \Longrightarrow \quad a_7 = \frac{a_3}{6} \quad \Longrightarrow \quad a_7 = \frac{a_0}{2*3*4}$

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$$y_2(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_2(x) = \sum_{n=0}^{+\infty} a_n\, x^{n-2}$$

$$y_2(x) = a_0x^{-2} + a_1x^{-1} + a_2 + a_3x + a_4x^2 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_2(x) = a_0x^{-2} + a_0x^{-1} + \frac{a_0}{2} + \frac{a_0}{2*3}x + \frac{a_0}{2*3*4}\ x^2 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_2(x) = a_0\left(x^{-2} + x^{-1} + \frac{1}{2} + \frac{x}{2*3} + \frac{x^2}{2*3*4} + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\right)$$

$$y_2(x) = a_0\sum_{n=0}^{+\infty}\frac{x^{n-2}}{n!} \quad \Rightarrow \quad y_2(x) = a_0\sum_{n=0}^{+\infty}\frac{x^n * x^{-2}}{n!}$$

$$y_2(x) = a_0\frac{1}{x^2}\sum_{n=0}^{+\infty}\frac{x^n}{n!} \quad \Rightarrow \quad y_2(x) = a_0\frac{e^x}{x^2}$$

$$y_1(x) = v(x)y_2(x)$$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_2}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int\frac{3-x}{x}dx}}{\left(\frac{e^x}{x^2}\right)^2}dx$$

$$v(x) = \int \frac{e^{-\int\frac{3-x}{x}dx}}{\left(\frac{e^x}{x^2}\right)^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{(-3\ ln|x|+x)}}{\frac{e^{2x}}{x^4}}dx$$

$$v(x) = \int \frac{x^4x^{-3}\ e^x}{e^{2x}}dx \quad \Rightarrow \quad v(x) = \int \frac{x}{e^x}dx$$

<u>Integrando por partes:</u>

$$u = x \qquad du = dx$$

$$dv = \frac{dx}{e^x} \qquad v = -\frac{1}{e^x}$$

$$v(x) = -\frac{x}{e^x} - \int -\frac{dx}{e^x} \quad \Rightarrow \quad v(x) = -\frac{x}{e^x} - \frac{1}{e^x}$$

$$y_1(x) = -\left(\frac{x}{e^x} + \frac{1}{e^x}\right)\frac{e^x}{x^2}$$

$$\boxed{\boldsymbol{y_1(x) = -\left(\frac{1}{x} + \frac{1}{x^2}\right)}}$$

$$\boxed{\boldsymbol{y_2(x) = \frac{e^x}{x^2}}}$$

4) $\boldsymbol{x(1-x)y'' - 3y' + 2y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{-3}{x(1-x)} = -3$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{2}{x(1-x)} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x(1-x)\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-2} - 3\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + 2\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-2} - x^2\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-2} - 3\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + 2\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} - 3\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + 2\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

Multiplicando por "x"

$$x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} - 3x\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + 2x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} - \sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r+1} - 3\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$M = n+1$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} - \sum_{M=1}^{+\infty}(M-1+r)(M+r-2)a_{M-1}\, x^{n+r} - 3\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + 2\sum_{M=1}^{+\infty} a_{M-1}\, x^{M+r} = 0$$

$$a_0(r)(r-1)x^r - 3a_0(r)x^r + \sum_{n=1}^{+\infty}[(n+r)(n+r-1)a_n - (n-1+r)(n+r-2)a_{n-1} - 3a_n(n+r) + 2a_{n-1}]x^{n+r} = 0$$

$a_0x^r[r(r-1)-3r] = 0 \;\Longrightarrow\; r(r-1-3) = 0 \;\Longrightarrow\; r_1 = 0 \;,\; r_2 = 4 \Longrightarrow \boldsymbol{a_0 \neq 0}$

$[(n+r)(n+r-1)a_n - (n-1+r)(n+r-2)a_{n-1} - 3a_n(n+r) + 2a_{n-1}]x^{n+r} = 0$

$$a_n(r) = \frac{a_{n-1}[(n-1+r)(n+r-2)-2]}{(n+r)[(n+r-1)-3]}\;;\; \forall n \geq 1$$

$$a_n(r) = \frac{a_{n-1}[(n+r)^2 - 3(n+r)]}{(n+r)(n+r-4)}\;;\; \forall n \geq 1 \quad\Rightarrow\quad \boldsymbol{a_n(r) = \frac{a_{n-1}(n+r-3)}{(n+r-4)}\;;\; \forall n \geq 1}$$

$\underline{Para\ \ r = 4}$

$a_n = \dfrac{a_{n-1}(n+1)}{n}\ ;\ \forall n \geq 1$

$n = 1 \quad \Rightarrow \quad a_1 = 2a_0$

$n = 2 \quad \Rightarrow \quad a_2 = \frac{3a_1}{2} \quad \Rightarrow \quad a_2 = 3a_0$

$n = 3 \quad \Rightarrow \quad a_3 = \frac{4a_2}{3} \quad \Rightarrow \quad a_3 = 4a_0$

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$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+4}$$

$y_1(x) = a_0x^4 + a_1x^5 + a_2x^6 + a_3x^7 + a_4x^8 + \ \dots\dots\dots\dots\dots\dots\dots.$

$y_1(x) = a_0x^4 + a_1x^5 + a_2x^6 + a_3x^7 + a_4x^8 + \ \dots\dots\dots\dots\dots\dots\dots.$

$y_1(x) = a_0x^4 + 2a_0x^5 + 3a_0x^6 + 4a_0x^7 + 5a_0x^8 + \ \dots\dots\dots\dots\dots\dots\dots.$

$y_1(x) = a_0(x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \ \dots\dots\dots\dots\dots\dots\dots.$

$Sabemos\ que$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ \dots\dots\dots.$$

Derivando tenemos:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + x^5 + \ \dots\dots\dots\dots.)$$

$$-\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ \dots\dots\dots.$$

$$-\frac{x^4}{(1-x)^2} = x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \ \dots\dots\dots.$$

$$y_1(x) = -a_0\left[\frac{x^4}{(1-x)^2}\right]$$

$$y_2(x) = v(x)y_1(x)$$

$$v(x)=\int\frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x)=\int\frac{e^{-\int -\frac{3}{x(x-1)}dx}}{\left[\frac{x^4}{(1-x)^2}\right]^2}dx$$

$$v(x)=\int\frac{e^{\int\left(\frac{3}{x}+\frac{3}{1-x}\right)dx}}{\frac{x^8}{(1-x)^4}}dx \quad \Rightarrow \quad v(x)=\int\frac{e^{(3\,ln|x|-3\,ln|1-x|)}}{\frac{x^8}{(1-x)^4}}dx$$

$$v(x)=\int\frac{(1-x)^4x^3(1-x)^{-3}}{x^8}dx \quad \Rightarrow \quad v(x)=\int\frac{(1-x)}{x^5}dx$$

$$v(x)=-\frac{1}{4x^4}+\frac{1}{3x^3}$$

$$y_2(x)=\left(\frac{1}{3x^3}-\frac{1}{4x^4}\right)\frac{x^4}{(1-x)^2}$$

$$\boxed{\boldsymbol{y_1(x)=\frac{x^4}{(1-x)^2}}}$$

$$\boxed{\boldsymbol{y_2(x)=\left[\frac{x}{3(1-x)^2}-\frac{1}{4(1-x)^2}\right]}}$$

5) $\boldsymbol{xy'' + 2y' - xy = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{2}{x} = 2$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{-x}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

Multiplicando por "x":

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 2x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} - \sum_{n=0}^{+\infty} a_n\, x^{n+r+2} = 0$$

$$M = n+2$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} - \sum_{M=2}^{+\infty} a_{M-2}\, x^{M+2} = 0$$

$$a_0(r)(r-1)x^r + a_1(r+1)(r)x^{r+1} + 2a_0(r)x^r + 2a_1(r+1)x^{r+1}\sum_{n=2}^{+\infty}[(n+r)(n+r-1)a_n + 2a_n(n+r) - a_{n-2}]x^{n+r} = 0$$

$$a_0x^r[r(r-1)+r] = 0$$

$$r(r-1+1) = 0 \;\Rightarrow\; r_1 = 0 \;\;,\;\; r_2 = 0 \Rightarrow \boldsymbol{a_0 \neq 0}$$

$$a_1x^{r+1}[r(r+1) + 2(r+1)] = 0$$

$$a_1x^{r+1}[0(0+1) + 2(0+1)] = 0 \Rightarrow a_1x^{r+1}(2) = 0 \Rightarrow \boldsymbol{a_1 = 0}$$

$$[(n+r)(n+r-1)a_n + 2a_n(n+r) - a_{n-2}]x^{n+r} = 0$$

$$(n+r)(n+r-1)a_n + 2a_n(n+r) - a_{n-2} = 0$$

$$a_n(r) = \frac{a_{n-2}}{(n+r)[(n+r-1)+2]} \ ; \ \forall n \geq 2$$

$$a_n(r) = \frac{a_{n-2}}{(n+r)(n+r+1)} \ ; \ \forall n \geq 2$$

$$\boldsymbol{a_n(r) = \frac{a_{n-2}}{(n+r)(n+r+1)} \ ; \ \forall n \geq 2}$$

$\underline{Para \ \ r = 0}$

$$a_n = \frac{a_{n-2}}{n(n+1)} \ ; \ \forall n \geq 2$$

$$n = 2 \quad \Longrightarrow \quad a_2 = \frac{a_0}{2*3}$$

$$n = 3 \quad \Longrightarrow \quad a_3 = \frac{a_1}{2*3} \quad \Longrightarrow \quad a_3 = 0$$

$$n = 4 \quad \Longrightarrow \quad a_4 = \frac{a_2}{4*5} \quad \Longrightarrow \quad a_4 = \frac{a_0}{2*3*4*5}$$

$$n = 5 \quad \Longrightarrow \quad a_5 = \frac{a_3}{5*6} \quad \Longrightarrow \quad a_5 = 0$$

$$n = 6 \quad \Longrightarrow \quad a_6 = \frac{a_4}{6*7} \quad \Longrightarrow \quad a_6 = \frac{a_0}{2*3*4*5*6*7}$$

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$$y_1(x) = \sum_{n=0}^{+\infty} a_n \, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n \, x^{n}$$

$$y_1(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ a_5 x^5 + a_6 6 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0 + \frac{a_0}{2*3} \ x^2 \ + \frac{a_0}{2*3*4*5} x^4 + \frac{a_0}{2*3*4*5*6*7} x^6 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0 \left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \right)$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n+1)!}$$

$$y_1(x) = a_0 \frac{1}{x} \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$y_1(x) = a_0 \frac{Senh \ x}{x}$$

$y_2(x) = v(x)y_1(x)$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{2}{x}dx}}{\left(\frac{Senh\ x}{x}\right)^2}dx$$

$$v(x) = \int \frac{e^{-2\ ln|x|}x^2}{(Senh\ x)^2}dx \quad \Rightarrow \quad v(x) = \int \frac{x^2x^{-2}}{(Senh\ x)^2}dx$$

$$v(x) = \int (Senh\ x)^{-2}\ dx$$

$$Pero\ Senh\ x = \frac{e^x - e^{-x}}{2} \quad , entonces:$$

$$v(x) = \int \left(\frac{2}{e^x - e^{-x}}\right)^2 dx \quad \Rightarrow \quad v(x) = \int \left(\frac{2e^x}{e^{2x} - 1}\right)^2 dx \quad \Rightarrow \quad v(x) = \int \frac{4e^{2x}}{(e^{2x} - 1)^2}dx$$

Integrando por cambio de variable:

$$u = e^x \qquad du = e^x$$

$$v(x) = \int \frac{4u^2}{(u^2 - 1)^2}dx \quad \Rightarrow \quad v(x) = \int \frac{4u^2}{[(u-1)(u+1)]^2}dx \quad \Rightarrow \quad v(x) = \int \frac{4u^2}{(u-1)^2(u+1)^2}dx$$

Integrando aplicando fracciones parciales:

$$\frac{u^2}{(u-1)^2(u+1)^2} = \frac{2A(u-1) + B}{(u-1)^2} + \frac{2C(u+1) + D}{(u+1)^2}$$

$$u^2 = [2A(u-1) + B](u+1)^2 + [2C(u+1) + D](u-1)^2$$

$$u^2 = 2A(u^3 + u^2 - u - 1) + B(u^2 + 2u + 1) + 2C(u^3 - u^2 - u + 1) + D(u^2 - 2u + 1)$$

$$u^2 = (2A + 2C)u^3 + (2A + B - 2C + D)u^2 + (-2A + 2B - 2C - 2D)u + (-2A + B + 2C + D)$$

$(1)\ 2A + 2C = 0$

$(2)\ 2A + B - 2C + D = 1$

$(3) - 2A + 2B - 2C - 2D = 0$

$(4) - 2A + B + 2C + D = 0$

$2A = -2C$ $\qquad$ $(1) + (3) \quad 2B = 2D$ $\qquad$ $(2) + (3) = 2B + 2D = 1$

$$B = \frac{1}{4} \qquad 2D + 2D = 1$$

$$D = \frac{1}{4}$$

$$2C + \frac{1}{4} + 2C + \frac{1}{4} = 0$$

$$C = \frac{1}{8} \qquad A = \frac{1}{8}$$

Entonces:

$$v(x) = \int \left[\frac{2A(u-1)}{(u-1)^2} + \frac{B}{(u-1)^2} + \frac{2C(u+1)}{(u+1)^2} + \frac{D}{(u+1)^2}\right] dx$$

$$v(x) = 2A\ ln|(u-1)^2| - \frac{B}{(u-1)} + 2C\ ln|(u+1)^2| - \frac{D}{(u+1)}$$

$$v(x) = 2\left(\frac{1}{8}\right)\ ln|(e^x-1)^2| - \frac{1}{4(e^x-1)} + 2\left(\frac{1}{8}\right)\ ln|(e^x+1)^2| - \frac{1}{4(e^x+1)}$$

$$y_2(x) = \left[\frac{1}{4}\ ln|(e^x-1)^2| - \frac{1}{4(e^x-1)} + \frac{1}{4}\ ln|(e^x+1)^2| - \frac{1}{4(e^x+1)}\right]\frac{Senh\ x}{x}$$

$$\boxed{\boldsymbol{y_1(x) = \frac{Senh\ x}{x}}}$$

$$\boxed{\boldsymbol{y_2(x) = \frac{1}{4}\left[\ ln\left|(e^x-1)^2\right| - \frac{1}{(e^x-1)} + \ ln\left|(e^x+1)^2\right| - \frac{1}{(e^x+1)}\right]\frac{Senh\ x}{x}}}$$

6) $\boldsymbol{xy'' + 3y' + 4x^3y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{3}{x} = 3$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{4x^3}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4x^3\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4\sum_{n=0}^{+\infty} a_n\, x^{n+r+3} = 0$$

Multiplicando por "x"

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 3x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4x\sum_{n=0}^{+\infty} a_n\, x^{n+r+3} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + 4\sum_{n=0}^{+\infty} a_n\, x^{n+r+4} = 0$$

$$M = n + 4$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + 4\sum_{M=4}^{+\infty} a_{M-4}\, x^{M+r} = 0$$

Generando términos hasta n=4

$a_0(r)(r-1)x^r + a_1(r+1)(r)x^{r+1} + a_2(r+2)(R+1)x^{r+2} + a_3(r+3)(r+2)x^{r+3} +$

$3a_0(r)x^r + 3a_1(r+1)x^{r+1} + 3a_2(r+2)x^{r+2} + 3a_3(r+3)x^{r+3} + \sum_{n=4}^{+\infty}[(n+r)(n+r-1)a_n + 3(n+r)a_n + 4a_{n-4}]x^{n+r} = 0$

$a_0x^r[r(r-1) + 3r] = 0$

$r(r-1+3) = 0 \;\Rightarrow\; r_1 = 0 \;\;,\;\; r_2 = -2 \;\Rightarrow \boldsymbol{a_0 \neq 0}$

$a_1x^{r+1}[r(r+1)+3(r+1)]=0$

$a_1x^{r+1}[0(0+1)+3(0+1)]=0 \Rightarrow a_1x^{r+1}(3)=0 \Rightarrow \boldsymbol{a_1=0}$

$a_2x^{r+2}[(r+1)(r+2)+3(r+2)]=0$

$a_2x^{r+2}[(0+1)(0+2)+3(0+2)]=0 \Rightarrow a_2x^{r+2}(6)=0 \Rightarrow \boldsymbol{a_2=0}$

$a_3x^{r+3}[(r+3)(r+2)+3(r+3)]=0$

$a_3x^{r+3}[(0+3)(0+2)+3(0+3)]=0 \Rightarrow a_2x^{r+2}(9)=0 \Rightarrow \boldsymbol{a_3=0}$

$[(n+r)(n+r-1)a_n+3(n+r)a_n+4a_{n-4}]x^{n+r}=0$

$(n+r)(n+r-1)a_n+3(n+r)a_n+4a_{n-4}=0$

$a_n(r)=-\frac{4a_{n-4}}{(n+r)(n+r-1+3)}\ ;\ \forall n\geq 4$

$\boldsymbol{a_n(r)=-\frac{4a_{n-4}}{(n+r)(n+r+2)}\ ;\ \forall n\geq 4}$

Para $r=0$

$a_n=-\frac{4a_{n-4}}{(n+2)(n)}\ ;\ \forall n\geq 4$

$n=4 \Rightarrow a_4=-\frac{4a_0}{4*6}$

$n=5 \Rightarrow a_5=-\frac{4a_1}{5*7} \Rightarrow a_5=0$

$n=6 \Rightarrow a_6=-\frac{4a_2}{6*8} \Rightarrow a_6=0$

$n=7 \Rightarrow a_7=-\frac{4a_3}{7*9} \Rightarrow a_7=0$

$n=8 \Rightarrow a_8=-\frac{4a_4}{8*10} \Rightarrow a_8=\frac{4*4a_0}{4*6*8*10}$

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$n=12 \Rightarrow a_{12}=-\frac{4a_8}{12*14} \Rightarrow a_{12}=\frac{4*4*4a_0}{4*6*8*10*12*14}$

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$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n}$$

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$y_1(x) = a_0 - \frac{4a_0}{4*6}x^4 + \frac{4*4a_0}{4*6*8*10}x^8 - \frac{4*4*4a_0}{4*6*8*10*12*14}x^{12} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$y_1(x) = a_0\left(1 - \frac{4}{3!*2^2}x^4 + \frac{4^2}{5!*2^4}x^8 - \frac{4^3}{7!*2^6}x^{12} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\right)$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{x^{4n}*4^n*(-1)^n}{(2n+1)!*2^{2n}}$$

$$y_1(x) = a_0 \frac{1}{x^2} \sum_{n=0}^{+\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$y_1(x) = a_0 \frac{1}{x^2} \sum_{n=0}^{+\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$y_1(x) = a_0\ \frac{Sen(x^2)}{x^2}$$

$$y_2(x) = v(x)y_1(x)$$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{3}{x}dx}}{\left[\frac{Sen(x^2)}{x^2}\right]^2}dx$$

$$v(x) = \int \frac{e^{-3\ ln|x|}}{\left[\frac{Sen(x^2)}{x^2}\right]^2}dx \quad \Rightarrow \quad v(x) = \int \frac{x^{-3}x^4}{Sen(x^2)^2}dx \quad \Rightarrow \quad v(x) = \int x\ Csc^2(x^2)$$

Integrando por cambio de variable:

$$u = x^2 \qquad du = 2x\ dx$$

$$v(x) = \int \frac{1}{2}Csc^2(u)du \quad \Rightarrow \quad v(x) = -\frac{1}{2}Cot(u) \quad \Rightarrow \quad v(x) = -\frac{1}{2}Cot(x^2)$$

$$y_2(x) = -\frac{1}{2}Cot(x^2)\ \frac{Sen(x^2)}{x^2} = -\frac{Cos(x^2)}{2x^2}$$

$$\boxed{\boldsymbol{y_1(x) = \frac{Sen(x^2)}{x^2}}}$$

$$\boxed{\boldsymbol{y_2(x) = \frac{Cos(x^2)}{2x^2}}}$$

7) $xy'' + 2y' + xy = 0$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \quad\Rightarrow\quad \lim_{x\to 0} x\,\frac{2}{x} = 2$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \quad\Rightarrow\quad \lim_{x\to 0} x^2\,\frac{x}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + \sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

Multiplicando por "x"

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 2x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + x\sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + \sum_{n=0}^{+\infty} a_n\, x^{n+r+2} = 0$$

$$M = n + 2$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + \sum_{M=2}^{+\infty} a_{M-2}\, x^{M+2} = 0$$

Generando términos hasta n=2

$$a_0(r)(r-1)x^r + a_1(r+1)(r)x^{r+1} + 2a_0(r)x^r + 2a_1(r+1)x^{r+1}\sum_{n=2}^{+\infty}[(n+r)(n+r-1)a_n + 2a_n(n+r) + a_{n-2}]x^{n+r} = 0$$

$$a_0x^r[r(r-1)+r] = 0$$

$$r(r-1+1) = 0 \quad\Rightarrow\quad r_1 = 0 \quad , \quad r_2 = 0 \Rightarrow \boldsymbol{a_0 \neq 0}$$

$$a_1x^{r+1}[r(r+1) + 2(r+1)] = 0$$

$$a_1x^{r+1}[0(0+1) + 2(0+1)] = 0 \Rightarrow a_1x^{r+1}(2) = 0 \Rightarrow \boldsymbol{a_1 = 0}$$

$[(n+r)(n+r-1)a_n + 2a_n(n+r) + a_{n-2}]x^{n+r} = 0$

$a_n(r) = -\frac{a_{n-2}}{(n+r)[(n+r-1)+2]}$; $\forall n \geq 2$

$a_n(r) = -\frac{a_{n-2}}{(n+r)(n+r+1)}$; $\forall n \geq 2$

$\boldsymbol{a_n(r) = -\frac{a_{n-2}}{(n+r)(n+r+1)}}$; $\boldsymbol{\forall n \geq 2}$

$\underline{Para \;\; r = 0}$

$a_n = -\frac{a_{n-2}}{n(n+1)}$; $\forall n \geq 2$

$n = 2 \;\; \Rightarrow \;\; a_2 = -\frac{a_0}{2*3}$

$n = 3 \;\; \Rightarrow \;\; a_3 = -\frac{a_1}{2*3} \;\; \Rightarrow \;\; a_3 = 0$

$n = 4 \;\; \Rightarrow \;\; a_4 = -\frac{a_2}{4*5} \;\; \Rightarrow \;\; a_4 = \frac{a_0}{2*3*4*5}$

$n = 5 \;\; \Rightarrow \;\; a_5 = -\frac{a_3}{5*6} \;\; \Rightarrow \;\; a_5 = 0$

$n = 6 \;\; \Rightarrow \;\; a_6 = -\frac{a_4}{6*7} \;\; \Rightarrow \;\; a_6 = \frac{a_0}{2*3*4*5*6*7}$

.
.
.

$$y_1(x) = \sum_{n=0}^{+\infty} a_n \, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n \, x^{n}$$

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_66 + \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0 - \frac{a_0}{2*3}\; x^2 + \frac{a_0}{2*3*4*5}x^4 - \frac{a_0}{2*3*4*5*6*7}x^6 + \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\right)$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{x^{2n}(-1)^n}{(2n+1)!}$$

$$y_1(x) = a_0 \frac{1}{x} \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$y_1(x) = a_0 \frac{Sen\, x}{x}$$

$y_2(x) = v(x)y_1(x)$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2} dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{2}{x}dx}}{\left(\frac{Sen\, x}{x}\right)^2} dx$$

$$v(x) = \int \frac{e^{-2\, ln|x|}x^2}{(Sen\, x)^2} dx \quad \Rightarrow \quad v(x) = \int \frac{x^2 x^{-2}}{(Sen\, x)^2} dx$$

$$v(x) = \int Csc^2(x)\, dx \quad \Rightarrow \quad v(x) = -Cot(x)$$

$$y_2(x) = -\frac{Sen\, x}{x}\, Cot(x) = -\frac{Cos\,(x)}{x}$$

$$\boxed{\boldsymbol{y_1(x) = \frac{Sen\, x}{x}}}$$

$$\boxed{\boldsymbol{y_2(x) = -\frac{Cos\,(x)}{x}}}$$

TRANSFORMADA DE LAPLACE

1) $\boldsymbol{\mathcal{L}\{Sen^5 t\}}$

Sabemos que:

① $e^{i\theta} = Cos\theta + i\, Sen\theta$

② $e^{-i\theta} = Cos\theta - i\, Sen\theta$

Entonces ① − ②

$$Sen\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\mathcal{L}\{Sen^5 t\} = \mathcal{L}\left\{\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^5\right\}$$

$$\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\mathcal{L}\left\{\frac{(e^{it})^5 - 5(e^{it})(e^{-it})^4 + 10(e^{it})^3(e^{-it})^2 - 10(e^{it})^2(e^{-it})^3 + 5(e^{it})(e^{-it})^4 - (e^{-it})^5}{2i}\right\}$$

$$\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\mathcal{L}\left\{\frac{e^{5it} - 5e^{3it} + 10e^{it} - 10e^{-it} + 5e^{-3it} - e^{-5it}}{2i}\right\}$$

$$\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\mathcal{L}\left\{\left(\frac{e^{5it} - e^{-5it}}{2i}\right) - 5\left(\frac{e^{3it} - e^{-3it}}{2i}\right) + 10\left(\frac{e^{it} - e^{-it}}{2i}\right)\right\}$$

$$\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\mathcal{L}\{Sen(5t) - 5\, Sen(3t) + 10\, Sen(t)\}$$

$$\boxed{\boldsymbol{\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\left(\frac{5}{s^2+25} - \frac{15}{s^2+9} + \frac{10}{s^2+1}\right)}}$$

2) $\boldsymbol{\mathcal{L}\{u(t-2\pi)Sen(t-2\pi)\}}$

Vamos a realizarlo paso a paso:

Como la función seno ya está desfasada, no hay problema, entonces, primero determinamos la trasformada de Laplace de la función seno: $\mathcal{L}\{Sen\ t\} = \frac{1}{s^2+1}$*, luego:*

$$\boxed{\boldsymbol{\mathcal{L}\{u(t-2\pi)Sen(t-2\pi)\} = e^{-2\pi s}\frac{1}{s^2+1}}}$$

$$3)\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)e^{-2\left(t-\frac{\pi}{2}\right)}Cosh\,4\left(t-\frac{\pi}{2}\right)\right\}$$

Determinamos la transformada de Laplace del coseno hiperbólico

$$\mathcal{L}\{Cosh\,4t\}=\frac{s}{s^2-16}$$

Luego:

$$\mathcal{L}\{e^{-2t}Cosh\,4t\}=\frac{s+2}{(s+2)^2-16}$$

Y finalmente:

$$\boxed{\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)e^{-2\left(t-\frac{\pi}{2}\right)}Cosh\,4\left(t-\frac{\pi}{2}\right)\right\}=e^{-\frac{\pi}{2}s}\left[\frac{s+2}{(s+2)^2-16}\right]}$$

$$4)\mathcal{L}\{u(t-2)t\}$$

Hay que desfasar la función

$$\mathcal{L}\{u(t-2)t\}=\mathcal{L}\{u(t-2)(t-2+2)\}$$

$$\mathcal{L}\{u(t-2)t\}=\mathcal{L}\{u(t-2)[(t-2)+2)]\}$$

$$\mathcal{L}\{u(t-2)t\}=\mathcal{L}\{u(t-2)(t-2)\}+2\mathcal{L}\{u(t-2)(1)\}$$

$$\boxed{\mathcal{L}\{u(t-2)t\}=e^{-2s}\frac{1}{s^2}+2e^{-2s}\frac{1}{s}}$$

$$5)\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}$$

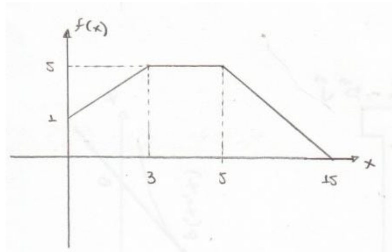
$$\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}=\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\left(\left(t-\frac{\pi}{2}\right)+\frac{\pi}{2}\right)\right\}$$

$$\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}=\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)\left[Sen\left(t-\frac{\pi}{2}\right)Cos\left(\frac{\pi}{2}\right)+Cos\left(t-\frac{\pi}{2}\right)Sen\left(\frac{\pi}{2}\right)\right]\right\}$$

$$\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}=\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Cos\left(t-\frac{\pi}{2}\right)\right\}$$

$$\boxed{\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}=e^{-\frac{\pi}{2}s}\frac{s}{s^2+1}}$$

6) $\boldsymbol{\mathcal{L}\{f(t)\}}$



$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{[u(t)-u(t-3)](t+2)+5[u(t-3)-u(t-5)]+[u(t-5)-u(t-15)]\left(\frac{1}{2}t-\frac{15}{2}\right)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t)(t+2)-u(t-3)(t+2)+5u(t-3)-5u(t-3)+u(t-5)\left(\frac{1}{2}t-\frac{15}{2}\right) - u(t-15)\left(\frac{1}{2}t-\frac{15}{2}\right)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t)(t+2-2+2)-u(t-3)(t+2-5+5)+5u(t-3)-5u(t-5) + \frac{1}{2}u(t-5)(t-15+10-10)-\frac{1}{2}u(t-15)(t-15)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t)t+2u(t)-u(t-3)(t-3)-5u(t-3)+5u(t-3)-5u(t-5)+\frac{1}{2}u(t-5)(t-5) - 10u(t-5)-\frac{1}{2}u(t-15)(t-15)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t)t+2u(t)-u(t-3)(t-3)-15u(t-5)+\frac{1}{2}u(t-5)(t-5)-\frac{1}{2}u(t-15)(t-15)\right\}$$

$$\boxed{\boldsymbol{\mathcal{L}\{f(t)\} = \frac{1}{s^2}+\frac{2}{s}-e^{-3s}\left(\frac{1}{s^2}\right)-15e^{-5s}\left(\frac{1}{s}\right)+\frac{e^{-5s}}{2}\left(\frac{1}{s^2}\right)-\frac{e^{-15s}}{2}\left(\frac{1}{s^2}\right)}}$$

7) $\boldsymbol{\mathcal{L}\left\{te^{-3t}Sen(4t)\right\}}$

$$\mathcal{L}\{Sen(4t)\} = \frac{4}{s^2+16}$$

$$\mathcal{L}\{e^{-3t}Sen(4t)\} = \frac{4}{(s+3)^2+16}$$

$$\mathcal{L}\{te^{-3t}Sen(4t)\} = -\frac{d}{ds}\left[\frac{4}{(s+3)^2+16}\right]$$

$$\mathcal{L}\{te^{-3t}Sen(4t)\} = -\frac{4}{[(s+3)^2+16]^2}[2(s+3)]$$

$$\boxed{\boldsymbol{\mathcal{L}\{te^{-3t}Sen(4t)\} = -\frac{8(s+3)}{[(s+3)^2+16]^2}}}$$

$$\mathbf{8)}\boldsymbol{\mathcal{L}}\left\{\boldsymbol{t}\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{Sen(\tau)d\tau}\right\}$$

$$\mathcal{L}\left\{\int_0^t f(x)g(t-x)dx\right\} = F(s)G(s)$$

$$\mathcal{L}\left\{\int_0^t \overbrace{(1)}^{g(t-x)}\ \overbrace{Sen(\tau)}^{f(x)}\,d\tau\right\} = \frac{1}{s}\left(\frac{1}{s^2+1}\right)$$

$$\mathcal{L}\left\{t\int_0^t Sen(\tau)d\tau\right\} = -\frac{d}{ds}\left[\frac{1}{s}\left(\frac{1}{s^2+1}\right)\right] = \frac{1}{s^2}\left(\frac{1}{s^2+1}\right) + \frac{2s}{[s^2+1]^2}\left(\frac{1}{s}\right)$$

$$\boxed{\boldsymbol{\mathcal{L}}\left\{\boldsymbol{t}\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{Sen(\tau)d\tau}\right\} = \frac{\mathbf{1}}{\boldsymbol{s}^{\mathbf{2}}}\left(\frac{\mathbf{1}}{\boldsymbol{s}^{\mathbf{2}}+\mathbf{1}}\right) + \frac{\mathbf{2}\boldsymbol{s}}{[\boldsymbol{s}^{\mathbf{2}}+\mathbf{1}]^{\mathbf{2}}}\left(\frac{\mathbf{1}}{\boldsymbol{s}}\right)}$$

$$\mathbf{9)}\boldsymbol{\mathcal{L}}\left\{\boldsymbol{e^{-2t}}\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{\tau e^{-2\tau}Sen(\tau)d\tau}\right\}$$

$$\mathcal{L}\{Sen(t)\} = \frac{1}{s^2+1}$$

$$\mathcal{L}\{e^{-2t}Sen(t)\} = \frac{1}{(s+2)^2+1}$$

$$\mathcal{L}\{te^{-2t}Sen(t)\} = -\frac{d}{ds}\left[\frac{1}{(s+2)^2+1}\right] = \frac{2(s+2)}{[(s+2)^2+1]^2}$$

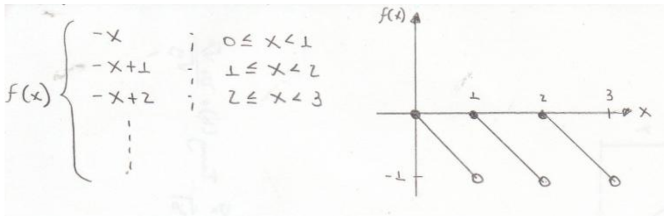
$$\mathcal{L}\left\{\int_0^t \overbrace{(1)}^{g(t-x)}\ \underbrace{\tau e^{-2\tau}Sen(\tau)}_{f(x)}\,d\tau\right\} = \frac{\frac{2(s+2)}{[(s+2)^2+1]^2}}{s}$$

$$\mathcal{L}\left\{e^{-2t}\int_0^t \tau e^{-2\tau}Sen(\tau)d\tau\right\} = \frac{\frac{2[(s+2)+2]}{\left[\left((s+2)+2\right)^2+1\right]^2}}{(s+2)}$$

$$\boxed{\boldsymbol{\mathcal{L}}\left\{\boldsymbol{e^{-2t}}\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{\tau e^{-2\tau}Sen(\tau)d\tau}\right\} = \frac{\mathbf{2(s+4)(s+2)}}{\left[\left(\boldsymbol{(s+2)+2}\right)^{\mathbf{2}}+\mathbf{1}\right]^{\mathbf{2}}}}$$

$\mathbf{10)}\boldsymbol{\mathcal{L}\{\|x\|-x\}}$

El gráfico correspondiente a esta función es:



$T=1$

$$\mathcal{L}\{\|x\|-x\}=\frac{1}{1-e^{-s}}\int_0^1 e^{-st}(-t)dt$$

$u=-t \quad \Rightarrow \quad du=-dt$

$dv=e^{-st}dt \quad \Rightarrow \quad v=-\frac{1}{s}e^{-st}$

$$\mathcal{L}\{\|x\|-x\}=\frac{1}{1-e^{-s}}\left[\frac{t}{s}e^{-st}+\frac{1}{s}\int e^{-st}dt\right]$$

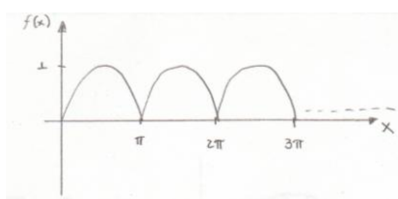
$$\mathcal{L}\{\|x\|-x\}=\frac{1}{1-e^{-s}}\left[\frac{t}{s}e^{-st}-\frac{e^{-st}}{s^2}\right]\begin{matrix}1\\0\end{matrix}$$

$$\mathcal{L}\{\|x\|-x\}=\frac{1}{1-e^{-s}}\left[\left(\frac{1}{s}e^{-s}-\frac{e^{-s}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]$$

$$\boxed{\boldsymbol{\mathcal{L}\{\|x\|-x\}=\frac{1}{1-e^{-s}}\left(\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right)}}$$

$\mathbf{11)}\boldsymbol{\mathcal{L}\{|Sen\,x|\}}$

El gráfico correspondiente a esta función es:



$T=\pi$

$$\mathcal{L}\{|Sen\,x|\}=\frac{1}{1-e^{-\pi s}}\int_0^\pi e^{-st}Sen(t)dt$$

$u=Sen(t) \quad \Rightarrow \quad du=-Cos(t)dt$

$dv=e^{-st}dt \quad \Rightarrow \quad v=-\frac{1}{s}e^{-st}$

$$\int e^{-st}Sen(t)dt = -\frac{Sen(t)}{s}e^{-st} - \frac{1}{s}\int e^{-st}Cos(t)dt$$

$$u = Cos(t) \quad \Rightarrow \quad du = Sen(t)dt$$

$$dv = e^{-st}dt \quad \Rightarrow \quad v = -\frac{1}{s}e^{-st}$$

$$\int e^{-st}Sen(t)dt = -\frac{Sen(t)}{s}e^{-st} - \frac{1}{s}\left[-\frac{Cos(t)}{s}e^{-st} + \frac{1}{s}\int e^{-st}Sen(t)dt\right]$$

$$\int e^{-st}Sen(t)dt = -\frac{Sen(t)}{s}e^{-st} + \frac{Cos(t)}{s^2}e^{-st} - \frac{1}{s^2}\int e^{-st}Sen(t)dt$$

$$\int e^{-st}Sen(t)dt = \frac{\left[\frac{Cos(t)}{s^2}e^{-st} - \frac{Sen(t)}{s}e^{-st}\right]}{\left(1+\frac{1}{s^2}\right)}$$

$$\int e^{-st}Sen(t)dt = \left(\frac{s^2}{s^2+1}\right)\left[\frac{Cos(t)}{s^2}e^{-st} - \frac{Sen(t)}{s}e^{-st}\right]$$

$$\mathcal{L}\{|Sen\,x|\} = \frac{1}{1-e^{-\pi s}}\left(\frac{s^2}{s^2+1}\right)\left[\frac{Cos(t)}{s^2}e^{-st} - \frac{Sen(t)}{s}e^{-st}\right]_0^{\pi}$$

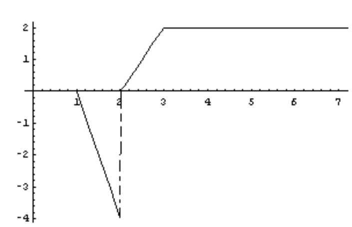
$$\mathcal{L}\{|Sen\,x|\} = \frac{1}{1-e^{-\pi s}}\left(\frac{s^2}{s^2+1}\right)\left[\left(\frac{Cos(\pi)}{s^2}e^{-s\pi} - \frac{Sen(\pi)}{s}e^{-s\pi}\right) - \left(\frac{Cos(0)}{s^2}e^{-s(0)} - \frac{Sen(0)}{s}e^{-s(0)}\right)\right]$$

$$\mathcal{L}\{|Sen\,x|\} = \frac{1}{1-e^{-\pi s}}\left(\frac{s^2}{s^2+1}\right)\left[\left(-\frac{1}{s^2}e^{-s\pi}\right) - \left(\frac{1}{s^2}\right)\right]$$

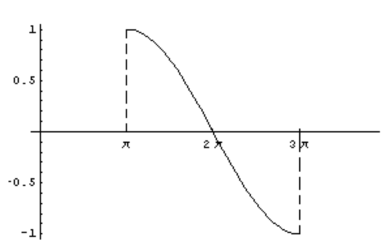
$$\boxed{\boldsymbol{\mathcal{L}\{|Sen\,x|\} = -\frac{1}{1-e^{-\pi s}}\left(\frac{s^2}{s^2+1}\right)\left(\frac{1}{s^2}e^{-s\pi} + \frac{1}{s^2}\right)}}$$

12)Encuentre la transformada de Laplace para las funciones cuyos gráficos se muestran a continuación:

a)



b)



Para a)

$P_1(1,0)$; $P_2(2,-4)$

$y_1 = mx + b$

$0 = m + b$; $-4 = 2m + b$

$y_1 = 4x + 4$

$P_1(2,0)$; $P_2(3,2)$

$y_2 = mx + b$

$0 = 2m + b$; $2 = 3m + b$

$y_2 = 2x - 4$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{[u(t-1) - u(t-2)]y_1 + [u(t-2) - u(t-3)]y_2 + u(t-3)y_3\}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{[u(t-1) - u(t-2)](4t+4) + [u(t-2) - u(t-3)](2t-4) + u(t-3)2\}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{4(t+1)u(t-1) - 4u(t-2)(t+1) + 2u(t-2)(t-2) - 2u(t-3)(t-2) + 2u(t-3)\}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4((t+1-2)+2)u(t-1) - 4u(t-2)((t+1-3)+3) + 2u(t-2)(t-2) - 2u(t-3)((t-2-1)+1) + 2u(t-3)\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4(t-1)u(t-1) + 8u(t-1) - 4u(t-2)(t-2) - 12u(t-2) + 2u(t-2)(t-2) - 2u(t-3)(t-3) - 2u(t-3) + 2u(t-3)\}$$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{4(t-1)u(t-1) + 8u(t-1) - 2u(t-2)(t-2) - 12u(t-2) - 2u(t-3)(t-3)\}$

$$\boxed{\boldsymbol{\mathcal{L}\{f(t)\} = \frac{e^{-t}}{s^2} + 8e^{-t} - 2\frac{e^{-2t}}{s^2} - 12e^{-2t} - 2\frac{e^{-3t}}{s^2}}}$$

Para b)

Sabemos que el período de la función Sen(Bx) es

$T = \frac{2\pi}{B}$, *entonces* $4\pi = \frac{2\pi}{B} \quad \Rightarrow \quad B = \frac{1}{2}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{[u(t-\pi) + u(t-3\pi)]Sen\left(\frac{x}{2}\right)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t-\pi)Sen\left(\frac{x}{2}\right) + u(t-3\pi)Sen\left(\frac{x}{2}\right)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t-\pi)Sen\left[\frac{1}{2}(t-\pi+\pi)\right] + u(t-3\pi)Sen\left[\frac{1}{2}(t-3\pi+3\pi)\right]\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t-\pi)Sen\left[\frac{1}{2}(t-\pi)+\frac{\pi}{2}\right] + u(t-3\pi)Sen\left[\frac{1}{2}(t-3\pi)+\frac{3\pi}{2}\right]\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\begin{matrix} u(t-\pi)\left[Sen\left[\frac{1}{2}(t-\pi)\right]Cos\left(\frac{\pi}{2}\right) + Cos\left[\frac{1}{2}(t-\pi)\right]Sen\left(\frac{\pi}{2}\right)\right] + \\ u(t-3\pi)\left[Sen\left[\frac{1}{2}(t-3\pi)\right]Cos\left(\frac{3\pi}{2}\right) + Cos\left[\frac{1}{2}(t-3\pi)\right]Sen\left(\frac{3\pi}{2}\right)\right]\end{matrix}\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t-\pi)Cos\left[\frac{1}{2}(t-\pi)\right] - u(t-3\pi)Cos\left[\frac{1}{2}(t-3\pi)\right]\right\}$$

$$\mathcal{L}\{f(t)\} = e^{-\pi s}\frac{s}{s^2+\frac{1}{4}} - e^{-3\pi s}\frac{s}{s^2+\frac{1}{4}}$$

$$\boxed{\mathcal{L}\{f(t)\} = e^{-\pi s}\frac{4s}{4s^2+1} - e^{-3\pi s}\frac{4s}{4s^2+1}}$$

TRANSFORMADA INVERSA DE LAPLACE

1) $\boldsymbol{\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+8}\right\}}$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+4s+4)+8-4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1}{(s+2)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{(s+1+1)-1}{(s+2)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\}-\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+4}\right\}$$

$$\boxed{\boldsymbol{\mathcal{L}^{-1}\{\frac{s+1}{s^2+4s+8}\}=e^{-2t}Cos(2t)-\frac{e^{-2t}}{2}Sen(2t)}}$$

2) $\boldsymbol{\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}}$

$$\frac{1}{(s^2+1)(s^2+4)}=\frac{A(2s)+B}{s^2+1}+\frac{C(2s)+D}{s^2+4}$$

$$1=(2As+B)(s^2+4)+(2Cs+D)(s^2+1)$$

$$1=2As^3+8As+Bs^2+4B+2Cs^3+2Cs+Ds^2+D$$

$$1=(2A+2C)s^3+(B+D)s^2+(8A+2C)s+(4B+D)$$

$$0=2A+2C$$

$$0=B+D$$

$$0=8A+2C$$

$$1=4B+D$$

Resolviendo el sistema A = 0, B = 1/3, C = 0, D = -1/3

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}=\mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{2As+B}{s^2+1}+\frac{2Cs+D}{s^2+4}\right)\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}=2A\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+1}\right\}+B\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\}+2C\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+4}\right\}+D\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\}-\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2+4}\right\}\quad\Rightarrow\quad\boxed{\boldsymbol{\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}=u(t-2)\left[\frac{1}{3}Sen\,(t-2)-\frac{1}{6}Sen(2(t-2))\right]}}$$

$$\mathbf{3)}\boldsymbol{\mathcal{L}^{-1}\left\{ln\left(\frac{s-1}{s^2+2s+5}\right)\right\}}$$

$$\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds}F(s)$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}F(s)\right\}$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\left[ln\left(\frac{s-1}{S^2+2s+5}\right)\right]\right\}$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln(s-1)\right\} + \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln[(S^2+2s+1)+5-1]\right\}$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln(s-1)\right\} + \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln[(s+1)^2+4]\right\}$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{-\frac{2(s+1)}{(s+1)^2+4}\right\}$$

$$t\, f(t) = -e^t - 2e^{-t}Cos(2t)$$

$$\boxed{\boldsymbol{f(t) = \frac{-e^t - 2e^{-t}Cos(2t)}{t}}}$$

$$\mathbf{4)}\boldsymbol{\mathcal{L}^{-1}\left\{ln\left(\frac{s^2+9}{s^2+1}\right)\right\}}$$

$$\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds}F(s)$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}F(s)\right\}$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\left[ln\left(\frac{s^2+9}{s^2+1}\right)\right]\right\}$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln(s^2+9)\right\} + \mathcal{L}^{-1}\left\{\frac{d}{ds}ln(s^2+1)\right\}$$

$$t\, f(t) = \mathcal{L}^{-1}\left\{-\frac{2s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\}$$

$$t\, f(t) = -2\, Cos(3t) + 2\, Cos(t)$$

$$\boxed{\boldsymbol{f(t) = \frac{2\, Cos(t) - 2\, Cos(3t)}{t}}}$$

$$5)\boldsymbol{\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}\right\}}$$

$$\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C(2s+2)+D}{(s^2+2s+2)}$$

$$s^3+3s^2+1=As(s^2+2s+2)+B(s^2+2s+2)+2Cs(s^2)+2C(s^2)+D(s^2)$$

$$s^3+3s^2+1=As^3+2As^2+2As+Bs^2+2Bs+2B+2Cs^3+2Cs^2+Ds^2$$

$$s^3+3s^2+1=(A+2C)s^3+(2A+B+2C+D)s^2+(2A+2B)s+2B$$

$$1=A+2C$$

$$3=2A+B+2C+D$$

$$0=2A+2B$$

$$1=2B$$

Resolviendo el sistema   A = -1/2, B = ½, C = ¾, D = 2

$$\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}\right\}=A\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+B\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+2C\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\}+D\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}\right\}=A+Bt+2Ce^{-t}Cos(t)+De^{-t}\,Sen(t)$$

$$\boxed{\boldsymbol{\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}\right\}=-\frac{1}{2}+\frac{t}{2}+\frac{3e^{-t}}{2}Cos(t)+2e^{-t}\,Sen(t)}}$$

$$6)\boldsymbol{\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^3}\right\}}$$

$$\mathcal{L}^{-1}\left\{\int_s^{+\infty}F(\sigma)d\sigma\right\}=\frac{f(t)}{t}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\int_s^{+\infty}F(\sigma)d\sigma$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\lim_{a\to+\infty}\int_s^{a}\frac{2s}{(s^2+1)^3}ds$$

$$u=s^2+1\quad\Rightarrow\quad du=2s$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\lim_{a\to+\infty}\int_s^{a}\frac{du}{(u)^3}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=-\lim_{a\to+\infty}\frac{1}{2u^2}\Bigg|_s^a$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = -\frac{1}{2}\lim_{a\to+\infty}\left[\frac{1}{(a^2+1)^2} - \frac{1}{(s^2+1)^2}\right]$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{1}{2(s^2+1)^2}$$

$$\frac{f(t)}{t} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$\frac{f(t)}{t} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} * \frac{1}{s^2+1}\right\}$$

$$\frac{f(t)}{t} = \frac{1}{2}\left[\int_0^t Sen(\tau)\, Sen(x-\tau)d\tau\right]$$

$$①\, Cos(a+b) = Cos(a)Cos(b) - Sen(a)Sen(b)$$

$$②\, Cos(a-b) = Cos(a)Cos(b) + Sen(a)Sen(b)$$

Multiplicando por (-1) la primera ecuación

$$① - Cos(a+b) = -Cos(a)Cos(b) + Sen(a)Sen(b)$$

$$②\, Cos(a-b) = Cos(a)Cos(b) + Sen(a)Sen(b)$$

Entonces ① + ②

$$Sen(a)Sen(b) = \frac{Cos(a-b) - Cos(a+b)}{2}$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\int_0^t [Cos(\tau - t + \tau) - Cos(\tau + t - \tau)]d\tau\right]$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\int_0^t [Cos(2\tau - t) - Cos(t)]d\tau\right]$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\int_0^t [Cos(2\tau - t) - Cos(t)]d\tau\right]$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\frac{1}{2}Sen(2\tau - t) - \tau\, Cos(t)\right]_0^t$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\frac{1}{2}Sen(2t - t) - t\, Cos(t) - \frac{1}{2}Sen(-t)\right]$$

$$f(t) = \frac{t}{4}\left[\frac{1}{2}Sen(t) - t\, Cos(t) + \frac{1}{2}Sen(t)\right]$$

$$\boxed{\boldsymbol{f(t) = \frac{t}{4}[Sen(t) - t\, Cos(t)]}}$$

$$\mathbf{7)}\boldsymbol{\mathcal{L}^{-1}\left\{\frac{\pi}{2}-Arctan\left(\frac{s}{2}\right)\right\}}$$

$$t\,f(t)=\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left[\frac{\pi}{2}-Arctan\left(\frac{s}{2}\right)\right]\right\}$$

$$t\,f(t)=\mathcal{L}^{-1}\left\{-\frac{1}{1+\left(\frac{s}{2}\right)^2}\right\}$$

$$t\,f(t)=\mathcal{L}^{-1}\left\{-\frac{4}{4+s^2}\right\}$$

$$t\,f(t)=-2\mathcal{L}^{-1}\left\{\frac{2}{4+s^2}\right\}$$

$$t\,f(t)=-2\,Sen(2t)$$

$$\boxed{\boldsymbol{f(t)=\frac{-2\,Sen(2t)}{t}}}$$

$$\mathbf{8)}\boldsymbol{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\}=\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{1}*\overbrace{\frac{1}{(s+2)^2+1}}^{e^{-2t}Sen(t)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\}=\int_0^t e^{-2x}Sen(x)dx$$

$$\int e^{-2x}Sen(x)dx$$

$$u=Sen(x)\;\Rightarrow\;du=Cos(x)$$

$$dv=e^{-2x}dx\;\Rightarrow\;v=-\frac{1}{2}e^{-2x}$$

$$\int e^{-2x}Sen(x)dx=-\frac{Sen(x)}{2}e^{-2x}+\frac{1}{2}\int e^{-2x}Cos(x)\,dx$$

$$u=Cos(x)\;\Rightarrow\;du=-Sen(x)$$

$$dv=e^{-2x}dx\;\Rightarrow\;v=-\frac{1}{2}e^{-2x}$$

$$\int e^{-2x}Sen(x)dx=-\frac{Sen(x)}{2}e^{-2x}+\frac{1}{2}\left[-\frac{Cos(x)}{2}e^{-2x}-\frac{1}{2}\int e^{-2x}Sen(x)dx\right]$$

$$\int e^{-2x}Sen(x)dx=-\frac{Sen(x)}{2}e^{-2x}-\frac{Cos(x)}{4}e^{-2x}-\frac{1}{4}\int e^{-2x}Sen(x)dx$$

$$\int e^{-2x} Sen(x)dx = -\frac{4}{5}e^{-2x}\left[\frac{Sen(x)}{2} + \frac{Cos(x)}{4}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\} = -\frac{4}{5}e^{-2x}\left[\frac{Sen(x)}{2} + \frac{Cos(x)}{4}\right]\Bigg|_{0}^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\} = -\frac{4}{5}e^{-2t}\left[\frac{Sen(t)}{2} + \frac{Cos(t)}{4}\right] + \frac{4}{5}\left[\frac{1}{4}\right]$$

$$\boxed{\boldsymbol{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\} = -\frac{1}{5}\left[2e^{-2t}Sen(t) + e^{-2t}Cos(t) - 1\right]}}$$

RESOLUCIÓN DE ECUACIONES DIFERENCIALES MEDIANTE LA TRANSFORMADA DE LAPLACE

1) $\boldsymbol{y'' - 6y' + 9y = t^2 e^{3t}}\,, \qquad \boldsymbol{y(0) = 2\,;\; y'(0) = 6}$

$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$

$[s^2Y - s\,y(0) - y'(0)] - 6[sY - y(0)] + 9Y = \dfrac{2!}{(s-3)^3}$

$Ys^2 - 2s - 6 - 6Ys + 12 + 9Y = \dfrac{2}{(s-3)^3}$

$Y[s^2 - 6s + 9] = \dfrac{2}{(s-3)^3} + 2s - 6$

$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^5} + \dfrac{2s}{(s-3)^2} - \dfrac{6}{(s-3)^2}\right\}$

$\mathcal{L}^{-1}\{Y\} = \dfrac{2}{4!}\mathcal{L}^{-1}\left\{\dfrac{4!}{(s-3)^5}\right\} + 2\mathcal{L}^{-1}\left\{\dfrac{s}{(s-3)^2}\right\} - 6\mathcal{L}^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}$

$\dfrac{s}{(s-3)^2} = \dfrac{A}{s-3} + \dfrac{B}{(s-3)^2}$

$s = A(s-3)^2 + B(s-3) = A(s^2 - 6s + 9) + Bs - 3B = As^2 - 6As + 9A + Bs - 3B$

$s = As^2 + (B - 6A)s + (9A - 3B)$

$0 = A$

$1 = B - 6A$

$0 = 9A - 3B$

Podemos notar que el sistema no tiene solución, entonces este método no funciona, pero sabemos que

$t\,f(t) = \mathcal{L}^{-1}\left\{-\dfrac{d}{ds}F(s)\right\}$

$t\,f(t) = \mathcal{L}^{-1}\left\{-\dfrac{d}{ds}\left[\dfrac{s}{(s-3)^2}\right]\right\}$

$t\,f(t) = \mathcal{L}^{-1}\left\{-\dfrac{1}{(s-3)^2} + \dfrac{2s}{(s-3)^3}\right\}$

$t\,f(t) = -\mathcal{L}^{-1}\left\{\dfrac{1}{(s-3)^2}\right\} + 2\mathcal{L}^{-1}\left\{\dfrac{s}{(s-3)^3}\right\}$

$\dfrac{s}{(s-3)^3} = \dfrac{A}{s-3} + \dfrac{B}{(s-3)^2} + \dfrac{C}{(s-3)^3}$

$s = A(s-3)^2 + B(s-3) + C = A(s^2 - 6s + 9) + Bs - 3B + C = As^2 - 6As + 9A + Bs - 3B + C$

$s = As^2 + (B - 6A)s + (9A - 3B + C)$

$0 = A$

$1 = B - 6A$

$0 = 9A - 3B + C$

Resolviendo el sistema A = 0, B = 1, C = 3

$$t\,f(t) = -\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{A}{s-3} + \frac{B}{(s-3)^2} + \frac{C}{(s-3)^3}\right\}$$

$$t\,f(t) = -\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{B}{(s-3)^2} + \frac{C}{(s-3)^3}\right\}$$

$$t\,f(t) = -\frac{\mathcal{L}^{-1}}{1!}\left\{\frac{1!}{(s-3)^2}\right\} + \frac{2B\mathcal{L}^{-1}}{1!}\left\{\frac{1!}{(s-3)^2}\right\} + \frac{2C\mathcal{L}^{-1}}{2!}\left\{\frac{2!}{(s-3)^3}\right\}$$

$$t\,f(t) = -te^{3t} + 2te^{3t} + 3t^2e^{3t}$$

$$f(t) = -e^{3t} + 2e^{3t} + 36te^{3t}$$

$$y(t) = \frac{t^4e^{3t}}{12} + 2e^{3t} + 72te^{3t} - 6te^{3t}$$

$$\boxed{\boldsymbol{y(t) = e^{3t}\left[\frac{t^4}{12} + 2 + 66t\right]}}$$

$2) y'' + 4y = u\left(t - \frac{\pi}{4}\right) Sen\,(t)\,, \qquad y(0) = 1\,;\; y'(0) = 0$

$$y'' + 4y = u\left(t - \frac{\pi}{4}\right) Sen\,\left[\left(t - \frac{\pi}{4}\right) + \frac{\pi}{4}\right]$$

$$y'' + 4y = u\left(t - \frac{\pi}{4}\right)\left[Sen\,\left(t - \frac{\pi}{4}\right) Cos\left(\frac{\pi}{4}\right) + Sen\left(\frac{\pi}{4}\right) Cos\left(t - \frac{\pi}{4}\right)\right]$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\left\{u\left(t - \frac{\pi}{4}\right)\left[Sen\,\left(t - \frac{\pi}{4}\right) Cos\left(\frac{\pi}{4}\right) + Sen\left(\frac{\pi}{4}\right) Cos\left(t - \frac{\pi}{4}\right)\right]\right\}$$

$$\left[s^2Y - s\,y(0) - y'(0)\right] + 4Y = e^{-\frac{\pi}{4}s}\left[\frac{\sqrt{2}}{2}\left(\frac{1}{s^2+1}\right) + \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1}\right)\right]$$

$$s^2Y - s + 4Y = e^{-\frac{\pi}{4}s}\left[\frac{\sqrt{2}}{2}\left(\frac{1}{s^2+1}\right) + \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1}\right)\right]$$

$$Y = \frac{e^{-\frac{\pi}{4}s}}{s^2+4}\left[\frac{\sqrt{2}}{2}\left(\frac{1}{s^2+1}\right) + \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1}\right)\right] + \frac{s}{s^2+4}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{4}s}}{s^2+4}\left[\frac{\sqrt{2}}{2}\left(\frac{1}{s^2+1}\right) + \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1}\right)\right] + \frac{s}{s^2+4}\right\}$$

$$y(t) = \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{4}s}}{(s^2+4)(s^2+1)}\right\} + \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{\frac{s\,e^{-\frac{\pi}{4}s}}{(s^2+4)(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\}$$

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{A(2s)+B}{s^2+4} + \frac{C(2s)+D}{s^2+1}$$

$1 = 2As(s^2+1) + B(s^2+1) + 2Cs(s^2+4) + D(s^2+4)$

$1 = 2As^3 + 2As + Bs^2 + B + 2Cs^3 + 8Cs + Ds^2 + 4D$

$1 = (2A+2C)s^3 + (B+D)s^2 + (2A+8C)s + (B+4D)$

$0 = 2A + 2C$

$0 = B + D$

$0 = 2A + 8C$

$1 = B + 4D$

Resolviendo el sistema A = 0, B = -1/3, C = 0, D = 1/3

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{A'(2s)+B'}{s^2+4} + \frac{C'(2s)+D'}{s^2+1}$$

$s = (2A'+2C')s^3 + (B'+D')s^2 + (2A'+8C')s + (B'+4D')$

$0 = 2A' + 2C'$

$0 = B' + D'$

$1 = 2A' + 8C'$

$0 = B' + 4D'$

Resolviendo el sistema A' = -1/6, B' = 0, C' = 1/6, D' = 0

$$y(t) = \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\left[\frac{2As+B}{s^2+4} + \frac{2Cs+D}{s^2+1}\right]\right\} + \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\left[\frac{2A's+B'}{s^2+4} + \frac{2C's+D'}{s^2+1}\right]\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\}$$

$$y(t) = \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\left[2A\frac{s}{s^2+4} + \left(\frac{B}{2}\right)\frac{1*2}{s^2+4} + 2C\frac{s}{s^2+1} + D\frac{1}{s^2+1}\right]\right\}$$
$$+ \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\left[2A'\frac{s}{s^2+4} + B'\frac{1}{s^2+4} + 2C'\frac{s}{s^2+1} + D'\frac{1}{s^2+1}\right]\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\}$$

$$\boxed{\boldsymbol{y(t) = \frac{\sqrt{2}}{2}u\left(t-\frac{\pi}{4}\right)\left[-\frac{1}{6}Sen\left(2\left(t-\frac{\pi}{4}\right)+\frac{1}{3}Sen\left(t-\frac{\pi}{4}\right)\right)\right] + \frac{\sqrt{2}}{2}u\left(t-\frac{\pi}{4}\right)\left[-\frac{1}{3}Cos\left(2\left(t-\frac{\pi}{4}\right)+\frac{1}{3}Cos\left(t-\frac{\pi}{4}\right)\right)\right] + Cos(2t)}}$$

3) $\boldsymbol{f(t) + 4\int_0^t Sen(\tau)f(t-\tau)d\tau = 2t}$

$$\mathcal{L}\{f(t)\} + 4\mathcal{L}\left\{\int_0^t Sen(\tau)f(t-\tau)d\tau\right\} = 2\mathcal{L}\{t\}$$

$$Y + 4Y\left(\frac{1}{s^2+1}\right) = 2\left(\frac{1}{s^2}\right)$$

$$Y=\frac{2\left(\frac{1}{s^2}\right)}{\left(1+\frac{4}{s^2+1}\right)}$$

$$Y=\frac{2(s^2+5)}{s^2(s^2+1)}$$

$$\mathcal{L}^{-1}\{Y\}=2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+10\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$$

$$y(t)=2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+10\int_0^t Sen(\tau)(t-\tau)d\tau$$

$$\int Sen(\tau)(t-\tau)d\tau$$

$$u=t-\tau \quad \Rightarrow \quad du=-d\tau$$

$$dv=Sen(\tau)d\tau \quad \Rightarrow \quad v=-Cos(\tau)$$

$$=-Cos(\tau)(t-\tau)-\int Cos(\tau)d\tau$$

$$=-Cos(\tau)(t-\tau)-Sen(\tau)$$

$$y(t)=2\,Sen(t)-10[Cos(\tau)(t-\tau)+Sen(\tau)]\Big|_0^t$$

$$y(t)=2\,Sen(t)-10[Sen(t)-t]$$

$$\boxed{\boldsymbol{y(t)=10t-8\,Sen(t)}}$$

4) $\boldsymbol{y''+2y'+2y=\delta(t-\pi),\qquad y(0)=\ y'(0)=0}$

$$\mathcal{L}\{y''\}+2\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\pi)\}$$

$$\left[s^2Y-s\,y(0)-y'(0)\right]+2[sY-y(0)]+2Y=e^{-\pi s}$$

$$Ys^2+2Ys+2Y=e^{-\pi s}$$

$$Y=\frac{e^{-\pi s}}{s^2+s+2}$$

$$\mathcal{L}^{-1}\{Y\}=2\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\}$$

$$\boxed{\boldsymbol{y(t)=2\,u(t-\pi)e^{(t-\pi)}Sen(t-\pi)}}$$

5) $y'' + 2y' + 2y = Cos(t)\delta(t - 3\pi), \quad y(0) = 1, \quad y'(0) = -1$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{Cos(t)\delta(t - 3\pi)\}$

$[s^2Y - s\,y(0) - y'(0)] + 2[sY - y(0)] + 2Y = Cos(3\pi)e^{-3\pi s}$

$Ys^2 - s + 1 + 2Ys - 2 + 2Y = -e^{-3\pi s}$

$Y(s^2 + 2s + 2) = -e^{-3\pi s} + (s + 1)$

$$\mathcal{L}^{-1}\{Y\} = -\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2 + 2s + 2}\right\} + \mathcal{L}^{-1}\left\{\frac{(s + 1)}{s^2 + 2s + 2}\right\}$$

$$y(t) = -\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{(s + 1)^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{(s + 1)}{(s + 1)^2 + 1}\right\}$$

$$\boxed{y(t) = -u(t - 3\pi)e^{(t-3\pi)}Sen(t - 3\pi) + e^{-t}Cos(t)}$$

6) $ty'' - ty' - y = 0 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 3$

$\mathcal{L}\{ty''\} - \mathcal{L}\{ty'\} - \mathcal{L}\{y\} = 0$

$$-\frac{d}{ds}[s^2Y - s\,y(0) - y'(0)] + \frac{d}{ds}[sY - y(0)] - Y = 0$$

$-2sY - s^2Y' + Y + sY' - Y = 0$

$Y'(1 - s^2) = 2sY$

$$\frac{dY}{ds}(1 - s^2) = 2sY$$

$$\frac{dY}{Y} = \frac{2s}{1 - s^2}ds \quad \Rightarrow \quad \int\frac{dY}{Y} = 2\int\frac{s}{1 - s^2}ds$$

$u = s^2 \quad \Rightarrow \quad u = 2s\,ds$

$$\int\frac{dY}{Y} = -\int\frac{1}{1 - u}ds$$

$ln|Y| = -ln|1 - u|$

$$e^{ln|Y|} = e^{-ln|1-u|} \quad \Rightarrow \quad Y = \frac{1}{1 - s^2}$$

$$\mathcal{L}^{-1}\{Y\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\}$$

$$\boxed{y(t) = -Senh(t)}$$

7) $\boldsymbol{y'' - 2y' + y = e^t \quad , \qquad y(0) = 0 \quad , \qquad y'(1) = \frac{11}{2}e}$

No conocemos el valor de y'(0), entonces vamos a realizar un artificio, multiplicaremos por "t"

$$ty'' - 2ty' + ty = te^t$$

$$\mathcal{L}\{ty''\} - 2\mathcal{L}\{ty'\} + \mathcal{L}\{ty\} = \mathcal{L}\{te^t\}$$

$$-\frac{d}{ds}\left[s^2Y - s\,y(0) - y'(0)\right] + \frac{d}{ds}[sY - y(0)] - \frac{d}{ds}[Y] = \frac{1}{(s-1)^2}$$

$$-2sY - s^2Y' + Y + sY' - Y' = \frac{1}{(s-1)^2}$$
$$Y'(-s^2 + s - 1) + (1 - 2s)Y = \frac{1}{(s-1)^2}$$
$$Y'(s^2 - s + 1) + (2s - 1)Y = -\frac{1}{(s-1)^2}$$
$$Y' + \frac{(2s-1)}{(s^2 - s + 1)}Y = -\frac{1}{(s-1)^2(s^2 - s + 1)}$$

$$u(s) = e^{\int p(s)ds} \quad \Rightarrow \quad u(s) = e^{\int \frac{(2s-1)}{(s^2-s+1)}ds}$$

Resolviendo la integral:

$$\int \frac{(2s-1)}{(s^2 - s + 1)}ds$$

$$u = s^2 - s \quad \Rightarrow \quad du = (2s - 1)ds$$

$$\int \frac{du}{u+1} \quad \Rightarrow \quad ln|u + 1| \quad \Rightarrow \quad ln|s^2 - s + 1|$$

Entonces:

$$u(s) = e^{ln|s^2-s+1|} \quad \Rightarrow \quad u(s) = s^2 - s + 1$$

$$\frac{d}{ds}[(s^2 - s + 1)Y] = -\frac{1}{(s-1)^2}$$

$$\int d[(s^2 - s + 1)Y] = -\int \frac{1}{(s-1)^2}ds$$

$$(s^2 - s + 1)Y = \frac{1}{s-1}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s^2 - s + 1)}\right\} \quad \Rightarrow \quad \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)\left[\left(s^2 - s + \frac{1}{4}\right) + 1 - \frac{1}{4}\right]}\right\}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)\left[\left(s - \frac{1}{2}\right)^2 + \frac{3}{4}\right]}\right\}$$

Aplicando convolución:

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s-1)}}_{e^t} * \overbrace{\frac{1}{\left[\left(s-\frac{1}{2}\right)^2+\frac{3}{4}\right]}}^{\frac{2}{\sqrt{3}}e^{\frac{1}{2}t}\,Sen\left(\frac{\sqrt{3}}{2}t\right)}\right\}$$

$$\frac{2}{\sqrt{3}}\int_0^t e^{(t-x)}\, e^{\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx$$

$$\frac{2}{\sqrt{3}}e^t\int_0^t e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx$$

Integrando por partes:

$$u = e^{-\frac{1}{2}x} \quad \Rightarrow \quad du = -\frac{1}{2}e^{-\frac{1}{2}x}dx$$

$$dv = Sen\left(\frac{\sqrt{3}}{2}x\right)dx \quad \Rightarrow \quad v = \frac{2}{\sqrt{3}}Cos\left(\frac{\sqrt{3}}{2}x\right)$$

Entonces:

$$\int e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right)+\frac{1}{\sqrt{3}}\int e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right)dx$$

Integrando nuevamente por partes:

$$u = e^{-\frac{1}{2}x} \quad \Rightarrow \quad du = -\frac{1}{2}e^{-\frac{1}{2}x}dx$$

$$dv = Cos\left(\frac{\sqrt{3}}{2}x\right)dx \quad \Rightarrow \quad v = -\frac{2}{\sqrt{3}}Sen\left(\frac{\sqrt{3}}{2}x\right)$$

Entonces:

$$\int e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right)+\frac{1}{\sqrt{3}}\left[-\frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)-\frac{1}{\sqrt{3}}\int e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)dx\right]$$

$$\int e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{2}{3}e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)-\frac{1}{3}\int e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)dx$$

$$\int e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx = \frac{3}{4}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{2}{3}e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Luego tenemos que:

$$\int_0^t e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx$$

$$\frac{3}{4}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{2}{3}e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)\right]\Bigg|_0^t$$

$$\frac{3}{4}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{3}e^{-\frac{1}{2}t}Sen\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{\sqrt{3}}e^{-\frac{1}{2}(0)}Cos\left(\frac{\sqrt{3}}{2}(0)\right)+\frac{2}{3}e^{-\frac{1}{2}(0)}Sen\left(\frac{\sqrt{3}}{2}(0)\right)\right]$$

$$\frac{3}{4}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{3}e^{-\frac{1}{2}t}Sen\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{\sqrt{3}}\right]$$

Entonces:

$$y(t)=\left(\frac{2}{\sqrt{3}}\right)\frac{3}{4}e^{t}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{3}e^{-\frac{1}{2}t}Sen\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{\sqrt{3}}\right]$$

$$\boxed{\boldsymbol{y(t)=\frac{3}{2\sqrt{3}}e^{t}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{3}e^{-\frac{1}{2}t}Sen\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{\sqrt{3}}\right]}}$$

SISTEMAS DE ECUACIONES DIFERENCIALES

$$\begin{cases} \boldsymbol{x_1' = 3x_1 - x_2} \\ \boldsymbol{x_2' = 4x_1 + 3x_2} \end{cases}$$

Derivando la primera ecuación:

$x_1'' = 3x'_1 - x'_2 \quad (3)$

(2) en (3)

$x_1'' = 3x'_1 - (4x_1 + 3x_2)$

$x_1'' = 3x'_1 - 4x_1 - 3x_2 \quad (4)$

(1) en (4)

$x_1'' = 3x'_1 - 4x_1 - 3(3x_1 - x_1')$

$x_1'' = 3x'_1 - 4x_1 - 9x_1 + 3x_1'$

$x_1'' - 6x_1' + 13x_1 = 0$

Entonces:

$x_1 = e^{rt}$

$x'_1 = re^{rt}$

$x''_1 = r^2e^{rt}$

Reemplazando:

$r^2e^{rt} - 6re^{rt} + 13e^{rt} = 0$

$e^{rt}(r^2 - 6r + 13) = 0 \quad \Rightarrow \quad r^2 - 6r + 13 = 0$

$$r_{1,2} = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2} = 3 \pm 2i$$

Entonces:

$$\boxed{\boldsymbol{x_1 = e^{3t}[C_1\, Cos(2t) + C_2\, Sen(2t)]}}$$

Pero:

$x_2 = 3x_1 - x_1'$

$$\boxed{\boldsymbol{x_2 = 3e^{3t}[C_1\, Cos(2t) + C_2\, Sen(2t)] - 3e^{3t}[C_1\, Cos(2t) + C_2\, Sen(2t)] - e^{3t}[2C_2\, Cos(2t) - 2C_1\, Sen(2t)]}}$$

OPERADORES DIFERENCIALES

$$\mathbf{1)}\begin{cases} \boldsymbol{x_1^{'} = x_1 + x_2} \\ \boldsymbol{x_2^{'} = 4x_1 - 2x_2} \end{cases}$$

$x_1^{'} = Dx_1 \quad ; \quad x_1^{''} = D^2x_1$

Entonces:

$Dx_1 = x_1 + x_2 \quad ; \quad Dx_2 = 4x_1 - 2x_2$

Luego:

$(D-1)x_1 - x_2 = 0 \ \ (1)$

$-4x_1 + (D+2)x_2 = 0 \ \ (2)$

Multiplicando por 4 a (1) y por (D+2) a (2), y luego sumamos (1)+(2):

$-4x_2 + (D-1)(D+2)x_2 = 0$

$-4x_2 + (D^2 - 3D + 2)x_2 = 0$

$-4x_2 + x''_2 - 3x'_2 + 2x_2 = 0$

$x''_2 - 3x'_2 - 2x_2 = 0$

Entonces:

$x_2 = e^{rt}$

$x'_2 = re^{rt}$

$x''_2 = r^2e^{rt}$

Reemplazando:

$r^2e^{rt} - 3re^{rt} - 2e^{rt} = 0$

$e^{rt}(r^2 - 3r - 2) = 0 \quad \Rightarrow \quad r^2 - 3r - 2 = 0$

$$r_{1,2} = \frac{3 \pm \sqrt{9 - 4(1)(-2)}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

Entonces:

$$\boxed{\boldsymbol{x_2 = C_1 e^{\frac{3+\sqrt{17}}{2}x} + C_2 e^{\frac{3-\sqrt{17}}{2}x}}}$$

Pero:

$$x_1 = \frac{1}{4}(x_2^{'} + 2x_2)$$

$$\boxed{\boldsymbol{x_1 = \frac{1}{4}\left[\left(\frac{3+\sqrt{17}}{2}\right)C_1 e^{\frac{3+\sqrt{17}}{2}x} + \left(\frac{3-\sqrt{17}}{2}\right)C_2 e^{\frac{3-\sqrt{17}}{2}x}\right] + 2\left[C_1 e^{\frac{3+\sqrt{17}}{2}x} + C_2 e^{\frac{3-\sqrt{17}}{2}x}\right]}}$$

$$\mathbf{2)}\begin{cases} \boldsymbol{x^{'} = 2x - 3y + 2\,Sen(2t)} \\ \boldsymbol{y^{'} = x - 2y - Cos(2t)} \end{cases}$$

$Dx = 2x - 3y + 2\,Sen(2t)$

$Dy = x - 2y - Cos(2t)$

Luego:

$(D-2)x + 3y = 2\,Sen(2t)$ (1)

$x - (D-2)y + 3y = Cos(2t)$ (2)

Multiplicando por –(D+2) a (2), y luego sumamos (1)+(2):

$3y + (D-2)(D+2)y = 2\,Sen(2t) + Cos(2t)$

$3y + (D^2-4)y = 2\,Sen(2t) + 2\,Sen(2t) + 2\,Cos(2t)$

Entonces:

$y^{''} - y = 4\,Sen(2t) + 2\,Cos(2t)$

<u>Encontrando la solución complementaria:</u>

$y^{''} - y = 0$

Luego:

$y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2e^{rt}$

Reemplazando:

$r^2e^{rt} - e^{rt} = 0 \quad \Rightarrow \quad e^{rt}(r^2-1) = 0$

$r_{1,2} = \pm 1$

Entonces:

$y_c = C_1e^t + C_2e^{-t}$

$\therefore C.F.S = \{e^t\,, e^{-t}\}$

<u>Encontrando la solución particular:</u>

$y_p = A\,Cos(2t) + B\,Sen(2t)$

$y^{'}{}_p = -2A\,Sen(2t) + 2B\,Cos(2t)$

$y''_p = -4A\,Cos(2t) - 4B\,Sen(2t)$

Reemplazando:

$-4A\ Cos(2t) - 4B\ Sen(2t) - A\ Cos(2t) - B\ Sen(2t) = 4\ Sen(2t) + 2\ Cos(2t)$

$-5A\ Cos(2t) - 5B\ Sen(2t) = 4\ Sen(2t) + 2\ Cos(2t)$

$-5A = 2 \quad \Rightarrow \quad A = -2/5$

$-5B = 4 \quad \Rightarrow \quad B = -4/5$

$y_p = -\frac{2}{5}Cos(2t) - \frac{4}{5}\ Sen(2t)$

Entonces:

$$\boxed{\boldsymbol{y(t) = C_1e^t + C_2e^{-t} - \frac{2}{5}Cos(2t) - \frac{4}{5}\ Sen(2t)}}$$

Pero:

$x(t) = y' + 2y + Cos(2t)$

$x(t) = C_1e^t - C_2e^{-t} + \frac{4}{5}Sen(2t) - \frac{8}{5}\ Cos(2t) + 2\left[C_1e^t + C_2e^{-t} - \frac{2}{5}Cos(2t) - \frac{4}{5}\ Sen(2t)\right] + Cos(2t)$

$x(t) = C_1e^t - C_2e^{-t} + \frac{4}{5}Sen(2t) - \frac{8}{5}\ Cos(2t) + 2C_1e^t + 2C_2e^{-t} - \frac{4}{5}Cos(2t) - \frac{8}{5}\ Sen(2t) + Cos(2t)$

$$\boxed{\boldsymbol{x(t) = 3C_1e^t + C_2e^{-t} - \frac{4}{5}Sen(2t) - \frac{11}{5}\ Cos(2t)}}$$

VALORES Y VECTORES PROPIOS

$$\mathbf{1)}\ \boldsymbol{X}' = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix}\boldsymbol{X}$$

$det(A - \lambda I) = 0$

$$\begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = 0$$

$$-\lambda(\lambda - 1)(\lambda + 1) + (\lambda + 1) + (\lambda + 1) = 0$$

$$(\lambda + 1)[-\lambda(\lambda - 1) + (\lambda + 1)] = 0$$

$$-(\lambda + 1)(\lambda^2 - \lambda - 2) = 0$$

$$-(\lambda + 1)(\lambda + 1)(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda_1 = -1 \ ; \ \lambda_2 = -1 \ ; \ \lambda_3 = 2$$

Entonces:

Para $\lambda_1 = -1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad a = -b - c$$

$$\varepsilon_{\lambda=-1} = \left\{\begin{pmatrix} a \\ b \\ c \end{pmatrix}\middle| a = -b - c\right\} \quad \Rightarrow \quad \beta_{\varepsilon_{\lambda=-1}} = \left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Para $\lambda_3 = 2$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad \begin{array}{c} -2a + b + c = 0 \\ b = c \end{array} \quad \rightarrow \quad a = c$$

$$\varepsilon_{\lambda=2} = \left\{\begin{pmatrix} a \\ b \\ c \end{pmatrix}\middle| b = c \ ; \ a = c \ ; \ c \in \mathbb{R}\right\} \quad \Rightarrow \quad \beta_{\varepsilon_{\lambda=2}} = \left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

Finalmente:

$$\boxed{\boldsymbol{x} = \boldsymbol{C_1}\begin{pmatrix} \mathbf{-1} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}\boldsymbol{e^{-t}} + \boldsymbol{C_2}\begin{pmatrix} \mathbf{-1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}\boldsymbol{e^{-t}} + \boldsymbol{C_3}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}\boldsymbol{e^{-2t}}}$$

$$\mathbf{2)}\ \boldsymbol{X}' = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \mathbf{3} & \mathbf{2} & \mathbf{1} \end{pmatrix}\boldsymbol{X} \quad ; \quad \boldsymbol{x}(\mathbf{0}) = \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{0} \end{pmatrix}$$

$det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)^2 + 4] = 0$$

$$(1-\lambda)(1 - 2\lambda + \lambda^2 + 4) = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$$\lambda_1 = 1 \quad ; \quad \lambda_{2,3} = \frac{2 \pm \sqrt{4 - 4(2)(5)}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Entonces:

Para $\lambda_1 = 1$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} 2a - 2c = 0 \quad \Rightarrow \quad a = c \\ 3a + 2b = 0 \quad \Rightarrow \quad b = -\frac{3}{2}a \end{array}$$

$$\varepsilon_{\lambda=-1} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a = c \ ; \ b = -\frac{3}{2}a \ ; \ a \in \mathbb{R} \right\} \quad \Rightarrow \quad \beta_{\varepsilon_{\lambda=-1}} = \left\{ \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right\}$$

Para $\lambda_2 = 1 + 2i$

$$\left(\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} -2ia = 0 \quad \Rightarrow \quad a = 0 \\ 2a - 2ib - 2c = 0 \quad \Rightarrow \quad c = -ib \end{array}$$

$$\varepsilon_{\lambda=1+2i} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a = 0 \ ; \ c = -ib \ ; \ b \in \mathbb{R} \right\} \quad \Rightarrow \quad \beta_{\varepsilon_{\lambda=1+2i}} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} \right\}$$

Para $\lambda_3 = 1 - 2i$

Es la conjugada de la segunda base, entonces:

$$\beta_{\varepsilon_{\lambda=1-2i}} = \left\{ \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \right\}$$

Entonces:

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + C_2\begin{pmatrix}0\\1\\-i\end{pmatrix}e^{(1+2i)t} + C_3\begin{pmatrix}0\\1\\i\end{pmatrix}e^{(1-2i)t}$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + e^t\left[C_2\begin{pmatrix}0\\1\\-i\end{pmatrix}e^{2it} + C_3\begin{pmatrix}0\\1\\i\end{pmatrix}e^{-2it}\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + e^t\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + i\begin{pmatrix}0\\0\\-1\end{pmatrix}\right](Cos\ 2t + i\ Sen\ 2t) + C_3\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + i\begin{pmatrix}0\\0\\1\end{pmatrix}\right](Cos\ 2t - i\ Sen\ 2t)\right]$$

Ahora, solo desarrollemos:

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + e^t\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + i\begin{pmatrix}0\\0\\-1\end{pmatrix}\right](Cos\ 2t + i\ Sen\ 2t)\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + e^t\left[C_2\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t + C_2 i\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t + C_2 i\begin{pmatrix}0\\0\\-1\end{pmatrix}Cos\ 2t + C_2 i^2\begin{pmatrix}0\\0\\-1\end{pmatrix}Sen\ 2t\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + e^t\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t - \begin{pmatrix}0\\0\\-1\end{pmatrix}Sen\ 2t\right] + C_2 i\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t + \begin{pmatrix}0\\0\\-1\end{pmatrix}Cos\ 2t\right]\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + e^t\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t + \begin{pmatrix}0\\0\\1\end{pmatrix}Sen\ 2t\right] + C_3\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t - \begin{pmatrix}0\\0\\1\end{pmatrix}Cos\ 2t\right]\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + e^t\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t + \begin{pmatrix}0\\0\\1\end{pmatrix}Sen\ 2t\right] + C_3\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t - \begin{pmatrix}0\\0\\1\end{pmatrix}Cos\ 2t\right]\right]$$

Sabemos que $x(0) = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$$\begin{pmatrix}1\\-1\\0\end{pmatrix} = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix} + C_2\begin{pmatrix}0\\1\\0\end{pmatrix} - C_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}2C_1\\-3C_1 + C_2\\2C_1 - C_3\end{pmatrix}$$

Resolviendo el sistema:

$$C_1 = \frac{1}{2} \quad ; \quad C_2 = \frac{1}{2} \quad ; \quad C_3 = 1$$

Finalmente:

$$\boxed{\boldsymbol{x = \frac{1}{2}\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t + e^t\left[\frac{1}{2}\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t + \begin{pmatrix}0\\0\\1\end{pmatrix}Sen\ 2t\right] + \left[\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t - \begin{pmatrix}0\\0\\1\end{pmatrix}Cos\ 2t\right]\right]}}$$

3) $\boldsymbol{X' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} X}$

$det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$(2-\lambda)^3 = 0$$

Cuando una matriz A solo tiene un vector propio asociado con un valor λ_1 de multiplicidad m, se puede determinar las soluciones de la siguiente forma:

$$x_m = K_{m1}\frac{t^{m-1}}{(m-1)!}e^{\lambda_1 t} + K_{m2}\frac{t^{m-2}}{(m-2)!}e^{\lambda_1 t} + \ \ldots\ldots\ldots + K_{mm}\,e^{\lambda_1 t}$$

En que K_{ij} son vectores columnas

Para nuestro caso la tercera solución se la determina de la siguiente manera:

$$x_3 = K\frac{t^2}{2}e^{\lambda_1 t} + P\,te^{\lambda_1 t} + Q\,e^{\lambda_1 t}$$

En donde:

$$K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \ , \ P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \ , \ Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

Al sustituir en el sistema X' = AX , los vectores columnas K, P, Q deben cumplir con:

$$(A - \lambda_1 I)K = 0$$
$$(A - \lambda_1 I)P = K$$
$$(A - \lambda_1 I)Q = P$$

La ecuación característica $(2-\lambda)^3 = 0$ indica que $\lambda_1 = 2$ es un valor de multiplicidad tres y al resolver tenemos:

Para $\lambda_1 = 2$

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} b + 6c = 0 \Rightarrow b = 0 \\ 5c = 0 \Rightarrow c = 0 \end{array}$$

$$\varepsilon_{\lambda=2} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \ b = 0 \ ; \ c = 0 \ ; \ a \in \mathbb{R} \right\} \Rightarrow \beta_{\varepsilon_{\lambda=2}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Entonces:

$$K = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Luego resolvemos los sistemas:

1er sistema

$$(A - \lambda_1 I)P = K$$

$$\left(\begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} p_2 + 6p_3 \\ 5p_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} p_2 + 6p_3 = 1 \\ 5p_3 = 0 \\ 0 = 0 \end{matrix}$$

Resolviendo tenemos que:

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \Rightarrow \quad P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

2do sistema

$$(A - \lambda_1 I)Q = P$$

$$\left(\begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} q_2 + 6q_3 \\ 5q_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} q_2 + 6q_3 = 0 \\ 5q_3 = 1 \\ 0 = 0 \end{matrix}$$

Resolviendo tenemos que:

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad \Rightarrow \quad Q = \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}$$

Finalmente las soluciones del sistema de ecuaciones diferenciales es:

$$x = C_1 K e^{2t} + C_2[K\ te^{2t} + P\ e^{2t}] + C_3\left[K\ \frac{t^2}{2}e^{2t} + P\ te^{2t} + Q\ e^{2t}\right]$$

$$\boxed{\boldsymbol{x = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{2t} + C_2\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{2t}\right] + C_3\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\frac{t^2}{2}e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}e^{2t}\right]}}$$

RESOLUCIÓN DE SISTEMAS DE ECUACIONES DIFERENCIALES UTILIZANDO TRANSFORMADA DE LAPLACE

$$\mathbf{1)}\ \begin{cases} x' + 2x + 6\displaystyle\int_0^t y(u)du = -2 \\ x' + y' + y = 0 \\ x(0) = -5\ ;\ y(0) = 6 \end{cases}$$

Aplicando transformada de Laplace a cada ecuación:

$$\mathcal{L}\{x'\} + 2\mathcal{L}\{x\} + 6\mathcal{L}\left\{\int_0^t y(u)du\right\} = -2\mathcal{L}\{1\}$$

$$\mathcal{L}\{x'\} + \mathcal{L}\{y'\} + \mathcal{L}\{y\} = 0$$

$$[sX - x(0)] + 2X + 6\frac{Y}{s} = -\frac{2}{s} \quad \Rightarrow \quad sX + 5 + 2X + 6\frac{Y}{s} = -\frac{2}{s}$$

$$[sX - x(0)] + [sY - y(0)] + Y = 0 \quad \Rightarrow \quad sX + 5 + sY - 6 + Y = 0$$

$$s^2X + 5s + 2sX + 6Y = -2 \quad \Rightarrow \quad s(s+2)X + 6Y = -2 - 5s$$

$$(s+1)Y + sX = 1$$

$$s(s+2)(s+1)X + 6(s+1)Y = -(5s+2)(s+1)$$

$$-6(s+1)Y - 6sX = -6$$

Sumando las dos ecuaciones tenemos:

$$s(s+2)(s+1)X - 6sX = -(5s+2)(s+1) - 6$$

$$Xs(s^2 + 3s + 2 - 6) = -(5s+2)(s+1) - 6$$

$$X = -\frac{5s^2 + 7s + 2}{s(s+4)(s-1)} - \frac{6}{s(s+4)(s-1)}$$

Descomponiendo en fracciones parciales:

$$\frac{5s^2 + 7s + 2}{s(s+4)(s-1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-1}$$

$$5s^2 + 7s + 2 = A(s+4)(s-1) + Bs(s-1) + Cs(s+4)$$

$$5s^2 + 7s + 2 = As^2 + 3As - 4A + Bs^2 - Bs + Cs^2 + 4Cs$$

$$5s^2 + 7s + 2 = (A + B + C)s^2 + (3A - B + 4C)s - 4A$$

$5 = A + B + C$

$7 = 3A - B + 4C$

$2 = -4A$

Resolviendo el sistema A = -1/2 , B = 27/10 , C = 14/5

Ahora:

$$\frac{6}{s(s+4)(s-1)} = \frac{A'}{s} + \frac{B'}{s+4} + \frac{C'}{s-1}$$

$6 = (A' + B' + C')s^2 + (3A' - B' + 4C')s - 4A'$

$0 = A' + B' + C'$

$0 = 3A' - B' + 4C'$

$6 = -4A'$

Resolviendo el sistema A' = -3/2 , B' = 3/10 , C' = 6/5

Entonces:

$$\mathcal{L}^{-1}\{X\} = -\mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{A'}{s} + \frac{B'}{s+4} + \frac{C'}{s-1}\right\}$$

$$x(t) = -(A + Be^{-4t} + Ce^{t}) - (A^{'} + B^{'}e^{-4t} + C'e^{t})$$

$$x(t) = -\left(-\frac{1}{2} + \frac{27}{10}e^{-4t} + \frac{14}{5}e^{t}\right) - \left(-\frac{3}{2} + \frac{3}{10}e^{-4t} + \frac{6}{5}e^{t}\right)$$

$$x(t) = -2 - 3e^{-4t} - 4e^{t}$$

Encontrando la segunda solución:

$(s+1)Y + sX = 1$

$$Y = \frac{1 - sX}{(s+1)}$$

$$Y = \frac{1}{s+1} - \frac{s}{(s+1)}\left[-\frac{5s^2 + 7s + 2}{s(s+4)(s-1)} - \frac{6}{s(s+4)(s-1)}\right]$$

$$Y = \frac{1}{s+1} + \frac{5s^2 + 7s + 2}{(s+4)(s-1)(s+1)} + \frac{6}{(s+4)(s-1)(s+1)}$$

Descomponiendo en fracciones parciales:

$$\frac{5s^2+7s+2}{(s+4)(s-1)(s+1)}=\frac{A}{s+4}+\frac{B}{s-1}+\frac{C}{s+1}$$

$$5s^2+7s+2=A(s-1)(s+1)+B(s+4)(s+1)+C(s+4)(s-1)$$

$$5s^2+7s+2=As^2-A+Bs^2+5Bs+4B+Cs^2+3Cs-4C$$

$$5s^2+7s+2=(A+B+C)s^2+(5B+3C)s+(4B-A-4C)$$

$$5=A+B+C$$

$$7=5B+3C$$

$$2=4B-A-4C$$

Resolviendo el sistema A = 18/5 , B = 7/5 , C = 0

$$\frac{6}{(s+4)(s-1)(s+1)}=\frac{A'}{s+4}+\frac{B'}{s-1}+\frac{C'}{s+1}$$

$$6=(A'+B'+C')s^2+(5B'+3C')s+(4B'-A'-4C')$$

$$0=A'+B'+C'$$

$$0=5B'+3C'$$

$$6=4B'-A'-4C'$$

Resolviendo el sistema A' = 6/15 , B' = 3/5 , C' = -1

Entonces:

$$\mathcal{L}^{-1}\{Y\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{A}{s+4}+\frac{B}{s-1}+\frac{C}{s+1}+\frac{A'}{s+4}+\frac{B'}{s-1}+\frac{C'}{s+1}\right\}$$

$$y(t)=e^{-t}+Ae^{-4t}+Be^{t}+Ce^{-t}+A'e^{-4t}+B'e^{t}+C'e^{-t}$$

$$y(t)=e^{-t}+\frac{18}{5}e^{-4t}+\frac{7}{5}e^{t}+\frac{6}{5}e^{-4t}+\frac{3}{5}e^{t}-e^{-t}$$

Finalmente:

$$\boxed{\boldsymbol{x(t)=-2-3e^{-4t}-4e^{t}}}$$

$$\boxed{\boldsymbol{y(t)=e^{-t}+\frac{24}{5}e^{-4t}+2e^{t}+\frac{3}{5}e^{t}-e^{-t}}}$$

$$\mathbf{2)}\ \begin{cases} \boldsymbol{x}' - \boldsymbol{y} = \begin{cases} \mathbf{0}\ , \mathbf{0} < t < 2 \\ \mathbf{1}\ , \mathbf{2} < t < 3 \\ \mathbf{0}\ , \quad \boldsymbol{t} \geq \mathbf{3} \end{cases} \\ \boldsymbol{y}' - \boldsymbol{x} = \mathbf{1} \\ \quad \boldsymbol{x}(\mathbf{1}) = \boldsymbol{y}(\mathbf{1}) = \mathbf{1} \end{cases}$$

$x' - y = u(t-2) - u(t-3)$
$y' - x = 1$

$\mathcal{L}\{x'\} - \mathcal{L}\{y\} = \mathcal{L}\{u(t-2)\} - \mathcal{L}\{u(t-3)\}$
$\mathcal{L}\{y'\} - \mathcal{L}\{x\} = \mathcal{L}\{1\}$

$[sX - x(0)] - Y = e^{-2s} - e^{-3s}$
$[sY - y(0)] - X = \frac{1}{s}$

No conocemos el valor de x(0) y de y(0), pero vamos a llamar x(0) = w y y(0) = z , entonces:

$sX - w - Y = e^{-2s} - e^{-3s}$

$s^2Y - zs - Xs = 1$

Entonces:

$$X = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{w}{s} + \frac{Y}{s}$$

$$Y = \frac{1}{s^2} + \frac{z}{s} + \frac{X}{s}$$

Reemplazando Y

$$X = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{w}{s} + \frac{1}{s}\left(\frac{1}{s^2} + \frac{z}{s} + \frac{X}{s}\right)$$

$$X = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{w}{s} + \frac{1}{s^3} + \frac{z}{s^2} + \frac{X}{s^2}$$

$$X\left(1 - \frac{1}{s^2}\right) = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{w}{s} + \frac{1}{s^3} + \frac{z}{s^2}$$

$$X = \frac{s}{s^2-1}e^{-2s} - \frac{s}{s^2-1}e^{-3s} + w\frac{s}{s^2-1} + \frac{1}{s(s^2-1)} + \frac{z}{s^2-1}$$

$$\mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}e^{-2s} - \frac{s}{s^2-1}e^{-3s} + w\frac{s}{s^2-1} + \frac{1}{s(s^2-1)} + \frac{z}{s^2-1}\right\}$$

Resolviendo cada transformada inversa:

$$* \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}e^{-2s}\right\} = u(t-2)\,Cosh(t-2)$$

$$* \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}e^{-3s}\right\} = u(t-3)\,Cosh(t-3)$$

$$* \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} = Cosh(t)$$

$$* \mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} * \frac{1}{s^2-1}\right\}$$

$$\int_0^t Senh(x)dx \quad ; \quad Pero\ Senh\ x = \frac{e^x - e^{-x}}{2}$$

$$\int_0^t \frac{e^x - e^{-x}}{2}dx = \left[\frac{1}{2}(e^x + e^{-x})\right]_0^t \quad \Rightarrow \quad \int_0^t \frac{e^x - e^{-x}}{2}dx = \frac{1}{2}(e^t + e^{-t} - 2)$$

$$\int_0^t \frac{e^x - e^{-x}}{2}dx = \frac{e^t + e^{-t}}{2} - 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \frac{e^t + e^{-t}}{2} - 1$$

$$* \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = Senh(t)$$

Entonces:

$$x(t) = u(t-2)\,Cosh(t-2) - u(t-3)\,Cosh(t-3) + w\,Cosh(t) + \frac{e^t + e^{-t}}{2} - 1 + z\,Senh(t)$$

Ahora:

$$Y = \frac{1}{s^2} + \frac{z}{s} + \frac{1}{s}\left(\frac{s}{s^2-1}e^{-2s} - \frac{s}{s^2-1}e^{-3s} + w\frac{s}{s^2-1} + \frac{1}{s(s^2-1)} + \frac{z}{s^2-1}\right)$$

$$Y = \frac{1}{s^2} + \frac{z}{s} + \frac{1}{s^2-1}e^{-2s} - \frac{1}{s^2-1}e^{-3s} + w\frac{1}{s^2-1} + \frac{1}{s^2(s^2-1)} + \frac{z}{s(s^2-1)}$$

Resolviendo cada transformada inversa:

$$* \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$* \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$* \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}e^{-2s}\right\} = u(t-2)\,Senh(t-2)$$

$$* \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}e^{-3s}\right\} = u(t-3)\,Senh(t-3)$$

$$* \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = Senh(t)$$

$$* \mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \frac{e^t+e^{-t}}{2} - 1$$

$$* \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} * \frac{1}{s^2-1}\right\}$$

$$\int_0^t (t-x)Senh(x)dx = \int_0^t (t-x)\left(\frac{e^x-e^{-x}}{2}\right)dx$$

$$t\int_0^t \frac{e^x-e^{-x}}{2}dx - \frac{1}{2}\left(\int_0^t x\,e^x dx - \int_0^t x\,e^{-x}dx\right)$$

$$\left[\frac{t}{2}(e^x+e^{-x}) - \frac{1}{2}[e^x(x-1)+e^{-x}(x+1)]\right]_0^t$$

Evaluando:

$$\frac{t}{2}(e^t+e^{-t}) - \frac{1}{2}[e^t(t-1)+e^{-t}(t+1)] - t = \frac{te^t}{2} + \frac{te^{-t}}{2} - \frac{te^t}{2} + \frac{e^t}{2} - \frac{te^{-t}}{2} - \frac{e^{-t}}{2} - t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \frac{e^t}{2} - \frac{e^{-t}}{2} - t$$

Por lo tanto:

$$x(t) = u(t-2)\,Cosh(t-2) - u(t-3)\,Cosh(t-3) + w\,Cosh(t) + \frac{e^t+e^{-t}}{2} - 1 + z\,Senh(t)$$

$$y(t) = t + z + u(t-2)\,Senh(t-2) - u(t-3)\,Senh(t-3) + w\,Senh(t) + \frac{e^t}{2} - \frac{e^{-t}}{2} - t + z\left(\frac{e^t+e^{-t}}{2} - 1\right)$$

Encontrando los valores de "w" y "z"

Sabemos que x(0) = w y y(0) = z

$$x(0) = u(0-2)\,Cosh(0-2) - u(0-3)\,Cosh(0-3) + w\,Cosh(0) + \frac{e^0+e^{-0}}{2} - 1 + z\,Senh(0)$$

$$w = u(-2)\,Cosh(-2) - u(-3)\,Cosh(-3) + w\left(\frac{e^0+e^{-0}}{2}\right) + \frac{e^0+e^{-0}}{2} - 1 + z\left(\frac{e^0-e^{-0}}{2}\right)$$

$$w = 0 - 0 + \frac{w}{2} + \frac{1}{2} - 1 + \frac{z}{2}$$

$$\frac{3w}{2} = \frac{z}{2} - \frac{1}{2}$$

$$w = \frac{1}{3}(z-1)$$

$$y(0) = 0 + z + u(0-2)\,Senh(0-2) - u(0-3)\,Senh(0-3) + w\,Senh(0) + \frac{e^0}{2} - \frac{e^{-0}}{2} - 0 + z\left(\frac{e^0 + e^{-0}}{2} - 1\right)$$

$$z = z + u(-2)\,Senh(-2) - u(-3)\,Senh(-3) + w\,\left(\frac{e^0 - e^{-0}}{2}\right) + \frac{e^0}{2} - \frac{e^{-0}}{2} + z\left(\frac{e^0 + e^{-0}}{2} - 1\right)$$

$$z = z + 0 - 0 + \frac{w}{2} + \frac{z}{2} - z$$

$$z = \frac{w}{2} + \frac{z}{2}$$

$$\frac{z}{2} = \frac{w}{2} \quad \Rightarrow \quad z = w$$

Reemplazando nos queda:

$$w = \frac{1}{3}(w-1) \quad \Rightarrow \quad \frac{2w}{3} = -\frac{1}{3} \quad \Rightarrow \quad w = -\frac{1}{2}$$

$$z = -\frac{1}{2}$$

Finalmente:

$$x(t) = u(t-2)\,Cosh(t-2) - u(t-3)\,Cosh(t-3) - \frac{1}{2}\,Cosh(t) + \frac{e^t + e^{-t}}{2} - 1 - \frac{1}{2}\,Senh(t)$$

$$y(t) = t - \frac{1}{2} + u(t-2)\,Senh(t-2) - u(t-3)\,Senh(t-3) - \frac{1}{2}\,Senh(t) + \frac{e^t}{2} - \frac{e^{-t}}{2} + \frac{1}{2} - \frac{1}{2}\left(\frac{e^t + e^{-t}}{2} - 1\right)$$

$$x(t) = u(t-2)\,Cosh(t-2) - u(t-3)\,Cosh(t-3) - \frac{1}{2}\,Cosh(t) + \,Cosh(t) - 1 - \frac{1}{2}\,Senh(t)$$

$$y(t) = t - \frac{1}{2} + u(t-2)\,Senh(t-2) - u(t-3)\,Senh(t-3) - \frac{1}{2}\,Senh(t) + Senh(t) + \frac{1}{2} - \frac{1}{2}(Cosh(t) - 1)$$

$$\boxed{\boldsymbol{x(t) = u(t-2)\,Cosh(t-2) - u(t-3)\,Cosh(t-3) + \frac{1}{2}\,Cosh(t) - 1 - \frac{1}{2}\,Senh(t)}}$$

$$\boxed{\boldsymbol{y(t) = t + u(t-2)\,Senh(t-2) - u(t-3)\,Senh(t-3) + \frac{1}{2}\,Senh(t) - \frac{1}{2}(Cosh(t) - 1)}}$$

$$3)\begin{cases} \boldsymbol{x'-y'=Sen(t)\,u(t-\pi)} \\ \boldsymbol{x+y'=0} \\ \boldsymbol{x(0)=y(0)=1} \end{cases}$$

$$x'-y'=Sen[(t-\pi)+\pi]\,u(t-\pi)$$
$$x+y'=0$$

$$x'-y'=[Sen(t-\pi)Cos\pi+Cos(t-\pi)Sen\pi]\,u(t-\pi)$$
$$x+y'=0$$

$$x'-y'=-Sen(t-\pi)\,u(t-\pi)$$
$$x+y'=0$$

$$\mathcal{L}\{x'\}-\mathcal{L}\{y'\}=-\mathcal{L}\{Sen(t-\pi)\,u(t-\pi)\}$$
$$\mathcal{L}\{x\}+\mathcal{L}\{y'\}=0$$

$$sX-x(0)-sY+y(0)=-e^{-\pi s}\frac{1}{s^2+1}$$
$$X+sY-y(0)=0$$

$$sX-sY=-e^{-\pi s}\frac{1}{s^2+1}$$
$$X+sY=1$$

Usando la regla de Kramer tenemos:

$$X=\frac{\begin{vmatrix}-e^{-\pi s}\frac{1}{s^2+1} & -s\\ 1 & s\end{vmatrix}}{\begin{vmatrix}s & -s\\ 1 & s\end{vmatrix}} \quad ; \quad Y=\frac{\begin{vmatrix}s & -e^{-\pi s}\frac{1}{s^2+1}\\ 1 & 1\end{vmatrix}}{\begin{vmatrix}s & -s\\ 1 & s\end{vmatrix}}$$

$$X=\frac{-se^{-\pi s}\frac{1}{s^2+1}+s}{s^2+s}=\frac{1}{s+1}-\frac{1}{(s^2+1)(s+1)}e^{-\pi s}$$

$$Y=\frac{s-e^{-\pi s}\frac{1}{s^2+1}}{s^2+s}=\frac{1}{s+1}-\frac{1}{s(s^2+1)(s+1)}e^{-\pi s}$$

<u>*Encontrando la 1era solución*</u>

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{X\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}-\frac{1}{(s^2+1)(s+1)}e^{-\pi s}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}=te^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}e^{-\pi s}\right\} \;;\; Aplicando\ convolución\ tenemos\ \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+1}*\frac{1}{s+1}}_{Sen(t)\,*e^{-t}}\right\}$$

$$\int_0^t e^{-(t-x)}Sen(x)dx=e^{-t}\int_0^t e^{x}Sen(x)dx$$

Resolviendo la integral por partes tenemos:

$$\int e^x Sen(x)dx = -e^x Cos(x) + \int e^x Cos(x)dx$$

$$\int e^x Sen(x)dx = -e^x Cos(x) + \left[e^x Sen(x) - \int e^x Sen(x)dx\right]$$

$$\int e^x Sen(x)dx = \frac{1}{2}[e^x Sen(x) - e^x Cos(x)]$$

Evaluando:

$$\left[\frac{1}{2}[e^x Sen(x) - e^x Cos(x)]\right]_0^t = \frac{1}{2}[e^t Sen(t) - e^t Cos(t) + 1]$$

Entonces:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \frac{e^{-t}}{2}[e^t Sen(t) - e^t Cos(t) + 1] = \frac{1}{2}[Sen(t) - Cos(t) + e^{-t}]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}e^{-\pi s}\right\} = \frac{1}{2}\left[Sen(t-\pi) - Cos(t-\pi) + e^{-(t-\pi)}\right]$$

Luego:

$$x(t) = te^{-t} - \frac{1}{2}\left[Sen(t-\pi) - Cos(t-\pi) + e^{-(t-\pi)}\right]$$

Encontrando la 2da solución

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s(s^2+1)(s+1)}e^{-\pi s}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = te^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)(s+1)}e^{-\pi s}\right\} \quad ; \quad Aplicando\ convolución\ tenemos\ \ \mathcal{L}^{-1}\left\{\frac{1}{s} * \frac{1}{s^2+1} * \frac{1}{s+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s} * \left(\frac{1}{s^2+1} * \frac{1}{s+1}\right)}_{1 * \frac{1}{2}[Sen(t) - Cos(t) + e^{-t}]}\right\}$$

$$\int_0^t \frac{1}{2}[Sen(x) - Cos(x) + e^{-x}]dx$$

$$\frac{1}{2}\int_0^t [Sen(x) - Cos(x) + e^{-x}]dx = \left[\frac{1}{2}[-Cos(x) - Sen(x) - e^{-x}]\right]_0^t$$

$$\frac{1}{2}\int_0^t [Sen(x) - Cos(x) + e^{-x}]dx = \frac{1}{2}[-Cos(t) - Sen(t) - e^{-t} + 2]$$

Entonces:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)(s+1)}e^{-\pi s}\right\} = \frac{1}{2}\left[-Cos(t-\pi) - Sen(t-\pi) - e^{-(t-\pi)} + 2\right]$$

Luego:

$$y(t) = te^{-t} + \frac{1}{2}\left[Cos(t-\pi) + Sen(t-\pi) + e^{-(t-\pi)} - 2\right]$$

Finalmente la solución del sistema es:

$$\boxed{\boldsymbol{x(t) = te^{-t} - \frac{1}{2}\left[Sen(t-\pi) - Cos(t-\pi) + e^{-(t-\pi)}\right]}}$$

$$\boxed{\boldsymbol{y(t) = te^{-t} + \frac{1}{2}\left[Cos(t-\pi) + Sen(t-\pi) + e^{-(t-\pi)} - 2\right]}}$$

APLICACIONES

SISTEMA MASA – RESORTE - AMORTIGUADOR

1) Una masa de 1 kg está unida a un resorte ligero que es estirado 2m por una fuerza de 8 N, la masa se encuentra inicialmente en reposo en su posición de equilibrio. Iniciando en el tiempo t = 0 seg se le aplica una fuerza externa f(t)=Cos(2t) a la masa pero en el instante t = 2π esta cesa abruptamente y la masa queda libre continuando con su movimiento, pero en el tiempo t = 4π, la masa es golpeada hacia abajo con un martillo con una fuerza de 10N. Determine la ecuación del movimiento, además la posición de la masa cuando t = 9π/4 seg.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

Nos dice que el resorte es estirado 2mpor una fuerza de 8N, entonces:

$$F = kx \quad \Rightarrow \quad k = \frac{F}{x} = \frac{8}{2} \quad \Rightarrow \quad k = 4\ N/m$$

Además nos dice, que en t=0 se le aplica una fuerza externa, y después cesa abruptamente, entonces f(t) nos queda:

$$f(t) = \begin{cases} Cos(2t)\ ; & 0 \leq t < 2\pi \\ 0 \quad ; & t > 2\pi \end{cases}$$

Pero en t = 4 π, es golpeado con un martillo, produciendo un impulso, entonces, nuestra ecuación nos queda:

$$\frac{d^2x}{dt^2} + 4x = (\mathcal{u}_0 - \mathcal{u}_{2\pi})Cos(2t) + 10\,\delta(t - 4\pi)$$

$$x'' + 4x = \mathcal{u}_0 Cos(2t) - \mathcal{u}_{2\pi} Cos(2t) + 10\delta(t - 4\pi)$$

La función coseno ya está desfasada, entonces aplicando transformada de Laplace, nos queda:

$$[s^2X - s\,x(0) - x'(0)] + 4X = \frac{s}{s^2+4} - \frac{s}{s^2+4}e^{-2\pi s} + 10e^{-4\pi s}$$

Sabemos que en t = 0 , x(0) = x'(0) = 0

$$s^2X + 4X = \frac{s}{s^2+4} - \frac{s}{s^2+4}e^{-2\pi s} + 10e^{-4\pi s}$$

$$X(s^2+4) = \frac{s}{s^2+4} - \frac{s}{s^2+4}e^{-2\pi s} + 10e^{-4\pi s}$$

$$X = \frac{s}{(s^2+4)^2} - \frac{s}{(s^2+4)^2}e^{-2\pi s} + 10\frac{e^{-4\pi s}}{s^2+4}$$

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}e^{-2\pi s}\right\} + 10\mathcal{L}^{-1}\left\{\frac{e^{-4\pi s}}{s^2+4}\right\}$$

Aplicando convolución:

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{s}{s^2+4}}_{Cos(2t)} * \overbrace{\frac{1}{s^2+4}}^{\frac{1}{2}Sen(2t)}\right\}$$

$$\frac{1}{2}\int_0^t Cos(2x)Sen[2(t-x)]dx$$

$$\frac{1}{4}\int_0^t [Sen(2x+2t-2x) - Sen(2x-2t+2x)]dx$$

$$\frac{1}{4}\int_0^t [Sen(2t) - Sen(4x-2t)]dx$$

$$\frac{1}{4}\left[x\,Sen(2t) + \frac{1}{4}Cos(4x-2t)\right]_0^t \quad \Rightarrow \quad \frac{1}{4}\left[t\,Sen(2t) + \frac{1}{4}Cos(2t) - \frac{1}{4}Cos(-2t)\right]$$

Sabemos que Cos(-x) = Cos(x), entonces:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}t\,Sen(2t)$$

Finalmente:

$$\boxed{\boldsymbol{x(t) = \frac{1}{4}t\,Sen(2t) - \frac{1}{4}u(t-2\pi)(t-2\pi)\,Sen[2(t-2\pi)] + 5\,u(t-4\pi)\,Sen[2(t-4\pi)]}}$$

<u>Encontrando la posición de la masa en t = 9π/4 seg</u>

$$x\left(\frac{9\pi}{4}\right) = \frac{1}{4}\left(\frac{9\pi}{4}\right)\,Sen\left[2\left(\frac{9\pi}{4}\right)\right] - \frac{1}{4}u\left[\left(\frac{9\pi}{4}\right) - 2\pi\right]\left(\frac{9\pi}{4} - 2\pi\right)\,Sen\left[2\left(\frac{9\pi}{4} - 2\pi\right)\right] + 5\,u\left(\frac{9\pi}{4} - 4\pi\right)\,Sen\left[2\left(\frac{9\pi}{4} - 4\pi\right)\right]$$

$$x\left(\frac{9\pi}{4}\right) = \frac{9\pi}{16}\,Sen\left(\frac{9\pi}{2}\right) - \frac{\pi}{16}u\left(\frac{\pi}{4}\right)Sen\left(\frac{\pi}{2}\right) + 5\,u\left(-\frac{7\pi}{4}\right)Sen\left(-\frac{7\pi}{2}\right)$$

$$x\left(\frac{9\pi}{4}\right) = \frac{9\pi}{16}(1) - \frac{\pi}{16}(1)(1) + 5(0)(1)$$

$$x\left(\frac{9\pi}{4}\right) = \frac{9\pi}{16} - \frac{\pi}{16}$$

$$\boxed{\boldsymbol{x\left(\frac{9\pi}{4}\right) = \frac{\pi}{2}[m]}}$$

2) En el extremo de un resorte espiral que está sujeto al techo se coloca un cuerpo de masa igual a 1 kg. El resorte se ha alargado 2m hasta quedar en reposo en su posición de equilibrio. En t = 0 el cuerpo es desplazado 50 cm por debajo de la posición de equilibrio y lanzado con una velocidad inicial de 1m/seg dirigida hacia arriba. El sistema consta también de un amortiguador cuyo coeficiente de amortiguamiento es de 2.5 N.seg/m. Desde t = 0, una fuerza externa es aplicada al cuerpo, la misma que está dada por f(t) = Sen$(\pi t/2)$. En t = 10 seg y en t = 20 seg el cuerpo es golpeado hacia abajo proporcionando una fuerza de 5N y de 10N, respectivamente. (use g = 10 m/seg^2). Determine la ecuación del movimiento

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$$

Nos dice que el resorte se ha alargado 2m hasta quedar en reposo al colocar una masa de 1 kg, entonces:

$$F=kx \quad \Rightarrow \quad k=\frac{mg}{x}=\frac{1(10)}{2} \quad \Rightarrow \quad k=5\ N/m$$

Además nos dice que en t=10 y en t=20 el cuerpo es golpeado hacia abajo, es decir recibe un impulso, entonces nuestra ecuación es la siguiente:

$$(1)\frac{d^2x}{dt^2}+(2.5)\frac{dx}{dt}+(5)x=Sen\left(\frac{\pi}{2}t\right)+5\ \delta(t-10)+10\ \delta(t-20)$$

$$x''+2.5x'+5x=Sen\left(\frac{\pi}{2}t\right)+5\ \delta(t-10)+10\ \delta(t-20)$$

Aplicando transformada de Laplace:

$$\mathcal{L}\{x''\}+2.5\ \mathcal{L}\{x'\}+5\ \mathcal{L}\{x\}=\mathcal{L}\left\{Sen\left(\frac{\pi}{2}t\right)\right\}+5\ \{\delta(t-10)\}+10\ \{\delta(t-20)\}$$

$$[s^2X-s\ x(0)-x'(0)]+2.5[sX-x(0)]+5X=\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}+5e^{-10s}+10e^{-20s}$$

Sabemos que en t = 0 el cuerpo es lanzado con una velocidad inicial hacia arriba y además es desplazado 50 cm por debajo de su posición de equilibrio, entonces:

$$[s^2X-0.5s+1]+2.5[sX-0.5]+5X=\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}+5e^{-10s}+10e^{-20s}$$

$$s^2X-0.5\ s+1+2.5\ sX-1.25+5X=\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}+5e^{-10s}+10e^{-20s}$$

$$X\left(s^2+\frac{5}{2}s+5\right)=\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}+5e^{-10s}+10e^{-20s}+\frac{1}{4}$$

$$X=\frac{\frac{\pi}{2}}{\left(s^2+\frac{\pi^2}{4}\right)\left[\left(s+\frac{5}{4}\right)^2+\frac{15}{8}\right]}+5\frac{e^{-10s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}+10\frac{e^{-20s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}+\frac{1}{4}\frac{1}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}$$

Aplicando la transformada inversa:

$$\mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\left\{\frac{\frac{\pi}{2}}{\left(s^2+\frac{\pi^2}{4}\right)\left[\left(s+\frac{5}{4}\right)^2+\frac{15}{8}\right]}\right\} + 5\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\} + 10\mathcal{L}^{-1}\left\{\frac{e^{-20s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}$$

Aplicando convolución:

$$\mathcal{L}^{-1}\left\{\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}} * \left(\sqrt{\frac{8}{15}}\right)\frac{\sqrt{\frac{15}{8}}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}$$

$$\left(\sqrt{\frac{8}{15}}\right)\mathcal{L}^{-1}\underbrace{\left\{\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}} * \frac{\sqrt{\frac{15}{8}}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}}_{Sen\left(\frac{\pi}{2}t\right)*e^{-\frac{5}{4}t}\,Sen\left(\sqrt{\frac{15}{8}}t\right)}$$

$$\int_0^t Sen\left[\frac{\pi}{2}(t-x)\right]e^{-\frac{5}{4}x}\,Sen\left(\sqrt{\frac{15}{8}}x\right)dx$$

Resolviendo la integral y para mayor comodidad, tenemos:

$$\int Sen[A(t-x)]e^{Bx}\,Sen(Cx)dx$$

$$\int e^{Bx}Sen(At-Ax)\,Sen(Cx)dx$$

Sabemos que:

$$Sen(mx)\,Sen(nx) = \frac{1}{2}[Cos(m-n)x - Cos(m+n)x]$$

Entonces:

$$\int e^{Bx}\frac{1}{2}[Cos(At-Ax-Cx) - Cos(At-Ax+Cx)]dx$$

$$\frac{1}{2}\left[\int e^{Bx}Cos[At-(A+C)x]dx - \int e^{Bx}Cos[At+(C-A)x]dx\right]$$

Resolviendo la integral por partes, tenemos:

$$u = e^{Bx} \quad \Rightarrow \quad du = Be^{Bx}dx$$

$$dv = Cos[At-(A+C)x]dx \quad \Rightarrow \quad v = -\frac{1}{A+C}Sen[At-(A+C)x]$$

$$\int e^{Bx}Cos[At-(A+C)x]dx = -\frac{e^{Bx}}{A+C}Sen[At-(A+C)x] + \frac{B}{A+C}\int e^{Bx}Sen[At-(A+C)x]dx$$

Integrando nuevamente por partes:

$u = e^{Bx} \quad \Rightarrow \quad du = Be^{Bx}dx$

$dv = Sen[At - (A+C)x]dx \quad \Rightarrow \quad v = \frac{1}{A+C}Cos[At - (A+C)x]$

$$\int e^{Bx}Cos[At-(A+C)x]dx = -\frac{e^{Bx}}{A+C}Sen[At-(A+C)x] + \frac{B}{A+C}\left[\frac{e^{Bx}}{A+C}Cos[At-(A+C)x] - \frac{B}{A+C}\int e^{Bx}Cos[At-(A+C)x]dx\right]$$

$$\int e^{Bx}Cos[At-(A+C)x]dx = -\frac{e^{Bx}}{A+C}Sen[At-(A+C)x] + \frac{Be^{Bx}}{(A+C)^2}Cos[At-(A+C)x] - \left(\frac{B}{A+C}\right)^2\int e^{Bx}Cos[At-(A+C)x]dx$$

$$\int e^{Bx}Cos[At-(A+C)x]dx = \left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bx}}{(A+C)^2}Cos[At-(A+C)x] - \frac{e^{Bx}}{A+C}Sen[At-(A+C)x]\right]$$

Luego tenemos que:

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bx}}{(A+C)^2}Cos[At-(A+C)x] - \frac{e^{Bx}}{A+C}Sen[At-(A+C)x]\right]_0^t$$

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\begin{array}{l}\frac{Be^{Bt}}{(A+C)^2}Cos[At-(A+C)t] - \frac{e^{Bt}}{A+C}Sen[At-(A+C)t] \\ -\frac{Be^{B(0)}}{(A+C)^2}Cos[At-(A+C)(0)] + \frac{e^{B(0)}}{A+C}Sen[At-(A+C)(0)]\end{array}\right]$$

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bt}}{(A+C)^2}Cos(-Ct) - \frac{e^{Bt}}{A+C}Sen(-Ct) - \frac{B}{(A+C)^2}Cos(At) + \frac{1}{A+C}Sen(At)\right]$$

Resolviendo la segunda integral:

$$\int e^{Bx}Cos[At+(A+C)x]dx = \left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bx}}{(A+C)^2}Cos[At+(A+C)x] - \frac{e^{Bx}}{A+C}Sen[At+(A+C)x]\right]$$

Luego tenemos que:

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bx}}{(A+C)^2}Cos[At+(A+C)x] - \frac{e^{Bx}}{A+C}Sen[At+(A+C)x]\right]_0^t$$

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\begin{array}{l}\frac{Be^{Bt}}{(A+C)^2}Cos[At+(A+C)t] - \frac{e^{Bt}}{A+C}Sen[At+(A+C)t] \\ -\frac{Be^{B(0)}}{(A+C)^2}Cos[At+(A+C)(0)] + \frac{e^{B(0)}}{A+C}Sen[At+(A+C)(0)]\end{array}\right]$$

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bt}}{(A+C)^2}Cos[(2A+C)t] - \frac{e^{Bt}}{A+C}Sen[(2A+C)t] - \frac{B}{(A+C)^2}Cos(At) + \frac{1}{A+C}Sen(At)\right]$$

Entonces, la transformada inversa nos queda:

$$\left(\sqrt{\frac{8}{15}}\right)\mathcal{L}^{-1}\left\{\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}*\frac{\sqrt{\frac{15}{8}}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}$$

$$\frac{1}{2}\left(\sqrt{\frac{8}{15}}\right)\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-2}\left[\frac{Be^{Bt}}{(A+C)^2}Cos(Ct)+\frac{e^{Bt}}{A+C}Sen(-Ct)-\frac{B}{(A+C)^2}Cos(At)+\frac{1}{A+C}Sen(At)\right]$$
$$\left[\frac{Be^{Bt}}{(A+C)^2}Cos[(2A+C)t]-\frac{e^{Bt}}{A+C}Sen[(2A+C)t]-\frac{B}{(A+C)^2}Cos(At)+\frac{1}{A+C}Sen(At)\right]$$

$$\frac{1}{2}\left(\sqrt{\frac{8}{15}}\right)\left[1+\left(-\frac{5/4}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}\right)^2\right]^{-2}\left[-\frac{5e^{-\frac{5}{4}t}}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\sqrt{\frac{15}{8}}t\right)+\frac{e^{-\frac{5}{4}t}}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(-\sqrt{\frac{15}{8}}t\right)+\frac{5}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\frac{\pi}{2}t\right)+\frac{1}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(\frac{\pi}{2}t\right)\right]$$
$$\left[-\frac{5e^{-\frac{5}{4}t}}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left[\left(\pi+\sqrt{\frac{15}{8}}\right)t\right]-\frac{e^{-\frac{5}{4}t}}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left[\left(\pi+\sqrt{\frac{15}{8}}\right)t\right]+\frac{5}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\frac{\pi}{2}t\right)+\frac{1}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(\frac{\pi}{2}t\right)\right]$$

Resolviendo las demás transformadas inversas:

$$\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}=\sqrt{\frac{8}{15}}e^{-\frac{5}{4}(t-10)}\,Sen\left[\sqrt{\frac{15}{8}}(t-10)\right]u(t-10)$$

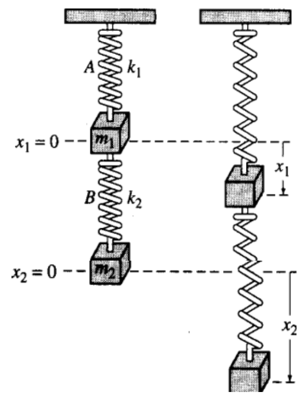
$$\mathcal{L}^{-1}\left\{\frac{e^{-20s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}=\sqrt{\frac{8}{15}}e^{-\frac{5}{4}(t-20)}\,Sen\left[\sqrt{\frac{15}{8}}(t-20)\right]u(t-20)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}=\sqrt{\frac{8}{15}}e^{-\frac{5}{4}t}\,Sen\left(\sqrt{\frac{15}{8}}t\right)$$

Finalmente la ecuación del movimiento es:

$$\mathbf{x(t)=\frac{1}{2}\left(\sqrt{\frac{8}{15}}\right)\left[1+\left(-\frac{5/4}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}\right)^2\right]^{-2}\left[-\frac{5e^{-\frac{5}{4}t}}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\sqrt{\frac{15}{8}}t\right)+\frac{e^{-\frac{5}{4}t}}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(-\sqrt{\frac{15}{8}}t\right)+\frac{5}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\frac{\pi}{2}t\right)+\frac{1}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(\frac{\pi}{2}t\right)\right]}$$
$$\mathbf{\left[-\frac{5e^{-\frac{5}{4}t}}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left[\left(\pi+\sqrt{\frac{15}{8}}\right)t\right]-\frac{e^{-\frac{5}{4}t}}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left[\left(\pi+\sqrt{\frac{15}{8}}\right)t\right]+\frac{5}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\frac{\pi}{2}t\right)+\frac{1}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(\frac{\pi}{2}t\right)\right]}$$
$$\mathbf{+5\sqrt{\frac{15}{8}}e^{-\frac{5}{4}(t-10)}\,Sen\left[\sqrt{\frac{15}{8}}(t-10)\right]u(t-10)+10\sqrt{\frac{15}{8}}e^{-\frac{5}{4}(t-20)}\,Sen\left[\sqrt{\frac{15}{8}}(t-20)\right]u(t-20)+\frac{1}{4}\sqrt{\frac{15}{8}}e^{-\frac{5}{4}t}\,Sen\left(\sqrt{\frac{15}{8}}t\right)}$$

3) Dos masas $m_1 = 1kg$ y $m_2 = 1kg$ estan unidas a dos resortes A y B, de masa insignificante cuyas constantes de resorte son $k_1 = 6\ N/m$ y $k_2 = 4N/m$ respectivamente, y los resortes se fijan como se ve en la figura. Además la masa m_1 parte con una velocidad de 1 m/seg hacia abajo y la masa m_2 parte con una velocidad de 1 m/seg hacia arriba, si no se aplica fuerzas externas al sistema, y en ausencia de fuerza de amortiguamiento encuentre los desplazamientos verticales de las masas respecto a sus posiciones de equilibrio.



Cuando el sistema está en movimiento, el resorte B es sometido a alargamiento y compresión, y en ausencia de fuerzas externas su alargamiento neto es $x_1 - x_2$ entonces según la ley de Hooke vemos que los resortes A y B ejercen fuerzas $-k_1x_1$ y $k_2(x_2 - x_1)$ respectivamente sobre m_1, entonces la fuerza neta sobre m_1 es $-k_1x_1 + k_2(x_2 - x_1)$, entonces según la 2da ley de Newton podemos escribir:

$$m_1\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1)$$

De igual forma, la fuerza neta ejercida sobre la masa m_2 sólo se debe al alargamiento neto de B; esto es, $-k_2(x_2 - x_1)$. En consecuencia:

$$m_2\frac{d^2x_1}{dt^2} = -k_2(x_2 - x_1)$$

En otras palabras, el movimiento del sistema acoplado se representa con el sistema de ecuaciones diferenciales simultáneas de 2do orden:

$$m_1x''_1 = -k_1x_1 + k_2(x_2 - x_1)$$

$$m_2x''_2 = -k_2(x_2 - x_1)$$

Entonces:

$m_1x''_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$

$m_2x''_2 + k_2x_2 - k_2x_1 = 0$

Aplicando transformada de Laplace:

$\mathcal{L}\{x''_1\} + 10\mathcal{L}\{x_1\} - 4\mathcal{L}\{x_2\} = 0$

$\mathcal{L}\{x''_2\} + 4\mathcal{L}\{x_2\} - 4\mathcal{L}\{x_1\} = 0$

Entonces:

$[s^2X_1 - sx_1(0) - x'_1(0)] + 10X_1 - 4X_2 = 0$

$[s^2X_2 - sx_2(0) - x'_2(0)] + 4X_2 - 4X_1 = 0$

Tenemos bien claro que, $x_1(0) = x'_1(0) = 1$ *y* $x_2(0) = x'_2(0) = -1$ *entonces:*

$s^2X_1 - 1 + 10X_1 - 4X_2 = 0 \quad \Rightarrow \quad (s^2 + 10)X_1 - 4X_2 = 1$

$s^2X_2 + 1 + 4X_2 - 4X_1 = 0 \quad \Rightarrow \quad (s^2 + 4)X_2 - 4X_1 = -1$

Despejando X_2 *de ambas ecuaciones e igualando tenemos:*

$$(s^2 + 10)X_1 - 1 = \frac{16X_1 - 4}{(s^2 + 4)}$$

$(s^2 + 10)(s^2 + 4)X_1 - (s^2 + 4) = 16X_1 - 4$

$[(s^2 + 10)(s^2 + 4) - 16]X_1 = s^2 + 4 - 4$

$$X_1 = \frac{s^2}{s^4 + 14s^2 + 24} \quad \Rightarrow \quad X_1 = \frac{s^2}{(s^2 + 2)(s^2 + 12)}$$

Descomponiendo en fracciones parciales:

$$\frac{s^2}{(s^2 + 2)(s^2 + 12)} = \frac{A(2s) + B}{s^2 + 2} + \frac{C(2s) + D}{s^2 + 12}$$

$s^2 = (2As + B)(s^2 + 12) + (2Cs + D)(s^2 + 2)$

$s^2 = 2As^3 + 24As + Bs^2 + 12B + 2Cs^3 + 4Cs + Ds^2 + 2D$

$s^2 = (2A + 2C)s^3 + (B + D)s^2 + (24A + 4C)s + (12B + 2D)$

$0 = 2A + 2C$

$1 = B + D$

$0 = 24A + 4C$

$0 = 12B + 2D$

Resolviendo el sistema A = 0 , B = -1/5 , C = 0 , D = 6/5

$$X_1 = -\frac{1/5}{s^2 + 2} + \frac{6/5}{s^2 + 12}$$

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{X_1\} = -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+12}\right\}$$

$$x_1(t) = -\frac{1}{5\sqrt{2}}Sen\left(\sqrt{2}t\right) + \frac{6}{5\sqrt{12}}Sen\left(\sqrt{12}t\right)$$

Sustituimos X_1 en cualquiera de las dos ecuaciones:

$$X_2 = \frac{(s^2+10)X_1 - 1}{4}$$

$$X_2 = \frac{(s^2+10)}{4}\left[\frac{6}{5(s^2+12)} - \frac{1}{5(s^2+2)}\right] - \frac{1}{4}$$

$$X_2 = \frac{(s^2+10)}{20}\left[\frac{6s^2+12-s^2-12}{(s^2+12)(s^2+2)}\right] - \frac{1}{4}$$

$$X_2 = \frac{s^2(s^2+10)}{4(s^2+12)(s^2+2)} - \frac{1}{4}$$

$$X_2 = \frac{s^4+10s^2-s^4-14s^2-24}{4(s^2+12)(s^2+2)}$$

$$X_2 = \frac{-4s^2-24}{4(s^2+12)(s^2+2)}$$

$$X_2 = -\frac{s^2+6}{(s^2+12)(s^2+2)}$$

Descomponiendo en fracciones parciales:

$$\frac{s^2+6}{(s^2+2)(s^2+12)} = \frac{A(2s)+B}{s^2+2} + \frac{C(2s)+D}{s^2+12}$$

$$s^2+6 = (2As+B)(s^2+12) + (2Cs+D)(s^2+2)$$

$$s^2+6 = 2As^3+24As+Bs^2+12B+2Cs^3+4Cs+Ds^2+2D$$

$$s^2+6 = (2A+2C)s^3+(B+D)s^2+(24A+4C)s+(12B+2D)$$

$$0 = 2A+2C$$

$$1 = B+D$$

$$0 = 24A+4C$$

$$6 = 12B+2D$$

Resolviendo el sistema A = 0 , B = 2/5 , C = 0 , D = 3/5

$$X_2 = -\frac{s^2+6}{(s^2+12)(s^2+2)}$$

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{X_2\} = -\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} - \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+12}\right\}$$

$$x_2(t) = -\frac{2}{5\sqrt{2}}Sen\left(\sqrt{2}t\right) - \frac{3}{5\sqrt{12}}Sen\left(\sqrt{12}t\right)$$

Finalmente las ecuaciones de los desplazamientos verticales para las masas son:

$$\boldsymbol{x_1(t) = -\frac{\sqrt{2}}{10}Sen\left(\sqrt{2}t\right) + \frac{\sqrt{3}}{5}Sen\left(2\sqrt{3}t\right)}$$

$$\boldsymbol{x_2(t) = -\frac{\sqrt{2}}{5}Sen\left(\sqrt{2}t\right) - \frac{\sqrt{3}}{10}Sen\left(2\sqrt{3}t\right)}$$

CIRCUITOS ELÉCTRICOS

1) Determine la corriente i(t) en un circuito RLC serie, cuando L = 1 h , R = 0 Ω , C = 10^{-4}F y el voltaje aplicado es:

$$E(t)\begin{cases} 100\ Sen(10t)\ ; & 0 \le t < \pi \\ 0 \ ; & t \ge \pi \end{cases}$$

La ecuación para este circuito es:

$$L\frac{di}{dt} + iR + \frac{1}{C}\int i\,dt = E(t)$$

Entonces tenemos:

$$\frac{di}{dt} + 10^4\int i\,dt = (u_0 - u_\pi)100\ Sen(10t)$$

$$i' + 10^4\int i\,dt = 100u_0 Sen(10t) - 100u_\pi Sen(10t)$$

Hay que desfasar la función Sen(10t)

$$Sen[10(t+\pi-\pi)] = Sen[10(t+\pi) - 10\pi]$$

$$Sen[10(t+\pi-\pi)] = Sen[10(t+\pi)]Cos(10\pi) + Cos[10(t+\pi)]Sen(10\pi)$$

$$Sen[10(t+\pi-\pi)] = Sen[10(t+\pi)]$$

Entonces:

$$i' + 10^4\int i\,dt = 100u_0 Sen(10t) - 100u_\pi Sen[10(t+\pi)]$$

Aplicando transformada de Laplace:

$$sI - i(0) + 10^4\frac{I}{s} = \frac{1000}{s^2+100} - \frac{1000e^{-\pi s}}{s^2+100}$$

Sabemos que i(0) = 0

$$s^2 I + 10^4 I = \frac{1000s}{s^2+100} - \frac{1000s}{s^2+100}e^{-\pi s}$$

$$I = \frac{1000s}{(s^2+100)(s^2+1000)} - \frac{1000s}{(s^2+100)(s^2+1000)}e^{-\pi s}$$

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{I\} = 10\mathcal{L}^{-1}\left\{\frac{100s}{(s^2+100)(s^2+1000)}\right\} - 10\mathcal{L}^{-1}\left\{\frac{100s}{(s^2+100)(s^2+1000)}e^{-\pi s}\right\}$$

Aplicando convolución:

$$\mathcal{L}^{-1}\left\{\frac{100s}{(s^2+100)(s^2+1000)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+100} * \frac{100}{s^2+1000}\right\}$$

$$\int_0^t Cos(10x)Sen[100(t-x)]dx$$

$$\frac{1}{2}\int_0^t [Sen(10x+100t-100x) - Sen(10x-100t+100x)]dx$$

$$\frac{1}{2}\int_0^t [Sen(-90x+100t) - Sen(110x-100t)]dx$$

$$\frac{1}{2}\left[\frac{1}{90}Cos(-90x+100t) + \frac{1}{110}Cos(110x-100t)\right]_0^t$$

$$\frac{1}{2}\left[\frac{1}{90}Cos(10t) + \frac{1}{110}Cos(10t) - \frac{1}{90}Cos(100t) - \frac{1}{110}Cos(100t)\right]$$

Entonces:

$$i(t) = 10\left(\frac{1}{2}\right)\left[\frac{1}{90}Cos(10t) + \frac{1}{110}Cos(10t) - \frac{1}{90}Cos(100t) - \frac{1}{110}Cos(100t)\right] -$$

$$10\left(\frac{1}{2}\right)u(t-\pi)\left[\frac{1}{90}Cos[10(t-\pi)] + \frac{1}{110}Cos[10(t-\pi)] - \frac{1}{90}Cos[100(t-\pi)] - \frac{1}{110}Cos[100(t-\pi)]\right]$$

Luego:

$$Cos(10t-10\pi) = Cos(10t)Cos(10\pi) + Sen(10t)Sen(10\pi)$$

$$Cos(10t-10\pi) = Cos(10t)$$

Así mismo:

$$Cos(100t-100\pi) = Cos(100t)$$

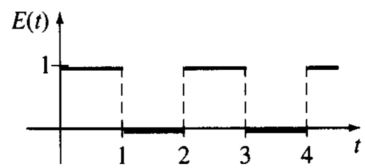
Finalmente:

$$i(t) = \frac{1}{18}Cos(10t) + \frac{1}{22}Cos(10t) - \frac{1}{18}Cos(100t) - \frac{1}{22}Cos(100t)$$

$$-u(t-\pi)\left[\frac{1}{18}Cos(10t) + \frac{1}{22}Cos(10t) - \frac{1}{18}Cos(100t) - \frac{1}{22}Cos(100t)\right]$$

$$\boxed{\boldsymbol{i(t) = \frac{1}{18}[Cos(10t) - Cos(100t)] + \frac{1}{22}[Cos(10t) - Cos(100t)] - u(t-\pi)\left[\frac{1}{18}[Cos(10t) - Cos(100t)]\right]}}$$

2) En un circuito LR, determine la corriente i(t) para cualquier tiempo t, sabemos, que cuando i(0) = 0 y E(t) es la función onda cuadrada que muestra la figura. Luego suponga que L = 1 y R = 1 y determine i(t) para el intervalo 0 ≤ t < 4.



Planteando la ecuación:

$$L\frac{di}{dt}+Ri=E(t)$$

Aplicando la transformada de Laplace:

$$L[sI-i(0)]+RI=\frac{1}{1-e^{-2s}}\left(\int_0^1(1)e^{-st}\,dt+\int_1^2(0)e^{-st}\,dt\right)$$

$$LsI+RI=\frac{1}{1-e^{-2s}}\left(-\frac{1}{s}e^{-st}\Big|_0^1\right)\quad\Rightarrow\quad LsI+RI=\frac{1}{1-e^{-2s}}\left(-\frac{1}{s}e^{-s}+\frac{1}{s}\right)$$

$$LsI+RI=\frac{1}{1-e^{-2s}}\left(\frac{1-e^{-s}}{s}\right)\quad\Rightarrow\quad LsI+RI=\frac{1}{(1-e^{-s})(1+e^{-s})}\left(\frac{1-e^{-s}}{s}\right)$$

$$LsI+RI=\frac{1}{s(1+e^{-s})}$$

$$I(Ls+R)=\frac{1}{s(1+e^{-s})}$$

$$I=\frac{1}{s(1+e^{-s})(Ls+R)}$$

$$I=\frac{{}^{1}/_{L}}{s(1+e^{-s})\left(s+{}^{R}/_{L}\right)}$$

Para determinar la transformada de Laplace de esta función, primero emplearemos una serie geométrica.

$$\frac{1}{1+x}=1-x+x^2-x^3+\ \dots\dots\dots\dots$$

Si $x=e^{-s}$ *, cuando s > 0 tenemos que:*

$$\frac{1}{1+e^{-s}}=1-e^{-s}+e^{-2s}-e^{-3s}+\ \dots\dots\dots\dots$$

Si escribimos:

$$I = \left(\frac{1}{R}\right)\frac{\left(\frac{R}{L}+s\right)-s}{s\left(s+{}^{R}/_{L}\right)}\left[\frac{1}{(1+e^{-s})}\right]$$

$$I = \frac{1}{R(1+e^{-s})}\left[\frac{\left(\frac{R}{L}+s\right)}{s\left(s+{}^{R}/_{L}\right)}-\frac{s}{s\left(s+{}^{R}/_{L}\right)}\right]$$

$$I = \frac{1}{R}\left[\frac{1}{s}-\frac{1}{\left(s+{}^{R}/_{L}\right)}\right](1+e^{-s})$$

Entonces:

$$I = \frac{1}{R}\left[\frac{1}{s}-\frac{1}{\left(s+{}^{R}/_{L}\right)}\right](1-e^{-s}+e^{-2s}-e^{-3s}+\ \dots\dots\dots)$$

$$I = \frac{1}{R}\left(\frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}+\ \dots\dots\right)-\frac{1}{R}\left(\frac{1}{s+{}^{R}/_{L}}-\frac{e^{-s}}{s+{}^{R}/_{L}}+\frac{e^{-2s}}{s+{}^{R}/_{L}}-\frac{e^{-3s}}{s+{}^{R}/_{L}}+\ \dots\dots\right)$$

Al aplicar la forma inversa del segundo teorema de traslación a cada término de ambas series tenemos:

$$i(t) = \frac{1}{R}[1-u(t-1)+u(t-2)-u(t-3)+\ \dots\dots]$$
$$-\frac{1}{R}\left[e^{-\frac{R}{L}t}-e^{-\frac{R}{L}(t-1)}u(t-1)+e^{-\frac{R}{L}(t-2)}u(t-2)-e^{-\frac{R}{L}(t-3)}u(t-3)+\ \dots\dots\right]$$

O, lo que es lo mismo:

$$\boldsymbol{i(t) = \frac{1}{R}\left(1-e^{-\frac{R}{L}t}\right)+\frac{1}{R}\sum_{n=1}^{+\infty}(-1)^n\left(1-e^{-\frac{R}{L}(t-n)}\right)u(t-n)}$$

Finalmente, encontrando i(t) para 0 ≤ t < 4 tenemos:

$$\boldsymbol{i(t) = 1-e^{-t}-\left[1-e^{-(t-1)}\right]u(t-1)+\left[1-e^{-(t-2)}\right]u(t-2)-\left[1-e^{-(t-3)}\right]u(t-3)}$$

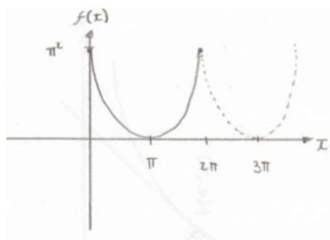
O, lo que es lo mismo:

$$\boldsymbol{i(t) = \begin{cases} 1-e^{-t} & 0 \le t < 1 \\ -e^{t}+e^{-(t-1)} & 1 \le t < 2 \\ 1-e^{-t}+e^{-(t-1)}-e^{-(t-2)} & 2 \le t < 3 \\ e^{-t}+e^{-(t-1)}-e^{-(t-2)}+e^{-(t-3)} & 3 \le t < 4 \end{cases}}$$

SERIES DE FOURIER

1) Obtenga la expansión en serie de Fourier de la función periódica $f(t)$ de periodo 2π definida sobre el periodo $0 \le t \le 2\pi$ por $f(t) = (\pi - t)^2$ y de allí demostrar que $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n Cos\left(\frac{n\pi}{l} t\right) + b_n Sen\left(\frac{n\pi}{l} t\right) \right] \quad ; \quad T = 2l = 2\pi$$



Dado que $f(t)$ es par:

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(t) dt \quad ; \quad a_n = \frac{1}{l} \int_{-l}^{l} f(t) Cos\left(\frac{n\pi}{l} t\right) dt \quad ; \quad b_n = 0$$

Encontrando a_0

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - t)^2 dt = \frac{2}{\pi} \left[-\frac{(\pi - t)^3}{3} \right]_0^{\pi} = -\frac{2}{\pi} \left(-\frac{\pi^3}{3} \right) = \frac{2}{3} \pi^2$$

Encontrando a_n

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi - t)^2 Cos(nt) dt$$

$u = (\pi - t)^2 \qquad du = -2(\pi - t)dt$

$dv = Cos(nt)dt \qquad v = \frac{1}{n} Sen(nt)$

$$a_n = \frac{2}{\pi} \left[\frac{(\pi - t)^2}{n} Sen(nt) - \int -\frac{2}{n} (\pi - t) Sen(nt) dt \right] = \frac{2}{\pi} \left[\frac{(\pi - t)^2}{n} Sen(nt) + \frac{2}{n} \int [\pi Sen(nt) - t\, Sen(nt)]\, dt \right]$$

$$a_n = \frac{2}{\pi} \left[\frac{(\pi - t)^2}{n} Sen(nt) - \frac{2\pi}{n^2} Cos(nt) - \frac{2}{n} \int t\, Sen(nt)\, dt \right]$$

$u = t \qquad du = dt$

$dv = Sen(nt)dt \qquad v = -\frac{1}{n} Cos(nt)$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi-t)^2}{n}Sen(nt) - \frac{2\pi}{n^2}Cos(nt) - \frac{2}{n}\left[-\frac{t}{n}Cos(nt) + \frac{1}{n}\int Cos(nt)dt\right]\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi-t)^2}{n}Sen(nt) - \frac{2\pi}{n^2}Cos(nt) + \frac{2t}{n^2}Cos(nt) - \frac{2}{n^3}Sen(nt)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[-\frac{2\pi}{n^2}Cos(n\pi) + \frac{2\pi}{n^2}Cos(n\pi) + \frac{2\pi}{n^2}\right] = \frac{4}{n^2}$$

Reemplazando:

$$\boldsymbol{f(t) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}Cos\ (nt)}$$

Si $t=\pi$, entonces:

$$f(\pi) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}Cos\ (n\pi) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$0 = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$-\frac{1}{3}\pi^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

Multiplicando por -1, finalmente tenemos:

$$\boldsymbol{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$$

2) Con respecto a la función $f(t)=\begin{cases}\pi^2, & -\pi<t<0\\ (t-\pi)^2, & 0<t<\pi\end{cases}, \quad f(t+2\pi)=f(t)$**:**

a) Pruebe que la serie de Fourier que representa la función periódica $f(t)$ **es:**

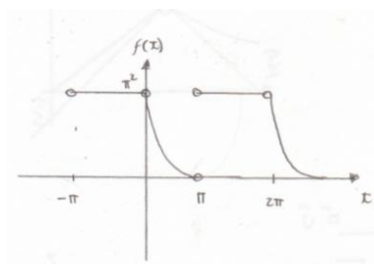
$$f(t)=\frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}\cos(nt)+\frac{(-1)^n}{n}\pi sen(nt)\right]-\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{sen((2n-1)t)}{(2n-1)^3}\right]$$

b) Utilice este resultado para probar que:

i) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

Para a)

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n Cos\left(\frac{n\pi}{l}t\right)+b_n Sen\left(\frac{n\pi}{l}t\right)\right] \quad ; \quad T=2l=2\pi$$



Dado que $f(t)$ no es impar y tampoco par:

$$a_0=\frac{1}{l}\int_{-l}^{l}f(t)dt \quad ; \quad a_n=\frac{1}{l}\int_{-l}^{l}f(t)Cos\left(\frac{n\pi}{l}t\right)dt \quad ; \quad b_n=\frac{1}{l}\int_{-l}^{l}f(t)Sen\left(\frac{n\pi}{l}t\right)dt$$

Encontrando a_0

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt=\frac{1}{\pi}\left[\int_{-\pi}^{0}\pi^2dt+\int_{0}^{\pi}(t-\pi)^2dt\right]=\frac{1}{\pi}\left[[\pi^2t]_{-\pi}^{0}+\left[\frac{(t-\pi)^3}{3}\right]_0^{\pi}\right]$$

$$a_0=\frac{1}{\pi}\left[\pi^3+\frac{\pi^3}{3}\right]=\frac{4\pi^2}{3}$$

Encontrando a_n

$$a_n = \frac{1}{\pi}\int_{-l}^{l} f(t)Cos\left(\frac{n\pi}{l}t\right)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}\pi^2 Cos(nt)dt + \int_{0}^{\pi}(t-\pi)^2 Cos(nt)dt\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\pi^2 Cos(nt)dt + \int_{0}^{\pi}(t-\pi)^2 Cos(nt)dt\right] = \frac{1}{\pi}\left[\frac{\pi^2}{n}[Sen(nt)]_{-\pi}^{0}\frac{0}{-\pi} + \underbrace{\int_{0}^{\pi}(t-\pi)^2 Cos(nt)dt}_{\text{Resuelta en el ejercicio anterior}}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{\pi^2}{n}Sen(n\pi) + \frac{2\pi}{n^2}\right] = \frac{2}{n^2}$$

Encontrando b_n

$$b_n = \frac{1}{\pi}\int_{-l}^{l} f(t)Sen\left(\frac{n\pi}{l}t\right)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}\pi^2 Sen(nt)dt + \int_{0}^{\pi}(t-\pi)^2 Sen(nt)dt\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi^2}{n}[Cos(nt)]_{-\pi}^{0} + \int_{0}^{\pi}(t-\pi)^2 Sen(nt)dt\right]$$

$$\int (t-\pi)^2 Sen(nt)dt$$

$$u = (t-\pi)^2 \qquad du = 2(t-\pi)dt$$

$$dv = Sen(nt)dt \qquad v = -\frac{1}{n}Cos(nt)$$

$$= -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2}{n}\int (t-\pi)Cos(nt)dt = -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2}{n}\left[\int t\,Cos(nt)dt - \pi\int Cos(nt)dt\right]$$

$$= -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2}{n}\int t\,Cos(nt)dt - \frac{2\pi}{n^2}Sen(nt)$$

$$u = t \qquad du = dt$$

$$dv = Cos(nt)dt \qquad v = \frac{1}{n}Sen(nt)$$

$$= -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2}{n}\left[\frac{t}{n}Sen(nt) - \frac{1}{n}\int Sen(nt)dt\right] - \frac{2\pi}{n^2}Sen(nt)$$

$$= -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2t}{n^2}Sen(nt) + \frac{2}{n^3}Cos(nt) - \frac{2\pi}{n^2}Sen(nt)$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi^2}{n}[Cos(nt)]_{-\pi}^{0} + \left[-\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2t}{n^2}Sen(nt) + \frac{2}{n^3}Cos(nt) - \frac{2\pi}{n^2}Sen(nt)\right]_{0}^{\pi}\right]$$

$$b_n = \frac{1}{\pi}\left[\left[-\frac{\pi^2}{n}+\frac{\pi^2}{n}Cos(n\pi)\right]+\left[\frac{2}{n^3}Cos(n\pi)+\frac{\pi^2}{n}-\frac{2}{n^3}\right]\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi^2}{n}+\frac{\pi^2}{n}Cos(n\pi)+\frac{2}{n^3}Cos(n\pi)+\frac{\pi^2}{n}+\frac{2}{n^3}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{n}Cos(n\pi)+\frac{2}{n^3}Cos(n\pi)-\frac{2}{n^3}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi^2}{n}(-1)^n+\frac{2}{n^3}(-1)^n-\frac{2}{n^3}\right]$$

Reemplazando:

$$f(t) = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt)+\left[\frac{\pi}{n}(-1)^n+\frac{2}{\pi n^3}(-1)^n-\frac{2}{\pi n^3}\right]Sen\,(nt)\right]$$

$$f(t) = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt)+\frac{(-1)^n}{n}\pi\,Sen\,(nt)\right]+\sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{\pi n^3}-\frac{2}{\pi n^3}\right]Sen\,(nt)$$

Hacemos un cambio de variable $\boldsymbol{n = 2m-1}$, *entonces*:

Cuando $n = 1 \;\Rightarrow\; m = 2(1)-1 = 1$ *y cuando* $n = \infty \;\Rightarrow\; m = 2(\infty)-1 = \infty$

$$f(t) = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt)+\frac{(-1)^n}{n}\pi\,Sen\,(nt)\right]+\sum_{m=1}^{\infty}\left[\frac{2(-1)^{\overbrace{2m-1}^{Siempre\ \ es\ impar}}}{\pi(2m-1)^3}-\frac{2}{\pi(2m-1)^3}\right]Sen\,[(2m-1)t]$$

$$f(t) = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt)+\frac{(-1)^n}{n}\pi\,Sen\,(nt)\right]+\sum_{m=1}^{\infty}\left[-\frac{2}{\pi(2m-1)^3}-\frac{2}{\pi(2m-1)^3}\right]Sen\,[(2m-1)t]$$

$$f(t) = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt)+\frac{(-1)^n}{n}\pi\,Sen\,(nt)\right]+\sum_{m=1}^{\infty}\left[-\frac{4}{\pi(2m-1)^3}\right]Sen\,[(2m-1)t]$$

Finalmente tenemos:

$$\boxed{\boldsymbol{f(t) = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt)+\frac{(-1)^n}{n}\pi\,Sen\,(nt)\right]-\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{Sen\,\left((2n-1)t\right)}{(2n-1)^3}\right]}}$$

<u>Para b)</u>

Si t = π

$$f(\pi) = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(n\pi)+\frac{(-1)^n}{n}\pi\,Sen\,(n\pi)\right]-\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{Sen\,\left((2n-1)\pi\right)}{(2n-1)^3}\right]$$

Generando términos:

$$f(\pi)=\frac{2\pi^2}{3}+\left[[2Cos\ (\pi)-\pi\ Sen\ (\pi)]+\left[\frac{2}{2^2}Cos\ (2\pi)+\frac{1}{2}\pi\ Sen\ (2\pi)\right]+\left[\frac{2}{3^2}Cos\ (3\pi)+\frac{1}{3}\pi\ Sen\ (3\pi)\right]+\ \dots\dots\right]$$

$$-\frac{4}{\pi}\left[[Sen\ (\pi)]+\left[\frac{Sen\ (3\pi)}{(3)^3}\right]+\left[\frac{Sen\ (5\pi)}{(5)^3}\right]+\ \dots\dots\right]$$

$$f(\pi)=\frac{2\pi^2}{3}+2\left(-1+\frac{1}{2^2}-\frac{1}{3^2}+\ \dots\dots\right)\ ;\ como\ f(\pi) es\ un\ punto\ de\ descontinuidad, entonces\ f(\pi)=\frac{1}{2}[f(t^+)+f(t^-)]$$

$$\frac{\pi^2}{2}=\frac{2\pi^2}{3}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\quad\Rightarrow\quad-\frac{\pi^2}{6}=2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$Multiplicando\ por-1\ tenemos:$

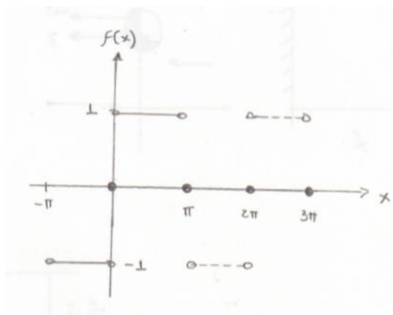
$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}}$$

3) Determine la representación en serie de Fourier de la función periódica f(t) de periodo 2π ***definida por la siguiente regla de correspondencia:*** $\boldsymbol{f(t)=\begin{cases}-1 & ;\ -\pi<x<0\\ 1 & ;\ \ 0<x<\pi\\ 0 & ;\ x=0\ ;\ x=\pi\end{cases}}$

Determine a que converge la serie:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}$$

Haciendo un bosquejo de la gráfica:



Podemos que f(t) es impar, entonces:

$$b_n=\frac{1}{l}\int_{-l}^{l}f(t)Sen\left(\frac{n\pi}{l}t\right)dt$$

<u>*Encontrando* b_n</u>

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}Sen(nt)f(t)dt=\frac{2}{\pi}\int_{0}^{\pi}Sen(nt)f(t)dt$$

$$b_n=\frac{2}{\pi}\int_{0}^{\pi}Sen(nt)dt$$

$$b_n=\frac{2}{\pi}\left[-\frac{1}{n}Cos(nt)\right]_0^{\pi}$$

$$b_n=\frac{2}{\pi}\left[\frac{1}{n}-\frac{1}{n}Cos(n\pi)\right]$$

Reemplazando:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_nCos(nt)+b_nSen(nt)]$$

$$f(t)=\sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{1}{n}-\frac{1}{n}Cos(n\pi)\right]Sen(nt)$$

Generando términos:

$$f(t) = \frac{2}{\pi}\left[[1 - Cos(\pi)]Sen(t) + \left[\frac{1}{2} - \frac{1}{2}Cos(2\pi)\right]Sen(2t) + \left[\frac{1}{3} - \frac{1}{3}Cos(3\pi)\right]Sen(3t) + \ \dots\dots\dots.\right]$$

$$f(t) = \frac{2}{\pi}\left[[1 + 1]Sen(t) + \left[\frac{1}{2} - \frac{1}{2}\right]Sen(2t) + \left[\frac{1}{3} + \frac{1}{3}\right]Sen(3t) + \ \dots\dots\dots.\right]$$

$$f(t) = \frac{2}{\pi}\left[2\ Sen(t) + \frac{2}{3}Sen(3t) + \frac{2}{5}Sen(5t) + \ \dots\dots\dots\right]$$

$$f(t) = \frac{4}{\pi}\left[\ Sen(t) + \frac{1}{3}Sen(3t) + \frac{1}{5}Sen(5t) + \ \dots\dots\dots\right]$$

Expresando en notación de sumatoria, por lo tanto la representación en serie de Fourier es:

$$\boxed{\boldsymbol{f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}Sen[(2n-1)t]}}$$

Encontrando la convergencia:

Si t = π/2

$$f(\pi/2) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}Sen\left[(2n-1)\frac{\pi}{2}\right]$$

Generando términos:

$$f(\pi/2) = \frac{4}{\pi}\left[Sen\left(\frac{\pi}{2}\right) + \frac{1}{3}Sen\left(\frac{3\pi}{2}\right) + \frac{1}{5}Sen\left(\frac{5\pi}{2}\right) + \ \dots\dots\dots\right]$$

$$f(\pi/2) = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ \dots\dots\dots\right]$$

Si notamos en el gráfico de f(π/2) = 1 , entonces:

$$1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ \dots\dots\dots\right]$$

Expresando en notación de sumatoria:

$$1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}$$

Entonces:

$$\boxed{\boldsymbol{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)} = \frac{\pi}{4}}}$$

EXTENSIONES PARES E IMPARES PERIÓDICAS DE UNA SERIE DE FOURIER

1) Con respecto a la función f, definida por: $\boldsymbol{f(x) = \begin{cases} 0.1x & ;\ 0 \leq x < 10 \\ 0.1(10-x); & 10 \leq x < 20 \end{cases}}$

determine:

a) La correspondiente serie impar de medio rango.

b) La suma de la serie numérica:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Para a)

$a_0 = a_n = 0$

Entonces:

$$b_n = \frac{1}{20}\int_{-20}^{20} f(t) Sen\left(\frac{n\pi}{20}t\right) dt = \frac{1}{10}\int_{0}^{20} f(t) Sen\left(\frac{n\pi}{20}t\right) dt$$

$$b_n = \frac{1}{10}\left[\int_{0}^{10} 0.1t\, Sen\left(\frac{n\pi}{20}t\right) dt + \int_{10}^{20} 0.1(10-t)\, Sen\left(\frac{n\pi}{20}t\right) dt\right]$$

Integrando en forma general, tenemos:

$$\int (at+b)\, Sen\left(\frac{n\pi}{20}t\right) dt$$

Por partes:

$$u = at + b \quad \Rightarrow \quad du = a\, dt$$

$$dv = Sen\left(\frac{n\pi}{20}t\right) dt \quad \Rightarrow \quad v = -\frac{20}{n\pi} Cos\left(\frac{n\pi}{20}t\right)$$

$$\int (at+b)\, Sen\left(\frac{n\pi}{20}t\right) dt = -\frac{20(at+b)}{n\pi} Cos\left(\frac{n\pi}{20}t\right) + \frac{20a}{n\pi}\int Cos\left(\frac{n\pi}{20}t\right) dt$$

$$\int (at+b)\, Sen\left(\frac{n\pi}{20}t\right) dt = -\frac{20}{n\pi}(at+b) Cos\left(\frac{n\pi}{20}t\right) + \left(\frac{20}{n\pi}\right)^2 a\, Sen\left(\frac{n\pi}{20}t\right)$$

Reemplazando:

$$\int_0^{10} 0.1t\,Sen\left(\frac{n\pi}{20}t\right)dt = \left[-\frac{20}{n\pi}(0.1t)Cos\left(\frac{n\pi}{20}t\right) + \left(\frac{20}{n\pi}\right)^2(0.1)Sen\left(\frac{n\pi}{20}t\right)\right]_0^{10}$$

$$\boldsymbol{\int_0^{10} 0.1t\,Sen\left(\frac{n\pi}{20}t\right)dt = \frac{1}{10}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]}$$

$$\int_{10}^{20} (1-0.1t)\,Sen\left(\frac{n\pi}{20}t\right)dt = \left[-\frac{20}{n\pi}(1-0.1t)Cos\left(\frac{n\pi}{20}t\right) - \left(\frac{20}{n\pi}\right)^2(0.1)Sen\left(\frac{n\pi}{20}t\right)\right]_{10}^{20}$$

$$\int_{10}^{20} (1-0.1t)\,Sen\left(\frac{n\pi}{20}t\right)dt = \frac{1}{10}\left[-\frac{20}{n\pi}(10-t)Cos\left(\frac{n\pi}{20}t\right) - \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{20}t\right)\right]_{10}^{20}$$

$$\boldsymbol{\int_{10}^{20} (1-0.1t)\,Sen\left(\frac{n\pi}{20}t\right)dt = \frac{1}{10}\left[\frac{200}{n\pi}Cos(n\pi) - \left(\frac{20}{n\pi}\right)^2 Sen(n\pi) + \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]}$$

Entonces b_n:

$$b_n = \frac{1}{100}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right) + \frac{200}{n\pi}Cos(n\pi) - \left(\frac{20}{n\pi}\right)^2 Sen(n\pi) + \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]$$

Podemos notar que $Sen(n\pi) = 0$, entonces:

$$b_n = \frac{1}{100}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right) + \frac{200}{n\pi}Cos(n\pi) + \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]$$

Luego:

$$f(t) = \sum_{n=1}^{\infty} b_n Sen\left(\frac{n\pi}{l}t\right)$$

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{100}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right) + \frac{200}{n\pi}Cos(n\pi) + \left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] Sen\left(\frac{n\pi}{20}t\right)$$

Si n es par, entonces $Sen\left(\frac{n\pi}{2}\right) = 0$ y $Cos(n\pi) = 1$ además cuando n es impar $Cos\left(\frac{n\pi}{2}\right) = 0$ y $Cos(n\pi) = -1$

Por lo tanto:

$$b_n = \begin{cases} \frac{1}{100}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \frac{200}{n\pi}\right] & ;\ n\ es\ par \\ \frac{1}{100}\left[-\frac{200}{n\pi} + 2\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] & ;n\ es\ impar \end{cases}$$

Luego expansión impar de medio rango de f(t) es:

$$\boldsymbol{f(t) = \frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{2n\pi}Cos\left(\frac{2n\pi}{2}\right)+\frac{200}{2n\pi}\right]Sen\left(\frac{2n\pi}{20}t\right)+\frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{(2n-1)\pi}+2\left[\frac{20}{(2n-1)\pi}\right]^2 Sen\left[\frac{(2n-1)\pi}{2}\right]\right]Sen\left[\frac{(2n-1)\pi}{20}t\right]}$$

Para b)

Si t = 10

$$f(10) = \frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{2n\pi}Cos\left(\frac{2n\pi}{2}\right)+\frac{200}{2n\pi}\right]Sen(n\pi)+\frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{(2n-1)\pi}+2\left[\frac{20}{(2n-1)\pi}\right]^2 Sen\left[\frac{(2n-1)\pi}{2}\right]\right]Sen\left[\frac{(2n-1)\pi}{2}\right]$$

Sabemos que $Sen(n\pi) = 0$ *, entonces:*

$$\frac{1}{2} = \frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{(2n-1)\pi}+2\left[\frac{20}{(2n-1)\pi}\right]^2 Sen\left[\frac{(2n-1)\pi}{2}\right]\right]Sen\left[\frac{(2n-1)\pi}{2}\right]$$

$$\frac{1}{2} = \frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{(2n-1)\pi}+2\left[\frac{20}{(2n-1)\pi}\right]^2 (-1)^{n+1}\right](-1)^{n+1}$$

$$\frac{1}{2} = \frac{1}{100}\left[\frac{200}{\pi}\sum_{n=1}^{\infty}\left[-\frac{(-1)^{n+1}}{(2n-1)}\right]+\sum_{n=1}^{\infty}2\left[\frac{20}{(2n-1)\pi}\right]^2(-1)^{2(n+1)}\right]$$

Como $2(n+1)$ *siempre es par, entonces:*

$$\frac{1}{2} = \frac{1}{100}\left[\frac{200}{\pi}\sum_{n=1}^{\infty}\left[-\frac{(-1)^{n+1}}{(2n-1)}\right]+\sum_{n=1}^{\infty}2\frac{400}{(2n-1)^2\pi^2}\right]$$

$$50 = -\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}+\frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

En el ejercicio anterior demostramos que:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)} = \frac{\pi}{4}$$

Entonces:

$$50 = -\frac{200}{\pi}\left(\frac{\pi}{4}\right)+\frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$100 = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Finalmente:

$$\boldsymbol{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

2) Con respecto a la función f definida por $f(x) = x(\pi - x), x \epsilon (0, \pi)$

a) Establezca la correspondiente expansión impar de medio rango de la función f

b) Determine la suma de la serie numérica:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

Para a)

$a_0 = a_n = 0$

Entonces:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\pi - t)Sen(nt)dt = \frac{2}{\pi}\int_{0}^{\pi} x(\pi - x)Sen(nt)dt$$

$$b_n = \frac{2}{\pi}\left[\pi\int_{0}^{\pi} t\, Sen(nt)dt - \int_{0}^{\pi} t^2\, Sen(nt)dt\right]$$

Integrando por partes:

$u = t \quad \Rightarrow \quad du = dt \qquad ; \qquad w = t^2 \quad \Rightarrow \quad dw = 2t\, dt$

$dv = Sen(nt)dt \quad \Rightarrow \quad v = -\frac{1}{n}Cos(nt)$

Entonces:

La primera integral nos queda:

$$\int t\, Sen(nt)dt = -\frac{t}{n}Cos(nt) + \frac{1}{n}\int Cos(nt)dt$$

$$\int t\, Sen(nt)dt = -\frac{t}{n}Cos(nt) + \frac{1}{n^2}Sen(nt)$$

La segunda integral nos queda:

$$\int t^2\, Sen(nt)dt = -\frac{t^2}{n}Cos(nt) + \frac{2}{n}\int t\, Cos(nt)dt$$

$$\int t^2\, Sen(nt)dt = -\frac{t^2}{n}Cos(nt) + \frac{2}{n}\left[\frac{t}{n}Sen(nt) - \frac{1}{n}\int Sen(nt)dt\right]$$

$$\int t^2\, Sen(nt)dt = -\frac{t^2}{n}Cos(nt) + \frac{2t}{n^2}Sen(nt) + \frac{2}{n^3}Cos(nt)$$

Reemplazando:

$$b_n = \frac{2}{\pi}\left[\pi\left[-\frac{t}{n}Cos(nt) + \frac{1}{n^2}Sen(nt)\right]_0^\pi - \left[-\frac{t^2}{n}Cos(nt) + \frac{2t}{n^2}Sen(nt) + \frac{2}{n^3}Cos(nt)\right]_0^\pi\right]$$

$$b_n = \frac{2}{\pi}\left[\pi\left[-\frac{\pi}{n}Cos(n\pi)\right] - \left[-\frac{\pi^2}{n}Cos(n\pi) + \frac{2}{n^3}Cos(n\pi) - \frac{2}{n^3}Cos(0)\right]\right]$$

$$b_n = \frac{2}{\pi}\left[-\frac{\pi^2}{n}Cos(n\pi) + \frac{\pi^2}{n}Cos(n\pi) - \frac{2}{n^3}Cos(n\pi) + \frac{2}{n^3}\right]$$

$$b_n = \frac{4}{\pi}\left[\frac{1}{n^3} - \frac{1}{n^3}Cos(n\pi)\right]$$

Entonces:

$$f(t) = \sum_{n=1}^{\infty} b_n Sen\left(\frac{n\pi}{l}t\right)$$

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^3}[1 - Cos(n\pi)]\right]Sen\,(nt)$$

Generando términos:

$$f(t) = \frac{4}{\pi}\left[[1 - Cos(\pi)]Sen(t) + \frac{1}{2^3}[1 - Cos(2\pi)]Sen(2t) + \frac{1}{3^3}[1 - Cos(3\pi)]Sen(3t) + \ \dots\dots\dots\right]$$

$$f(t) = \frac{4}{\pi}\left[2\,Sen(t) + \frac{2}{3^3}Sen(3t) + \frac{2}{5^3}Sen(5t) + \ \dots\dots\dots\dots\right]$$

Finalmente la representación la expansión impar de medio rango de f, nos queda:

$$\boxed{\boldsymbol{f(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}Sen[(2n-1)t]}}$$

<u>Para b)</u>

Si t = π/2

$$f(\pi/2) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}Sen\left[(2n-1)\frac{\pi}{2}\right]$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{(2n-1)^3}$$

Finalmente:

$$\boxed{\boldsymbol{\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}}}$$

3) A la función f(x) = Sen(x) , 0 < x < π expresarla mediante un desarrollo de series de cosenos, y utilizando la serie obtenida y aplicando el teorema de convergencia de las series de Fourier determine la suma de la serie numérica:

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}$$

Para a)

$b_n = 0$

Entonces:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} Sen(x)dx = \frac{2}{\pi}\int_{0}^{\pi} Sen(x)dx$$

$$a_0 = -\frac{2}{\pi}Cos(x)\Big|_0^{\pi} = -\frac{2}{\pi}(-1-1) \quad \Rightarrow \quad \boldsymbol{a_0 = \frac{4}{\pi}}$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} Sen(x)\,Cos(nx)dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi}\frac{1}{2}[Sen(x+nx)+Sen(x-nx)]dx$$

$$a_n = \frac{1}{\pi}\left[\int_{0}^{\pi} Sen[x(1+n)]dx + \int_{0}^{\pi} Sen[x(1-n)]dx\right]$$

$$a_n = \frac{1}{\pi}\left[-\frac{1}{1+n}Cos[x(1+n)] - \frac{1}{1-n}Cos[x(1-n)]\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[-\frac{1}{1+n}Cos[x(1+n)] - \frac{1}{1-n}Cos[-x(n-1)]\right]_0^{\pi}$$

Sabemos que:

$$Cos[-x(n-1)] = Cos[x(n-1)]$$

Entonces:

$$a_n = \frac{1}{\pi}\left[-\frac{1}{n+1}Cos[x(1+n)] - \frac{1}{1-n}Cos[x(n-1)]\right]_0^{\pi}$$

$$a_n = -\frac{1}{\pi}\left[\frac{1}{n+1}Cos[x(1+n)]\frac{1}{1-n}Cos[x(n-1)]\right]_0^{\pi}$$

$$a_n = -\frac{1}{\pi}\left[\frac{1}{n+1}Cos[x(1+n)] - \frac{1}{n-1}Cos[x(n-1)]\right]_0^{\pi}$$

Evaluando:

$$a_n = -\frac{1}{\pi}\left[\frac{1}{n+1}Cos[\pi(1+n)] - \frac{1}{n-1}Cos[\pi(n-1)] - \frac{1}{n+1} + \frac{1}{n-1}\right]$$

Si resolvemos:

$$Cos(\pi + \pi n) = Cos(\pi)Cos(\pi n) - Sen(\pi)Sen(\pi n)$$

$$Cos(\pi + \pi n) = -Cos(\pi n)$$

Entonces:

$$a_n = -\frac{1}{\pi}\left[-\frac{1}{n+1}Cos(\pi n) + \frac{1}{n-1}Cos(\pi n) + \frac{2}{n^2-1}\right]$$

$$a_n = -\frac{1}{\pi}\left[-\frac{1}{n+1}Cos(\pi n) + \frac{1}{n-1}Cos(\pi n) + \frac{2}{n^2-1}\right]$$

$$a_n = -\frac{1}{\pi}\left[Cos(\pi n)\left[\frac{1}{n-1} - \frac{1}{n+1}\right] + \frac{2}{n^2-1}\right]$$

$$a_n = -\frac{1}{\pi}\left[Cos(\pi n)\left[\frac{n+1-n+1}{n^2-1}\right] + \frac{2}{n^2-1}\right]$$

$$a_n = -\frac{1}{\pi}\left[Cos(\pi n)\left[\frac{2}{n^2-1}\right] + \frac{2}{n^2-1}\right]$$

$$\boldsymbol{a_n = -\frac{1}{\pi}\left[\frac{2}{n^2-1}\right][Cos(\pi n) + 1]}$$

Reemplazando:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, Cos\left(\frac{n\pi}{l}x\right)$$

$$f(x) = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{1}{\pi}\left[\frac{2}{n^2-1}\right][Cos(\pi n) + 1]\, Cos(nx)$$

Cuando n es par:

$$Cos(\pi n) + 1 = 2$$

Cuando n es impar:

$$Cos(\pi n) + 1 = 0$$

Entonces solamente generemos términos pares:

$$f(x) = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{1}{\pi}\left[\frac{2}{(2n)^2 - 1}\right] 2\, Cos(2nx)$$

Finalmente la expansión par de f es:

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}\; Cos(2nx)$$

Encontrando la convergencia:

Si t = π

$$f(\pi) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}\; Cos(2\pi n)$$

Si generamos términos:

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\left[Cos(2\pi) + \frac{1}{15}\; Cos(4\pi) + \frac{1}{35}\; Cos(6\pi) + \;\ldots\ldots\ldots\right]$$

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\left[1 + \frac{1}{15} + \frac{1}{35} + \;\ldots\ldots\ldots\right]$$

Podemos notar que $Cos(2\pi n) = 1$

Entonces:

$$f(2\pi) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}$$

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}$$

$$-\frac{2}{\pi} = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}$$

Finalmente tenemos que:

$$\boxed{\sum_{n=1}^{\infty} \frac{\mathbf{1}}{(\mathbf{2n})^2 - \mathbf{1}} = \frac{\mathbf{1}}{\mathbf{2}}}$$

ECUACIONES EN DERIVADAS PARCIALES

ECUACIÓN DE CALOR

La ecuación de calor es:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Resolución de ecuación de calor

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}\, Sen\left(\frac{n\pi}{l}x\right)$$

Donde f(x) es la temperatura en cualquier punto de la varilla:

$$f(x) = \sum_{n=1}^{\infty} C_n\, Sen\left(\frac{n\pi}{l}x\right)$$

Y C_n se lo calcula de la siguiente forma:

$$C_n = \frac{2}{l}\int_0^l f(x) Sen\left(\frac{n\pi}{l}x\right) dx$$

1) Determine la solución de la siguiente ecuación con derivadas parciales

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; ; \;\; 0 < x < 100 \;\; ; \;\; t > 0 \;\; ; u(0,t) = u(100,t) = 0 \;\; ; \;\; u(x,0) = x(100 - x)$$

Determinando C_n

$$C_n = \frac{2}{100}\int_0^{100} x(100 - x) Sen\left(\frac{n\pi}{100}x\right) dx \quad \Rightarrow \quad C_n = \frac{2}{100}\left[\int_0^{100} (100x - x^2) Sen\left(\frac{n\pi}{100}x\right) dx\right]$$

Integrando por partes:

$$u = 100x - x^2 \quad \Rightarrow \quad du = (100 - 2x)dx \qquad ; \qquad dv = Sen\left(\frac{n\pi}{100}x\right) dx \quad \Rightarrow \quad v = -\frac{100}{n\pi} Cos\left(\frac{n\pi}{100}x\right)$$

Entonces:

$$\int (100x - x^2) Sen\left(\frac{n\pi}{100}x\right) dx = -\frac{100}{n\pi}(100x - x^2) Cos\left(\frac{n\pi}{100}x\right) + \frac{100}{n\pi}\int (100 - 2x) Cos\left(\frac{n\pi}{100}x\right) dx$$

Integrando nuevamente por partes:

$$u = 100 - 2x \quad \Rightarrow \quad du = -2\, dx$$

$$dv = Cos\left(\frac{n\pi}{100}x\right) dx \quad \Rightarrow \quad v = \frac{100}{n\pi} Sen\left(\frac{n\pi}{100}x\right)$$

$$\int (100x - x^2)Sen\left(\frac{n\pi}{100}x\right)dx = -\frac{100}{n\pi}(100x - x^2)Cos\left(\frac{n\pi}{100}x\right) + \frac{100}{n\pi}\left[\frac{100}{n\pi}(100 - 2x)Sen\left(\frac{n\pi}{100}x\right) + \frac{200}{n\pi}\int Sen\left(\frac{n\pi}{100}x\right)dx\right]$$

$$\int (100x - x^2)Sen\left(\frac{n\pi}{100}x\right)dx = -\frac{100}{n\pi}(100x - x^2)Cos\left(\frac{n\pi}{100}x\right) + \frac{100}{n\pi}\left[\frac{100}{n\pi}(100 - 2x)Sen\left(\frac{n\pi}{100}x\right) - 2\left(\frac{100}{n\pi}\right)^2 Cos\left(\frac{n\pi}{100}x\right)\right]$$

$$\int (100x - x^2)Sen\left(\frac{n\pi}{100}x\right)dx = -\frac{100}{n\pi}(100x - x^2)Cos\left(\frac{n\pi}{100}x\right) + \left(\frac{100}{n\pi}\right)^2(100 - 2x)Sen\left(\frac{n\pi}{100}x\right) - 2\left(\frac{100}{n\pi}\right)^3 Cos\left(\frac{n\pi}{100}x\right)$$

Evaluando:

$$\left[-\frac{100}{n\pi}x(100 - x)Cos\left(\frac{n\pi}{100}x\right) + \left(\frac{100}{n\pi}\right)^2(100 - 2x)Sen\left(\frac{n\pi}{100}x\right) - 2\left(\frac{100}{n\pi}\right)^3 Cos\left(\frac{n\pi}{100}x\right)\right]_0^{100}$$

$$-2\left(\frac{100}{n\pi}\right)^3 Cos(n\pi) + 2\left(\frac{100}{n\pi}\right)^3$$

Ahora:

Cuando n es par tenemos que:

$$-2\left(\frac{100}{2n\pi}\right)^3 Cos(2n\pi) + 2\left(\frac{100}{2n\pi}\right)^3$$

$$-2\left(\frac{100}{2n\pi}\right)^3 + 2\left(\frac{100}{2n\pi}\right)^3 = 0$$

Cuando n es impar:

$$-2\left[\frac{100}{(2n-1)\pi}\right]^3 Cos[(2n-1)\pi] + 2\left[\frac{100}{(2n-1)\pi}\right]^3$$

$$2\left[\frac{100}{(2n-1)\pi}\right]^3 + 2\left[\frac{100}{(2n-1)\pi}\right]^3 = 4\left[\frac{100}{(2n-1)\pi}\right]^3$$

Ahora:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{100}\right)^2 t} Sen\left(\frac{n\pi}{100}x\right)$$

Sabemos que los términos pares es igual a cero, por lo tanto solo generemos los términos impares:

$$u(x,t) = \frac{4}{50}\sum_{n=1}^{\infty} C_{(2n-1)} e^{-\left[\frac{(2n-1)\pi}{100}\right]^2 t} Sen\left[\frac{(2n-1)\pi}{100}x\right]$$

Finalmente la ecuación del calor es:

$$\boxed{\boldsymbol{u(x,t) = \frac{4}{50}\sum_{n=1}^{\infty}\left[\frac{100}{(2n-1)\pi}\right]^3 e^{-\left[\frac{(2n-1)\pi}{100}\right]^2 t} Sen\left[\frac{(2n-1)\pi}{100}x\right]}}$$

2) Determine la solución de la siguiente ecuación con derivadas parciales

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; ; \;\; 0 < x < \pi \;\; ; \;\; t > 0$$

Sujeta a las siguientes condiciones:

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = 4\,Sen(4x)Cos(2x)$$

Determinando C_n

$$C_n = \frac{2}{\pi}\int_0^{\pi} 4\,Sen(4x)Cos(2x)\,Sen\,(n\pi)dx$$

Resolver aquella integral resulta complicado, pero vamos a realizar un artificio

Sabemos que:

$$Sen(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad ; \qquad Cos(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

Por lo tanto:

$$4\,Sen(4x)Cos(2x) = 4\left(\frac{e^{i4x} - e^{-i4x}}{2i}\right)\left(\frac{e^{i2x} - e^{-i2x}}{2}\right)$$

$$4\,Sen(4x)Cos(2x) = 2\left(\frac{e^{i6x} + e^{i2x} - e^{-i2x} - e^{-i6x}}{2i}\right)$$

$$4\,Sen(4x)Cos(2x) = 2\left(\frac{e^{i6x} - e^{-i6x}}{2i} + \frac{e^{i2x} - e^{-i2x}}{2i}\right)$$

$$4\,Sen(4x)Cos(2x) = 2\,Sen(6x) + 2\,Sen(2x)$$

Además sabemos que:

$$f(x) = \sum_{n=1}^{\infty} C_n\,Sen(nx)$$

$$2\,Sen(6x) + 2\,Sen(2x) = C_1 Sen(x) + C_2 Sen(2x) + C_3 Sen(3x) + C_4 Sen(4x) + C_5 Sen(5x) + C_6 Sen(6x) + \;\ldots\ldots\ldots$$

Entonces:

$$C_2 = 2 \quad ; \quad C_6 = 2 \quad ; \quad C_n = 0 \quad , \quad \forall n \in N - \{2{,}6\}$$

Finalmente la ecuación de calor es:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(n)^2 t}\,Sen(nx)$$

$$\boxed{u(x,t) = 2e^{-4t}Sen(2x) + 2e^{-36t}Sen(6x)}$$

3) Determine la solución:

$$\mathbf{2\,u_t = u_{xx} \quad ; \quad 0 < x < 1 \quad ; \quad t > 0}$$

$$\mathbf{u(0,t) = u(1,t) = 0}$$

$$\mathbf{u(x,0) = 4\,Sen(\pi x)Cos^3(\pi x)}$$

Determinando C_n

$$C_n = 2\int_0^1 4\,Sen(\pi x)Cos^3(\pi x)\,Sen\,(n\pi)dx$$

Resulta complicado resolver la integral, entonces:

$$4\,Sen(\pi x)Cos^3(\pi x) = 4\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2i}\right)\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2}\right)^3$$

$$4\,Sen(\pi x)Cos^3(\pi x) = \frac{1}{2}\left[\frac{\left(e^{i\pi x} - e^{-i\pi x}\right)\left(e^{i3\pi x} + 3e^{i\pi x} + 3e^{-i\pi x} + e^{-i3\pi x}\right)}{2i}\right]$$

$$4\,Sen(\pi x)Cos^3(\pi x) = \frac{1}{2}\left(\frac{e^{i4\pi x} + 3e^{i2\pi x} + 3e^{0} + e^{-i2\pi x} - e^{-i2\pi x} - 3e^{0} - 3e^{-i2\pi x} - e^{-i4\pi x}}{2i}\right)$$

$$4\,Sen(\pi x)Cos^3(\pi x) = \frac{1}{2}\left(\frac{e^{i4\pi x} - e^{-i4\pi x}}{2i} + 2\frac{e^{i2\pi x} - e^{-i2\pi x}}{2i}\right)$$

$$4\,Sen(\pi x)Cos^3(\pi x) = \frac{1}{2}[Sen(4\pi x) + 2\,Sen(2\pi x)]$$

Ahora:

$$Sen(2\pi x) + \frac{1}{2}Sen(4\pi x) = C_1Sen(\pi x) + C_2Sen(2\pi x) + C_3Sen(3\pi x) + C_4Sen(4\pi x) + C_5Sen(5\pi x) + \dots\dots$$

Entonces:

$$C_2 = 1 \quad ; \quad C_4 = \frac{1}{2} \quad ; \quad C_n = 0 \quad , \quad \forall n \in N - \{2,4\}$$

Finalmente la ecuación de calor es:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{\sqrt{2}}\right)^2 t}\,Sen(n\pi x)$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n\,e^{-\frac{(n\pi)^2}{2}t}\,Sen(n\pi x)$$

$$\boxed{\mathbf{u(x,t) = e^{-2\pi^2 t}Sen(2\pi x) + \frac{1}{2}e^{-8\pi^2 t}Sen(4\pi x)}}$$

ECUACIÓN DE LA ONDA

La ecuación de la onda es:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Resolución de la ecuación de la onda

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \, Cos\left(\frac{n\pi c}{l} t\right) + B_n \, Sen\left(\frac{n\pi c}{l} t\right) \right] Sen\left(\frac{n\pi}{l} x\right)$$

Encontrando A_n

$$A_n = \frac{2}{l} \int_0^l f(x) \, Sen\left(\frac{n\pi}{l} x\right) dx$$

Donde f(x) es el desplazamiento de la cuerda

$$f(x) = \sum_{n=1}^{\infty} A_n \, Sen\left(\frac{n\pi}{l} x\right)$$

Encontrando B_n

$$B_n = \frac{2}{n\pi c} \int_0^l g(x) \, Sen\left(\frac{n\pi}{l} x\right) dx$$

Donde g(x) es la velocidad de la cuerda

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l} B_n\right) Sen\left(\frac{n\pi}{l} x\right)$$

1) La figura muestra la función de la posición inicial f(x) de una cuerda de la longitud L = 2m estirada, que es puesta en movimiento al colocar su punto medio x = L/2 = 1m a una distancia de 2m desde la línea de referencia horizontal y soltándola de esa posición de reposo a partir del tiempo t = 0. Considere que la constante $c^2 = 4$

Pero f(x) presenta dos tramos, por lo que vamos a hallar las correspondientes reglas de correspondencia

Ecuación de la recta conociendo 2 puntos:

$$\frac{y-y_2}{x-x_2}=\frac{y_2-y_1}{x_2-x_1}$$

Ecuación de la recta 1: Q(1,2) ; R(2,0) :

$$\frac{y-0}{x-2}=\frac{0-2}{2-1} \quad \Rightarrow \quad y=-2x+4$$

Ecuación de la recta 2: P(0,0) ; Q(1,2) :

$$\frac{y-2}{x-1}=\frac{2-0}{1-0} \quad \Rightarrow \quad y=2x$$

Entonces:

$$f(x)=\begin{cases} 2x & ;\ 0\le x\le 1 \\ -2x+4 & ;\ 1<x\le 2 \end{cases}$$

Encontrando A_n

$$A_n=\frac{2}{2}\int_0^2 f(x)\,Sen\left(\frac{n\pi}{2}x\right)dx$$

$$A_n=\int_0^1 2x\,Sen\left(\frac{n\pi}{2}x\right)dx+\int_1^2(-2x+4)\,Sen\left(\frac{n\pi}{2}x\right)dx$$

$$A_n=2\int_0^1 x\,Sen\left(\frac{n\pi}{2}x\right)dx-2\int_1^2 x\,Sen\left(\frac{n\pi}{2}x\right)dx+4\int_1^2 Sen\left(\frac{n\pi}{2}x\right)dx$$

Resolviendo:

$$\int x\,Sen\left(\frac{n\pi}{2}x\right)dx$$

Integrando por partes:

$$u=x \quad \Rightarrow \quad du=dx$$

$$dv=Sen\left(\frac{n\pi}{2}x\right)dx \quad \Rightarrow \quad v=-\frac{2}{n\pi}Cos\left(\frac{n\pi}{2}x\right)$$

Luego:

$$\int x\,Sen\left(\frac{n\pi}{2}x\right)dx = -\frac{2x}{n\pi}Cos\left(\frac{n\pi}{2}x\right) + \frac{2}{n\pi}\int Cos\left(\frac{n\pi}{2}x\right)dx$$

$$\int x\,Sen\left(\frac{n\pi}{2}x\right)dx = -\frac{2x}{n\pi}Cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}x\right)$$

Por lo tanto:

$$A_n = 2\left[-\frac{2x}{n\pi}Cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}x\right)\right]_0^1 - 2\left[-\frac{2x}{n\pi}Cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}x\right)\right]_1^2 - \frac{8}{n\pi}\left[Cos\left(\frac{n\pi}{2}x\right)\right]_1^2$$

$$A_n = 2\left[-\frac{2}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] - 2\left[-\frac{4}{n\pi}Cos(n\pi) + \left(\frac{2}{n\pi}\right)^2 Sen(n\pi) + \frac{2}{n\pi}Cos\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] - \frac{8}{n\pi}\left[Cos(n\pi) - Cos\left(\frac{n\pi}{2}\right)\right]$$

$$A_n = 2\left[-\frac{2}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] - 2\left[-\frac{4}{n\pi}Cos(n\pi) + \frac{2}{n\pi}Cos\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] - \frac{8}{n\pi}\left[Cos(n\pi) - Cos\left(\frac{n\pi}{2}\right)\right]$$

$$A_n = -\frac{4}{n\pi}Cos\left(\frac{n\pi}{2}\right) + 2\left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}Cos(n\pi) - \frac{4}{n\pi}Cos\left(\frac{n\pi}{2}\right) + 2\left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right) - \frac{8}{n\pi}Cos(n\pi) + \frac{8}{n\pi}Cos\left(\frac{n\pi}{2}\right)$$

$$A_n = -\frac{8}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}Cos(n\pi) - \frac{8}{n\pi}Cos(n\pi) + 4\left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{16}{(n\pi)^2}Sen\left(\frac{n\pi}{2}\right)$$

Cuando n es par tenemos que:

$$A_{2n} = \frac{16}{(2n\pi)^2}Sen\left(\frac{2n\pi}{2}\right)$$

$$A_{2n} = \frac{16}{(2n\pi)^2}Sen(n\pi) \quad \Rightarrow \quad A_{2n} = 0$$

Cuando n es impar tenemos que:

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}Sen\left[\frac{(2n-1)\pi}{2}\right]$$

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}Sen\left(n\pi - \frac{\pi}{2}\right)$$

Si resolvemos:

$$Sen\left(n\pi - \frac{\pi}{2}\right) = Sen(n\pi)\,Cos\left(\frac{\pi}{2}\right) - Cos(n\pi)\,Sen\left(\frac{\pi}{2}\right)$$

$$Sen\left(n\pi - \frac{\pi}{2}\right) = -Cos(n\pi)$$

Entonces:

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}[-Cos(n\pi)]$$

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}[-(-1)^n]$$

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}(-1)^{n+1}$$

Ojo $B_n = 0$ *, debido a que la cuerda parte del reposo, es decir g(x)=0*

Entonces la solución de la ecuación de la onda descrita por la cuerda está dada por:

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\, Cos\left(\frac{n\pi(2)}{2}t\right)\right] Sen\left(\frac{n\pi}{2}x\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_{2n-1}\, Cos[(2n-1)\pi\, t]\right] Sen\left[\frac{(2n-1)\pi}{2}x\right]$$

$$\boldsymbol{u(x,t) = \sum_{n=1}^{\infty}\frac{16(-1)^{n+1}}{[(2n-1)\pi]^2} Sen\left[\frac{(2n-1)\pi}{2}x\right] Cos[(2n-1)\pi\, t]}$$