CHAPTER 1 Limits

1.1 Concepts Review

1. L; c

2. 6

3. L; right

4. $\lim_{x \to c} f(x) = M$

Problem Set 1.1

1. $\lim_{x \to 3}(x-5) = -2$

2. $\lim_{t \to -1}(1-2t) = 3$

3. $\lim_{x \to -2}(x^2+2x-1) = (-2)^2 + 2(-2) - 1 = -1$

4. $\lim_{x \to -2}(x^2+2t-1) = (-2)^2 + 2t - 1 = 3 + 2t$

5. $\lim_{t \to -1}\left(t^2-1\right) = \left((-1)^2-1\right) = 0$

6. $\lim_{t \to -1}\left(t^2-x^2\right) = \left((-1)^2-x^2\right) = 1-x^2$

7. $$\lim_{x \to 2}\frac{x^2-4}{x-2} = \lim_{x \to 2}\frac{(x-2)(x+2)}{x-2}$$
$$= \lim_{x \to 2}(x+2)$$
$$= 2 + 2 = 4$$

8. $$\lim_{t \to -7}\frac{t^2+4t-21}{t+7}$$
$$= \lim_{t \to -7}\frac{(t+7)(t-3)}{t+7}$$
$$= \lim_{t \to -7}(t-3)$$
$$= -7-3 = -10$$

9. $$\lim_{x \to -1}\frac{x^3-4x^2+x+6}{x+1}$$
$$= \lim_{x \to -1}\frac{(x+1)(x^2-5x+6)}{x+1}$$
$$= \lim_{x \to -1}(x^2-5x+6)$$
$$= (-1)^2 - 5(-1) + 6$$
$$= 12$$

10. $$\lim_{x \to 0}\frac{x^4+2x^3-x^2}{x^2}$$
$$= \lim_{x \to 0}(x^2+2x-1) = -1$$

11. $$\lim_{x \to -t}\frac{x^2-t^2}{x+t} = \lim_{x \to -t}\frac{(x+t)(x-t)}{x+t}$$
$$= \lim_{x \to -t}(x-t)$$
$$= -t - t = -2t$$

12. $$\lim_{x \to 3}\frac{x^2-9}{x-3}$$
$$= \lim_{x \to 3}\frac{(x-3)(x+3)}{x-3}$$
$$= \lim_{x \to 3}(x+3)$$
$$= 3 + 3 = 6$$

13. $$\lim_{t \to 2}\frac{\sqrt{(t+4)(t-2)^4}}{(3t-6)^2}$$
$$= \lim_{t \to 2}\frac{(t-2)^2\sqrt{t+4}}{9(t-2)^2}$$
$$= \lim_{t \to 2}\frac{\sqrt{t+4}}{9}$$
$$= \frac{\sqrt{2+4}}{9} = \frac{\sqrt{6}}{9}$$

14. $$\lim_{t \to 7^+}\frac{\sqrt{(t-7)^3}}{t-7}$$
$$= \lim_{t \to 7^+}\frac{(t-7)\sqrt{t-7}}{t-7}$$
$$= \lim_{t \to 7^+}\sqrt{t-7}$$
$$= \sqrt{7-7} = 0$$

15. $\lim_{x\to 3}\frac{x^4-18x^2+81}{(x-3)^2}=\lim_{x\to 3}\frac{(x^2-9)^2}{(x-3)^2}$

$=\lim_{x\to 3}\frac{(x-3)^2(x+3)^2}{(x-3)^2}=\lim_{x\to 3}(x+3)^2=(3+3)^2$

$=36$

16. $\lim_{u\to 1}\frac{(3u+4)(2u-2)^3}{(u-1)^2}=\lim_{u\to 1}\frac{8(3u+4)(u-1)^3}{(u-1)^2}$

$=\lim_{u\to 1}8(3u+4)(u-1)=8[3(1)+4](1-1)=0$

17. $\lim_{h\to 0}\frac{(2+h)^2-4}{h}=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}$

$=\lim_{h\to 0}\frac{h^2+4h}{h}=\lim_{h\to 0}(h+4)=4$

18. $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$

$=\lim_{h\to 0}\frac{h^2+2xh}{h}=\lim_{h\to 0}(h+2x)=2x$

19.

x	$\frac{\sin x}{2x}$
1.	0.420735
0.1	0.499167
0.01	0.499992
0.001	0.49999992
–1.	0.420735
–0.1	0.499167
–0.01	0.499992
–0.001	0.49999992

$\lim_{x\to 0}\frac{\sin x}{2x}=0.5$

20.

t	$\frac{1-\cos t}{2t}$
1.	0.229849
0.1	0.0249792
0.01	0.00249998
0.001	0.00024999998
–1.	–0.229849
–0.1	–0.0249792
–0.01	–0.00249998
–0.001	–0.00024999998

$\lim_{t\to 0}\frac{1-\cos t}{2t}=0$

21.

x	$(x-\sin x)^2/x^2$
1.	0.0251314
0.1	2.775×10^{-6}
0.01	2.77775×10^{-10}
0.001	2.77778×10^{-14}
–1.	0.0251314
–0.1	2.775×10^{-6}
–0.01	2.77775×10^{-10}
–0.001	2.77778×10^{-14}

$\lim_{x\to 0}\frac{(x-\sin x)^2}{x^2}=0$

22.

x	$(1-\cos x)^2/x^2$
1.	0.211322
0.1	0.00249584
0.01	0.0000249996
0.001	2.5×10^{-7}
–1.	0.211322
–0.1	0.00249584
–0.01	0.0000249996
–0.001	2.5×10^{-7}

$\lim_{x\to 0}\frac{(1-\cos x)^2}{x^2}=0$

23.

t	$(t^2-1)/(\sin(t-1))$
2.	3.56519
1.1	2.1035
1.01	2.01003
1.001	2.001
0	1.1884
0.9	1.90317
0.99	1.99003
0.999	1.999

$\lim_{t\to 1}\frac{t^2-1}{\sin(t-1)}=2$

24.

x	$\frac{x-\sin(x-3)-3}{x-3}$
4.	0.158529
3.1	0.00166583
3.01	0.0000166666
3.001	1.66667×10^{-7}
2.	0.158529
2.9	0.00166583
2.99	0.0000166666
2.999	1.66667×10^{-7}

$$\lim_{x\to3}\frac{x-\sin(x-3)-3}{x-3}=0$$

25.

x	$(1+\sin(x-3\pi/2))/(x-\pi)$
$1.+\pi$	0.4597
$0.1+\pi$	0.0500
$0.01+\pi$	0.0050
$0.001+\pi$	0.0005
$-1.+\pi$	–0.4597
$-0.1+\pi$	–0.0500
$-0.01+\pi$	–0.0050
$-0.001+\pi$	–0.0005

$$\lim_{x\to\pi}\frac{1+\sin\left(x-\frac{3\pi}{2}\right)}{x-\pi}=0$$

26.

t	$(1-\cot t)/(1/t)$
1.	0.357907
0.1	–0.896664
0.01	–0.989967
0.001	–0.999
–1.	–1.64209
–0.1	–1.09666
–0.01	–1.00997
–0.001	–1.001

$$\lim_{t\to0}\frac{1-\cot t}{\frac{1}{t}}=-1$$

27.

x	$(x-\pi/4)^2/(\tan x-1)^2$
$1.+\frac{\pi}{4}$	0.0320244
$0.1+\frac{\pi}{4}$	0.201002
$0.01+\frac{\pi}{4}$	0.245009
$0.001+\frac{\pi}{4}$	0.2495
$-1.+\frac{\pi}{4}$	0.674117
$-0.1+\frac{\pi}{4}$	0.300668
$-0.01+\frac{\pi}{4}$	0.255008
$-0.001+\frac{\pi}{4}$	0.2505

$$\lim_{x\to\frac{\pi}{4}}\frac{\left(x-\frac{\pi}{4}\right)^2}{(\tan x-1)^2}=0.25$$

28.

u	$(2-2\sin u)/3u$
$1.+\frac{\pi}{2}$	0.11921
$0.1+\frac{\pi}{2}$	0.00199339
$0.01+\frac{\pi}{2}$	0.0000210862
$0.001+\frac{\pi}{2}$	2.12072×10^{-7}
$-1.+\frac{\pi}{2}$	0.536908
$-0.1+\frac{\pi}{2}$	0.00226446
$-0.01+\frac{\pi}{2}$	0.0000213564
$-0.001+\frac{\pi}{2}$	2.12342×10^{-7}

$$\lim_{u\to\frac{\pi}{2}}\frac{2-2\sin u}{3u}=0$$

29. **a.** $\lim_{x\to-3} f(x)=2$

b. $f(-3)=1$

c. $f(-1)$ does not exist.

d. $\lim_{x\to-1} f(x)=\frac{5}{2}$

e. $f(1)=2$

f. $\lim_{x\to1} f(x)$ does not exist.

g. $\lim_{x\to1^-} f(x)=2$

h. $\lim_{x\to1^+} f(x)=1$

i. $\lim_{x\to-1^+} f\left(x\right)=\frac{5}{2}$

30. **a.** $\lim_{x\to -3} f(x)$ does not exist.

b. $f(-3) = 1$

c. $f(-1) = 1$

d. $\lim_{x\to -1} f(x) = 2$

e. $f(1) = 1$

f. $\lim_{x\to 1} f(x)$ does not exist.

g. $\lim_{x\to 1^-} f(x) = 1$

h. $\lim_{x\to 1^+} f(x)$ does not exist.

i. $\lim_{x\to -1^+} f\left(x\right) = 2$

31. **a.** $f(-3) = 2$

b. $f(3)$ is undefined.

c. $\lim_{x\to -3^-} f(x) = 2$

d. $\lim_{x\to -3^+} f(x) = 4$

e. $\lim_{x\to -3} f(x)$ does not exist.

f. $\lim_{x\to 3^+} f(x)$ does not exist.

32. **a.** $\lim_{x\to -1^-} f(x) = -2$

b. $\lim_{x\to -1^+} f(x) = -2$

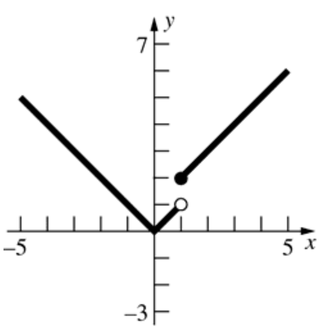
c. $\lim_{x\to -1} f(x) = -2$

d. $f(-1) = -2$

e. $\lim_{x\to 1} f(x) = 0$

f. $f(1) = 0$

33.



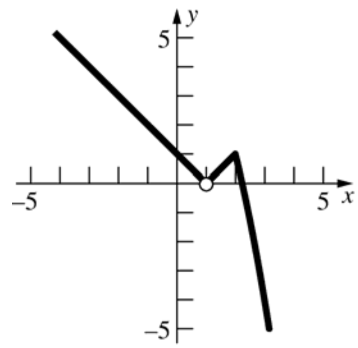
a. $\lim_{x\to 0} f(x) = 0$

b. $\lim_{x\to 1} f(x)$ does not exist.

c. $f(1) = 2$

d. $\lim_{x\to 1^+} f(x) = 2$

34.



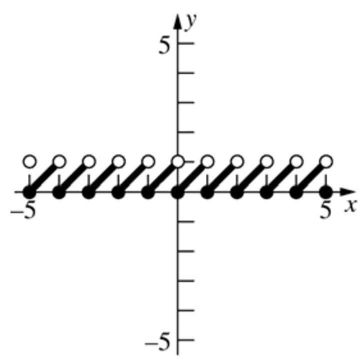
a. $\lim_{x\to 1} g(x) = 0$

b. $g(1)$ does not exist.

c. $\lim_{x\to 2} g(x) = 1$

d. $\lim_{x\to 2^+} g(x) = 1$

35. $f(x) = x - [[x]]$



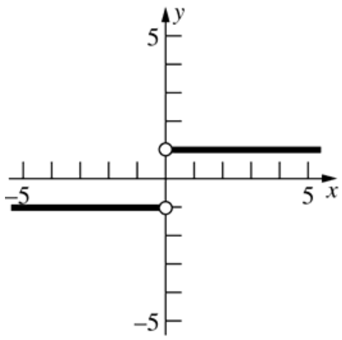
a. $f(0) = 0$

b. $\lim_{x\to 0} f(x)$ does not exist.

c. $\lim_{x\to 0^-} f(x) = 1$

d. $\lim_{x\to \frac{1}{2}} f(x) = \frac{1}{2}$

36. $f(x) = \frac{x}{|x|}$



a. $f(0)$ does not exist.

b. $\lim_{x\to 0} f(x)$ does not exist.

c. $\lim_{x\to 0^-} f(x) = -1$

d. $\lim_{x\to \frac{1}{2}} f(x) = 1$

37. $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$ does not exist.

$\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = -2$ and $\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = 2$

38. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$

$= \lim_{x\to 0} \frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})}$

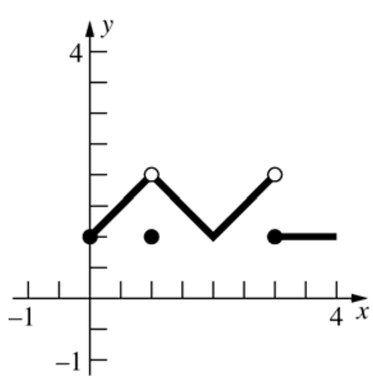
$= \lim_{x\to 0} \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{\sqrt{0+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

39. a. $\lim_{x\to 1} f(x)$ does not exist.

b. $\lim_{x\to 0} f(x) = 0$

40.



41. $\lim_{x\to a} f(x)$ exists for $a = -1, 0, 1$.

42. The changed values will not change $\lim_{x\to a} f(x)$ at any a. As x approaches a, the limit is still a^2.

43. a. $\lim_{x\to 1} \frac{|x-1|}{x-1}$ does not exist.

$\lim_{x\to 1^-} \frac{|x-1|}{x-1} = -1$ and $\lim_{x\to 1^+} \frac{|x-1|}{x-1} = 1$

b. $\lim_{x\to 1^-} \frac{|x-1|}{x-1} = -1$

c. $\lim_{x\to 1^-} \frac{x^2-|x-1|-1}{|x-1|} = -3$

d. $\lim_{x\to 1^-} \left[\frac{1}{x-1} - \frac{1}{|x-1|}\right]$ does not exist.

44. a. $\lim_{x\to 1^+} \sqrt{x - [\![x]\!]} = 0$

b. $\lim_{x\to 0^+} \left[\!\!\left[\frac{1}{x}\right]\!\!\right]$ does not exist.

c. $\lim_{x\to 0^+} x(-1)^{[\![1/x]\!]} = 0$

d. $\lim_{x\to 0^+} [\![x]\!](-1)^{[\![1/x]\!]} = 0$

45. a) 1 **b)** 0

c) –1 **d)** –1

46. a) Does not exist **b)** 0

c) 1 **d)** 0.556

47. $\lim_{x\to 0} \sqrt{x}$ does not exist since $\sqrt{x}$ is not defined for $x < 0$.

48. $\lim_{x\to 0^+} x^x = 1$

49. $\lim_{x\to 0} \sqrt{|x|} = 0$

50. $\lim_{x\to 0} |x|^x = 1$

51. $\lim_{x\to 0} \frac{\sin 2x}{4x} = \frac{1}{2}$

52. $\lim_{x\to 0}\frac{\sin 5x}{3x}=\frac{5}{3}$

53. $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist.

54. $\lim_{x\to 0}x\cos\left(\frac{1}{x}\right)=0$

55. $\lim_{x\to 1}\frac{x^3-1}{\sqrt{2x+2}-2}=6$

56. $\lim_{x\to 0}\frac{x\sin 2x}{\sin(x^2)}=2$

57. $\lim_{x\to 2^-}\frac{x^2-x-2}{|x-2|}=-3$

58. $\lim_{x\to 1^+}\frac{2}{1+2^{1/(x-1)}}=0$

59. $\lim_{x\to 0}\sqrt{x}$; The computer gives a value of 0, but $\lim_{x\to 0^-}\sqrt{x}$ does not exist.

1.2 Concepts Review

1. $L-\varepsilon$; $L+\varepsilon$

2. $0<|x-a|<\delta$; $|f(x)-L|<\varepsilon$

3. $\frac{\varepsilon}{3}$

4. $ma+b$

Problem Set 1.2

1. $0<|t-a|<\delta\Rightarrow|f(t)-M|<\varepsilon$

2. $0<|u-b|<\delta\Rightarrow|g(u)-L|<\varepsilon$

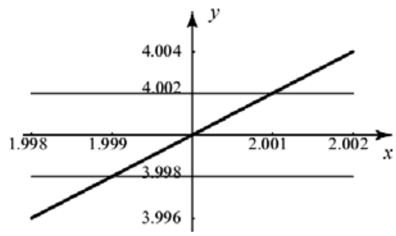
3. $0<|z-d|<\delta\Rightarrow|h(z)-P|<\varepsilon$

4. $0<|y-e|<\delta\Rightarrow|\phi(y)-B|<\varepsilon$

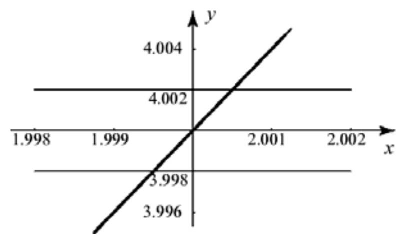
5. $0<c-x<\delta\Rightarrow|f(x)-L|<\varepsilon$

6. $0<t-a<\delta\Rightarrow|g(t)-D|<\varepsilon$

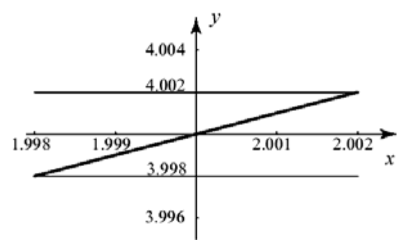
7. If x is within 0.001 of 2, then $2x$ *is* within 0.002 of 4.



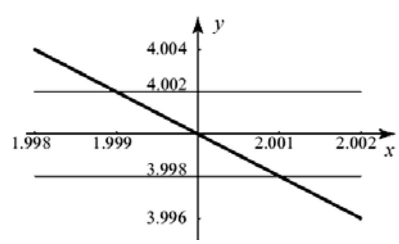
8. If x is within 0.0005 of 2, then x^2 is within 0.002 of 4.



9. If x is within 0.0019 of 2, then $\sqrt{8x}$ is within 0.002 of 4.



10. If x is within 0.001 of 2, then $\frac{8}{x}$ is within 0.002 of 4.



11. $0<|x-0|<\delta\Rightarrow|(2x-1)-(-1)|<\varepsilon$

$|2x-1+1|<\varepsilon\Leftrightarrow|2x|<\varepsilon$

$\Leftrightarrow 2|x|<\varepsilon$

$\Leftrightarrow |x|<\frac{\varepsilon}{2}$

$\delta=\frac{\varepsilon}{2}$; $0<|x-0|<\delta$

$|(2x-1)-(-1)|=|2x|=2|x|<2\delta=\varepsilon$

12. $0<|x+21|<\delta \Rightarrow |(3x-1)-(-64)|<\varepsilon$

$|3x-1+64|<\varepsilon \Leftrightarrow |3x+63|<\varepsilon$

$\Leftrightarrow |3(x+21)|<\varepsilon$

$\Leftrightarrow 3|x+21|<\varepsilon$

$\Leftrightarrow |x+21|<\dfrac{\varepsilon}{3}$

$\delta=\dfrac{\varepsilon}{3};\ 0<|x+21|<\delta$

$|(3x-1)-(-64)|=|3x+63|=3|x+21|<3\delta=\varepsilon$

13. $0<|x-5|<\delta \Rightarrow \left|\dfrac{x^2-25}{x-5}-10\right|<\varepsilon$

$\left|\dfrac{x^2-25}{x-5}-10\right|<\varepsilon \Leftrightarrow \left|\dfrac{(x-5)(x+5)}{x-5}-10\right|<\varepsilon$

$\Leftrightarrow |x+5-10|<\varepsilon$

$\Leftrightarrow |x-5|<\varepsilon$

$\delta=\varepsilon;\ 0<|x-5|<\delta$

$\left|\dfrac{x^2-25}{x-5}-10\right|=\left|\dfrac{(x-5)(x+5)}{x-5}-10\right|=|x+5-10|$

$=|x-5|<\delta=\varepsilon$

14. $0<|x-0|<\delta \Rightarrow \left|\dfrac{2x^2-x}{x}-(-1)\right|<\varepsilon$

$\left|\dfrac{2x^2-x}{x}+1\right|<\varepsilon \Leftrightarrow \left|\dfrac{x(2x-1)}{x}+1\right|<\varepsilon$

$\Leftrightarrow |2x-1+1|<\varepsilon$

$\Leftrightarrow |2x|<\varepsilon$

$\Leftrightarrow 2|x|<\varepsilon$

$\Leftrightarrow |x|<\dfrac{\varepsilon}{2}$

$\delta=\dfrac{\varepsilon}{2};\ 0<|x-0|<\delta$

$\left|\dfrac{2x^2-x}{x}-(-1)\right|=\left|\dfrac{x(2x-1)}{x}+1\right|=|2x-1+1|$

$=|2x|=2|x|<2\delta=\varepsilon$

15. $0<|x-5|<\delta \Rightarrow \left|\dfrac{2x^2-11x+5}{x-5}-9\right|<\varepsilon$

$\left|\dfrac{2x^2-11x+5}{x-5}-9\right|<\varepsilon \Leftrightarrow \left|\dfrac{(2x-1)(x-5)}{x-5}-9\right|<\varepsilon$

$\Leftrightarrow |2x-1-9|<\varepsilon$

$\Leftrightarrow |2(x-5)|<\varepsilon$

$\Leftrightarrow |x-5|<\dfrac{\varepsilon}{2}$

$\delta=\dfrac{\varepsilon}{2};\ 0<|x-5|<\delta$

$\left|\dfrac{2x^2-11x+5}{x-5}-9\right|=\left|\dfrac{(2x-1)(x-5)}{x-5}-9\right|$

$=|2x-1-9|=|2(x-5)|=2|x-5|<2\delta=\varepsilon$

16. $0<|x-1|<\delta \Rightarrow \left|\sqrt{2x}-\sqrt{2}\right|<\varepsilon$

$\left|\sqrt{2x}-\sqrt{2}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{2x-2}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\Leftrightarrow 2\left|\dfrac{x-1}{\sqrt{2x}+\sqrt{2}}\right|<\varepsilon$

$\delta=\dfrac{\sqrt{2}\varepsilon}{2};\ 0<|x-1|<\delta$

$\left|\sqrt{2x}-\sqrt{2}\right|=\left|\dfrac{(\sqrt{2x}-\sqrt{2})(\sqrt{2x}+\sqrt{2})}{\sqrt{2x}+\sqrt{2}}\right|$

$=\left|\dfrac{2x-2}{\sqrt{2x}+\sqrt{2}}\right|$

$\dfrac{2|x-1|}{\sqrt{2x}+\sqrt{2}}\le\dfrac{2|x-1|}{\sqrt{2}}<\dfrac{2\delta}{\sqrt{2}}=\varepsilon$

17. $0<|x-4|<\delta \Rightarrow \left|\dfrac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right|<\varepsilon$

$\left|\dfrac{\sqrt{2x-1}}{\sqrt{x-3}}-\sqrt{7}\right|<\varepsilon \Leftrightarrow \left|\dfrac{\sqrt{2x-1}-\sqrt{7(x-3)}}{\sqrt{x-3}}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{(\sqrt{2x-1}-\sqrt{7(x-3)})(\sqrt{2x-1}+\sqrt{7(x-3)})}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{2x-1-(7x-21)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow \left|\dfrac{-5(x-4)}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}\right|<\varepsilon$

$\Leftrightarrow |x-4| \cdot \dfrac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < \varepsilon$

To bound $\dfrac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$, agree that $\delta \le \frac{1}{2}$. If $\delta \le \frac{1}{2}$, then $\frac{7}{2} < x < \frac{9}{2}$, so

$0.65 < \dfrac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < 1.65$ and

hence $|x-4| \cdot \dfrac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})} < \varepsilon$

$\Leftrightarrow |x-4| < \dfrac{\varepsilon}{1.65}$

For whatever ε is chosen, let δ be the smaller of $\frac{1}{2}$ and $\frac{\varepsilon}{1.65}$.

$\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{1.65}\right\}$, $0 < |x-4| < \delta$

$\left|\dfrac{\sqrt{2x-1}}{\sqrt{x-3}} - \sqrt{7}\right| = |x-4| \cdot \dfrac{5}{\sqrt{x-3}(\sqrt{2x-1}+\sqrt{7(x-3)})}$

$< |x-4|(1.65) < 1.65\delta \le \varepsilon$

since $\delta = \frac{1}{2}$ only when $\frac{1}{2} \le \frac{\varepsilon}{1.65}$ so $1.65\delta \le \varepsilon$.

18. $0 < |x-1| < \delta \Rightarrow \left|\dfrac{14x^2-20x+6}{x-1} - 8\right| < \varepsilon$

$\left|\dfrac{14x^2-20x+6}{x-1} - 8\right| < \varepsilon \Leftrightarrow \left|\dfrac{2(7x-3)(x-1)}{x-1} - 8\right| < \varepsilon$

$\Leftrightarrow |2(7x-3)-8| < \varepsilon$

$\Leftrightarrow |14(x-1)| < \varepsilon$

$\Leftrightarrow 14|x-1| < \varepsilon$

$\Leftrightarrow |x-1| < \dfrac{\varepsilon}{14}$

$\delta = \dfrac{\varepsilon}{14}; 0 < |x-1| < \delta$

$\left|\dfrac{14x^2-20x+6}{x-1} - 8\right| = \left|\dfrac{2(7x-3)(x-1)}{x-1} - 8\right|$

$= |2(7x-3)-8|$

$= |14(x-1)| = 14|x-1| < 14\delta = \varepsilon$

19. $0 < |x-1| < \delta \Rightarrow \left|\dfrac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| < \varepsilon$

$\left|\dfrac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| < \varepsilon$

$\Leftrightarrow \left|\dfrac{(10x-6)(x-1)^2}{(x-1)^2} - 4\right| < \varepsilon$

$\Leftrightarrow |10x-6-4| < \varepsilon$

$\Leftrightarrow |10(x-1)| < \varepsilon$

$\Leftrightarrow 10|x-1| < \varepsilon$

$\Leftrightarrow |x-1| < \dfrac{\varepsilon}{10}$

$\delta = \dfrac{\varepsilon}{10}; 0 < |x-1| < \delta$

$\left|\dfrac{10x^3-26x^2+22x-6}{(x-1)^2} - 4\right| = \left|\dfrac{(10x-6)(x-1)^2}{(x-1)^2} - 4\right|$

$= |10x-6-4| = |10(x-1)|$

$= 10|x-1| < 10\delta = \varepsilon$

20. $0 < |x-1| < \delta \Rightarrow |(2x^2+1)-3| < \varepsilon$

$|2x^2+1-3| = |2x^2-2| = 2|x+1||x-1|$

To bound $|2x+2|$, agree that $\delta \le 1$.

$|x-1| < \delta$ implies

$|2x+2| = |2x-2+4|$

$\le |2x-2| + |4|$

$< 2 + 4 = 6$

$\delta \le \dfrac{\varepsilon}{6}; \delta = \min\left\{1, \dfrac{\varepsilon}{6}\right\}; 0 < |x-1| < \delta$

$|(2x^2+1)-3| = |2x^2-2|$

$= |2x+2||x-1| < 6 \cdot \left(\dfrac{\varepsilon}{6}\right) = \varepsilon$

21. $0<|x+1|<\delta \Rightarrow |(x^2-2x-1)-2|<\varepsilon$

$|x^2-2x-1-2| = |x^2-2x-3| = |x+1||x-3|$

To bound $|x-3|$, agree that $\delta \le 1$.

$|x+1|<\delta$ implies

$|x-3| = |x+1-4| \le |x+1|+|-4| < 1 + 4 = 5$

$\delta \le \frac{\varepsilon}{5}$; $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$; $0<|x+1|<\delta$

$|(x^2-2x-1)-2| = |x^2-2x-3|$

$= |x+1||x-3| < 5\cdot\frac{\varepsilon}{5} = \varepsilon$

22. $0<|x|<\delta \Rightarrow |x^4-0| = |x^4| < \varepsilon$

$|x^4| = |x||x^3|$. To bound $|x^3|$, agree that $\delta \le 1$. $|x|<\delta \le 1$ implies $|x^3| = |x|^3 \le 1$ so $\delta \le \varepsilon$.

$\delta = \min\{1, \varepsilon\}$; $0<|x|<\delta \Rightarrow |x^4| = |x||x^3| < \varepsilon \cdot 1 = \varepsilon$

23. Choose $\varepsilon > 0$. Then since $\lim_{x\to c} f(x) = L$, there is some $\delta_1 > 0$ such that

$0<|x-c|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon$.

Since $\lim_{x\to c} f(x) = M$, there is some $\delta_2 > 0$ such that $0<|x-c|<\delta_2 \Rightarrow |f(x)-M|<\varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and choose x_0 such that $0<|x_0-c|<\delta$.

Thus, $|f(x_0)-L|<\varepsilon \Rightarrow -\varepsilon < f(x_0)-L < \varepsilon$

$\Rightarrow -f(x_0)-\varepsilon < -L < -f(x_0)+\varepsilon$

$\Rightarrow f(x_0)-\varepsilon < L < f(x_0)+\varepsilon.$

Similarly,

$f(x_0)-\varepsilon < M < f(x_0)+\varepsilon$.

Thus,

$-2\varepsilon < L-M < 2\varepsilon$. As $\varepsilon \Rightarrow 0$, $L-M \to 0$, so $L = M$.

24. Since $\lim_{x\to c} G(x) = 0$, then given any $\varepsilon > 0$, we can find $\delta > 0$ such that whenever $|x-c|<\delta, |G(x)|<\varepsilon$.

Take any $\varepsilon > 0$ and the corresponding δ that works for $G(x)$, then $|x-c|<\delta$ implies

$|F(x)-0| = |F(x)| \le |G(x)| < \varepsilon$ since $\lim_{x\to c} G(x) = 0.$

Thus, $\lim_{x\to c} F(x) = 0.$

25. For all $x \ne 0$, $0 \le \sin^2\left(\frac{1}{x}\right) \le 1$ so

$x^4 \sin^2\left(\frac{1}{x}\right) \le x^4$ for all $x \ne 0$. By Problem 18,

$\lim_{x\to 0} x^4 = 0$, so, by Problem 20,

$\lim_{x\to 0} x^4 \sin^2\left(\frac{1}{x}\right) = 0.$

26. $0<x<\delta \Rightarrow |\sqrt{x}-0| = |\sqrt{x}| = \sqrt{x} < \varepsilon$

For $x > 0$, $(\sqrt{x})^2 = x$.

$\sqrt{x} < \varepsilon \Leftrightarrow (\sqrt{x})^2 = x < \varepsilon^2$

$\delta = \varepsilon^2$; $0<x<\delta \Rightarrow \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$

27. $\lim_{x\to 0^+} |x| : 0<x<\delta \Rightarrow ||x|-0| < \varepsilon$

For $x \ge 0$, $|x| = x$.

$\delta = \varepsilon$; $0<x<\delta \Rightarrow ||x|-0| = |x| = x < \delta = \varepsilon$

Thus, $\lim_{x\to 0^+} |x| = 0.$

$\lim_{x\to 0^-} |x| : 0 < 0-x < \delta \Rightarrow ||x|-0| < \varepsilon$

For $x < 0$, $|x| = -x$; note also that $||x|| = |x|$ since $|x| \ge 0$.

$\delta = \varepsilon$; $0<-x<\delta \Rightarrow ||x|| = |x| = -x < \delta = \varepsilon$

Thus, $\lim_{x\to 0^-} |x| = 0$,

since $\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x| = 0$, $\lim_{x\to 0} |x| = 0.$

28. Choose $\varepsilon > 0$. Since $\lim_{x\to a} g(x) = 0$ there is some $\delta_1 > 0$ such that

$0<|x-a|<\delta_1 \Rightarrow |g(x)-0| < \frac{\varepsilon}{B}$.

Let $\delta = \min\{1, \delta_1\}$, then $|f(x)| < B$ for $|x-a|<\delta$ or $|x-a|<\delta \Rightarrow |f(x)| < B$. Thus,

$|x-a|<\delta \Rightarrow |f(x)g(x)-0| = |f(x)g(x)|$

$= |f(x)||g(x)| < B\cdot\frac{\varepsilon}{B} = \varepsilon$ so $\lim_{x\to a} f(x)g(x) = 0.$

29. Choose $\varepsilon > 0$. Since $\lim_{x \to a} f(x) = L$, there is a $\delta > 0$ such that for $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.
That is, for
$a - \delta < x < a$ or $a < x < a + \delta$,
$L - \varepsilon < f(x) < L + \varepsilon$.
Let $f(a) = A$,
$M = \max\{|L - \varepsilon|, |L + \varepsilon|, |A|\}$, $c = a - \delta$,
$d = a + \delta$. Then for x in (c, d), $|f(x)| \le M$, since either $x = a$, in which case
$|f(x)| = |f(a)| = |A| \le M$ or $0 < |x - a| < \delta$ so
$L - \varepsilon < f(x) < L + \varepsilon$ and $|f(x)| < M$.

30. Suppose that $L > M$. Then $L - M = \alpha > 0$. Now take $\varepsilon < \frac{\alpha}{2}$ and $\delta = \min\{\delta_1, \delta_2\}$ where
$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$ and
$0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon$.
Thus, for $0 < |x - a| < \delta$,
$L - \varepsilon < f(x) < L + \varepsilon$ and $M - \varepsilon < g(x) < M + \varepsilon$.
Combine the inequalities and use the fact that $f(x) \le g(x)$ to get
$L - \varepsilon < f(x) \le g(x) < M + \varepsilon$ which leads to
$L - \varepsilon < M + \varepsilon$ or $L - M < 2\varepsilon$.
However,
$L - M = \alpha > 2\varepsilon$
which is a contradiction.
Thus $L \le M$.

31. (b) and (c) are equivalent to the definition of limit.

32. For every $\varepsilon > 0$ and $\delta > 0$ there is some x with $0 < |x - c| < \delta$ such that $|f(x) - L| > \varepsilon$.

33. **a.** $g(x) = \dfrac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}$

b. No, because $\dfrac{x + 6}{x^4 - 4x^3 + x^2 + x + 6} + 1$ has an asymptote at $x \approx 3.49$.

c. If $\delta \le \frac{1}{4}$, then $2.75 < x < 3$ or $3 < x < 3.25$ and by graphing
$$y = |g(x)| = \left|\frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}\right|$$
on the interval [2.75, 3.25], we see that
$$0 < \left|\frac{x^3 - x^2 - 2x - 4}{x^4 - 4x^3 + x^2 + x + 6}\right| < 3$$
so m must be at least three.

1.3 Concepts Review

1. 48

2. 4

3. – 8; – 4 + 5c

4. 0

Problem Set 1.3

1. $\lim_{x \to 1}(2x + 1)$ 4
$= \lim_{x \to 1} 2x + \lim_{x \to 1} 1$ 3
$= 2\lim_{x \to 1} x + \lim_{x \to 1} 1$ 2,1
$= 2(1) + 1 = 3$

2. $\lim_{x \to -1}(3x^2 - 1)$ 5
$= \lim_{x \to -1} 3x^2 - \lim_{x \to -1} 1$ 3
$= 3\lim_{x \to -1} x^2 - \lim_{x \to -1} 1$ 8
$= 3\left(\lim_{x \to -1} x\right)^2 - \lim_{x \to -1} 1$ 2, 1
$= 3(-1)^2 - 1 = 2$

3. $\lim_{x \to 0}[(2x + 1)(x - 3)]$ 6
$= \lim_{x \to 0}(2x + 1) \cdot \lim_{x \to 0}(x - 3)$ 4, 5
$= \left(\lim_{x \to 0} 2x + \lim_{x \to 0} 1\right) \cdot \left(\lim_{x \to 0} x - \lim_{x \to 0} 3\right)$ 3
$= \left(2\lim_{x \to 0} x + \lim_{x \to 0} 1\right) \cdot \left(\lim_{x \to 0} x - \lim_{x \to 0} 3\right)$ 2, 1
$= [2(0) + 1](0 - 3) = -3$

4. $\lim_{x \to \sqrt{2}}[(2x^2 + 1)(7x^2 + 13)]$ 6
$= \lim_{x \to \sqrt{2}}(2x^2 + 1) \cdot \lim_{x \to \sqrt{2}}(7x^2 + 13)$ 4, 3
$= \left(2\lim_{x \to \sqrt{2}} x^2 + \lim_{x \to \sqrt{2}} 1\right) \cdot \left(7\lim_{x \to \sqrt{2}} x^2 + \lim_{x \to \sqrt{2}} 13\right)$ 8,1
$= \left[2\left(\lim_{x \to \sqrt{2}} x\right)^2 + 1\right]\left[7\left(\lim_{x \to \sqrt{2}} x\right)^2 + 13\right]$ 2
$= [2(\sqrt{2})^2 + 1][7(\sqrt{2})^2 + 13] = 135$

5. $\lim_{x \to 2} \dfrac{2x+1}{5-3x}$ 7

$= \dfrac{\lim_{x \to 2}(2x+1)}{\lim_{x \to 2}(5-3x)}$ 4, 5

$= \dfrac{\lim_{x \to 2} 2x + \lim_{x \to 2} 1}{\lim_{x \to 2} 5 - \lim_{x \to 2} 3x}$ 3, 1

$= \dfrac{2\lim_{x \to 2} x + 1}{5 - 3\lim_{x \to 2} x}$ 2

$= \dfrac{2(2)+1}{5-3(2)} = -5$

6. $\lim_{x \to -3} \dfrac{4x^3+1}{7-2x^2}$ 7

$= \dfrac{\lim_{x \to -3}(4x^3+1)}{\lim_{x \to -3}(7-2x^2)}$ 4, 5

$= \dfrac{\lim_{x \to -3} 4x^3 + \lim_{x \to -3} 1}{\lim_{x \to -3} 7 - \lim_{x \to -3} 2x^2}$ 3, 1

$= \dfrac{4\lim_{x \to -3} x^3 + 1}{7 - 2\lim_{x \to -3} x^2}$ 8

$= \dfrac{4\left(\lim_{x \to -3} x\right)^3 + 1}{7 - 2\left(\lim_{x \to -3} x\right)^2}$ 2

$= \dfrac{4(-3)^3+1}{7-2(-3)^2} = \dfrac{107}{11}$

7. $\lim_{x \to 3} \sqrt{3x-5}$ 9

$= \sqrt{\lim_{x \to 3}(3x-5)}$ 5, 3

$= \sqrt{3\lim_{x \to 3} x - \lim_{x \to 3} 5}$ 2, 1

$= \sqrt{3(3)-5} = 2$

8. $\lim_{x \to -3} \sqrt{5x^2+2x}$ 9

$= \sqrt{\lim_{x \to -3}(5x^2+2x)}$ 4, 3

$= \sqrt{5\lim_{x \to -3} x^2 + 2\lim_{x \to -3} x}$ 8

$= \sqrt{5\left(\lim_{x \to -3} x\right)^2 + 2\lim_{x \to -3} x}$ 2

$= \sqrt{5(-3)^2+2(-3)} = \sqrt{39}$

9. $\lim_{t \to -2} (2t^3+15)^{13}$ 8

$= \left[\lim_{t \to -2}(2t^3+15)\right]^{13}$ 4, 3

$= \left[2\lim_{t \to -2} t^3 + \lim_{t \to -2} 15\right]^{13}$ 8

$= \left[2\left(\lim_{t \to -2} t\right)^3 + \lim_{t \to -2} 15\right]^{13}$ 2, 1

$= [2(-2)^3+15]^{13} = -1$

10. $\lim_{w \to -2} \sqrt{-3w^3+7w^2}$ 9

$= \sqrt{\lim_{w \to -2}(-3w^3+7w^2)}$ 4, 3

$= \sqrt{-3\lim_{w \to -2} w^3 + 7\lim_{w \to -2} w^2}$ 8

$= \sqrt{-3\left(\lim_{w \to -2} w\right)^3 + 7\left(\lim_{w \to -2} w\right)^2}$ 2

$= \sqrt{-3(-2)^3+7(-2)^2} = 2\sqrt{13}$

11. $\lim_{y \to 2} \left(\dfrac{4y^3+8y}{y+4}\right)^{1/3}$ 9

$= \left(\lim_{y \to 2} \dfrac{4y^3+8y}{y+4}\right)^{1/3}$ 7

$= \left[\dfrac{\lim_{y \to 2}(4y^3+8y)}{\lim_{y \to 2}(y+4)}\right]^{1/3}$ 4, 3

$= \left(\dfrac{4\lim_{y \to 2} y^3 + 8\lim_{y \to 2} y}{\lim_{y \to 2} y + \lim_{y \to 2} 4}\right)^{1/3}$ 8, 1

$= \left[\dfrac{4\left(\lim_{y \to 2} y\right)^3 + 8\lim_{y \to 2} y}{\lim_{y \to 2} y + 4}\right]^{1/3}$ 2

$= \left[\dfrac{4(2)^3+8(2)}{2+4}\right]^{1/3} = 2$

12. $\lim_{w\to 5}(2w^4-9w^3+19)^{-1/2}$

$= \lim_{w\to 5}\dfrac{1}{\sqrt{2w^4-9w^3+19}}$ 7

$= \dfrac{\lim_{w\to 5} 1}{\lim_{w\to 5}\sqrt{2w^4-9w^3+19}}$ 1, 9

$= \dfrac{1}{\sqrt{\lim_{w\to 5}(2w^4-9w^3+19)}}$ 4,5

$= \dfrac{1}{\sqrt{\lim_{w\to 5} 2w^4 - \lim_{w\to 5} 9w^3 + \lim_{w\to 5} 19}}$ 1,3

$= \dfrac{1}{\sqrt{2\lim_{w\to 5} w^4 - 9\lim_{w\to 5} w^3 + 19}}$ 8

$= \dfrac{1}{\sqrt{2\left(\lim_{w\to 5} w\right)^4 - 9\left(\lim_{w\to 5} w\right)^3 + 19}}$ 2

$= \dfrac{1}{\sqrt{2(5)^4-9(5)^3+19}}$

$= \dfrac{1}{\sqrt{144}} = \dfrac{1}{12}$

13. $\lim_{x\to 2}\dfrac{x^2-4}{x^2+4} = \dfrac{\lim_{x\to 2}\left(x^2-4\right)}{\lim_{x\to 2}\left(x^2+4\right)} = \dfrac{4-4}{4+4} = 0$

14. $\lim_{x\to 2}\dfrac{x^2-5x+6}{x-2} = \lim_{x\to 2}\dfrac{(x-3)(x-2)}{(x-2)}$

$= \lim_{x\to 2}(x-3) = -1$

15. $\lim_{x\to -1}\dfrac{x^2-2x-3}{x+1} = \lim_{x\to -1}\dfrac{(x-3)(x+1)}{(x+1)}$

$= \lim_{x\to -1}(x-3) = -4$

16. $\lim_{x\to -1}\dfrac{x^2+x}{x^2+1} = \dfrac{\lim_{x\to -1}\left(x^2+x\right)}{\lim_{x\to -1}\left(x^2+1\right)} = \dfrac{0}{2} = 0$

17. $\lim_{x\to -1}\dfrac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+7)} = \lim_{x\to -1}\dfrac{x-3}{x+7}$

$= \dfrac{-1-3}{-1+7} = -\dfrac{2}{3}$

18. $\lim_{x\to 2}\dfrac{x^2+7x+10}{x+2} = \lim_{x\to 2}\dfrac{(x+2)(x+5)}{x+2}$

$= \lim_{x\to 2}(x+5) = 7$

19. $\lim_{x\to 1}\dfrac{x^2+x-2}{x^2-1} = \lim_{x\to 1}\dfrac{(x+2)(x-1)}{(x+1)(x-1)}$

$= \lim_{x\to 1}\dfrac{x+2}{x+1} = \dfrac{1+2}{1+1} = \dfrac{3}{2}$

20. $\lim_{x\to -3}\dfrac{x^2-14x-51}{x^2-4x-21} = \lim_{x\to -3}\dfrac{(x+3)(x-17)}{(x+3)(x-7)}$

$= \lim_{x\to -3}\dfrac{x-17}{x-7} = \dfrac{-3-17}{-3-7} = 2$

21. $\lim_{u\to -2}\dfrac{u^2-ux+2u-2x}{u^2-u-6} = \lim_{u\to -2}\dfrac{(u+2)(u-x)}{(u+2)(u-3)}$

$= \lim_{u\to -2}\dfrac{u-x}{u-3} = \dfrac{x+2}{5}$

22. $\lim_{x\to 1}\dfrac{x^2+ux-x-u}{x^2+2x-3} = \lim_{x\to 1}\dfrac{(x-1)(x+u)}{(x-1)(x+3)}$

$= \lim_{x\to 1}\dfrac{x+u}{x+3} = \dfrac{1+u}{1+3} = \dfrac{u+1}{4}$

23. $\lim_{x\to \pi}\dfrac{2x^2-6x\pi+4\pi^2}{x^2-\pi^2} = \lim_{x\to \pi}\dfrac{2(x-\pi)(x-2\pi)}{(x-\pi)(x+\pi)}$

$= \lim_{x\to \pi}\dfrac{2(x-2\pi)}{x+\pi} = \dfrac{2(\pi-2\pi)}{\pi+\pi} = -1$

24. $\lim_{w\to -2}\dfrac{(w+2)(w^2-w-6)}{w^2+4w+4}$

$= \lim_{w\to -2}\dfrac{(w+2)^2(w-3)}{(w+2)^2} = \lim_{w\to -2}(w-3)$

$= -2-3 = -5$

25. $\lim_{x\to a}\sqrt{f^2(x)+g^2(x)}$

$= \sqrt{\lim_{x\to a} f^2(x) + \lim_{x\to a} g^2(x)}$

$= \sqrt{\left(\lim_{x\to a} f(x)\right)^2 + \left(\lim_{x\to a} g(x)\right)^2}$

$= \sqrt{(3)^2+(-1)^2} = \sqrt{10}$

26. $\lim_{x\to a}\dfrac{2f(x)-3g(x)}{f(x)+g(x)} = \dfrac{\lim_{x\to a}[2f(x)-3g(x)]}{\lim_{x\to a}[f(x)+g(x)]}$

$= \dfrac{2\lim_{x\to a} f(x) - 3\lim_{x\to a} g(x)}{\lim_{x\to a} f(x) + \lim_{x\to a} g(x)} = \dfrac{2(3)-3(-1)}{3+(-1)} = \dfrac{9}{2}$

27. $\lim_{x\to a}\sqrt[3]{g(x)}[f(x)+3] = \lim_{x\to a}\sqrt[3]{g(x)}\cdot\lim_{x\to a}[f(x)+3]$

$= \sqrt[3]{\lim_{x\to a} g(x)}\cdot\left[\lim_{x\to a} f(x)+\lim_{x\to a} 3\right] = \sqrt[3]{-1}\cdot(3+3)$

$= -6$

28. $\lim_{x\to a}[f(x)-3]^4 = \left[\lim_{x\to a}(f(x)-3)\right]^4$

$= \left[\lim_{x\to a} f(x)-\lim_{x\to a} 3\right]^4 = (3-3)^4 = 0$

29. $\lim_{t\to a}\left[|f(t)|+|3g(t)|\right] = \lim_{t\to a}|f(t)| + 3\lim_{t\to a}|g(t)|$

$= \left|\lim_{t\to a} f(t)\right| + 3\left|\lim_{t\to a} g(t)\right|$

$= |3| + 3|-1| = 6$

30. $\lim_{u\to a}[f(u)+3g(u)]^3 = \left(\lim_{u\to a}[f(u)+3g(u)]\right)^3$

$= \left[\lim_{u\to a} f(u) + 3\lim_{u\to a} g(u)\right]^3 = [3+3(-1)]^3 = 0$

31. $\lim_{x\to 2}\frac{3x^2-12}{x-2} = \lim_{x\to 2}\frac{3(x-2)(x+2)}{x-2}$

$= 3\lim_{x\to 2}(x+2) = 3(2+2) = 12$

32. $\lim_{x\to 2}\frac{(3x^2+2x+1)-17}{x-2} = \lim_{x\to 2}\frac{3x^2+2x-16}{x-2}$

$= \lim_{x\to 2}\frac{(3x+8)(x-2)}{x-2} = \lim_{x\to 2}(3x+8)$

$= 3\lim_{x\to 2} x + 8 = 3(2)+8 = 14$

33. $\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim_{x\to 2}\frac{\frac{2-x}{2x}}{x-2} = \lim_{x\to 2}\frac{-\frac{x-2}{2x}}{x-2}$

$= \lim_{x\to 2} -\frac{1}{2x} = \frac{-1}{2\lim_{x\to 2} x} = \frac{-1}{2(2)} = -\frac{1}{4}$

34. $\lim_{x\to 2}\frac{\frac{3}{x^2}-\frac{3}{4}}{x-2} = \lim_{x\to 2}\frac{\frac{3(4-x^2)}{4x^2}}{x-2} = \lim_{x\to 2}\frac{\frac{-3(x+2)(x-2)}{4x^2}}{x-2}$

$= \lim_{x\to 2}\frac{-3(x+2)}{4x^2} = \frac{-3\left(\lim_{x\to 2} x+2\right)}{4\left(\lim_{x\to 2} x\right)^2} = \frac{-3(2+2)}{4(2)^2}$

$= -\frac{3}{4}$

35. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$.

$|f(x)g(x)-LM| \le |g(x)||f(x)-L| + |L||g(x)-M|$

as shown in the text. Choose $\varepsilon_1 = 1$. Since $\lim_{x\to c} g(x) = M$, there is some $\delta_1 > 0$ such that if $0 < |x-c| < \delta_1$, $|g(x)-M| < \varepsilon_1 = 1$ or

$M - 1 < g(x) < M + 1$

$|M-1| \le |M|+1$ and $|M+1| \le |M|+1$ so for $0 < |x-c| < \delta_1, |g(x)| < |M|+1$. Choose $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, there exist δ_2 and δ_3 such that $0 < |x-c| < \delta_2 \Rightarrow$

$|f(x)-L| < \frac{\varepsilon}{|L|+|M|+1}$ and $0 < |x-c| < \delta_3 \Rightarrow$

$|g(x)-M| < \frac{\varepsilon}{|L|+|M|+1}$. Let

$\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then $0 < |x-c| < \delta \Rightarrow$

$|f(x)g(x)-LM| \le |g(x)||f(x)-L| + |L||g(x)-M|$

$< (|M|+1)\frac{\varepsilon}{|L|+|M|+1} + |L|\frac{\varepsilon}{|L|+|M|+1} = \varepsilon$

Hence,

$\lim_{x\to c} f(x)g(x) = LM = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

36. Say $\lim_{x\to c} g(x) = M$, $M \ne 0$, and choose

$\varepsilon_1 = \frac{1}{2}|M|$.

There is some $\delta_1 > 0$ such that

$0 < |x-c| < \delta_1 \Rightarrow |g(x)-M| < \varepsilon_1 = \frac{1}{2}|M|$ or

$M - \frac{1}{2}|M| < g(x) < M + \frac{1}{2}|M|$.

$\left|M - \frac{1}{2}|M|\right| \ge \left|\frac{1}{2}|M|\right|$ and $\left|M + \frac{1}{2}|M|\right| \ge \left|\frac{1}{2}|M|\right|$

so $|g(x)| > \frac{1}{2}|M|$ and $\frac{1}{|g(x)|} < \frac{2}{|M|}$

Choose $\varepsilon > 0$.

Since $\lim_{x\to c} g(x) = M$ there is $\delta_2 > 0$ such that

$0 < |x-c| < \delta_2 \Rightarrow |g(x)-M| < \frac{1}{2}M^2\varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$, then

$0 < |x-c| < \delta \Rightarrow \left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M-g(x)}{g(x)M}\right|$

$= \frac{1}{|M||g(x)|}|g(x)-M| < \frac{2}{M^2}|g(x)-M| = \frac{2}{M^2}\cdot\frac{1}{2}M^2\varepsilon$

$= \varepsilon$

Thus, $\lim\limits_{x \to c} \frac{1}{g(x)} = \frac{1}{M} = \frac{1}{\lim\limits_{x \to c} g(x)}$.

Using statement 6 and the above result,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} f(x) \cdot \lim_{x \to c} \frac{1}{g(x)}$$

$$= \lim_{x \to c} f(x) \cdot \frac{1}{\lim\limits_{x \to c} g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}.$$

37. $\lim\limits_{x \to c} f(x) = L \Leftrightarrow \lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} L$

$\Leftrightarrow \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} L = 0$

$\Leftrightarrow \lim\limits_{x \to c} [f(x) - L] = 0$

38. $\lim\limits_{x \to c} f(x) = 0 \Leftrightarrow \left[\lim\limits_{x \to c} f(x)\right]^2 = 0$

$\Leftrightarrow \lim\limits_{x \to c} f^2(x) = 0$

$\Leftrightarrow \sqrt{\lim\limits_{x \to c} f^2(x)} = 0$

$\Leftrightarrow \lim\limits_{x \to c} \sqrt{f^2(x)} = 0$

$\Leftrightarrow \lim\limits_{x \to c} |f(x)| = 0$

39. $\lim\limits_{x \to c} |x| = \sqrt{\left(\lim\limits_{x \to c} |x|\right)^2} = \sqrt{\lim\limits_{x \to c} |x|^2} = \sqrt{\lim\limits_{x \to c} x^2}$

$= \sqrt{\left(\lim\limits_{x \to c} x\right)^2} = \sqrt{c^2} = |c|$

40. **a.** If $f(x) = \frac{x+1}{x-2}, g(x) = \frac{x-5}{x-2}$ and $c = 2$, then $\lim\limits_{x \to c} [f(x) + g(x)]$ exists, but neither $\lim\limits_{x \to c} f(x)$ nor $\lim\limits_{x \to c} g(x)$ exists.

b. If $f(x) = \frac{2}{x}, g(x) = x,$ and $c = 0$, then $\lim\limits_{x \to c} [f(x) \cdot g(x)]$ exists, but $\lim\limits_{x \to c} f(x)$ does not exist.

41. $\lim\limits_{x \to -3^+} \frac{\sqrt{3+x}}{x} = \frac{\sqrt{3-3}}{-3} = 0$

42. $\lim\limits_{x \to -\pi^+} \frac{\sqrt{\pi^3 + x^3}}{x} = \frac{\sqrt{\pi^3 + (-\pi)^3}}{-\pi} = 0$

43. $\lim\limits_{x \to 3^+} \frac{x-3}{\sqrt{x^2-9}} = \lim\limits_{x \to 3^+} \frac{(x-3)\sqrt{x^2-9}}{x^2-9}$

$= \lim\limits_{x \to 3^+} \frac{(x-3)\sqrt{x^2-9}}{(x-3)(x+3)} = \lim\limits_{x \to 3^+} \frac{\sqrt{x^2-9}}{x+3}$

$= \frac{\sqrt{3^2-9}}{3+3} = 0$

44. $\lim\limits_{x \to 1^-} \frac{\sqrt{1+x}}{4+4x} = \frac{\sqrt{1+1}}{4+4(1)} = \frac{\sqrt{2}}{8}$

45. $\lim\limits_{x \to 2^+} \frac{(x^2+1)[\![x]\!]}{(3x-1)^2} = \frac{(2^2+1)[\![2]\!]}{(3 \cdot 2 - 1)^2} = \frac{5 \cdot 2}{5^2} = \frac{2}{5}$

46. $\lim\limits_{x \to 3^-} (x - [\![x]\!]) = \lim\limits_{x \to 3^-} x - \lim\limits_{x \to 3^-} [\![x]\!] = 3 - 2 = 1$

47. $\lim\limits_{x \to 0^-} \frac{x}{|x|} = -1$

48. $\lim\limits_{x \to 3^+} [\![x^2 + 2x]\!] = [\![3^2 + 2 \cdot 3]\!] = 15$

49. $f(x)g(x) = 1; g(x) = \frac{1}{f(x)}$

$\lim\limits_{x \to a} g(x) = 0 \Leftrightarrow \lim\limits_{x \to a} \frac{1}{f(x)} = 0$

$\Leftrightarrow \frac{1}{\lim\limits_{x \to a} f(x)} = 0$

No value satisfies this equation, so $\lim\limits_{x \to a} f(x)$ must not exist.

50. R has the vertices $\left(\pm\frac{x}{2}, \pm\frac{1}{2}\right)$

Each side of Q has length $\sqrt{x^2+1}$ so the perimeter of Q is $4\sqrt{x^2+1}$. R has two sides of length 1 and two sides of length $\sqrt{x^2}$ so the perimeter of R is $2 + 2\sqrt{x^2}$.

$\lim\limits_{x \to 0^+} \frac{\text{perimeter of } R}{\text{perimeter of } Q} = \lim\limits_{x \to 0^+} \frac{2\sqrt{x^2}+2}{4\sqrt{x^2+1}}$

$= \frac{2\sqrt{0^2}+2}{4\sqrt{0^2+1}} = \frac{2}{4} = \frac{1}{2}$

51. a. $NO = \sqrt{(0-0)^2 + (1-0)^2} = 1$

$OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$

$= \sqrt{x^2 + x}$

$NP = \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{x^2 + y^2 - 2y + 1}$

$= \sqrt{x^2 + x - 2\sqrt{x} + 1}$

$MO = \sqrt{(1-0)^2 + (0-0)^2} = 1$

$MP = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{y^2 + x^2 - 2x + 1}$

$= \sqrt{x^2 - x + 1}$

$$\lim_{x\to 0^+} \frac{\text{perimeter of } \Delta NOP}{\text{perimeter of } \Delta MOP}$$

$$= \lim_{x\to 0^+} \frac{1 + \sqrt{x^2 + x} + \sqrt{x^2 + x - 2\sqrt{x} + 1}}{1 + \sqrt{x^2 + x} + \sqrt{x^2 - x + 1}}$$

$$= \frac{1 + \sqrt{1}}{1 + \sqrt{1}} = 1$$

b. Area of $\Delta NOP = \frac{1}{2}(1)(x) = \frac{x}{2}$

Area of $\Delta MOP = \frac{1}{2}(1)(y) = \frac{\sqrt{x}}{2}$

$$\lim_{x\to 0^+} \frac{\text{area of } \Delta NOP}{\text{area of } \Delta MOP} = \lim_{x\to 0^+} \frac{\frac{x}{2}}{\frac{\sqrt{x}}{2}} = \lim_{x\to 0^+} \frac{x}{\sqrt{x}}$$

$$= \lim_{x\to 0^+} \sqrt{x} = 0$$

1.4 Concepts Review

1. 0

2. 1

3. the denominator is 0 when $t = 0$.

4. 1

Problem Set 1.4

1. $\lim_{x\to 0} \frac{\cos x}{x+1} = \frac{1}{1} = 1$

2. $\lim_{\theta\to \pi/2} \theta\cos\theta = \frac{\pi}{2}\cdot 0 = 0$

3. $\lim_{t\to 0} \frac{\cos^2 t}{1+\sin t} = \frac{\cos^2 0}{1+\sin 0} = \frac{1}{1+0} = 1$

4. $\lim_{x\to 0} \frac{3x\tan x}{\sin x} = \lim_{x\to 0} \frac{3x(\sin x/\cos x)}{\sin x} = \lim_{x\to 0} \frac{3x}{\cos x}$

$= \frac{0}{1} = 0$

5. $\lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}\cdot 1 = \frac{1}{2}$

6. $\lim_{\theta\to 0} \frac{\sin 3\theta}{2\theta} = \lim_{\theta\to 0} \frac{3}{2}\cdot\frac{\sin 3\theta}{3\theta} = \frac{3}{2}\lim_{\theta\to 0} \frac{\sin 3\theta}{3\theta}$

$= \frac{3}{2}\cdot 1 = \frac{3}{2}$

7. $\lim_{\theta\to 0} \frac{\sin 3\theta}{\tan\theta} = \lim_{\theta\to 0} \frac{\sin 3\theta}{\frac{\sin\theta}{\cos\theta}} = \lim_{\theta\to 0} \frac{\cos\theta\sin 3\theta}{\sin\theta}$

$$= \lim_{\theta\to 0}\left[\cos\theta\cdot 3\cdot\frac{\sin 3\theta}{3\theta}\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right]$$

$$= 3\lim_{\theta\to 0}\left[\cos\theta\cdot\frac{\sin 3\theta}{3\theta}\cdot\frac{1}{\frac{\sin\theta}{\theta}}\right] = 3\cdot 1\cdot 1\cdot 1 = 3$$

8. $\lim_{\theta\to 0} \frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\frac{\sin 5\theta}{\cos 5\theta}}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\sin 5\theta}{\cos 5\theta\sin 2\theta}$

$$= \lim_{\theta\to 0}\left[\frac{1}{\cos 5\theta}\cdot 5\cdot\frac{\sin 5\theta}{5\theta}\cdot\frac{1}{2}\cdot\frac{2\theta}{\sin 2\theta}\right]$$

$$= \frac{5}{2}\lim_{\theta\to 0}\left[\frac{1}{\cos 5\theta}\cdot\frac{\sin 5\theta}{5\theta}\cdot\frac{2\theta}{\sin 2\theta}\right]$$

$= \frac{5}{2}\cdot 1\cdot 1\cdot 1 = \frac{5}{2}$

9. $\lim_{\theta\to 0} \frac{\cot\pi\theta\sin\theta}{2\sec\theta} = \lim_{\theta\to 0} \frac{\frac{\cos\pi\theta}{\sin\pi\theta}\sin\theta}{\frac{2}{\cos\theta}}$

$$= \lim_{\theta\to 0} \frac{\cos\pi\theta\sin\theta\cos\theta}{2\sin\pi\theta}$$

$$= \lim_{\theta\to 0}\left[\frac{\cos\pi\theta\cos\theta}{2}\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{\pi}\cdot\frac{\pi\theta}{\sin\pi\theta}\right]$$

$$= \frac{1}{2\pi}\lim_{\theta\to 0}\left[\cos\pi\theta\cos\theta\cdot\frac{\sin\theta}{\theta}\cdot\frac{\pi\theta}{\sin\pi\theta}\right]$$

$= \frac{1}{2\pi}\cdot 1\cdot 1\cdot 1\cdot 1 = \frac{1}{2\pi}$

10. $\lim_{t\to 0} \frac{\sin^2 3t}{2t} = \lim_{t\to 0} \frac{9t}{2}\cdot\frac{\sin 3t}{3t}\cdot\frac{\sin 3t}{3t} = 0\cdot 1\cdot 1 = 0$

11. $\lim_{t\to 0} \frac{\tan^2 3t}{2t} = \lim_{t\to 0} \frac{\sin^2 3t}{(2t)(\cos^2 3t)}$

$= \lim_{t\to 0} \frac{3(\sin 3t)}{2\cos^2 3t}\cdot\frac{\sin 3t}{3t} = 0\cdot 1 = 0$

12. $\lim_{t\to 0} \frac{\tan 2t}{\sin 2t - 1} = \frac{0}{-1} = 0$

13. $\lim\limits_{t\to 0}\frac{\sin(3t)+4t}{t\sec t}=\lim\limits_{t\to 0}\left(\frac{\sin 3t}{t\sec t}+\frac{4t}{t\sec t}\right)$

$=\lim\limits_{t\to 0}\frac{\sin 3t}{t\sec t}+\lim\limits_{t\to 0}\frac{4t}{t\sec t}$

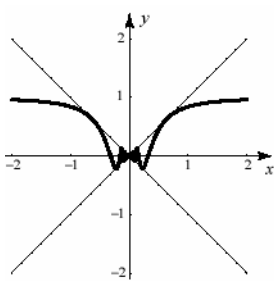
$=\lim\limits_{t\to 0}3\cos t\cdot\frac{\sin 3t}{3t}+\lim\limits_{t\to 0}4\cos t$

$=3\cdot 1+4=7$

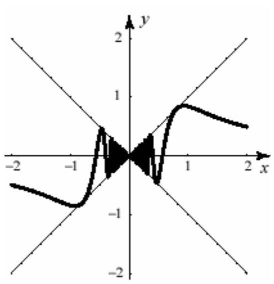
14. $\lim\limits_{\theta\to 0}\frac{\sin^2\theta}{\theta^2}=\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{\sin\theta}{\theta}$

$=\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}\times\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1\times 1=1$

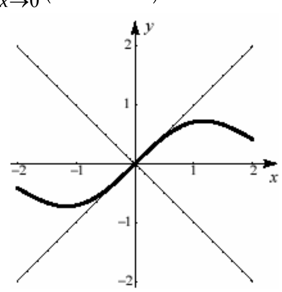
15. $\lim\limits_{x\to 0}x\sin(1/x)=0$



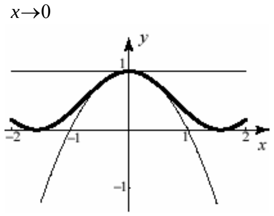
16. $\lim\limits_{x\to 0}x\sin\left(1/x^2\right)=0$



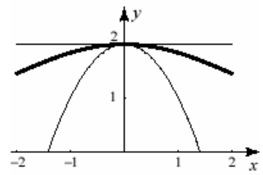
17. $\lim\limits_{x\to 0}\left(1-\cos^2 x\right)/x=0$



18. $\lim\limits_{x\to 0}\cos^2 x=1$



19. $\lim\limits_{x\to 0}1+\frac{\sin x}{x}=2$



20. The result that $\lim\limits_{t\to 0}\cos t=1$ was established in the proof of the theorem. Then

$\lim\limits_{t\to c}\cos t=\lim\limits_{h\to 0}\cos(c+h)$

$=\lim\limits_{h\to 0}(\cos c\cos h-\sin c\sin h)$

$=\lim\limits_{h\to 0}\cos c\lim\limits_{h\to 0}\cos h-\sin c\lim\limits_{h\to 0}\sin h$

$=\cos c$

21. $\lim\limits_{t\to c}\tan t=\lim\limits_{t\to c}\frac{\sin t}{\cos t}=\frac{\lim\limits_{t\to c}\sin t}{\lim\limits_{t\to c}\cos t}=\frac{\sin c}{\cos c}=\tan c$

$\lim\limits_{t\to c}\cot t=\lim\limits_{t\to c}\frac{\cos t}{\sin t}=\frac{\lim\limits_{t\to c}\cos t}{\lim\limits_{t\to c}\sin t}=\frac{\cos c}{\sin c}=\cot c$

22. $\lim\limits_{t\to c}\sec t=\lim\limits_{t\to c}\frac{1}{\cos t}=\frac{1}{\cos c}=\sec c$

$\lim\limits_{t\to c}\csc t=\lim\limits_{t\to c}\frac{1}{\sin t}=\frac{1}{\sin c}=\csc c$

23. $\overline{BP}=\sin t,\ \overline{OB}=\cos t$

$\text{area}(\Delta OBP)\le\text{area (sector }OAP)$

$\le\text{area }(\Delta OBP)+\text{area}(ABPQ)$

$\frac{1}{2}\overline{OB}\cdot\overline{BP}\le\frac{1}{2}t(1)^2\le\frac{1}{2}\overline{OB}\cdot\overline{BP}+(1-\overline{OB})\overline{BP}$

$\frac{1}{2}\sin t\cos t\le\frac{1}{2}t\le\frac{1}{2}\sin t\cos t+(1-\cos t)\sin t$

$\cos t\le\frac{t}{\sin t}\le 2-\cos t$

$\frac{1}{2-\cos t}\le\frac{\sin t}{t}\le\frac{1}{\cos t}$ for $-\frac{\pi}{2}<t<\frac{\pi}{2}$.

$\lim\limits_{t\to 0}\frac{1}{2-\cos t}\le\lim\limits_{t\to 0}\frac{\sin t}{t}\le\lim\limits_{t\to 0}\frac{1}{\cos t}$

$1\le\lim\limits_{t\to 0}\frac{\sin t}{t}\le 1$

Thus, $\lim\limits_{t\to 0}\frac{\sin t}{t}=1$.

24. a. Written response

b. $D = \frac{1}{2}\overline{AB} \cdot \overline{BP} = \frac{1}{2}(1-\cos t)\sin t$

$= \frac{\sin t(1-\cos t)}{2}$

$E = \frac{1}{2}t(1)^2 - \frac{1}{2}\overline{OB} \cdot \overline{BP} = \frac{t}{2} - \frac{\sin t \cos t}{2}$

$\frac{D}{E} = \frac{\sin t(1-\cos t)}{t - \sin t \cos t}$

c. $\lim_{t \to 0^+}\left(\frac{D}{E}\right) = 0.75$

1.5 Concepts Review

1. x increases without bound; $f(x)$ gets close to L as x increases without bound

2. $f(x)$ increases without bound as x approaches c from the right; $f(x)$ decreases without bound as x approaches c from the left

3. $y = 6$; horizontal

4. $x = 6$; vertical

Problem Set 1.5

1. $\lim_{x\to\infty} \frac{x}{x-5} = \lim_{x\to\infty} \frac{1}{1-\frac{5}{x}} = 1$

2. $\lim_{x\to\infty} \frac{x^2}{5-x^3} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{5}{x^3}-1} = 0$

3. $\lim_{t\to-\infty} \frac{t^2}{7-t^2} = \lim_{t\to-\infty} \frac{1}{\frac{7}{t^2}-1} = -1$

4. $\lim_{t\to-\infty} \frac{t}{t-5} = \lim_{t\to-\infty} \frac{1}{1-\frac{5}{t}} = 1$

5. $\lim_{x\to\infty} \frac{x^2}{(x-5)(3-x)} = \lim_{x\to\infty} \frac{x^2}{-x^2+8x-15}$

$= \lim_{x\to\infty} \frac{1}{-1+\frac{8}{x}-\frac{15}{x^2}} = -1$

6. $\lim_{x\to\infty} \frac{x^2}{x^2-8x+15} = \lim_{x\to\infty} \frac{1}{1-\frac{8}{x}+\frac{15}{x^2}} = 1$

7. $\lim_{x\to\infty} \frac{x^3}{2x^3-100x^2} = \lim_{x\to\infty} \frac{1}{2-\frac{100}{x}} = \frac{1}{2}$

8. $\lim_{\theta\to-\infty} \frac{\pi\theta^5}{\theta^5-5\theta^4} = \lim_{\theta\to-\infty} \frac{\pi}{1-\frac{5}{\theta}} = \pi$

9. $\lim_{x\to\infty} \frac{3x^3-x^2}{\pi x^3-5x^2} = \lim_{x\to\infty} \frac{3-\frac{1}{x}}{\pi-\frac{5}{x}} = \frac{3}{\pi}$

10. $\lim_{\theta\to\infty} \frac{\sin^2\theta}{\theta^2-5}$; $0 \le \sin^2\theta \le 1$ for all θ and

$\lim_{\theta\to\infty} \frac{1}{\theta^2-5} = \lim_{\theta\to\infty} \frac{\frac{1}{\theta^2}}{1-\frac{5}{\theta^2}} = 0$ so $\lim_{\theta\to\infty} \frac{\sin^2\theta}{\theta^2-5} = 0$

11. $\lim_{x\to\infty} \frac{3\sqrt{x^3}+3x}{\sqrt{2x^3}} = \lim_{x\to\infty} \frac{3x^{3/2}+3x}{\sqrt{2}x^{3/2}}$

$= \lim_{x\to\infty} \frac{3+\frac{3}{\sqrt{x}}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

12. $\lim_{x\to\infty} \sqrt[3]{\frac{\pi x^3+3x}{\sqrt{2}x^3+7x}} = \sqrt[3]{\lim_{x\to\infty} \frac{\pi x^3+3x}{\sqrt{2}x^3+7x}}$

$= \sqrt[3]{\lim_{x\to\infty} \frac{\pi+\frac{3}{x^2}}{\sqrt{2}+\frac{7}{x^2}}} = \sqrt[3]{\frac{\pi}{\sqrt{2}}}$

13. $\lim_{x\to\infty} \sqrt[3]{\frac{1+8x^2}{x^2+4}} = \sqrt[3]{\lim_{x\to\infty} \frac{1+8x^2}{x^2+4}}$

$= \sqrt[3]{\lim_{x\to\infty} \frac{\frac{1}{x^2}+8}{1+\frac{4}{x^2}}} = \sqrt[3]{8} = 2$

14. $\lim_{x\to\infty} \sqrt{\frac{x^2+x+3}{(x-1)(x+1)}} = \sqrt{\lim_{x\to\infty} \frac{x^2+x+3}{x^2-1}}$

$= \sqrt{\lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{3}{x^2}}{1-\frac{1}{x^2}}} = \sqrt{1} = 1$

15. $\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}$

16. $\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+0} = 1$

17. $\lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} n}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)} = \frac{\infty}{1+0} = \infty$

18. $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \frac{0}{1+0} = 0$

19. For $x > 0$, $x = \sqrt{x^2}$.

$\lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+3}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\frac{\sqrt{x^2+3}}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{3}{x^2}}}$

$= \frac{2}{\sqrt{1}} = 2$

20. $\lim_{x\to\infty} \frac{\sqrt{2x+1}}{x+4} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x+1}}{\sqrt{x^2}}}{1+\frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{2}{x}+\frac{1}{x^2}}}{1+\frac{4}{x}} = 0$

21. $\lim_{x\to\infty}\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)$

$= \lim_{x\to\infty} \frac{\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)\left(\sqrt{2x^2+3}+\sqrt{2x^2-5}\right)}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$

$= \lim_{x\to\infty} \frac{2x^2+3-(2x^2-5)}{\sqrt{2x^2+3}+\sqrt{2x^2-5}}$

$= \lim_{x\to\infty} \frac{8}{\sqrt{2x^2+3}+\sqrt{2x^2-5}} = \lim_{x\to\infty} \frac{\frac{8}{x}}{\frac{\sqrt{2x^2+3}+\sqrt{2x^2-5}}{\sqrt{x^2}}}$

$= \lim_{x\to\infty} \frac{\frac{8}{x}}{\sqrt{2+\frac{3}{x^2}}+\sqrt{2-\frac{5}{x^2}}} = 0$

22.

$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$

$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x}-x\right)\left(\sqrt{x^2+2x}+x\right)}{\sqrt{x^2+2x}+x}$

$= \lim_{x\to\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+2x}+x}$

$= \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{2}{x}}+1} = \frac{2}{2} = 1$

23. $\lim_{y\to-\infty} \frac{9y^3+1}{y^2-2y+2} = \lim_{y\to-\infty} \frac{9y+\frac{1}{y^2}}{1-\frac{2}{y}+\frac{2}{y^2}} = -\infty$

24. $\lim_{x\to\infty} \frac{a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n}{b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n}$

$= \lim_{x\to\infty} \frac{a_0+\frac{a_1}{x}+\ldots+\frac{a_{n-1}}{x^{n-1}}+\frac{a_n}{x^n}}{b_0+\frac{b_1}{x}+\ldots+\frac{b_{n-1}}{x^{n-1}}+\frac{b_n}{x^n}} = \frac{a_0}{b_0}$

25. $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+0}} = 1$

26. $\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+2n+1}} = \lim_{n\to\infty} \frac{\frac{n^2}{n^{3/2}}}{\sqrt{1+\frac{2}{n^2}+\frac{1}{n^3}}} = \frac{\infty}{1} = \infty$

27. As $x \to 4^+, x \to 4$ while $x-4 \to 0^+$.

$\lim_{x\to 4^+} \frac{x}{x-4} = \infty$

28. $\lim_{t\to -3^+} \frac{t^2-9}{t+3} = \lim_{t\to -3^+} \frac{(t+3)(t-3)}{t+3}$

$= \lim_{t\to -3^+} (t-3) = -6$

29. As $t \to 3^-, t^2 \to 9$ while $9-t^2 \to 0^+$.

$\lim_{t\to 3^-} \frac{t^2}{9-t^2} = \infty$

30. As $x \to \sqrt[3]{5}^+, x^2 \to 5^{2/3}$ while $5-x^3 \to 0^-$.

$\lim_{x\to \sqrt[3]{5}^+} \frac{x^2}{5-x^3} = -\infty$

31. As $x \to 5^-, x^2 \to 25, x-5 \to 0^-$, and $3-x \to -2$.

$\lim_{x\to 5^-} \frac{x^2}{(x-5)(3-x)} = \infty$

32. As $\theta \to \pi^+, \theta^2 \to \pi^2$ while $\sin\theta \to 0^-$.

$\lim_{\theta\to \pi^+} \frac{\theta^2}{\sin\theta} = -\infty$

33. As $x \to 3^-$, $x^3 \to 27$, while $x - 3 \to 0^-$.

$$\lim_{x \to 3^-} \frac{x^3}{x-3} = -\infty$$

34. As $\theta \to \frac{\pi}{2}^+$, $\pi\theta \to \frac{\pi^2}{2}$ while $\cos\theta \to 0^-$.

$$\lim_{\theta \to \frac{\pi}{2}^+} \frac{\pi\theta}{\cos\theta} = -\infty$$

35. $\displaystyle\lim_{x \to 3^-} \frac{x^2 - x - 6}{x-3} = \lim_{x \to 3^-} \frac{(x+2)(x-3)}{x-3}$

$\displaystyle= \lim_{x \to 3^-} (x+2) = 5$

36. $\displaystyle\lim_{x \to 2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \lim_{x \to 2^+} \frac{(x+4)(x-2)}{(x+2)(x-2)}$

$\displaystyle= \lim_{x \to 2^+} \frac{x+4}{x+2} = \frac{6}{4} = \frac{3}{2}$

37. For $0 \le x < 1$, $[\![x]\!] = 0$, so for $0 < x < 1$, $\dfrac{[\![x]\!]}{x} = 0$

thus $\displaystyle\lim_{x \to 0^+} \frac{[\![x]\!]}{x} = 0$

38. For $-1 \le x < 0$, $[\![x]\!] = -1$, so for $-1 < x < 0$,

$\dfrac{[\![x]\!]}{x} = -\dfrac{1}{x}$ thus $\displaystyle\lim_{x \to 0^-} \frac{[\![x]\!]}{x} = \infty$.

(Since $x < 0$, $-\dfrac{1}{x} > 0$.)

39. For $x < 0$, $|x| = -x$, thus

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

40. For $x > 0$, $|x| = x$, thus $\displaystyle\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$

41. As $x \to 0^-$, $1 + \cos x \to 2$ while $\sin x \to 0^-$.

$$\lim_{x \to 0^-} \frac{1 + \cos x}{\sin x} = -\infty$$

42. $-1 \le \sin x \le 1$ for all x, and

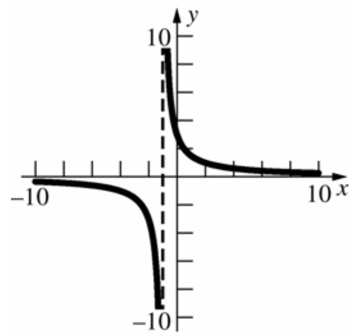
$\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0$, so $\displaystyle\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

43. $\displaystyle\lim_{x \to \infty} \frac{3}{x+1} = 0$, $\displaystyle\lim_{x \to -\infty} \frac{3}{x+1} = 0$;

Horizontal asymptote $y = 0$.

$\displaystyle\lim_{x \to -1^+} \frac{3}{x+1} = \infty$, $\displaystyle\lim_{x \to -1^-} \frac{3}{x+1} = -\infty$;

Vertical asymptote $x = -1$

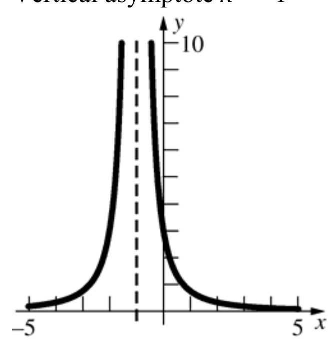


44. $\displaystyle\lim_{x \to \infty} \frac{3}{(x+1)^2} = 0$, $\displaystyle\lim_{x \to -\infty} \frac{3}{(x+1)^2} = 0$;

Horizontal asymptote $y = 0$.

$\displaystyle\lim_{x \to -1^+} \frac{3}{(x+1)^2} = \infty$, $\displaystyle\lim_{x \to -1^-} \frac{3}{(x+1)^2} = \infty$;

Vertical asymptote $x = -1$



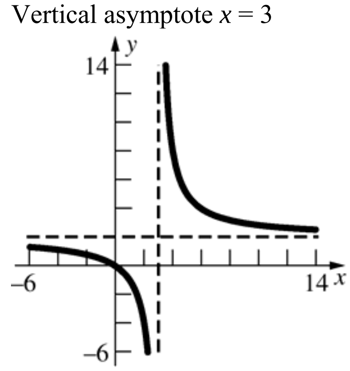
45. $\displaystyle\lim_{x \to \infty} \frac{2x}{x-3} = \lim_{x \to \infty} \frac{2}{1 - \frac{3}{x}} = 2$,

$\displaystyle\lim_{x \to -\infty} \frac{2x}{x-3} = \lim_{x \to -\infty} \frac{2}{1 - \frac{3}{x}} = 2$,

Horizontal asymptote $y = 2$

$\displaystyle\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$, $\displaystyle\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$;

Vertical asymptote $x = 3$



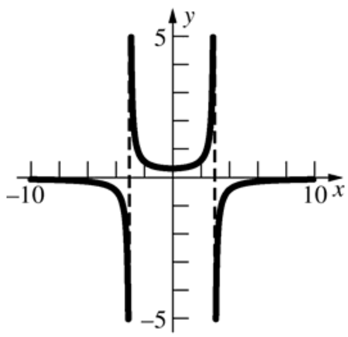
46. $\lim\limits_{x\to\infty}\dfrac{3}{9-x^2}=0,\ \lim\limits_{x\to-\infty}\dfrac{3}{9-x^2}=0;$

Horizontal asymptote $y = 0$

$\lim\limits_{x\to3^+}\dfrac{3}{9-x^2}=-\infty,\ \lim\limits_{x\to3^-}\dfrac{3}{9-x^2}=\infty,$

$\lim\limits_{x\to-3^+}\dfrac{3}{9-x^2}=\infty,\ \lim\limits_{x\to-3^-}\dfrac{3}{9-x^2}=-\infty;$

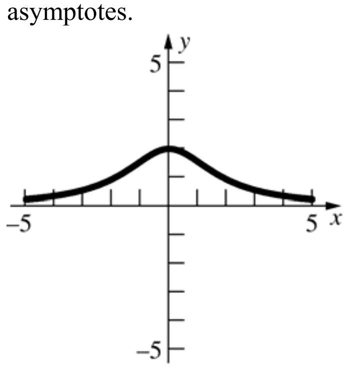
Vertical asymptotes $x = –3$, $x = 3$



47. $\lim\limits_{x\to\infty}\dfrac{14}{2x^2+7}=0,\ \lim\limits_{x\to-\infty}\dfrac{14}{2x^2+7}=0;$

Horizontal asymptote $y = 0$

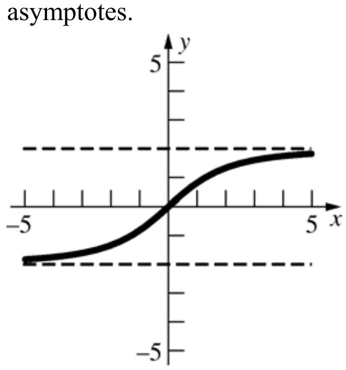
Since $2x^2+7>0$ for all x, $g(x)$ has no vertical asymptotes.



48. $\lim\limits_{x\to\infty}\dfrac{2x}{\sqrt{x^2+5}}=\lim\limits_{x\to\infty}\dfrac{2}{\sqrt{1+\frac{5}{x^2}}}=\dfrac{2}{\sqrt{1}}=2,$

$\lim\limits_{x\to-\infty}\dfrac{2x}{\sqrt{x^2+5}}=\lim\limits_{x\to-\infty}\dfrac{2}{-\sqrt{1+\frac{5}{x^2}}}=\dfrac{2}{-\sqrt{1}}=-2$

Since $\sqrt{x^2+5}>0$ for all x, $g(x)$ has no vertical asymptotes.



49. $f(x)=2x+3-\dfrac{1}{x^3-1}$, thus

$\lim\limits_{x\to\infty}[f(x)-(2x+3)]=\lim\limits_{x\to\infty}\left[-\dfrac{1}{x^3-1}\right]=0$

The oblique asymptote is $y = 2x + 3$.

50. $f(x)=3x+4-\dfrac{4x+3}{x^2+1}$, thus

$\lim\limits_{x\to\infty}[f(x)-(3x+4)]=\lim\limits_{x\to\infty}\left[-\dfrac{4x+3}{x^2+1}\right]$

$=\lim\limits_{x\to\infty}\left[-\dfrac{\frac{4}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}}\right]=0\,.$

The oblique asymptote is $y = 3x + 4$.

51. **a.** We say that $\lim\limits_{x\to c^+}f(x)=-\infty$ if to each negative number M there corresponds a $\delta > 0$ such that $0 < x - c < \delta \Rightarrow f(x) < M$.

b. We say that $\lim\limits_{x\to c^-}f(x)=\infty$ if to each positive number M there corresponds a $\delta > 0$ such that $0 < c - x < \delta \Rightarrow f(x) > M$.

52. **a.** We say that $\lim\limits_{x\to\infty}f(x)=\infty$ if to each positive number M there corresponds an $N > 0$ such that $N < x \Rightarrow f(x) > M$.

b. We say that $\lim\limits_{x\to-\infty}f(x)=\infty$ if to each positive number M there corresponds an $N < 0$ such that $x < N \Rightarrow f(x) > M$.

53. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to\infty}f(x)=A$, there is a corresponding number M_1 such that

$x>M_1\Rightarrow|f(x)-A|<\dfrac{\varepsilon}{2}$. Similarly, there is a

number M_2 such that $x>M_2\Rightarrow|g(x)-B|<\dfrac{\varepsilon}{2}$.

Let $M=\max\{M_1, M_2\}$, then

$x>M\Rightarrow|f(x)+g(x)-(A+B)|$

$=|f(x)-A+g(x)-B|\le|f(x)-A|+|g(x)-B|$

$<\dfrac{\varepsilon}{2}+\dfrac{\varepsilon}{2}=\varepsilon$

Thus, $\lim\limits_{x\to\infty}[f(x)+g(x)]=A+B$

54. Written response

55. a. $\lim_{x\to\infty} \sin x$ does not exist as $\sin x$ oscillates between –1 and 1 as x increases.

b. Let $u = \frac{1}{x}$, then as $x \to \infty, u \to 0^+$.

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{u\to 0^+} \sin u = 0$$

c. Let $u = \frac{1}{x}$, then as $x \to \infty, u \to 0^+$.

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{u\to 0^+} \frac{1}{u}\sin u = \lim_{u\to 0^+} \frac{\sin u}{u} = 1$$

d. Let $u = \frac{1}{x}$, then

$$\lim_{x\to\infty} x^{3/2} \sin\frac{1}{x} = \lim_{u\to 0^+} \left(\frac{1}{u}\right)^{3/2} \sin u$$

$$= \lim_{u\to 0^+} \left[\left(\frac{1}{\sqrt{u}}\right)\left(\frac{\sin u}{u}\right)\right] = \infty$$

e. As $x \to \infty$, $\sin x$ oscillates between –1 and 1, while $x^{-1/2} = \frac{1}{\sqrt{x}} \to 0$.

$$\lim_{x\to\infty} x^{-1/2} \sin x = 0$$

f. Let $u = \frac{1}{x}$, then

$$\lim_{x\to\infty} \sin\left(\frac{\pi}{6} + \frac{1}{x}\right) = \lim_{u\to 0^+} \sin\left(\frac{\pi}{6} + u\right)$$

$$= \sin\frac{\pi}{6} = \frac{1}{2}$$

g. As $x \to \infty, x + \frac{1}{x} \to \infty$, so $\lim_{x\to\infty} \sin\left(x + \frac{1}{x}\right)$ does not exist. (See part a.)

h. $\sin\left(x + \frac{1}{x}\right) = \sin x \cos\frac{1}{x} + \cos x \sin\frac{1}{x}$

$$\lim_{x\to\infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right]$$

$$= \lim_{x\to\infty} \left[\sin x\left(\cos\frac{1}{x} - 1\right) + \cos x \sin\frac{1}{x}\right]$$

As $x \to \infty, \cos\frac{1}{x} \to 1$ so $\cos\frac{1}{x} - 1 \to 0$.

From part **b**., $\lim_{x\to\infty} \sin\frac{1}{x} = 0$.

As $x \to \infty$ both $\sin x$ and $\cos x$ oscillate between –1 and 1.

$$\lim_{x\to\infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right] = 0.$$

56. $\lim_{v\to c^-} m(v) = \lim_{v\to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \infty$

57. $\lim_{x\to\infty} \frac{3x^2 + x + 1}{2x^2 - 1} = \frac{3}{2}$

58. $\lim_{x\to -\infty} \sqrt{\frac{2x^2 - 3x}{5x^2 + 1}} = \sqrt{\frac{2}{5}}$

59. $\lim_{x\to -\infty} \left(\sqrt{2x^2 + 3x} - \sqrt{2x^2 - 5}\right) = -\frac{3}{2\sqrt{2}}$

60. $\lim_{x\to\infty} \frac{2x + 1}{\sqrt{3x^2 + 1}} = \frac{2}{\sqrt{3}}$

61. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{10} = 1$

62. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x} = e \approx 2.718$

63. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x^2} = \infty$

64. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{\sin x} = 1$

65. $\lim_{x\to 3^-} \frac{\sin|x - 3|}{x - 3} = -1$

66. $\lim_{x\to 3^-} \frac{\sin|x - 3|}{\tan(x - 3)} = -1$

67. $\lim_{x\to 3^-} \frac{\cos(x - 3)}{x - 3} = -\infty$

68. $\lim_{x\to \frac{\pi}{2}^+} \frac{\cos x}{x - \frac{\pi}{2}} = -1$

69. $\lim_{x\to 0^+} (1 + \sqrt{x})^{\frac{1}{\sqrt{x}}} = e \approx 2.718$

70. $\lim_{x\to 0^+} (1 + \sqrt{x})^{1/x} = \infty$

71. $\lim_{x\to 0^+} (1 + \sqrt{x})^{x} = 1$

1.6 Concepts Review

1. $\lim_{x \to c} f(x)$

2. every integer

3. $\lim_{x \to a^+} f(x) = f(a);\ \lim_{x \to b^-} f(x) = f(b)$

4. a; b; $f(c) = W$

Problem Set 1.6

1. $\lim_{x \to 3}[(x-3)(x-4)] = 0 = f(3);$ continuous

2. $\lim_{x \to 3}(x^2 - 9) = 0 = g(3);$ continuous

3. $\lim_{x \to 3} \frac{3}{x-3}$ and $h(3)$ do not exist, so $h(x)$ is not continuous at 3.

4. $\lim_{t \to 3} \sqrt{t-4}$ and $g(3)$ do not exist, so $g(t)$ is not continuous at 3.

5. $\lim_{t \to 3} \frac{|t-3|}{t-3}$ and $h(3)$ do not exist, so $h(t)$ is not continuous at 3.

6. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

7. $\lim_{t \to 3} |t| = 3 = f(3);$ continuous

8. $\lim_{t \to 3} |t-2| = 1 = g(3);$ continuous

9. $h(3)$ does not exist, so $h(t)$ is not continuous at 3.

10. $f(3)$ does not exist, so $f(x)$ is not continuous at 3.

11. $\lim_{t \to 3} \frac{t^3 - 27}{t-3} = \lim_{t \to 3} \frac{(t-3)(t^2+3t+9)}{t-3}$
$= \lim_{t \to 3}(t^2 + 3t + 9) = (3)^2 + 3(3) + 9 = 27 = r(3)$
continuous

12. From Problem 11, $\lim_{t \to 3} r(t) = 27,$ so $r(t)$ is not continuous at 3 because $\lim_{t \to 3} r(t) \neq r(3).$

13. $\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+} (3-t) = 0$
$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-} (t-3) = 0$
$\lim_{t \to 3} f(t) = f(3);$ continuous

14. $\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+} (3-t)^2 = 0$
$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-} (t^2 - 9) = 0$
$\lim_{t \to 3} f(t) = f(3);$ continuous

15. $\lim_{t \to 3} f(x) = -2 = f(3);$ continuous

16. g is discontinuous at $x = -3, 4, 6, 8$; g is left continuous at $x = 4, 8$; g is right continuous at $x = -3, 6$

17. h is continuous on the intervals $(-\infty, -5),\ [-5, 4],\ (4, 6),\ [6, 8],\ (8, \infty)$

18. $\lim_{x \to 7} \frac{x^2 - 49}{x-7} = \lim_{x \to 7} \frac{(x-7)(x+7)}{x-7} = \lim_{x \to 7}(x+7)$
$= 7 + 7 = 14$
Define $f(7) = 14$.

19. $\lim_{x \to 3} \frac{2x^2 - 18}{3-x} = \lim_{x \to 3} \frac{2(x+3)(x-3)}{3-x}$
$= \lim_{x \to 3}[-2(x+3)] = -2(3+3) = -12$
Define $f(3) = -12$.

20. $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$
Define $g(0) = 1$

21. $\lim_{t \to 1} \frac{\sqrt{t}-1}{t-1} = \lim_{t \to 1} \frac{(\sqrt{t}-1)(\sqrt{t}+1)}{(t-1)(\sqrt{t}+1)}$
$= \lim_{t \to 1} \frac{t-1}{(t-1)(\sqrt{t}+1)} = \lim_{t \to 1} \frac{1}{\sqrt{t}+1} = \frac{1}{2}$
Define $H(1) = \frac{1}{2}$.

22. $\lim_{x \to -1} \frac{x^4 + 2x^2 - 3}{x+1} = \lim_{x \to -1} \frac{(x^2-1)(x^2+3)}{x+1}$
$= \lim_{x \to -1} \frac{(x+1)(x-1)(x^2+3)}{x+1}$
$= \lim_{x \to -1} [(x-1)(x^2+3)]$
$= (-1-1)[(-1)^2 + 3] = -8$
Define $\phi(-1) = -8$.

23. $\lim_{x \to -1} \sin\left(\frac{x^2-1}{x+1}\right) = \lim_{x \to -1} \sin\left(\frac{(x-1)(x+1)}{x+1}\right)$

$= \lim_{x \to -1} \sin(x-1) = \sin(-1-1) = \sin(-2) = -\sin 2$

Define $F(-1) = -\sin 2$.

24. Discontinuous at $x = \pi, 30$

25. $f(x) = \frac{33-x^2}{(\pi - x)(x-3)}$

Discontinuous at $x = 3, \pi$

26. Continuous at all points

27. Discontinuous at all $\theta = n\pi + \frac{\pi}{2}$ where n is any integer.

28. Discontinuous at all $u \le -5$

29. Discontinuous at $u = -1$

30. Continuous at all points

31. $G(x) = \frac{1}{\sqrt{(2-x)(2+x)}}$

Discontinuous on $(-\infty, -2] \cup [2, \infty)$

32. Continuous at all points since $\lim_{x \to 0} f(x) = 0 = f(0)$ and $\lim_{x \to 1} f(x) = 1 = f(1)$.

33. $\lim_{x \to 0} g(x) = 0 = g(0)$

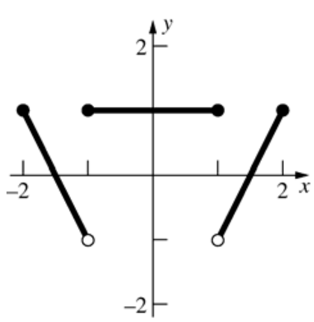
$\lim_{x \to 1^+} g(x) = 1,\ \lim_{x \to 1^-} g(x) = -1$

$\lim_{x \to 1} g(x)$ does not exist, so $g(x)$ is discontinuous at $x = 1$.

34. Discontinuous at every integer

35. Discontinuous at $t = n + \frac{1}{2}$ where n is any integer

36.



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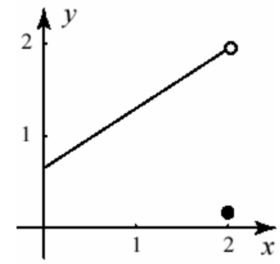

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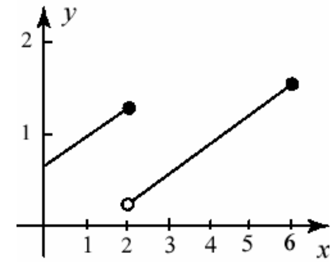

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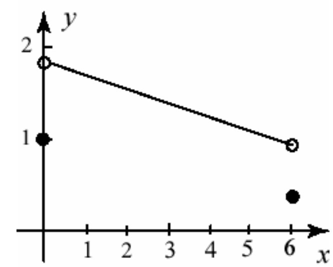

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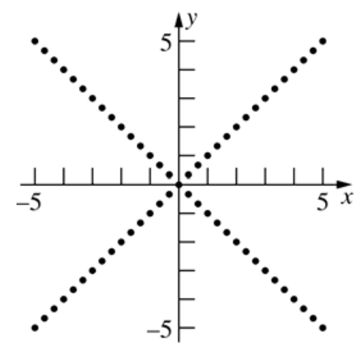


Discontinuous at all points except $x = 0$, because $\lim_{x \to c} f(x) \ne f(c)$ for $c \ne 0$. $\lim_{x \to c} f(x)$ exists only at $c = 0$ and $\lim_{x \to 0} f(x) = 0 = f(0)$.

41. Continuous.

42. Discontinuous: removable, define $f(10) = 20$

43. Discontinuous: removable, define $f(0) = 1$

44. Discontinuous: nonremovable.

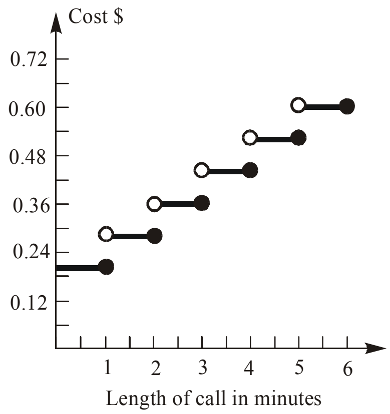
45. Discontinuous, removable, redefine $g(0) = 1$

46. Discontinuous: removable, define $F(0) = 0$

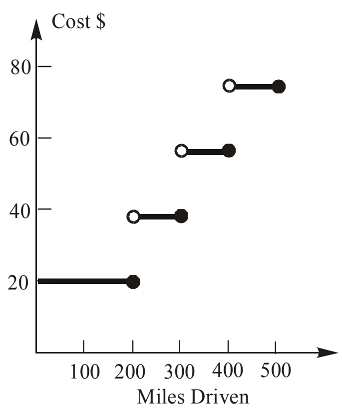
47. Discontinuous: nonremovable.

48. Discontinuous: removable, define $f(4) = 4$

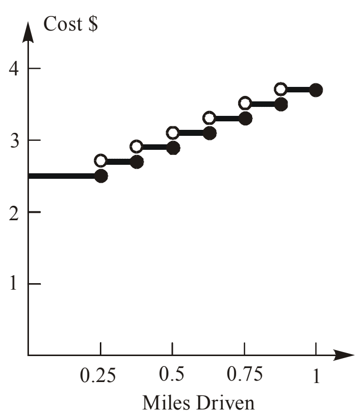
49. The function is continuous on the intervals $(0,1], (1,2], (2,3], \ldots$



50. The function is continuous on the intervals $[0,200], (200,300], (300,400], \ldots$



51. The function is continuous on the intervals $(0,0.25], (0.25,0.375], (0.375,0.5], \ldots$



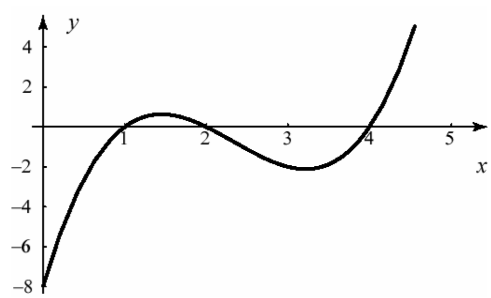
52. Let $f(x) = x^3 + 3x - 2$. f is continuous on [0, 1]. $f(0) = -2 < 0$ and $f(1) = 2 > 0$. Thus, there is at least one number c between 0 and 1 such that $x^3 + 3x - 2 = 0$.

53. Because the function is continuous on $[0, 2\pi]$ and

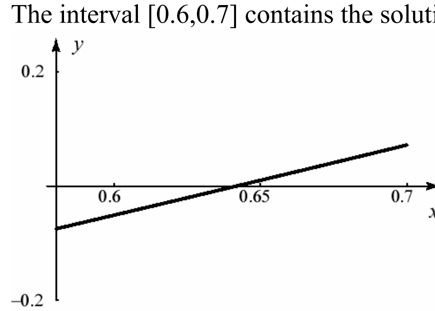
$(\cos 0)0^3 + 6\sin^5 0 - 3 = -3 < 0,$

$(\cos 2\pi)(2\pi)^3 + 6\sin^5(2\pi) - 3 = 8\pi^3 - 3 > 0,$ there is at least one number c between 0 and 2π such that $(\cos t)t^3 + 6\sin^5 t - 3 = 0$.

54. Let $f(x) = x^3 - 7x^2 + 14x - 8$. $f(x)$ is continuous at all values of x.
$f(0) = -8$, $f(5) = 12$
Because 0 is between –8 and 12, there is at least one number c between 0 and 5 such that
$f(x) = x^3 - 7x^2 + 14x - 8 = 0$.
This equation has three solutions ($x = 1,2,4$)



55. Let $f(x) = \sqrt{x} - \cos x$.. $f(x)$ is continuous at all values of $x \geq 0$. $f(0) = -1$, $f(\pi/2) = \sqrt{\pi/2}$
Because 0 is between –1 and $\sqrt{\pi/2}$, there is at least one number c between 0 and $\pi/2$ such that
$f(x) = \sqrt{x} - \cos x = 0$.
The interval [0.6,0.7] contains the solution.



56. Let $f(x) = x^5 + 4x^3 - 7x + 14$
$f(x)$ is continuous at all values of x.
$f(-2) = -36$, $f(0) = 14$
Because 0 is between –36 and 14, there is at least one number c between –2 and 0 such that
$f(x) = x^5 + 4x^3 - 7x + 14 = 0$.

57. Suppose that f is continuous at c, so $\lim_{x\to c} f(x) = f(c)$. Let $x = c + t$, so $t = x – c$, then as $x \to c$, $t \to 0$ and the statement $\lim_{x\to c} f(x) = f(c)$ becomes $\lim_{t\to 0} f(t+c) = f(c)$. Suppose that $\lim_{t\to 0} f(t+c) = f(c)$ and let $x = t + c$, so $t = x – c$. Since c is fixed, $t \to 0$ means that $x \to c$ and the statement $\lim_{t\to 0} f(t+c) = f(c)$ becomes $\lim_{x\to c} f(x) = f(c)$, so f is continuous at c.

58. Since $f(x)$ is continuous at c, $\lim_{x\to c} f(x) = f(c) > 0$. Choose $\varepsilon = f(c)$, then there exists a $\delta > 0$ such that

$0 < |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

Thus, $f(x) - f(c) > -\varepsilon = -f(c)$, or $f(x) > 0$.

Since also $f(c) > 0$, $f(x) > 0$ for all x in $(c-\delta, c+\delta)$.

59. Let $g(x) = x – f(x)$. Then, $g(0) = 0 – f(0) = –f(0) \le 0$ and $g(1) = 1 – f(1) \ge 0$ since $0 \le f(x) \le 1$ on $[0, 1]$. If $g(0) = 0$, then $f(0) = 0$ and $c = 0$ is a fixed point of f. If $g(1) = 0$, then $f(1) = 1$ and $c = 1$ is a fixed point of f. If neither $g(0) = 0$ nor $g(1) = 0$, then $g(0) < 0$ and $g(1) > 0$ so there is some c in [0, 1] such that $g(c) = 0$. If $g(c) = 0$ then $c – f(c) = 0$ or $f(c) = c$ and c is a fixed point of f.

60. For $f(x)$ to be continuous everywhere, $f(1) = a(1) + b = 2$ and $f(2) = 6 = a(2) + b$

$$\begin{array}{r} a+b=2 \\ \underline{2a+b=6} \\ -a=-4 \\ a=4,\quad b=-2 \end{array}$$

61. For x in [0, 1], let $f(x)$ indicate where the string originally at x ends up. Thus $f(0) = a$, $f(1) = b$. $f(x)$ is continuous since the string is unbroken. Since $0 \le a,\ b \le 1$, $f(x)$ satisfies the conditions of Problem 59, so there is some c in [0, 1] with $f(c) = c$, i.e., the point of string originally at c ends up at c.

62. The Intermediate Value Theorem does not imply the existence of a number c between –2 and 2 such that $f(c) = 0$. The reason is that the function $f(x)$ is not continuous on $[-2, 2]$.

63. Let $f(x)$ be the difference in times on the hiker's watch where x is a point on the path, and suppose $x = 0$ at the bottom and $x = 1$ at the top of the mountain.
So $f(x)$ = (time on watch on the way up) – (time on watch on the way down).
$f(0) = 4 – 11 = –7$, $f(1) = 12 – 5 = 7$. Since time is continuous, $f(x)$ is continuous, hence there is some c between 0 and 1 where $f(c) = 0$. This c is the point where the hiker's watch showed the same time on both days.

64. Let f be the function on $\left[0, \frac{\pi}{2}\right]$ such that $f(\theta)$ is the length of the side of the rectangle which makes angle θ with the x-axis minus the length of the sides perpendicular to it. f is continuous on $\left[0, \frac{\pi}{2}\right]$. If $f(0) = 0$ then the region is circumscribed by a square. If $f(0) \neq 0$, then observe that $f(0) = -f\left(\frac{\pi}{2}\right)$. Thus, by the Intermediate Value Theorem, there is an angle θ_0 between 0 and $\frac{\pi}{2}$ such that $f(\theta_0) = 0$. Hence, D can be circumscribed by a square.

65. Yes, g is continuous at R.

$$\lim_{r\to R^-} g(r) = \frac{GMm}{R^2} = \lim_{r\to R^+} g(r)$$

66. No. By the Intermediate Value Theorem, if f were to change signs on $[a,b]$, then f must be 0 at some c in $[a,b]$. Therefore, f cannot change sign.

67. **a.** $f(x) = f(x + 0) = f(x) + f(0)$, so $f(0) = 0$. We want to prove that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, $\lim_{x\to c} [f(x) - f(c)] = 0$. But $f(x) – f(c) = f(x – c)$, so $\lim_{x\to c} [f(x) - f(c)] = \lim_{x\to c} f(x-c)$. Let $h = x – c$ then as $x \to c$, $h \to 0$ and $\lim_{x\to c} f(x-c) = \lim_{h\to 0} f(h) = f(0) = 0$. Hence $\lim_{x\to c} f(x) = f(c)$ and f is continuous at c. Thus, f is continuous everywhere, since c was arbitrary.

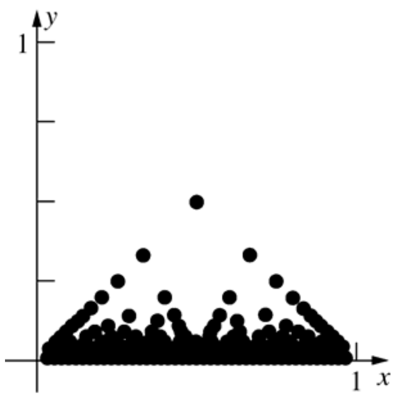
b. By Problem 43 of Section 0.5, $f(t) = mt$ for all t in Q. Since $g(t) = mt$ is a polynomial function, it is continuous for all real numbers. $f(t) = g(t)$ for all t in Q, thus $f(t) = g(t)$ for all t in R, i.e. $f(t) = mt$.

68. If $f(x)$ is continuous on an interval then $\lim_{x \to c} f(x) = f(c)$ for all points in the interval:

$$\lim_{x \to c} f(x) = f(c) \Rightarrow \lim_{x \to c} |f(x)|$$
$$= \lim_{x \to c} \sqrt{f^2(x)} = \sqrt{\left(\lim_{x \to c} f(x)\right)^2}$$
$$= \sqrt{(f(c))^2} = |f(c)|$$

69. Suppose $f(x) = \begin{cases} 1 \text{ if } x \ge 0 \\ -1 \text{ if } x < 0 \end{cases}$. $f(x)$ is discontinuous at $x = 0$, but $g(x) = |f(x)| = 1$ is continuous everywhere.

70. **a.**



b. If r is any rational number, then any deleted interval about r contains an irrational number. Thus, if $f(r) = \frac{1}{q}$, any deleted interval about r contains at least one point c such that $|f(r) - f(c)| = \left|\frac{1}{q} - 0\right| = \frac{1}{q}$. Hence, $\lim_{x \to r} f(x)$ does not exist.
If c is any irrational number in (0, 1), then as $x = \frac{p}{q} \to c$ (where $\frac{p}{q}$ is the reduced form of the rational number) $q \to \infty$, so $f(x) \to 0$ as $x \to c$. Thus, $\lim_{x \to c} f(x) = 0 = f(c)$ for any irrational number c.

71. **a.** Suppose the block rotates to the left. Using geometry, $f(x) = -\frac{3}{4}$. Suppose the block rotates to the right. Using geometry, $f(x) = \frac{3}{4}$. If $x = 0$, the block does not rotate, so $f(x) = 0$.

Domain: $\left[-\frac{3}{4}, \frac{3}{4}\right]$;

Range: $\left\{-\frac{3}{4}, 0, \frac{3}{4}\right\}$

b. At $x = 0$

c. If $x = 0$, $f(x) = 0$, if $x = -\frac{3}{4}$, $f(x) = -\frac{3}{4}$ and if $x = \frac{3}{4}$, $f(x) = \frac{3}{4}$, so $x = -\frac{3}{4}, 0, \frac{3}{4}$ are fixed points of f.

1.7 Chapter Review

Concepts Test

1. False. Consider $f(x) = [\![x]\!]$ at $x = 2$.

2. False: c may not be in the domain of $f(x)$, or it may be defined separately.

3. False: c may not be in the domain of $f(x)$, or it may be defined separately.

4. True. By definition, where $c = 0, L = 0$.

5. False: If $f(c)$ is not defined, $\lim_{x \to c} f(x)$ might exist; e.g., $f(x) = \frac{x^2 - 4}{x + 2}$.

$f(-2)$ does not exist, but $\lim_{x \to -2} \frac{x^2 - 4}{x + 2} = -4$.

6. True: $\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5}$
$= \lim_{x \to 5} (x+5) = 5 + 5 = 10$

7. True: Substitution Theorem

8. False: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

9. False: The tangent function is not defined for all values of c.

10. True: If x is in the domain of $\tan x = \frac{\sin x}{\cos x}$, then $\cos x \ne 0$, and Theorem A.7 applies..

11. True: Since both $\sin x$ and $\cos x$ are continuous for all real numbers, by Theorem C we can conclude that $f(x) = 2\sin^2 x - \cos x$ is also continuous for all real numbers.

12. True. By definition, $\lim_{x\to c} f(x) = f(c)$.

13. True. $2 \in [1,3]$

14. False: $\lim_{x\to 0^-}$ may not exist

15. False: Consider $f(x) = \sin x$.

16. True. By the definition of continuity on an interval.

17. False: Since $-1 \le \sin x \le 1$ for all x and $\lim_{x\to\infty} \frac{1}{x} = 0$, we get $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

18. False. It could be the case where $\lim_{x\to-\infty} f(x) = 2$

19. False: The graph has many vertical asymptotes; e.g., $x = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

20. True: $x = 2$; $x = -2$

21. True: As $x \to 1^+$ both the numerator and denominator are positive. Since the numerator approaches a constant and the denominator approaches zero, the limit goes to $+\infty$.

22. False: $\lim_{x\to c} f(x)$ must equal $f(c)$ for f to be continuous at $x = c$.

23. True: $\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right) = f(c)$, so f is continuous at $x = c$.

24. True: $\lim_{x\to 2.3} \left[\!\left[\frac{x}{2} \right]\!\right] = 1 = f(2.3)$

25. True: Choose $\varepsilon = 0.001 f(2)$ then since $\lim_{x\to 2} f(x) = f(2)$, there is some δ such that $0 < |x-2| < \delta \Rightarrow$ $|f(x) - f(2)| < 0.001 f(2)$, or $-0.001 f(2) < f(x) - f(2)$ $< 0.001 f(2)$
Thus, $0.999 f(2) < f(x) < 1.001 f(2)$ and $f(x) < 1.001 f(2)$ for $0 < |x-2| < \delta$.
Since $f(2) < 1.001 f(2)$, as $f(2) > 0$, $f(x) < 1.001 f(2)$ on $(2-\delta,\ 2+\delta)$.

26. False: That $\lim_{x\to c} [f(x) + g(x)]$ exists does not imply that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist; e.g., $f(x) = \frac{x-3}{x+2}$ and $g(x) = \frac{x+7}{x+2}$ for $c = -2$.

27. True: Squeeze Theorem

28. True: A function has only one limit at a point, so if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, $L = M$

29. False: That $f(x) \neq g(x)$ for all x does not imply that $\lim_{x\to c} f(x) \neq \lim_{x\to c} g(x)$. For example, if $f(x) = \frac{x^2 + x - 6}{x-2}$ and $g(x) = \frac{5}{2}x$, then $f(x) \neq g(x)$ for all x, but $\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = 5$.

30. False: If $f(x) < 10$, $\lim_{x\to 2} f(x)$ could equal 10 if there is a discontinuity point (2, 10). For example,
$$f(x) = \frac{-x^3 + 6x^2 - 2x - 12}{x-2} < 10 \text{ for}$$
all x, but $\lim_{x\to 2} f(x) = 10$.

31. True: $\lim_{x\to a} |f(x)| = \lim_{x\to a} \sqrt{f^2(x)}$
$$= \sqrt{\left[\lim_{x\to a} f(x)\right]^2} = \sqrt{(b)^2} = |b|$$

32. True: If f is continuous and positive on $[a, b]$, the reciprocal is also continuous, so it will assume all values between $\frac{1}{f(a)}$ and $\frac{1}{f(b)}$.

Sample Test Problems

1. $\lim_{x\to 2}\frac{x-2}{x+2}=\frac{2-2}{2+2}=\frac{0}{4}=0$

2. $\lim_{u\to 1}\frac{u^2-1}{u+1}=\frac{1^2-1}{1+1}=0$

3. $\lim_{u\to 1}\frac{u^2-1}{u-1}=\lim_{u\to 1}\frac{(u-1)(u+1)}{u-1}=\lim_{u\to 1}(u+1)$
$= 1 + 1 = 2$

4. $\lim_{u\to 1}\frac{u+1}{u^2-1}=\lim_{u\to 1}\frac{u+1}{(u+1)(u-1)}=\lim_{u\to 1}\frac{1}{u-1}$;
does not exist

5. $\lim_{x\to 2}\frac{1-\frac{2}{x}}{x^2-4}=\lim_{x\to 2}\frac{\frac{x-2}{x}}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x(x+2)}$
$=\frac{1}{2(2+2)}=\frac{1}{8}$

6. $\lim_{z\to 2}\frac{z^2-4}{z^2+z-6}=\lim_{z\to 2}\frac{(z+2)(z-2)}{(z+3)(z-2)}$
$=\lim_{z\to 2}\frac{z+2}{z+3}=\frac{2+2}{2+3}=\frac{4}{5}$

7. $\lim_{x\to 0}\frac{\tan x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{2\sin x\cos x}=\lim_{x\to 0}\frac{1}{2\cos^2 x}$
$=\frac{1}{2\cos^2 0}=\frac{1}{2}$

8. $\lim_{y\to 1}\frac{y^3-1}{y^2-1}=\lim_{y\to 1}\frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}$
$=\lim_{y\to 1}\frac{y^2+y+1}{y+1}=\frac{1^2+1+1}{1+1}=\frac{3}{2}$

9. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2}$
$=\lim_{x\to 4}(\sqrt{x}+2)=\sqrt{4}+2=4$

10. $\lim_{x\to 0}\frac{\cos x}{x}$ does not exist.

11. $\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$

12. $\lim_{x\to (1/2)^+}[\![4x]\!]=2$

13. $\lim_{t\to 2^-}\left([\![t]\!]-t\right)=\lim_{t\to 2^-}[\![t]\!]-\lim_{t\to 2^-}t=1-2=-1$

14. $\lim_{x\to 1^-}\frac{|x-1|}{x-1}=\lim_{x\to 1^-}\frac{1-x}{x-1}=-1$ since $x-1<0$ as
$x\to 1^-$

15. $\lim_{x\to 0}\frac{\sin 5x}{3x}=\lim_{x\to 0}\frac{5}{3}\frac{\sin 5x}{5x}$
$=\frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x}=\frac{5}{3}\times 1=\frac{5}{3}$

16. $\lim_{x\to 0}\frac{1-\cos 2x}{3x}=\lim_{x\to 0}\frac{2}{3}\frac{1-\cos 2x}{2x}$
$=\frac{2}{3}\lim_{x\to 0}\frac{1-\cos 2x}{2x}=\frac{2}{3}\times 0=0$

17. $\lim_{x\to\infty}\frac{x-1}{x+2}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{2}{x}}=\frac{1+0}{1+0}=1$

18. Since $-1\le \sin t\le 1$ for all t and $\lim_{t\to\infty}\frac{1}{t}=0$, we
get $\lim_{t\to\infty}\frac{\sin t}{t}=0$.

19. $\lim_{t\to 2}\frac{t+2}{(t-2)^2}=\infty$ because as $t\to 0,\ t+2\to 4$
while the denominator goes to 0 from the right.

20. $\lim_{x\to 0^+}\frac{\cos x}{x}=\infty$, because as $x\to 0^+$, $\cos x\to 1$
while the denominator goes to 0 from the right.

21. $\lim_{x\to \pi/4^-}\tan 2x=\infty$ because as $x\to(\pi/4)^-$,
$2x\to(\pi/2)^-$, so $\tan 2x\to\infty$.

22. $\lim_{x\to 0^+}\frac{1+\sin x}{x}=\infty$, because as $x\to 0^+$,
$1+\sin x\to 1$ while the denominator goes to 0 from the right.

23. Preliminary analysis: Let $\varepsilon > 0$. We need to find a $\delta > 0$ such that
$0<|x-3|<\delta \Rightarrow |(2x+1)-7|<\varepsilon.$
$|2x-6|<\varepsilon \Leftrightarrow 2|x-3|<\varepsilon$
$\Leftrightarrow |x-3|<\frac{\varepsilon}{2}.$ Choose $\delta=\frac{\varepsilon}{2}.$

Let $\varepsilon > 0$. Choose $\delta=\varepsilon/2$. Thus,

$|(2x+1)-7|=|2x-6|=2|x-3|<2(\varepsilon/2)=\varepsilon.$

24. **a.** $f(1) = 0$

b. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1-x) = 0$

c. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$

d. $\lim_{x\to -1} f(x) = -1$ because

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^3 = -1$ and

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = -1$

25. **a.** f is discontinuous at $x = 1$ because $f(1) = 0$, but $\lim_{x\to 1} f(x)$ does not exist. f is discontinuous at $x = -1$ because $f(-1)$ does not exist.

b. Define $f(-1) = -1$

26. **a.** $0 < |u-a| < \delta \Rightarrow |g(u) - M| < \varepsilon$

b. $0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon$

27. **a.** $\lim_{x\to 3}[2f(x) - 4g(x)]$
$= 2\lim_{x\to 3} f(x) - 4\lim_{x\to 3} g(x)$
$= 2(3) - 4(-2) = 14$

b. $\lim_{x\to 3} g(x)\dfrac{x^2-9}{x-3} = \lim_{x\to 3} g(x)(x+3)$
$= \lim_{x\to 3} g(x)\cdot \lim_{x\to 3}(x+3) = -2\cdot(3+3) = -12$

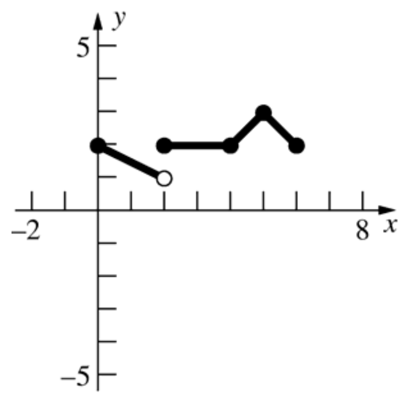
c. $g(3) = -2$

d. $\lim_{x\to 3} g(f(x)) = g\left(\lim_{x\to 3} f(x)\right) = g(3) = -2$

e. $\lim_{x\to 3}\sqrt{f^2(x) - 8g(x)}$
$= \sqrt{\left[\lim_{x\to 3} f(x)\right]^2 - 8\lim_{x\to 3} g(x)}$
$= \sqrt{(3)^2 - 8(-2)} = 5$

f. $\lim_{x\to 3}\dfrac{|g(x)-g(3)|}{f(x)} = \dfrac{|-2-g(3)|}{3} = \dfrac{|-2-(-2)|}{3}$
$= 0$

28.



29. $a(0) + b = -1$ and $a(1) + b = 1$
$b = -1;\quad a + b = 1$
$a - 1 = 1$
$a = 2$

30. Let $f(x) = x^5 - 4x^3 - 3x + 1$
$f(2) = -5$, $f(3) = 127$
Because $f(x)$ is continuous on $[2, 3]$ and $f(2) < 0 < f(3)$, there exists some number c between 2 and 3 such that $f(c) = 0$.

31. Vertical: None, denominator is never 0.

Horizontal: $\lim_{x\to\infty}\dfrac{x}{x^2+1} = \lim_{x\to -\infty}\dfrac{x}{x^2+1} = 0$, so $y = 0$ is a horizontal asymptote.

32. Vertical: None, denominator is never 0.

Horizontal: $\lim_{x\to\infty}\dfrac{x^2}{x^2+1} = \lim_{x\to -\infty}\dfrac{x^2}{x^2+1} = 1$, so $y = 1$ is a horizontal asymptote.

33. Vertical: $x = 1, x = -1$ because $\lim_{x\to 1^+}\dfrac{x^2}{x^2-1} = \infty$

and $\lim_{x\to -1^-}\dfrac{x^2}{x^2-1} = \infty$

Horizontal: $\lim_{x\to\infty}\dfrac{x^2}{x^2-1} = \lim_{x\to -\infty}\dfrac{x^2}{x^2-1} = 1$, so $y = 1$ is a horizontal asymptote.

34. Vertical: $x = 2, x = -2$ because

$$\lim_{x\to 2^+} \frac{x^3}{x^2-4} = \infty \text{ and } \lim_{x\to -2^-} \frac{x^3}{x^2-4} = \infty$$

Horizontal: $\lim_{x\to\infty} \frac{x^3}{x^2-4} = \infty$ and

$\lim_{x\to-\infty} \frac{x^3}{x^2-4} = -\infty$, so there are no horizontal asymptotes.

35. Vertical: $x = \pm\pi/4, \pm 3\pi/4, \pm 5\pi/4, \ldots$ because $\lim_{x\to\pi/4^-} \tan 2x = \infty$ and similarly for other odd multiples of $\pi/4$.

Horizontal: None, because $\lim_{x\to\infty} \tan 2x$ and $\lim_{x\to-\infty} \tan 2x$ do not exist.

36. Vertical: $x = 0$, because

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{1}{x}\frac{\sin x}{x} = \infty .$$

Horizontal: $y = 0$, because

$$\lim_{x\to\infty} \frac{\sin x}{x^2} = \lim_{x\to-\infty} \frac{\sin x}{x^2} = 0.$$

Review and Preview Problems

1. **a.** $f(2) = 2^2 = 4$

b. $f(2.1) = 2.1^2 = 4.41$

c. $f(2.1) - f(2) = 4.41 - 4 = 0.41$

d. $\frac{f(2.1) - f(2)}{2.1-2} = \frac{0.41}{0.1} = 4.1$

e. $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2$

f. $f(a+h) - f(a) = a^2 + 2ah + h^2 - a^2 = 2ah + h^2$

g. $\frac{f(a+h) - f(a)}{(a+h)-a} = \frac{2ah+h^2}{h} = 2a + h$

h. $\lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h)-a} = \lim_{h\to 0}(2a+h) = 2a$

2. **a.** $g(2) = 1/2$

b. $g(2.1) = 1/2.1 \approx 0.476$

c. $g(2.1) - g(2) = 0.476 - 0.5 = -0.024$

d. $\frac{g(2.1) - g(2)}{2.1-2} = \frac{-0.024}{0.1} = -0.24$

e. $g(a+h) = 1/(a+h)$

f. $g(a+h) - g(a) = 1/(a+h) - 1/a = \frac{-h}{a(a+h)}$

g. $\frac{g(a+h) - g(a)}{(a+h)-a} = \frac{\frac{-h}{a(a+h)}}{h} = \frac{-1}{a(a+h)}$

h. $\lim_{h\to 0} \frac{g(a+h) - g(a)}{(a+h)-a} = \frac{-1}{a^2}$

3. **a.** $F(2) = \sqrt{2} \approx 1.414$

b. $F(2.1) = \sqrt{2.1} \approx 1.449$

c. $F(2.1) - F(2) = 1.449 - 1.414 = 0.035$

d. $\frac{F(2.1) - F(2)}{2.1-2} = \frac{0.035}{0.1} = 0.35$

e. $F(a+h) = \sqrt{a+h}$

f. $F(a+h) - F(a) = \sqrt{a+h} - \sqrt{a}$

g. $\frac{F(a+h) - F(a)}{(a+h)-a} = \frac{\sqrt{a+h} - \sqrt{a}}{h}$

h. $$\lim_{h\to 0} \frac{F(a+h) - F(a)}{(a+h)-a} = \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{a+h} - \sqrt{a}\right)\left(\sqrt{a+h} + \sqrt{a}\right)}{h\left(\sqrt{a+h} + \sqrt{a}\right)}$$

$$= \lim_{h\to 0} \frac{a+h-a}{h\left(\sqrt{a+h} + \sqrt{a}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{a+h} + \sqrt{a}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}} = \frac{\sqrt{a}}{2a}$$

4. a. $G(2)=(2)^3+1=8+1=9$

b. $G(2.1)=(2.1)^3+1=9.261+1=10.261$

c. $G(2.1)-G(2)=10.261-9=1.261$

d. $\dfrac{G(2.1)-G(2)}{2.1-2}=\dfrac{1.261}{0.1}=12.61$

e.
$$\begin{aligned}G(a+h)&=(a+h)^3+1\\&=a^3+3a^2h+3ah^2+h^3+1\end{aligned}$$

f.
$$\begin{aligned}G(a+h)-G(a)&=\left[(a+h)^3+1\right]-\left[a^3+1\right]\\&=\left(a^3+3a^2h+3ah^2+h^3+1\right)-\left(a^3+1\right)\\&=3a^2h+3ah^2+h^3\end{aligned}$$

g.
$$\begin{aligned}\frac{G(a+h)-G(a)}{(a+h)-a}&=\frac{3a^2h+3ah^2+h^3}{h}\\&=3a^2+3ah+h^2\end{aligned}$$

h.
$$\begin{aligned}\lim_{h\to0}\frac{G(a+h)-G(a)}{(a+h)-a}&=\lim_{h\to0}3a^2+3ah+h^2\\&=3a^2\end{aligned}$$

5. a. $(a+b)^3=a^3+3a^2b+\cdots$

b. $(a+b)^4=a^4+4a^3b+\cdots$

c. $(a+b)^5=a^5+5a^4b+\cdots$

6. $(a+b)^n=a^n+na^{n-1}b+\cdots$

7. $\sin(x+h)=\sin x\cos h+\cos x\sin h$

8. $\cos(x+h)=\cos x\cos h-\sin x\sin h$

9. a. The point will be at position $(10,0)$ in all three cases ($t=1,2,3$) because it will have made 4, 8, and 12 revolutions respectively.

b. Since the point is rotating at a rate of 4 revolutions per second, it will complete 1 revolution after $\frac{1}{4}$ second. Therefore, the point will first return to its starting position at time $t=\frac{1}{4}$.

10.
$$\begin{aligned}V_0&=\frac{4}{3}\pi(2)^3=\frac{32\pi}{3}\text{cm}^3\\V_1&=\frac{4}{3}\pi(2.5)^3=\frac{62.5\pi}{3}=\frac{125\pi}{6}\text{cm}^3\\\Delta V&=V_1-V_0=\frac{125\pi}{6}\text{cm}^3-\frac{32\pi}{3}\text{cm}^3\\&=\frac{61}{6}\pi\,\text{cm}^3\approx31.940\,\text{cm}^3\end{aligned}$$

11. a. North plane has traveled 600miles. East plane has traveled 400 miles.

b.
$$\begin{aligned}d&=\sqrt{600^2+400^2}\\&=721\ \text{ miles}\end{aligned}$$

c.
$$\begin{aligned}d&=\sqrt{675^2+500^2}\\&=840\ \text{ miles}\end{aligned}$$