CHAPTER 2 The Derivative

2.1 Concepts Review

1. tangent line

2. secant line

3. $\dfrac{f(c+h)-f(c)}{h}$

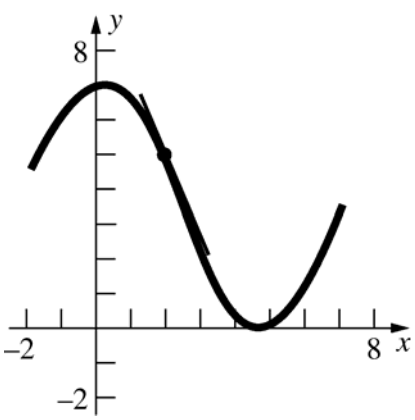
4. average velocity

Problem Set 2.1

1. Slope $=\dfrac{5-3}{2-\frac{3}{2}}=4$

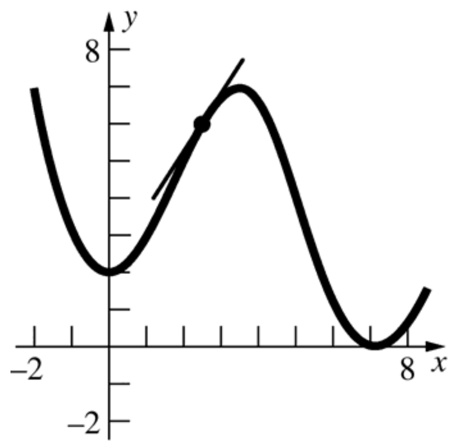
2. Slope $=\dfrac{6-4}{4-6}=-2$

3.



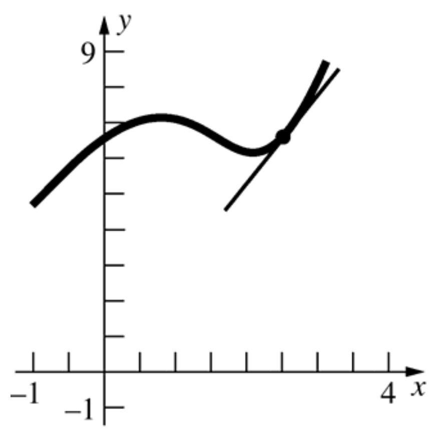
Slope ≈ -2

4.



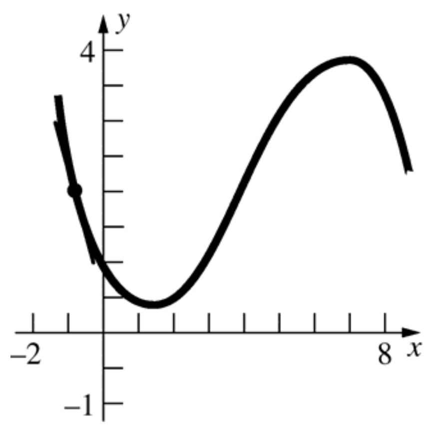
Slope ≈ 1.5

5.



Slope $\approx \dfrac{5}{2}$

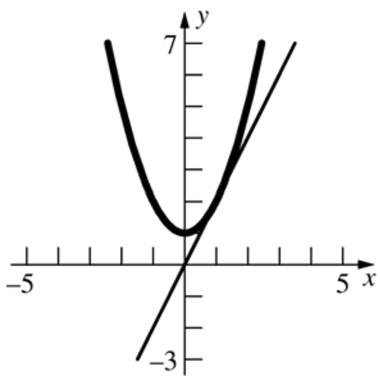
6.



Slope $\approx -\dfrac{3}{2}$

7. $y = x^2 + 1$

a., b.



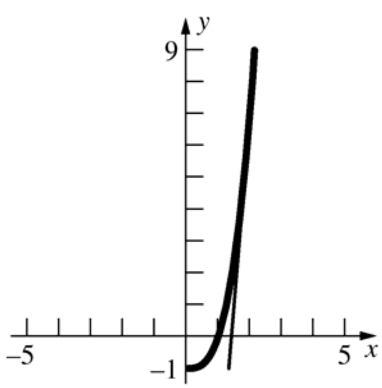
c. $m_{\tan} = 2$

d. $m_{\sec} = \frac{(1.01)^2 + 1.0 - 2}{1.01 - 1}$
$= \frac{0.0201}{.01}$
$= 2.01$

e. $m_{\tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$
$= \lim_{h \to 0} \frac{[(1+h)^2 + 1] - (1^2 + 1)}{h}$
$= \lim_{h \to 0} \frac{2 + 2h + h^2 - 2}{h}$
$= \lim_{h \to 0} \frac{h(2+h)}{h}$
$= \lim_{h \to 0} (2+h) = 2$

8. $y = x^3 - 1$

a., b.



c. $m_{\tan} = 12$

d. $m_{\sec} = \frac{[(2.01)^3 - 1.0] - 7}{2.01 - 2}$
$= \frac{0.120601}{0.01}$
$= 12.0601$

e. $m_{\tan} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$
$= \lim_{h \to 0} \frac{[(2+h)^3 - 1] - (2^3 - 1)}{h}$
$= \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h}$
$= \lim_{h \to 0} \frac{h(12 + 6h + h^2)}{h}$
$= 12$

9. $f(x) = x^2 - 1$

$m_{\tan} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$
$= \lim_{h \to 0} \frac{[(c+h)^2 - 1] - (c^2 - 1)}{h}$
$= \lim_{h \to 0} \frac{c^2 + 2ch + h^2 - 1 - c^2 + 1}{h}$
$= \lim_{h \to 0} \frac{h(2c+h)}{h} = 2c$

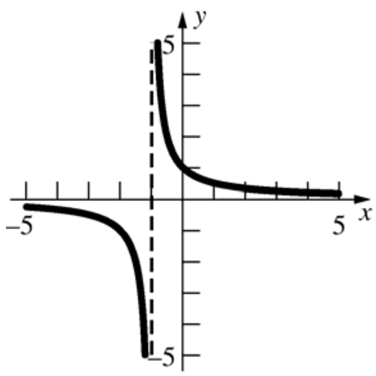
At $x = -2$, $m_{\tan} = -4$
$x = -1$, $m_{\tan} = -2$
$x = 1$, $m_{\tan} = 2$
$x = 2$, $m_{\tan} = 4$

10. $f(x) = x^3 - 3x$

$m_{\tan} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$
$= \lim_{h \to 0} \frac{[(c+h)^3 - 3(c+h)] - (c^3 - 3c)}{h}$
$= \lim_{h \to 0} \frac{c^3 + 3c^2h + 3ch^2 + h^3 - 3c - 3h - c^3 + 3c}{h}$
$= \lim_{h \to 0} \frac{h(3c^2 + 3ch + h^2 - 3)}{h} = 3c^2 - 3$

At $x = -2$, $m_{\tan} = 9$
$x = -1$, $m_{\tan} = 0$
$x = 0$, $m_{\tan} = -3$
$x = 1$, $m_{\tan} = 0$
$x = 2$, $m_{\tan} = 9$

11.



$$f(x)=\frac{1}{x+1}$$

$$m_{\tan}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}$$

$$=\lim_{h\to 0}\frac{-\frac{h}{2(2+h)}}{h}$$

$$=\lim_{h\to 0}-\frac{1}{2(2+h)}$$

$$=-\frac{1}{4}$$

$$y-\frac{1}{2}=-\frac{1}{4}(x-1)$$

12. $f(x)=\frac{1}{x-1}$

$$m_{\tan}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{h-1}+1}{h}$$

$$=\lim_{h\to 0}\frac{\frac{h}{h-1}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h-1}$$

$$=-1$$

$y + 1 = –1(x – 0)$; $y = –x – 1$

13. a. $16(1^2)-16(0^2)=16$ ft

b. $16(2^2)-16(1^2)=48$ ft

c. $V_{\text{ave}}=\frac{144-64}{3-2}=80$ ft/sec

d. $V_{\text{ave}}=\frac{16(3.01)^2-16(3)^2}{3.01-3}$

$$=\frac{0.9616}{0.01}$$

$= 96.16$ ft/s

e. $f(t)=16t^2$; $v=32c$

$v = 32(3) = 96$ ft/s

14. a. $V_{\text{ave}}=\frac{(3^2+1)-(2^2+1)}{3-2}=5$ m/sec

b. $V_{\text{ave}}=\frac{[(2.003)^2+1]-(2^2+1)}{2.003-2}$

$$=\frac{0.012009}{0.003}$$

$= 4.003$ m/sec

c. $V_{\text{ave}}=\frac{[(2+h)^2+1]-(2^2+1)}{2+h-2}$

$$=\frac{4h+h^2}{h}$$

$= 4+h$

d. $f(t)=t^2+1$

$$v=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{[(2+h)^2+1]-(2^2+1)}{h}$$

$$=\lim_{h\to 0}\frac{4h+h^2}{h}$$

$$=\lim_{h\to 0}(4+h)$$

$$=4$$

15. a. $v = \lim_{h\to 0} \frac{f(\alpha+h)-f(\alpha)}{h}$

$= \lim_{h\to 0} \frac{\sqrt{2(\alpha+h)+1}-\sqrt{2\alpha+1}}{h}$

$= \lim_{h\to 0} \frac{\sqrt{2\alpha+2h+1}-\sqrt{2\alpha+1}}{h}$

$= \lim_{h\to 0} \frac{(\sqrt{2\alpha+2h+1}-\sqrt{2\alpha+1})(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}{h(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}$

$= \lim_{h\to 0} \frac{2h}{h(\sqrt{2\alpha+2h+1}+\sqrt{2\alpha+1})}$

$= \frac{2}{\sqrt{2\alpha+1}+\sqrt{2\alpha+1}} = \frac{1}{\sqrt{2\alpha+1}}$ ft/s

b. $\frac{1}{\sqrt{2\alpha+1}} = \frac{1}{2}$

$\sqrt{2\alpha+1} = 2$

$2\alpha+1 = 4;\ \alpha = \frac{3}{2}$

The object reaches a velocity of $\frac{1}{2}$ ft/s when $t = \frac{3}{2}$.

16. $f(t) = -t^2 + 4t$

$v = \lim_{h\to 0} \frac{[-(c+h)^2+4(c+h)]-(-c^2+4c)}{h}$

$= \lim_{h\to 0} \frac{-c^2-2ch-h^2+4c+4h+c^2-4c}{h}$

$= \lim_{h\to 0} \frac{h(-2c-h+4)}{h} = -2c+4$

$-2c + 4 = 0$ when $c = 2$

The particle comes to a momentary stop at $t = 2$.

17. a. $\left[\frac{1}{2}(2.01)^2+1\right]-\left[\frac{1}{2}(2)^2+1\right] = 0.02005$ g

b. $r_{ave} = \frac{0.02005}{2.01-2} = 2.005$ g/hr

c. $f(t) = \frac{1}{2}t^2 + 1$

$r = \lim_{h\to 0} \frac{\left[\frac{1}{2}(2+h)^2+1\right]-\left[\frac{1}{2}2^2+1\right]}{h}$

$= \lim_{h\to 0} \frac{2+2h+\frac{1}{2}h^2+1-2-1}{h}$

$= \lim_{h\to 0} \frac{h\left(2+\frac{1}{2}h\right)}{h} = 2$

At $t = 2$, $r = 2$

18. a. $1000(3)^2 - 1000(2)^2 = 5000$

b. $\frac{1000(2.5)^2-1000(2)^2}{2.5-2} = \frac{2250}{0.5} = 4500$

c. $f(t) = 1000t^2$

$r = \lim_{h\to 0} \frac{1000(2+h)^2-1000(2)^2}{h}$

$= \lim_{h\to 0} \frac{4000+4000h+1000h^2-4000}{h}$

$= \lim_{h\to 0} \frac{h(4000+1000h)}{h} = 4000$

19. a. $d_{ave} = \frac{5^3-3^3}{5-3} = \frac{98}{2} = 49$ g/cm

b. $f(x) = x^3$

$d = \lim_{h\to 0} \frac{(3+h)^3-3^3}{h}$

$= \lim_{h\to 0} \frac{27+27h+9h^2+h^3-27}{h}$

$= \lim_{h\to 0} \frac{h(27+9h+h^2)}{h} = 27$ g/cm

20. $MR = \lim_{h\to 0} \frac{R(c+h)-R(c)}{h}$

$= \lim_{h\to 0} \frac{[0.4(c+h)-0.001(c+h)^2]-(0.4c-0.001c^2)}{h}$

$= \lim_{h\to 0} \frac{0.4c+0.4h-0.001c^2-0.002ch-0.001h^2-0.4c+0.001c^2}{h}$

$= \lim_{h\to 0} \frac{h(0.4-0.002c-0.001h)}{h} = 0.4-0.002c$

When n = 10, MR = 0.38; when n = 100, MR = 0.2

21. $a = \lim_{h\to 0} \frac{2(1+h)^2-2(1)^2}{h}$

$= \lim_{h\to 0} \frac{2+4h+2h^2-2}{h}$

$= \lim_{h\to 0} \frac{h(4+2h)}{h} = 4$

22. $r = \lim_{h\to 0} \frac{p(c+h)-p(c)}{h}$

$= \lim_{h\to 0} \frac{[120(c+h)^2-2(c+h)^3]-(120c^2-2c^3)}{h}$

$= \lim_{h\to 0} \frac{h(240c-6c^2+120h-6ch-2h^2)}{h}$

$= 240c-6c^2$

When $t = 10,\ r = 240(10)-6(10)^2 = 1800$

$t = 20,\ r = 240(20)-6(20)^2 = 2400$

$t = 40,\ r = 240(40)-6(40)^2 = 0$

23. $r_{\text{ave}} = \frac{100-800}{24-0} = -\frac{175}{6} \approx -29.167$

29,167 gal/hr

At 8 o'clock, $r \approx \frac{700-400}{6-10} \approx -75$

75,000 gal/hr

24. a. The elevator reached the seventh floor at time $t = 80$. The average velocity is $v_{avg} = (84-0)/80 = 1.05$ feet per second

b. The slope of the line is approximately $\frac{60-12}{55-15} = 1.2$. The velocity is approximately 1.2 feet per second.

c. The building averages 84/7=12 feet from floor to floor. Since the velocity is zero for two intervals between time 0 and time 85, the elevator stopped twice. The heights are approximately 12 and 60. Thus, the elevator stopped at floors 1 and 5.

25. a. A tangent line at $t = 91$ has slope approximately $(63-48)/(91-61) = 0.5$. The normal high temperature increases at the rate of 0.5 degree F per day.

b. A tangent line at $t = 191$ has approximate slope $(90-88)/30 \approx 0.067$. The normal high temperature increases at the rate of 0.067 degree per day.

c. There is a time in January, about January 15, when the rate of change is zero. There is also a time in July, about July 15, when the rate of change is zero.

d. The greatest rate of increase occurs around day 61, that is, some time in March. The greatest rate of decrease occurs between day 301 and 331, that is, sometime in November.

26. The slope of the tangent line at $t = 1930$ is approximately $(8-6)/(1945-1930) \approx 0.13$. The rate of growth in 1930 is approximately 0.13 million, or 130,000, persons per year. In 1990, the tangent line has approximate slope $(24-16)/(20000-1980) \approx 0.4$. Thus, the rate of growth in 1990 is 0.4 million, or 400,000, persons per year. The approximate percentage growth in 1930 is $0.107/6 \approx 0.018$ and in 1990 it is approximately $0.4/20 \approx 0.02$.

27. In both (a) and (b), the tangent line is always positive. In (a) the tangent line becomes steeper and steeper as *t* increases; thus, the velocity is increasing. In (b) the tangent line becomes flatter and flatter as *t* increases; thus, the velocity is decreasing.

28. $f(t)=\frac{1}{3}t^3+t$

$$\text{current} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$
$$= \lim_{h\to 0}\frac{\left[\frac{1}{3}(c+h)^3+(c+h)\right]-\left(\frac{1}{3}c^3+c\right)}{h}$$
$$= \lim_{h\to 0}\frac{h\left(c^2+ch+\frac{1}{3}h^2+1\right)}{h}=c^2+1$$

When $t = 3$, the current $=10$

$c^2+1=20$

$c^2=19$

$c=\sqrt{19}\approx 4.4$

A 20-amp fuse will blow at $t = 4.4$ s.

29. $A=\pi r^2,\ r=2t$

$A=4\pi t^2$

$$\text{rate} = \lim_{h\to 0}\frac{4\pi(3+h)^2-4\pi(3)^2}{h}$$
$$= \lim_{h\to 0}\frac{h(24\pi+4\pi h)}{h}=24\pi \text{ km}^2\text{/day}$$

30. $V=\frac{4}{3}\pi r^3, r=\frac{1}{4}t$

$V=\frac{1}{48}\pi t^3$

$$\text{rate} = \frac{1}{48}\pi\lim_{h\to 0}\frac{(3+h)^3-3^3}{h}=\frac{27}{48}\pi$$
$$=\frac{9}{16}\pi \text{ inch}^3/\text{sec}$$

31. $y=f(x)=x^3-2x^2+1$

a. $m_{\tan}=7$ **b.** $m_{\tan}=0$

c. $m_{\tan}=-1$ **d.** $m_{\tan}=17.92$

32. $y=f(x)=\sin x\sin^2 2x$

a. $m_{\tan}=-1.125$ **b.** $m_{\tan}\approx -1.0315$

c. $m_{\tan}=0$ **d.** $m_{\tan}\approx 1.1891$

33. $s=f(t)=t+t\cos^2 t$

At $t = 3$, $v\approx 2.818$

34. $s=f(t)=\frac{(t+1)^3}{t+2}$

At $t = 1.6$, $v\approx 4.277$

2.2 Concepts Review

1. $\frac{f(c+h)-f(c)}{h};\frac{f(t)-f(c)}{t-c}$

2. $f'(c)$

3. continuous; $f(x)=|x|$

4. $f'(x);\frac{dy}{dx}$

Problem Set 2.2

1. $$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{(1+h)^2-1^2}{h}=\lim_{h\to 0}\frac{2h+h^2}{h}$$
$$=\lim_{h\to 0}(2+h)=2$$

2. $$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0}\frac{[2(2+h)]^2-[2(2)]^2}{h}$$
$$=\lim_{h\to 0}\frac{16h+4h^2}{h}=\lim_{h\to 0}(16+4h)=16$$

3. $$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$
$$=\lim_{h\to 0}\frac{[(3+h)^2-(3+h)]-(3^2-3)}{h}$$
$$=\lim_{h\to 0}\frac{5h+h^2}{h}=\lim_{h\to 0}(5+h)=5$$

4. $$f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{4-1}}{h}=\lim_{h\to 0}\frac{\frac{3-(3+h)}{3(3+h)}}{h}=\lim_{h\to 0}\frac{-1}{3(3+h)}$$
$$=-\frac{1}{9}$$

5. $$s'(x)=\lim_{h\to 0}\frac{s(x+h)-s(x)}{h}$$
$$=\lim_{h\to 0}\frac{[2(x+h)+1]-(2x+1)}{h}$$
$$=\lim_{h\to 0}\frac{2h}{h}=2$$

6. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$=\lim_{h\to 0}\frac{[\alpha(x+h)+\beta]-(\alpha x+\beta)}{h}$
$=\lim_{h\to 0}\frac{\alpha h}{h}=\alpha$

7. $r'(x)=\lim_{h\to 0}\frac{r(x+h)-r(x)}{h}$
$=\lim_{h\to 0}\frac{[3(x+h)^2+4]-(3x^2+4)}{h}$
$=\lim_{h\to 0}\frac{6xh+3h^2}{h}=\lim_{h\to 0}(6x+3h)=6x$

8. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$=\lim_{h\to 0}\frac{[(x+h)^2+(x+h)+1]-(x^2+x+1)}{h}$
$=\lim_{h\to 0}\frac{2xh+h^2+h}{h}=\lim_{h\to 0}(2x+h+1)=2x+1$

9. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$=\lim_{h\to 0}\frac{[a(x+h)^2+b(x+h)+c]-(ax^2+bx+c)}{h}$
$=\lim_{h\to 0}\frac{2axh+ah^2+bh}{h}=\lim_{h\to 0}(2ax+ah+b)$
$=2ax+b$

10. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}$
$=\lim_{h\to 0}\frac{4hx^3+6h^2x^2+4h^3x+h^4}{h}$
$=\lim_{h\to 0}(4x^3+6hx^2+4h^2x+h^3)=4x^3$

11. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$=\lim_{h\to 0}\frac{[(x+h)^3+2(x+h)^2+1]-(x^3+2x^2+1)}{h}$
$=\lim_{h\to 0}\frac{3hx^2+3h^2x+h^3+4hx+2h^2}{h}$
$=\lim_{h\to 0}(3x^2+3hx+h^2+4x+2h)=3x^2+4x$

12. $g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$
$=\lim_{h\to 0}\frac{[(x+h)^4+(x+h)^2]-(x^4+x^2)}{h}$
$=\lim_{h\to 0}\frac{4hx^3+6h^2x^2+4h^3x+h^4+2hx+h^2}{h}$
$=\lim_{h\to 0}(4x^3+6hx^2+4h^2x+h^3+2x+h)$
$=4x^3+2x$

13. $h'(x)=\lim_{h\to 0}\frac{h(x+h)-h(x)}{h}$
$=\lim_{h\to 0}\left[\left(\frac{2}{x+h}-\frac{2}{x}\right)\cdot\frac{1}{h}\right]$
$=\lim_{h\to 0}\left[\frac{-2h}{x(x+h)}\cdot\frac{1}{h}\right]=\lim_{h\to 0}\frac{-2}{x(x+h)}=-\frac{2}{x^2}$

14. $S'(x)=\lim_{h\to 0}\frac{S(x+h)-S(x)}{h}$
$=\lim_{h\to 0}\left[\left(\frac{1}{x+h+1}-\frac{1}{x+1}\right)\cdot\frac{1}{h}\right]$
$=\lim_{h\to 0}\left[\frac{-h}{(x+1)(x+h+1)}\cdot\frac{1}{h}\right]$
$=\lim_{h\to 0}\frac{-1}{(x+1)(x+h+1)}=-\frac{1}{(x+1)^2}$

15. $F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$
$=\lim_{h\to 0}\left[\left(\frac{6}{(x+h)^2+1}-\frac{6}{x^2+1}\right)\cdot\frac{1}{h}\right]$
$=\lim_{h\to 0}\left[\frac{6(x^2+1)-6(x^2+2hx+h^2+1)}{(x^2+1)(x^2+2hx+h^2+1)}\cdot\frac{1}{h}\right]$
$=\lim_{h\to 0}\left[\frac{-12hx-6h^2}{(x^2+1)(x^2+2hx+h^2+1)}\cdot\frac{1}{h}\right]$
$=\lim_{h\to 0}\frac{-12x-6h}{(x^2+1)(x^2+2hx+h^2+1)}=-\frac{12x}{(x^2+1)^2}$

16. $F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$
$=\lim_{h\to 0}\left[\left(\frac{x+h-1}{x+h+1}-\frac{x-1}{x+1}\right)\cdot\frac{1}{h}\right]$
$=\lim_{h\to 0}\left[\frac{x^2+hx+h-1-(x^2+hx-h-1)}{(x+h+1)(x+1)}\cdot\frac{1}{h}\right]$
$=\lim_{h\to 0}\left[\frac{2h}{(x+h+1)(x+1)}\cdot\frac{1}{h}\right]=\frac{2}{(x+1)^2}$

17. $G'(x)=\lim\limits_{h\to 0}\frac{G(x+h)-G(x)}{h}$

$=\lim\limits_{h\to 0}\left[\left(\frac{2(x+h)-1}{x+h-4}-\frac{2x-1}{x-4}\right)\cdot\frac{1}{h}\right]$

$=\lim\limits_{h\to 0}\left[\frac{2x^2+2hx-9x-8h+4-(2x^2+2hx-9x-h+4)}{(x+h-4)(x-4)}\cdot\frac{1}{h}\right]=\lim\limits_{h\to 0}\left[\frac{-7h}{(x+h-4)(x-4)}\cdot\frac{1}{h}\right]$

$=\lim\limits_{h\to 0}\frac{-7}{(x+h-4)(x-4)}=-\frac{7}{(x-4)^2}$

18. $G'(x)=\lim\limits_{h\to 0}\frac{G(x+h)-G(x)}{h}$

$=\lim\limits_{h\to 0}\left[\left(\frac{2(x+h)}{(x+h)^2-(x+h)}-\frac{2x}{x^2-x}\right)\cdot\frac{1}{h}\right]=\lim\limits_{h\to 0}\left[\frac{(2x+2h)(x^2-x)-2x(x^2+2xh+h^2-x-h)}{(x^2+2hx+h^2-x-h)(x^2-x)}\cdot\frac{1}{h}\right]$

$=\lim\limits_{h\to 0}\left[\frac{-2h^2x-2hx^2}{(x^2+2hx+h^2-x-h)(x^2-x)}\cdot\frac{1}{h}\right]$

$=\lim\limits_{h\to 0}\frac{-2hx-2x^2}{(x^2+2hx+h^2-x-h)(x^2-x)}$

$=\frac{-2x^2}{(x^2-x)^2}=-\frac{2}{(x-1)^2}$

19. $g'(x)=\lim\limits_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$=\lim\limits_{h\to 0}\frac{\sqrt{3(x+h)}-\sqrt{3x}}{h}$

$=\lim\limits_{h\to 0}\frac{(\sqrt{3x+3h}-\sqrt{3x})(\sqrt{3x+3h}+\sqrt{3x})}{h(\sqrt{3x+3h}+\sqrt{3x})}$

$=\lim\limits_{h\to 0}\frac{3h}{h(\sqrt{3x+3h}+\sqrt{3x})}=\lim\limits_{h\to 0}\frac{3}{\sqrt{3x+3h}+\sqrt{3x}}=\frac{3}{2\sqrt{3x}}$

20. $g'(x)=\lim\limits_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$=\lim\limits_{h\to 0}\left[\left(\frac{1}{\sqrt{3(x+h)}}-\frac{1}{\sqrt{3x}}\right)\cdot\frac{1}{h}\right]$

$=\lim\limits_{h\to 0}\left[\frac{\sqrt{3x}-\sqrt{3x+3h}}{\sqrt{9x(x+h)}}\cdot\frac{1}{h}\right]$

$=\lim\limits_{h\to 0}\left[\frac{(\sqrt{3x}-\sqrt{3x+3h})(\sqrt{3x}+\sqrt{3x+3h})}{\sqrt{9x(x+h)}(\sqrt{3x}+\sqrt{3x+3h})}\cdot\frac{1}{h}\right]$

$=\lim\limits_{h\to 0}\frac{-3h}{h\sqrt{9x(x+h)}(\sqrt{3x}+\sqrt{3x+3h})}=\frac{-3}{3x\cdot 2\sqrt{3x}}=-\frac{1}{2x\sqrt{3x}}$

21. $H'(x) = \lim_{h\to 0} \frac{H(x+h) - H(x)}{h}$

$$= \lim_{h\to 0}\left[\left(\frac{3}{\sqrt{x+h-2}} - \frac{3}{\sqrt{x-2}}\right)\cdot\frac{1}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{3\sqrt{x-2} - 3\sqrt{x+h-2}}{\sqrt{(x+h-2)(x-2)}}\cdot\frac{1}{h}\right]$$

$$= \lim_{h\to 0}\frac{3(\sqrt{x-2} - \sqrt{x+h-2})(\sqrt{x-2} + \sqrt{x+h-2})}{h\sqrt{(x+h-2)(x-2)}(\sqrt{x-2} + \sqrt{x+h-2})}$$

$$= \lim_{h\to 0}\frac{-3h}{h[(x-2)\sqrt{x+h-2} + (x+h-2)\sqrt{x-2}]}$$

$$= \lim_{h\to 0}\frac{-3}{(x-2)\sqrt{x+h-2} + (x+h-2)\sqrt{x-2}}$$

$$= -\frac{3}{2(x-2)\sqrt{x-2}} = -\frac{3}{2(x-2)^{3/2}}$$

22. $H'(x) = \lim_{h\to 0} \frac{H(x+h) - H(x)}{h}$

$$= \lim_{h\to 0}\frac{\sqrt{(x+h)^2+4} - \sqrt{x^2+4}}{h}$$

$$= \lim_{h\to 0}\frac{\left(\sqrt{x^2+2hx+h^2+4} - \sqrt{x^2+4}\right)\left(\sqrt{x^2+2hx+h^2+4} + \sqrt{x^2+4}\right)}{h\left(\sqrt{x^2+2hx+h^2+4} + \sqrt{x^2+4}\right)}$$

$$= \lim_{h\to 0}\frac{2hx+h^2}{h\left(\sqrt{x^2+2hx+h^2+4} + \sqrt{x^2+4}\right)}$$

$$= \lim_{h\to 0}\frac{2x+h}{\sqrt{x^2+2hx+h^2+4} + \sqrt{x^2+4}}$$

$$= \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}}$$

23. $f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$

$$= \lim_{t\to x}\frac{(t^2-3t) - (x^2-3x)}{t-x}$$

$$= \lim_{t\to x}\frac{t^2 - x^2 - (3t - 3x)}{t-x}$$

$$= \lim_{t\to x}\frac{(t-x)(t+x) - 3(t-x)}{t-x}$$

$$= \lim_{t\to x}\frac{(t-x)(t+x-3)}{t-x} = \lim_{t\to x}(t+x-3)$$

$$= 2x - 3$$

24. $f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$

$$= \lim_{t\to x}\frac{(t^3+5t) - (x^3+5x)}{t-x}$$

$$= \lim_{t\to x}\frac{t^3 - x^3 + 5t - 5x}{t-x}$$

$$= \lim_{t\to x}\frac{(t-x)(t^2+tx+x^2) + 5(t-x)}{t-x}$$

$$= \lim_{t\to x}\frac{(t-x)(t^2+tx+x^2+5)}{t-x}$$

$$= \lim_{t\to x}(t^2+tx+x^2+5) = 3x^2+5$$

25. $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$

$= \lim_{t \to x} \left[\left(\frac{t}{t-5} - \frac{x}{x-5} \right) \left(\frac{1}{t-x} \right) \right]$

$= \lim_{t \to x} \frac{tx - 5t - tx + 5x}{(t-5)(x-5)(t-x)}$

$= \lim_{t \to x} \frac{-5(t-x)}{(t-5)(x-5)(t-x)} = \lim_{t \to x} \frac{-5}{(t-5)(x-5)}$

$= -\frac{5}{(x-5)^2}$

26. $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$

$= \lim_{t \to x} \left[\left(\frac{t+3}{t} - \frac{x+3}{x} \right) \left(\frac{1}{t-x} \right) \right]$

$= \lim_{t \to x} \frac{3x - 3t}{xt(t-x)} = \lim_{t \to x} \frac{-3}{xt} = -\frac{3}{x^2}$

27. $f(x) = 2x^3$ at $x = 5$

28. $f(x) = x^2 + 2x$ at $x = 3$

29. $f(x) = x^2$ at $x = 2$

30. $f(x) = x^3 + x$ at $x = 3$

31. $f(x) = x^2$ at x

32. $f(x) = x^3$ at x

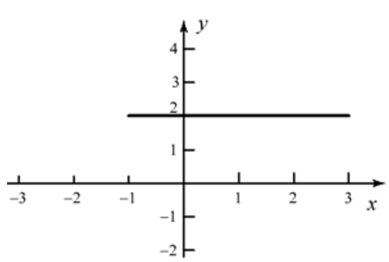
33. $f(t) = \frac{2}{t}$ at t

34. $f(y) = \sin y$ at y

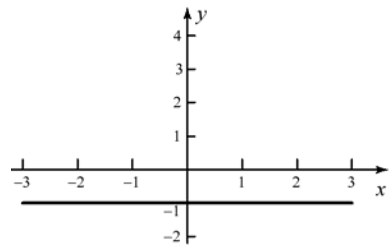
35. $f(x) = \cos x$ at x

36. $f(t) = \tan t$ at t

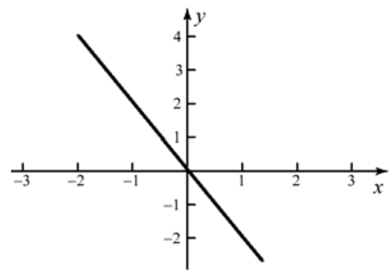
37. The slope of the tangent line is always 2.



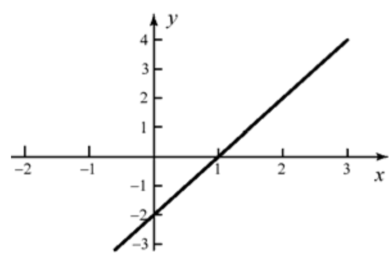
38. The slope of the tangent line is always -1.



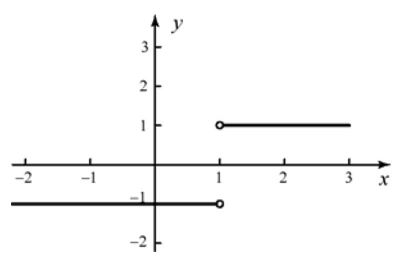
39. The derivative is positive until $x = 0$, then becomes negative.



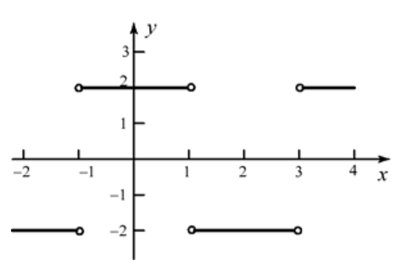
40. The derivative is negative until $x = 1$, then becomes positive.



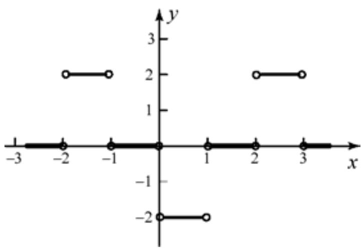
41. The derivative is -1 until $x = 1$. To the right of $x = 1$, the derivative is 1. The derivative is undefined at $x = 1$.



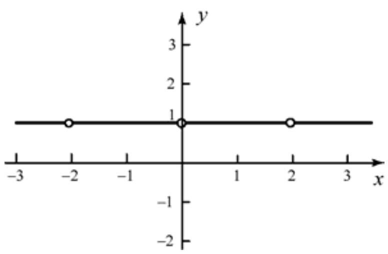
42. The derivative is -2 to the left of $x = -1$; from -1 to 1, the derivative is 2, etc. The derivative is not defined at $x = -1, 1, 3$.



43. The derivative is 0 on $(-3,-2)$, 2 on $(-2,-1)$, 0 on $(-1,0)$, -2 on $(0,1)$, 0 on $(1,2)$, 2 on $(2,3)$ and 0 on $(3,4)$. The derivative is undefined at $x=-2,\ -1,\ 0,\ 1,\ 2,\ 3$.



44. The derivative is 1 except at $x=-2,\ 0,\ 2$ where it is undefined.



45. $\Delta y = [3(1.5) + 2] - [3(1) + 2] = 1.5$

46. $\Delta y = [3(0.1)^2 + 2(0.1) + 1] - [3(0.0)^2 + 2(0.0) + 1]$ $= 0.23$

47. $\Delta y = 1/1.2 - 1/1 = -0.1667$

48. $\Delta y = 2/(0.1+1) - 2/(0+1) = -0.1818$

49. $\Delta y = \dfrac{3}{2.31+1} - \dfrac{3}{2.34+1} \approx 0.0081$

50. $\Delta y = \cos[2(0.573)] - \cos[2(0.571)] \approx -0.0036$

51. $\dfrac{\Delta y}{\Delta x} = \dfrac{(x+\Delta x)^2 - x^2}{\Delta x} = \dfrac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}(2x + \Delta x) = 2x$$

52. $\dfrac{\Delta y}{\Delta x} = \dfrac{[(x+\Delta x)^3 - 3(x+\Delta x)^2] - (x^3 - 3x^2)}{\Delta x}$

$$= \frac{3x^2\Delta x + 3x(\Delta x)^2 - 6x\Delta x - 3(\Delta x)^2 + \Delta x^3}{\Delta x}$$

$$= 3x^2 + 3x\Delta x - 6x - 3\Delta x + (\Delta x)^2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}(3x^2 + 3x\Delta x - 6x - 3\Delta x + (\Delta x)^2)$$

$$= 3x^2 - 6x$$

53. $\dfrac{\Delta y}{\Delta x} = \dfrac{\frac{1}{x+\Delta x+1} - \frac{1}{x+1}}{\Delta x}$

$$= \left(\frac{x+1-(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}\right)\left(\frac{1}{\Delta x}\right)$$

$$= \frac{-\Delta x}{(x+\Delta x+1)(x+1)\Delta x}$$

$$= -\frac{1}{(x+\Delta x+1)(x+1)}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[-\frac{1}{(x+\Delta x+1)(x+1)}\right] = -\frac{1}{(x+1)^2}$$

54. $\dfrac{\Delta y}{\Delta x} = \dfrac{1 + \frac{1}{x+\Delta x} - \left(1 + \frac{1}{x}\right)}{\Delta x}$

$$= \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \frac{\frac{-\Delta x}{x(x+\Delta x)}}{\Delta x} = -\frac{1}{x(x+\Delta x)}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} -\frac{1}{x(x+\Delta x)} = -\frac{1}{x^2}$$

55.

$$\frac{\Delta y}{\Delta x} = \frac{\frac{x+\Delta x-1}{x+\Delta x+1} - \frac{x-1}{x+1}}{\Delta x}$$

$$= \frac{(x+1)(x+\Delta x-1) - (x-1)(x+\Delta x+1)}{(x+\Delta x+1)(x+1)} \times \frac{1}{\Delta x}$$

$$= \frac{x^2 + x\Delta x - x + x + \Delta x - 1 - \left[x^2 + x\Delta x - x + x - \Delta x - 1\right]}{x^2 + x\Delta x + x + x + \Delta x + 1} \times \frac{1}{\Delta x}$$

$$= \frac{2\Delta x}{x^2 + x\Delta x + x + x + \Delta x + 1} \times \frac{1}{\Delta x} = \frac{2}{x^2 + x\Delta x + x + x + \Delta x + 1}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2}{x^2 + x\Delta x + x + x + \Delta x + 1} = \frac{2}{x^2 + 2x + 1} = \frac{2}{(x+1)^2}$$

56. $\dfrac{\Delta y}{\Delta x} = \dfrac{\frac{(x+\Delta x)^2 - 1}{x+\Delta x} - \frac{x^2 - 1}{x}}{\Delta x}$

$$= \left[\frac{x(x+\Delta x)^2 - x - (x+\Delta x)(x^2 - 1)}{x(x+\Delta x)}\right] \times \frac{1}{\Delta x}$$

$$= \left[\frac{x\left(x^2 + 2x\Delta x + \left(\Delta x^2\right)\right) - x - \left(x^3 + x^2\Delta x - x - \Delta x\right)}{x^2 + x\Delta x}\right] \times \frac{1}{\Delta x}$$

$$= \frac{x^2\Delta x + x(\Delta x)^2 + \Delta x}{x^2 + x\Delta x} \times \frac{1}{\Delta x} = \frac{x^2 + x\Delta x + 1}{x^2 + x\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{x^2 + x\Delta x + 1}{x^2 + x\Delta x} = \frac{x^2 + 1}{x^2}$$

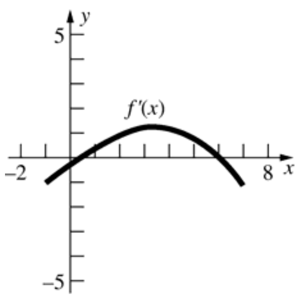
57. $f'(0) \approx -\frac{1}{2}$; $f'(2) \approx 1$

$f'(5) \approx \frac{2}{3}$; $f'(7) \approx -3$

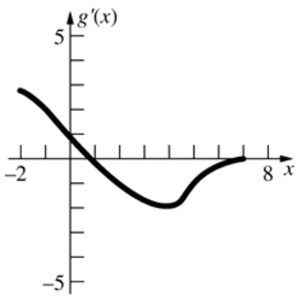
58. $g'(-1) \approx 2$; $g'(1) \approx 0$

$g'(4) \approx -2$; $g'(6) \approx -\frac{1}{3}$

59.



60.



61. a. $f(2) \approx \frac{5}{2}$; $f'(2) \approx \frac{3}{2}$

$f(0.5) \approx 1.8$; $f'(0.5) \approx -0.6$

b. $\frac{2.9-1.9}{2.5-0.5} = 0.5$

c. $x = 5$

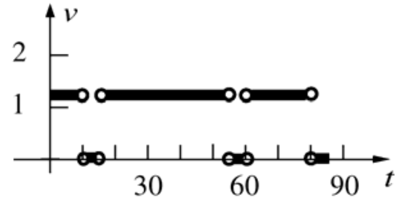
d. $x = 3, 5$

e. $x = 1, 3, 5$

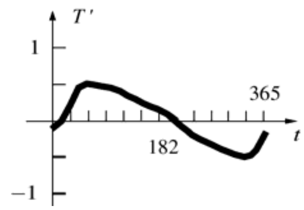
f. $x = 0$

g. $x \approx -0.7, \frac{3}{2}$ and $5 < x < 7$

62. The derivative fails to exist at the corners of the graph; that is, at $t = 10, 15, 55, 60, 80$. The derivative exists at all other points on the interval $(0,85)$.



63. The derivative is 0 at approximately $t = 15$ and $t = 201$. The greatest rate of increase occurs at about $t = 61$ and it is about 0.5 degree F per day. The greatest rate of decrease occurs at about $t = 320$ and it is about 0.5 degree F per day. The derivative is positive on (15,201) and negative on (0,15) and (201,365).



64. The slope of a tangent line for the dashed function is zero when *x* is approximately 0.3 or 1.9. The solid function is zero at both of these points. The graph indicates that the solid function is negative when the dashed function has a tangent line with a negative slope and positive when the dashed function has a tangent line with a positive slope. Thus, the solid function is the derivative of the dashed function.

65. The short-dash function has a tangent line with zero slope at about $x = 2.1$, where the solid function is zero. The solid function has a tangent line with zero slope at about $x = 0.4, 1.2$ and 3.5. The long-dash function is zero at these points. The graph shows that the solid function is positive (negative) when the slope of the tangent line of the short-dash function is positive (negative). Also, the long-dash function is positive (negative) when the slope of the tangent line of the solid function is positive (negative). Thus, the short-dash function is *f*, the solid function is $f' = g$, and the dash function is g'.

66. Note that since $x = 0 + x$, $f(x) = f(0 + x) = f(0)f(x)$, hence $f(0) = 1$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h\to 0} \frac{f(a)f(h) - f(a)}{h}$$
$$= f(a)\lim_{h\to 0} \frac{f(h)-1}{h} = f(a)\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$
$$= f(a)f'(0)$$

$f'(a)$ exists since $f'(0)$ exists.

67. If f is differentiable everywhere, then it is continuous everywhere, so
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(mx+b) = 2m+b = f(2) = 4$$
and $b = 4 - 2m$.
For f to be differentiable everywhere,
$$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} \text{ must exist.}$$
$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+}\frac{x^2-4}{x-2} = \lim_{x\to 2^+}(x+2) = 4$$
$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-}\frac{mx+b-4}{x-2}$$
$$= \lim_{x\to 2^-}\frac{mx+4-2m-4}{x-2} = \lim_{x\to 2^-}\frac{m(x-2)}{x-2} = m$$
Thus $m = 4$ and $b = 4 - 2(4) = -4$

68. $$f_s(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)+f(x)-f(x-h)}{2h}$$
$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{2h} + \frac{f(x-h)-f(x)}{-2h}\right]$$
$$= \frac{1}{2}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \frac{1}{2}\lim_{-h\to 0}\frac{f[x+(-h)]-f(x)}{-h}$$
$$= \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = f'(x).$$
For the converse, let $f(x) = |x|$. Then
$$f_s(0) = \lim_{h\to 0}\frac{|h|-|-h|}{2h} = \lim_{h\to 0}\frac{|h|-|h|}{2h} = 0$$
but $f'(0)$ does not exist.

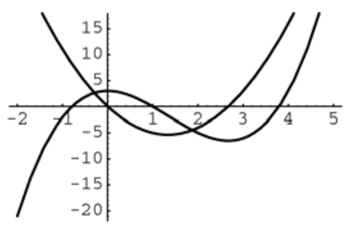
69. $$f'(x_0) = \lim_{t\to x_0}\frac{f(t)-f(x_0)}{t-x_0}, \text{ so}$$
$$f'(-x_0) = \lim_{t\to -x_0}\frac{f(t)-f(-x_0)}{t-(-x_0)}$$
$$= \lim_{t\to -x_0}\frac{f(t)-f(-x_0)}{t+x_0}$$

a. If f is an odd function,
$$f'(-x_0) = \lim_{t\to -x_0}\frac{f(t)-[-f(-x_0)]}{t+x_0}$$
$$= \lim_{t\to -x_0}\frac{f(t)+f(-x_0)}{t+x_0}.$$
Let $u = -t$. As $t \to -x_0$, $u \to x_0$ and so
$$f'(-x_0) = \lim_{u\to x_0}\frac{f(-u)+f(x_0)}{-u+x_0}$$
$$= \lim_{u\to x_0}\frac{-f(u)+f(x_0)}{-(u-x_0)} = \lim_{u\to x_0}\frac{-[f(u)-f(x_0)]}{-(u-x_0)}$$
$$= \lim_{u\to x_0}\frac{f(u)-f(x_0)}{u-x_0} = f'(x_0) = m.$$

b. If f is an even function,
$$f'(-x_0) = \lim_{t\to -x_0}\frac{f(t)-f(x_0)}{t+x_0}.$$ Let $u = -t$, as above, then $$f'(-x_0) = \lim_{u\to x_0}\frac{f(-u)-f(x_0)}{-u+x_0}$$
$$= \lim_{u\to x_0}\frac{f(u)-f(x_0)}{-(u-x_0)} = -\lim_{u\to x_0}\frac{f(u)-f(x_0)}{u-x_0}$$
$$= -f'(x_0) = -m.$$

70. Say $f(-x) = -f(x)$. Then
$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h}$$
$$= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} = -\lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$$
$$= \lim_{-h\to 0}\frac{f[x+(-h)]-f(x)}{-h} = f'(x) \text{ so } f'(x) \text{ is}$$
an even function if $f(x)$ is an odd function.
Say $f(-x) = f(x)$. Then
$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$$
$$= -\lim_{-h\to 0}\frac{f[x+(-h)]-f(x)}{-h} = -f'(x) \text{ so } f'(x)$$
is an odd function if $f(x)$ is an even function.

71.

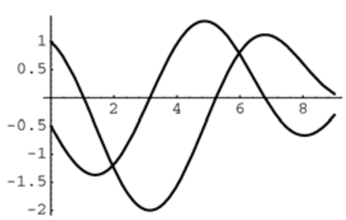


a. $0 < x < \frac{8}{3}$; $\left(0, \frac{8}{3}\right)$

b. $0 \le x \le \frac{8}{3}$; $\left[0, \frac{8}{3}\right]$

c. A function $f(x)$ decreases as x increases when $f'(x) < 0$.

72.



a. $\pi < x < 6.8$ **b.** $\pi < x < 6.8$

c. A function $f(x)$ increases as x increases when $f'(x) > 0$.

2.3 Concepts Review

1. the derivative of the second; second; $f(x)g'(x)+g(x)f'(x)$

2. denominator; denominator; square of the denominator; $\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$

3. $nx^{n-1}h$; nx^{n-1}

4. $kL(f)$; $L(f)+L(g)$; D_x

Problem Set 2.3

1. $D_x(2x^2)=2D_x(x^2)=2\cdot 2x=4x$

2. $D_x(3x^3)=3D_x(x^3)=3\cdot 3x^2=9x^2$

3. $D_x(\pi x)=\pi D_x(x)=\pi\cdot 1=\pi$

4. $D_x(\pi x^3)=\pi D_x(x^3)=\pi\cdot 3x^2=3\pi x^2$

5. $D_x(2x^{-2})=2D_x(x^{-2})=2(-2x^{-3})=-4x^{-3}$

6. $D_x(-3x^{-4})=-3D_x(x^{-4})=-3(-4x^{-5})=12x^{-5}$

7. $D_x\left(\frac{\pi}{x}\right)=\pi D_x(x^{-1})=\pi(-1x^{-2})=-\pi x^{-2}$
$=-\frac{\pi}{x^2}$

8. $D_x\left(\frac{\alpha}{x^3}\right)=\alpha D_x(x^{-3})=\alpha(-3x^{-4})=-3\alpha x^{-4}$
$=-\frac{3\alpha}{x^4}$

9. $D_x\left(\frac{100}{x^5}\right)=100D_x(x^{-5})=100(-5x^{-6})$
$=-500x^{-6}=-\frac{500}{x^6}$

10. $D_x\left(\frac{3\alpha}{4x^5}\right)=\frac{3\alpha}{4}D_x(x^{-5})=\frac{3\alpha}{4}(-5x^{-6})$
$=-\frac{15\alpha}{4}x^{-6}=-\frac{15\alpha}{4x^6}$

11. $D_x(x^2+2x)=D_x(x^2)+2D_x(x)=2x+2$

12. $D_x(3x^4+x^3)=3D_x(x^4)+D_x(x^3)$
$=3(4x^3)+3x^2=12x^3+3x^2$

13. $D_x(x^4+x^3+x^2+x+1)$
$=D_x(x^4)+D_x(x^3)+D_x(x^2)+D_x(x)+D_x(1)$
$=4x^3+3x^2+2x+1$

14. $D_x(3x^4-2x^3-5x^2+\pi x+\pi^2)$
$=3D_x(x^4)-2D_x(x^3)-5D_x(x^2)$
$+\pi D_x(x)+D_x(\pi^2)$
$=3(4x^3)-2(3x^2)-5(2x)+\pi(1)+0$
$=12x^3-6x^2-10x+\pi$

15. $D_x(\pi x^7-2x^5-5x^{-2})$
$=\pi D_x(x^7)-2D_x(x^5)-5D_x(x^{-2})$
$=\pi(7x^6)-2(5x^4)-5(-2x^{-3})$
$=7\pi x^6-10x^4+10x^{-3}$

16. $D_x(x^{12}+5x^{-2}-\pi x^{-10})$
$=D_x(x^{12})+5D_x(x^{-2})-\pi D_x(x^{-10})$
$=12x^{11}+5(-2x^{-3})-\pi(-10x^{-11})$
$=12x^{11}-10x^{-3}+10\pi x^{-11}$

17. $D_x\left(\frac{3}{x^3}+x^{-4}\right)=3D_x(x^{-3})+D_x(x^{-4})$
$=3(-3x^{-4})+(-4x^{-5})=-\frac{9}{x^4}-4x^{-5}$

18. $D_x(2x^{-6}+x^{-1})=2D_x(x^{-6})+D_x(x^{-1})$
$=2(-6x^{-7})+(-1x^{-2})=-12x^{-7}-x^{-2}$

19. $D_x\left(\frac{2}{x}-\frac{1}{x^2}\right)=2D_x(x^{-1})-D_x(x^{-2})$
$=2(-1x^{-2})-(-2x^{-3})=-\frac{2}{x^2}+\frac{2}{x^3}$

20. $D_x\left(\frac{3}{x^3}-\frac{1}{x^4}\right)=3D_x(x^{-3})-D_x(x^{-4})$
$=3(-3x^{-4})-(-4x^{-5})=-\frac{9}{x^4}+\frac{4}{x^5}$

21. $D_x\left(\frac{1}{2x}+2x\right)=\frac{1}{2}D_x(x^{-1})+2D_x(x)$
$=\frac{1}{2}(-1x^{-2})+2(1)=-\frac{1}{2x^2}+2$

22. $D_x\left(\frac{2}{3x}-\frac{2}{3}\right)=\frac{2}{3}D_x(x^{-1})-D_x\left(\frac{2}{3}\right)$

$=\frac{2}{3}(-1x^{-2})-0=-\frac{2}{3x^2}$

23. $D_x[x(x^2+1)]=x\,D_x(x^2+1)+(x^2+1)D_x(x)$

$=x(2x)+(x^2+1)(1)=3x^2+1$

24. $D_x[3x(x^3-1)]=3x\,D_x(x^3-1)+(x^3-1)D_x(3x)$

$=3x(3x^2)+(x^3-1)(3)=12x^3-3$

25. $D_x[(2x+1)^2]$

$=(2x+1)D_x(2x+1)+(2x+1)D_x(2x+1)$

$=(2x+1)(2)+(2x+1)(2)=8x+4$

26. $D_x[(-3x+2)^2]$

$=(-3x+2)D_x(-3x+2)+(-3x+2)D_x(-3x+2)$

$=(-3x+2)(-3)+(-3x+2)(-3)=18x-12$

27. $D_x[(x^2+2)(x^3+1)]$

$=(x^2+2)D_x(x^3+1)+(x^3+1)D_x(x^2+2)$

$=(x^2+2)(3x^2)+(x^3+1)(2x)$

$=3x^4+6x^2+2x^4+2x$

$=5x^4+6x^2+2x$

28. $D_x[(x^4-1)(x^2+1)]$

$=(x^4-1)D_x(x^2+1)+(x^2+1)D_x(x^4-1)$

$=(x^4-1)(2x)+(x^2+1)(4x^3)$

$=2x^5-2x+4x^5+4x^3=6x^5+4x^3-2x$

29. $D_x[(x^2+17)(x^3-3x+1)]$

$=(x^2+17)D_x(x^3-3x+1)+(x^3-3x+1)D_x(x^2+17)$

$=(x^2+17)(3x^2-3)+(x^3-3x+1)(2x)$

$=3x^4+48x^2-51+2x^4-6x^2+2x$

$=5x^4+42x^2+2x-51$

30. $D_x[(x^4+2x)(x^3+2x^2+1)]=(x^4+2x)D_x(x^3+2x^2+1)+(x^3+2x^2+1)D_x(x^4+2x)$

$=(x^4+2x)(3x^2+4x)+(x^3+2x^2+1)(4x^3+2)$

$=7x^6+12x^5+12x^3+12x^2+2$

31. $D_x[(5x^2-7)(3x^2-2x+1)]=(5x^2-7)D_x(3x^2-2x+1)+(3x^2-2x+1)D_x(5x^2-7)$

$=(5x^2-7)(6x-2)+(3x^2-2x+1)(10x)$

$=60x^3-30x^2-32x+14$

32. $D_x[(3x^2+2x)(x^4-3x+1)]=(3x^2+2x)D_x(x^4-3x+1)+(x^4-3x+1)D_x(3x^2+2x)$

$=(3x^2+2x)(4x^3-3)+(x^4-3x+1)(6x+2)$

$=18x^5+10x^4-27x^2-6x+2$

33. $D_x\left(\frac{1}{3x^2+1}\right)=\frac{(3x^2+1)D_x(1)-(1)D_x(3x^2+1)}{(3x^2+1)^2}$

$=\frac{(3x^2+1)(0)-(6x)}{(3x^2+1)^2}=-\frac{6x}{(3x^2+1)^2}$

34. $D_x\left(\frac{2}{5x^2-1}\right)=\frac{(5x^2-1)D_x(2)-(2)D_x(5x^2-1)}{(5x^2-1)^2}$

$=\frac{(5x^2-1)(0)-2(10x)}{(5x^2-1)^2}=-\frac{20x}{(5x^2-1)^2}$

35. $D_x\left(\frac{1}{4x^2-3x+9}\right)=\frac{(4x^2-3x+9)D_x(1)-(1)D_x(4x^2-3x+9)}{(4x^2-3x+9)^2}$

$=\frac{(4x^2-3x+9)(0)-(8x-3)}{(4x^2-3x+9)^2}=-\frac{8x-3}{(4x^2-3x+9)^2}$

$=\frac{-8x+3}{(4x^2-3x+9)^2}$

36. $D_x\left(\frac{4}{2x^3-3x}\right)=\frac{(2x^3-3x)D_x(4)-(4)D_x(2x^3-3x)}{(2x^3-3x)^2}$

$=\frac{(2x^3-3x)(0)-4(6x^2-3)}{(2x^3-3x)^2}=\frac{-24x^2+12}{(2x^3-3x)^2}$

37. $D_x\left(\frac{x-1}{x+1}\right)=\frac{(x+1)D_x(x-1)-(x-1)D_x(x+1)}{(x+1)^2}$

$=\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}=\frac{2}{(x+1)^2}$

38. $D_x\left(\frac{2x-1}{x-1}\right)=\frac{(x-1)D_x(2x-1)-(2x-1)D_x(x-1)}{(x-1)^2}$

$=\frac{(x-1)(2)-(2x-1)(1)}{(x-1)^2}=-\frac{1}{(x-1)^2}$

39. $D_x\left(\frac{2x^2-1}{3x+5}\right)=\frac{(3x+5)D_x(2x^2-1)-(2x^2-1)D_x(3x+5)}{(3x+5)^2}$

$=\frac{(3x+5)(4x)-(2x^2-1)(3)}{(3x+5)^2}$

$=\frac{6x^2+20x+3}{(3x+5)^2}$

40. $D_x\left(\frac{5x-4}{3x^2+1}\right)=\frac{(3x^2+1)D_x(5x-4)-(5x-4)D_x(3x^2+1)}{(3x^2+1)^2}$

$=\frac{(3x^2+1)(5)-(5x-4)(6x)}{(3x^2+1)^2}$

$=\frac{-15x^2+24x+5}{(3x^2+1)^2}$

41. $D_x\left(\frac{2x^2-3x+1}{2x+1}\right)=\frac{(2x+1)D_x(2x^2-3x+1)-(2x^2-3x+1)D_x(2x+1)}{(2x+1)^2}$

$=\frac{(2x+1)(4x-3)-(2x^2-3x+1)(2)}{(2x+1)^2}$

$=\frac{4x^2+4x-5}{(2x+1)^2}$

42. $D_x\left(\frac{5x^2+2x-6}{3x-1}\right)=\frac{(3x-1)D_x(5x^2+2x-6)-(5x^2+2x-6)D_x(3x-1)}{(3x-1)^2}$

$=\frac{(3x-1)(10x+2)-(5x^2+2x-6)(3)}{(3x-1)^2}$

$=\frac{15x^2-10x+16}{(3x-1)^2}$

43. $D_x\left(\frac{x^2-x+1}{x^2+1}\right)=\frac{(x^2+1)D_x(x^2-x+1)-(x^2-x+1)D_x(x^2+1)}{(x^2+1)^2}$

$=\frac{(x^2+1)(2x-1)-(x^2-x+1)(2x)}{(x^2+1)^2}$

$=\frac{x^2-1}{(x^2+1)^2}$

44. $D_x\left(\frac{x^2-2x+5}{x^2+2x-3}\right)=\frac{(x^2+2x-3)D_x(x^2-2x+5)-(x^2-2x+5)D_x(x^2+2x-3)}{(x^2+2x-3)^2}$

$=\frac{(x^2+2x-3)(2x-2)-(x^2-2x+5)(2x+2)}{(x^2+2x-3)^2}$

$=\frac{4x^2-16x-4}{(x^2+2x-3)^2}$

45. **a.** $(f\cdot g)'(0)=f(0)g'(0)+g(0)f'(0)$
$= 4(5) + (–3)(–1) = 23$

b. $(f+g)'(0)=f'(0)+g'(0)=-1+5=4$

c. $(f/g)'(0)=\frac{g(0)f'(0)-f(0)g'(0)}{g^2(0)}$

$=\frac{-3(-1)-4(5)}{(-3)^2}=-\frac{17}{9}$

46. **a.** $(f-g)'(3)=f'(3)-g'(3)=2-(-10)=12$

b. $(f\cdot g)'(3)=f(3)g'(3)+g(3)f'(3) = 7(-10) + 6(2) = -58$

c. $(g/f)'(3)=\frac{f(3)g'(3)-g(3)f'(3)}{f^2(3)}=\frac{7(-10)-6(2)}{(7)^2}=-\frac{82}{49}$

47. $D_x[f(x)]^2=D_x[f(x)f(x)]$
$=f(x)D_x[f(x)]+f(x)D_x[f(x)]$
$=2\cdot f(x)\cdot D_x f(x)$

48. $D_x[f(x)g(x)h(x)]=f(x)D_x[g(x)h(x)]+g(x)h(x)D_x f(x)$
$=f(x)[g(x)D_x h(x)+h(x)D_x g(x)]+g(x)h(x)D_x f(x)$
$=f(x)g(x)D_x h(x)+f(x)h(x)D_x g(x)+g(x)h(x)D_x f(x)$

49. $D_x(x^2 - 2x + 2) = 2x - 2$
At $x = 1$: $m_{\tan} = 2(1) - 2 = 0$
Tangent line: $y = 1$

50. $D_x\left(\frac{1}{x^2+4}\right) = \frac{(x^2+4)D_x(1) - (1)D_x(x^2+4)}{(x^2+4)^2}$

$= \frac{(x^2+4)(0) - (2x)}{(x^2+4)^2} = -\frac{2x}{(x^2+4)^2}$

At $x = 1$: $m_{\tan} = -\frac{2(1)}{(1^2+4)^2} = -\frac{2}{25}$

Tangent line: $y - \frac{1}{5} = -\frac{2}{25}(x-1)$

$y = -\frac{2}{25}x + \frac{7}{25}$

51. $D_x(x^3 - x^2) = 3x^2 - 2x$
The tangent line is horizontal when $m_{\tan} = 0$:
$m_{\tan} = 3x^2 - 2x = 0$
$x(3x-2) = 0$
$x = 0$ and $x = \frac{2}{3}$
(0, 0) and $\left(\frac{2}{3}, -\frac{4}{27}\right)$

52. $D_x\left(\frac{1}{3}x^3 + x^2 - x\right) = x^2 + 2x - 1$

$m_{\tan} = x^2 + 2x - 1 = 1$
$x^2 + 2x - 2 = 0$
$x = \frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2} = \frac{-2 \pm \sqrt{12}}{2}$
$= -1 - \sqrt{3}, -1 + \sqrt{3}$
$x = -1 \pm \sqrt{3}$
$\left(-1+\sqrt{3}, \frac{5}{3} - \sqrt{3}\right), \left(-1-\sqrt{3}, \frac{5}{3} + \sqrt{3}\right)$

53. $y = 100/x^5 = 100x^{-5}$
$y' = -500x^{-6}$

Set y' equal to -1, the negative reciprocal of the slope of the line $y = x$. Solving for x gives
$x = \pm 500^{1/6} \approx \pm 2.817$
$y = \pm 100(500)^{-5/6} \approx \pm 0.563$

The points are (2.817,0.563) and (–2.817,–0.563).

54. Proof #1:

$$D_x[f(x) - g(x)] = D_x[f(x) + (-1)g(x)]$$
$$= D_x[f(x)] + D_x[(-1)g(x)]$$
$$= D_x f(x) - D_x g(x)$$

Proof #2:

Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = \lim_{h \to 0} \frac{[f(x+h) - g(x+h)] - [f(x) - g(x)]}{h}$$
$$= \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h}\right]$$
$$= f'(x) - g'(x)$$

55. **a.** $D_t(-16t^2 + 40t + 100) = -32t + 40$
$v = -32(2) + 40 = -24$ ft/s

b. $v = -32t + 40 = 0$
$t = \frac{5}{4}$ s

56. $D_t(4.5t^2 + 2t) = 9t + 2$
$9t + 2 = 30$
$t = \frac{28}{9}$ s

57. $m_{\tan} = D_x(4x - x^2) = 4 - 2x$
The line through (2,5) and (x_0, y_0) has slope $\frac{y_0 - 5}{x_0 - 2}$.

$4 - 2x_0 = \frac{4x_0 - x_0^2 - 5}{x_0 - 2}$
$-2x_0^2 + 8x_0 - 8 = -x_0^2 + 4x_0 - 5$
$x_0^2 - 4x_0 + 3 = 0$
$(x_0 - 3)(x_0 - 1) = 0$
$x_0 = 1, \ x_0 = 3$
At $x_0 = 1$: $y_0 = 4(1) - (1)^2 = 3$
$m_{\tan} = 4 - 2(1) = 2$
Tangent line: $y - 3 = 2(x - 1)$; $y = 2x + 1$
At $x_0 = 3$: $y_0 = 4(3) - (3)^2 = 3$
$m_{\tan} = 4 - 2(3) = -2$
Tangent line: $y - 3 = -2(x - 3)$; $y = -2x + 9$

58. $D_x(x^2) = 2x$

The line through (4, 15) and (x_0, y_0) has slope $\frac{y_0 - 15}{x_0 - 4}$. If (x_0, y_0) is on the curve $y = x^2$, then

$m_{\tan} = 2x_0 = \frac{x_0^2 - 15}{x_0 - 4}$.

$2x_0^2 - 8x_0 = x_0^2 - 15$

$x_0^2 - 8x_0 + 15 = 0$

$(x_0 - 3)(x_0 - 5) = 0$

At $x_0 = 3: y_0 = (3)^2 = 9$

She should shut off the engines at (3, 9). (At $x_0 = 5$ she would not go to (4, 15) since she is moving left to right.)

59. $D_x(7 - x^2) = -2x$

The line through (4, 0) and (x_0, y_0) has slope $\frac{y_0 - 0}{x_0 - 4}$. If the fly is at (x_0, y_0) when the spider sees it, then $m_{\tan} = -2x_0 = \frac{7 - x_0^2 - 0}{x_0 - 4}$.

$-2x_0^2 + 8x_0 = 7 - x_0^2$

$x_0^2 - 8x_0 + 7 = 0$

$(x_0 - 7)(x_0 - 1) = 0$

At $x_0 = 1: y_0 = 6$

$d = \sqrt{(4-1)^2 + (0-6)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$
≈ 6.7

They are 6.7 units apart when they see each other.

60. $P(a, b)$ is $\left(a, \frac{1}{a}\right)$. $D_x y = -\frac{1}{x^2}$ so the slope of the tangent line at P is $-\frac{1}{a^2}$. The tangent line is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$ or $y = -\frac{1}{a^2}(x - 2a)$ which has x-intercept $(2a, 0)$.

$d(O,P) = \sqrt{a^2 + \frac{1}{a^2}}, d(P,A) = \sqrt{(a-2a)^2 + \frac{1}{a^2}}$
$= \sqrt{a^2 + \frac{1}{a^2}} = d(O,P)$ so AOP is an isosceles triangle. The height of AOP is a while the base, $\overline{OA}$ has length $2a$, so the area is $\frac{1}{2}(2a)(a) = a^2$.

61. The watermelon has volume $\frac{4}{3}\pi r^3$; the volume of the rind is

$V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi\left(r - \frac{r}{10}\right)^3 = \frac{271}{750}\pi r^3.$

At the end of the fifth week $r = 10$, so

$D_r V = \frac{271}{250}\pi r^2 = \frac{271}{250}\pi(10)^2 = \frac{542\pi}{5} \approx 340 \text{ cm}^3$

per cm of radius growth. Since the radius is growing 2 cm per week, the volume of the rind is growing at the rate of $\frac{542\pi}{5}(2) \approx 681 \text{ cm}^3$ per week.

2.4 Concepts Review

1. $\frac{\sin(x+h) - \sin(x)}{h}$

2. 0; 1

3. $\cos x$; $-\sin x$

4. $\cos\frac{\pi}{3} = \frac{1}{2}; y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$

Problem Set 2.4

1. $D_x(2 \sin x + 3 \cos x) = 2 D_x(\sin x) + 3 D_x(\cos x)$
$= 2 \cos x - 3 \sin x$

2. $D_x(\sin^2 x) = \sin x\, D_x(\sin x) + \sin x\, D_x(\sin x)$
$= \sin x \cos x + \sin x \cos x = 2 \sin x \cos x = \sin 2x$

3. $D_x(\sin^2 x + \cos^2 x) = D_x(1) = 0$

4. $D_x(1 - \cos^2 x) = D_x(\sin^2 x)$
$= \sin x\, D_x(\sin x) + \sin x\, D_x(\sin x)$
$= \sin x \cos x + \sin x \cos x$
$= 2 \sin x \cos x = \sin 2x$

5. $D_x(\sec x) = D_x\left(\frac{1}{\cos x}\right)$
$= \frac{\cos x\, D_x(1) - (1) D_x(\cos x)}{\cos^2 x}$
$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$

6. $D_x(\csc x) = D_x\left(\frac{1}{\sin x}\right)$

$= \frac{\sin x\, D_x(1) - (1)D_x(\sin x)}{\sin^2 x}$

$= \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$

7. $D_x(\tan x) = D_x\left(\frac{\sin x}{\cos x}\right)$

$= \frac{\cos x\, D_x(\sin x) - \sin x\, D_x(\cos x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

8. $D_x(\cot x) = D_x\left(\frac{\cos x}{\sin x}\right)$

$= \frac{\sin x\, D_x(\cos x) - \cos x\, D_x(\sin x)}{\sin^2 x}$

$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= -\frac{1}{\sin^2 x} = -\csc^2 x$

9. $D_x\left(\frac{\sin x + \cos x}{\cos x}\right)$

$= \frac{\cos x\, D_x(\sin x + \cos x) - (\sin x + \cos x)D_x(\cos x)}{\cos^2 x}$

$= \frac{\cos x(\cos x - \sin x) - (-\sin^2 x - \sin x \cos x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

10. $D_x\left(\frac{\sin x + \cos x}{\tan x}\right)$

$= \frac{\tan x\, D_x(\sin x + \cos x) - (\sin x + \cos x)D_x(\tan x)}{\tan^2 x}$

$= \frac{\tan x(\cos x - \sin x) - \sec^2 x(\sin x + \cos x)}{\tan^2 x}$

$= \left(\sin x - \frac{\sin^2 x}{\cos x} - \frac{\sin x}{\cos^2 x} - \frac{1}{\cos x}\right) \div \left(\frac{\sin^2 x}{\cos^2 x}\right)$

$= \left(\sin x - \frac{\sin^2 x}{\cos x} - \frac{\sin x}{\cos^2 x} - \frac{1}{\cos x}\right)\left(\frac{\cos^2 x}{\sin^2 x}\right)$

$= \frac{\cos^2 x}{\sin x} - \cos x - \frac{1}{\sin x} - \frac{\cos x}{\sin^2 x}$

11. $D_x(\sin x \cos x) = \sin x D_x[\cos x] + \cos x D_x[\sin x]$

$= \sin x(-\sin x) + \cos x(\cos x) = \cos^2 x - \sin^2 x$

12. $D_x(\sin x \tan x) = \sin x D_x[\tan x] + \tan x D_x[\sin x]$

$= \sin x(\sec^2 x) + \tan x(\cos x)$

$= \sin x\left(\frac{1}{\cos^2 x}\right) + \frac{\sin x}{\cos x}(\cos x)$

$= \tan x \sec x + \sin x$

13. $D_x\left(\frac{\sin x}{x}\right) = \frac{x D_x(\sin x) - \sin x D_x(x)}{x^2}$

$= \frac{x \cos x - \sin x}{x^2}$

14. $D_x\left(\frac{1 - \cos x}{x}\right) = \frac{x D_x(1 - \cos x) - (1 - \cos x) D_x(x)}{x^2}$

$= \frac{x \sin x + \cos x - 1}{x^2}$

15. $D_x(x^2 \cos x) = x^2 D_x(\cos x) + \cos x\, D_x(x^2)$

$= -x^2 \sin x + 2x \cos x$

16. $D_x\left(\frac{x \cos x + \sin x}{x^2 + 1}\right)$

$= \frac{(x^2+1)D_x(x \cos x + \sin x) - (x \cos x + \sin x)D_x(x^2+1)}{(x^2+1)^2}$

$= \frac{(x^2+1)(-x \sin x + \cos x + \cos x) - 2x(x \cos x + \sin x)}{(x^2+1)^2}$

$= \frac{-x^3 \sin x - 3x \sin x + 2\cos x}{(x^2+1)^2}$

17. $y = \tan^2 x = (\tan x)(\tan x)$

$D_x y = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$

$= 2\tan x \sec^2 x$

18. $y = \sec^3 x = (\sec^2 x)(\sec x)$

$D_x y = (\sec^2 x)\sec x \tan x + (\sec x) D_x (\sec^2 x)$

$= \sec^3 x \tan x + \sec x(\sec x \cdot \sec x \tan x$

$+ \sec x \cdot \sec x \tan x)$

$= \sec^3 x \tan x + 2\sec^3 x \tan x$

$= 3\sec^2 x \tan x$

19. $D_x(\cos x) = -\sin x$

At $x = 1$: $m_{\tan} = -\sin 1 \approx -0.8415$

$y = \cos 1 \approx 0.5403$

Tangent line: $y - 0.5403 = -0.8415(x - 1)$

20. $D_x(\cot x) = -\csc^2 x$

At $x = \frac{\pi}{4}$: $m_{\tan} = -2$;

$y = 1$

Tangent line: $y - 1 = -2\left(x - \frac{\pi}{4}\right)$

21. $D_x \sin 2x = D_x(2\sin x \cos x)$

$= 2\left[\sin x D_x \cos x + \cos x D_x \sin x\right]$

$= -2\sin^2 x + 2\cos^2 x$

22. $D_x \cos 2x = D_x(2\cos^2 x - 1) = 2D_x \cos^2 x - D_x 1$

$= -2\sin x \cos x$

23. $D_t(30\sin 2t) = 30 D_t(2\sin t \cos t)$

$= 30\left(-2\sin^2 t + 2\cos^2 t\right)$

$= 60\cos 2t$

$30\sin 2t = 15$

$\sin 2t = \frac{1}{2}$

$2t = \frac{\pi}{6} \quad \to \quad t = \frac{\pi}{12}$

At $t = \frac{\pi}{12}$; $60\cos\left(2 \cdot \frac{\pi}{12}\right) = 30\sqrt{3}$ ft/sec

The seat is moving to the left at the rate of $30\sqrt{3}$ ft/s.

24. The coordinates of the seat at time t are $(20 \cos t, 20 \sin t)$.

a. $\left(20\cos\frac{\pi}{6}, 20\sin\frac{\pi}{6}\right) = (10\sqrt{3}, 10)$

$\approx (17.32,\ 10)$

b. $D_t(20 \sin t) = 20 \cos t$

At $t = \frac{\pi}{6}$: rate $= 20\cos\frac{\pi}{6} = 10\sqrt{3} \approx 17.32$ ft/s

c. The fastest rate $20 \cos t$ can obtain is 20 ft/s.

25. $y = \tan x$

$y' = \sec^2 x$

When $y = 0$, $y = \tan 0 = 0$ and $y' = \sec^2 0 = 1$.

The tangent line at $x = 0$ is $y = x$.

26. $y = \tan^2 x = (\tan x)(\tan x)$

$y' = (\tan x)(\sec^2 x) + (\tan x)(\sec^2 x)$

$= 2\tan x \sec^2 x$

Now, $\sec^2 x$ is never 0, but $\tan x = 0$ at $x = k\pi$ where k is an integer.

27. $y = 9\sin x \cos x$

$y' = 9\left[\sin x(-\sin x) + \cos x(\cos x)\right]$

$= 9\left[\sin^2 x - \cos^2 x\right]$

$= 9\left[-\cos 2x\right]$

The tangent line is horizontal when $y' = 0$ or, in this case, where $\cos 2x = 0$. This occurs when $x = \frac{\pi}{4} + k\frac{\pi}{2}$ where k is an integer.

28. $f(x) = x - \sin x$

$f'(x) = 1 - \cos x$

$f'(x) = 0$ when $\cos x = 1$; i.e. when $x = 2k\pi$ where k is an integer.

$f'(x) = 2$ when $x = (2k+1)\pi$ where k is an integer.

29. The curves intersect when $\sqrt{2}\sin x = \sqrt{2}\cos x$, $\sin x = \cos x$ at $x = \frac{\pi}{4}$ for $0 < x < \frac{\pi}{2}$.

$D_x(\sqrt{2}\sin x) = \sqrt{2}\cos x$; $\sqrt{2}\cos\frac{\pi}{4} = 1$

$D_x(\sqrt{2}\cos x) = -\sqrt{2}\sin x$; $-\sqrt{2}\sin\frac{\pi}{4} = -1$

$1(-1) = -1$ so the curves intersect at right angles.

30. $v = D_t(3\sin 2t) = 6\cos 2t$

At $t = 0$: $v = 6$ cm/s

$t = \frac{\pi}{2}$: $v = -6$ cm/s

$t = \pi$: $v = 6$ cm/s

31. $D_x(\sin x^2) = \lim_{h\to 0}\frac{\sin(x+h)^2 - \sin x^2}{h}$

$= \lim_{h\to 0}\frac{\sin(x^2+2xh+h^2) - \sin x^2}{h}$

$= \lim_{h\to 0}\frac{\sin x^2 \cos(2xh+h^2) + \cos x^2 \sin(2xh+h^2) - \sin x^2}{h} = \lim_{h\to 0}\frac{\sin x^2[\cos(2xh+h^2)-1] + \cos x^2 \sin(2xh+h^2)}{h}$

$= \lim_{h\to 0}(2x+h)\left[\sin x^2 \frac{\cos(2xh+h^2)-1}{2xh+h^2} + \cos x^2 \frac{\sin(2xh+h^2)}{2xh+h^2}\right] = 2x(\sin x^2 \cdot 0 + \cos x^2 \cdot 1) = 2x\cos x^2$

32. $D_x(\sin 5x) = \lim_{h\to 0}\frac{\sin(5(x+h)) - \sin 5x}{h}$

$= \lim_{h\to 0}\frac{\sin(5x+5h) - \sin 5x}{h}$

$= \lim_{h\to 0}\frac{\sin 5x\cos 5h + \cos 5x \sin 5h - \sin 5x}{h}$

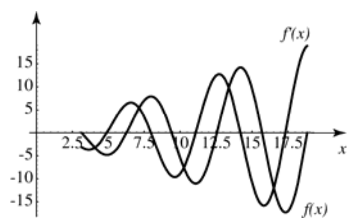
$= \lim_{h\to 0}\left[\sin 5x \frac{\cos 5h - 1}{h} + \cos 5x \frac{\sin 5h}{h}\right]$

$= \lim_{h\to 0}\left[5\sin 5x \frac{\cos 5h - 1}{5h} + 5\cos 5x \frac{\sin 5h}{5h}\right]$

$= 0 + 5\cos 5x \cdot 1 = 5\cos 5x$

33. $f(x) = x \sin x$

a.

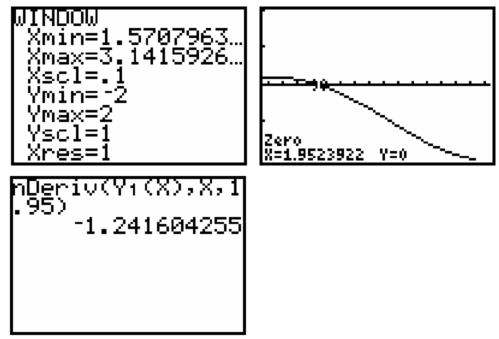


b. $f(x) = 0$ has 6 solutions on $[\pi, 6\pi]$
$f'(x) = 0$ has 5 solutions on $[\pi, 6\pi]$

c. $f(x) = x \sin x$ is a counterexample.
Consider the interval $[0, \pi]$.
$f(-\pi) = f(\pi) = 0$ and $f(x) = 0$ has exactly two solutions in the interval (at 0 and π). However, $f'(x) = 0$ has two solutions in the interval, not 1 as the conjecture indicates it should have.

d. The maximum value of $|f(x) - f'(x)|$ on $[\pi, 6\pi]$ is about 24.93.

34. $f(x) = \cos^3 x - 1.25\cos^2 x + 0.225$



$x_0 \approx 1.95$

$f'(x_0) \approx -1.24$

2.5 Concepts Review

1. $D_t u$; $f'(g(t))g'(t)$

2. $D_v w$; $G'(H(s))H'(s)$

3. $(f(x))^2$; $(f(x))^2$

4. $2x\cos(x^2)$; $6(2x+1)^2$

Problem Set 2.5

1. $y = u^{15}$ and $u = 1 + x$

$D_x y = D_u y \cdot D_x u$

$= (15u^{14})(1)$

$= 15(1+x)^{14}$

2. $y = u^5$ and $u = 7 + x$

$D_x y = D_u y \cdot D_x u$

$= (5u^4)(1)$

$= 5(7+x)^4$

3. $y = u^5$ and $u = 3 - 2x$

$D_x y = D_u y \cdot D_x u$

$= (5u^4)(-2) = -10(3-2x)^4$

4. $y = u^7$ and $u = 4 + 2x^2$
$D_x y = D_u y \cdot D_x u$
$= (7u^6)(4x) = 28x(4+2x^2)^6$

5. $y = u^{11}$ and $u = x^3 - 2x^2 + 3x + 1$
$D_x y = D_u y \cdot D_x u$
$= (11u^{10})(3x^2 - 4x + 3)$
$= 11(3x^2 - 4x + 3)(x^3 - 2x^2 + 3x + 1)^{10}$

6. $y = u^{-7}$ and $u = x^2 - x + 1$
$D_x y = D_u y \cdot D_x u$
$= (-7u^{-8})(2x - 1)$
$= -7(2x-1)(x^2 - x + 1)^{-8}$

7. $y = u^{-5}$ and $u = x + 3$
$D_x y = D_u y \cdot D_x u$
$= (-5u^{-6})(1) = -5(x+3)^{-6} = -\dfrac{5}{(x+3)^6}$

8. $y = u^{-9}$ and $u = 3x^2 + x - 3$
$D_x y = D_u y \cdot D_x u$
$= (-9u^{-10})(6x + 1)$
$= -9(6x+1)(3x^2 + x - 3)^{-10}$
$= -\dfrac{9(6x+1)}{(3x^2 + x - 3)^{10}}$

9. $y = \sin u$ and $u = x^2 + x$
$D_x y = D_u y \cdot D_x u$
$= (\cos u)(2x + 1)$
$= (2x+1)\cos(x^2 + x)$

10. $y = \cos u$ and $u = 3x^2 - 2x$
$D_x y = D_u y \cdot D_x u$
$= (-\sin u)(6x - 2)$
$= -(6x-2)\sin(3x^2 - 2x)$

11. $y = u^3$ and $u = \cos x$
$D_x y = D_u y \cdot D_x u$
$= (3u^2)(-\sin x)$
$= -3\sin x \cos^2 x$

12. $y = u^4$, $u = \sin v$, and $v = 3x^2$
$D_x y = D_u y \cdot D_v u \cdot D_x v$
$= (4u^3)(\cos v)(6x)$
$= 24x \sin^3(3x^2)\cos(3x^2)$

13. $y = u^3$ and $u = \dfrac{x+1}{x-1}$
$D_x y = D_u y \cdot D_x u$
$= (3u^2)\dfrac{(x-1)D_x(x+1) - (x+1)D_x(x-1)}{(x-1)^2}$
$= 3\left(\dfrac{x+1}{x-1}\right)^2\left(\dfrac{-2}{(x-1)^2}\right) = -\dfrac{6(x+1)^2}{(x-1)^4}$

14. $y = u^{-3}$ and $u = \dfrac{x-2}{x-\pi}$
$D_x y = D_u y \cdot D_x u$
$= (-3u^{-4}) \cdot \dfrac{(x-\pi)D_x(x-2) - (x-2)D_x(x-\pi)}{(x-\pi)^2}$
$= -3\left(\dfrac{x-2}{x-\pi}\right)^{-4}\dfrac{(2-\pi)}{(x-\pi)^2} = -3\dfrac{(x-\pi)^2}{(x-2)^4}(2-\pi)$

15. $y = \cos u$ and $u = \dfrac{3x^2}{x+2}$
$D_x y = D_u y \cdot D_x u = (-\sin u)\dfrac{(x+2)D_x(3x^2) - (3x^2)D_x(x+2)}{(x+2)^2}$
$= -\sin\left(\dfrac{3x^2}{x+2}\right)\dfrac{(x+2)(6x) - (3x^2)(1)}{(x+2)^2} = -\dfrac{3x^2 + 12x}{(x+2)^2}\sin\left(\dfrac{3x^2}{x+2}\right)$

16. $y = u^3$, $u = \cos v$, and $v = \dfrac{x^2}{1-x}$
$D_x y = D_u y \cdot D_v u \cdot D_x v = (3u^2)(-\sin v)\dfrac{(1-x)D_x(x^2) - (x^2)D_x(1-x)}{(1-x)^2}$
$= -3\cos^2\left(\dfrac{x^2}{1-x}\right)\sin\left(\dfrac{x^2}{1-x}\right)\dfrac{(1-x)(2x) - (x^2)(-1)}{(1-x)^2} = \dfrac{-3(2x - x^2)}{(1-x)^2}\cos^2\left(\dfrac{x^2}{1-x}\right)\sin\left(\dfrac{x^2}{1-x}\right)$

17. $D_x[(3x-2)^2(3-x^2)^2] = (3x-2)^2 D_x(3-x^2)^2 + (3-x^2)^2 D_x(3x-2)^2$

$= (3x-2)^2(2)(3-x^2)(-2x) + (3-x^2)^2(2)(3x-2)(3)$

$= 2(3x-2)(3-x^2)[(3x-2)(-2x)+(3-x^2)(3)] = 2(3x-2)(3-x^2)(9+4x-9x^2)$

18. $D_x[(2-3x^2)^4(x^7+3)^3] = (2-3x^2)^4 D_x(x^7+3)^3 + (x^7+3)^3 D_x(2-3x^2)^4$

$= (2-3x^2)^4(3)(x^7+3)^2(7x^6) + (x^7+3)^3(4)(2-3x^2)^3(-6x) = 3x(3x^2-2)^3(x^7+3)^2(29x^7-14x^5+24)$

19. $D_x\left[\frac{(x+1)^2}{3x-4}\right] = \frac{(3x-4)D_x(x+1)^2-(x+1)^2 D_x(3x-4)}{(3x-4)^2} = \frac{(3x-4)(2)(x+1)(1)-(x+1)^2(3)}{(3x-4)^2} = \frac{3x^2-8x-11}{(3x-4)^2}$

$= \frac{(x+1)(3x-11)}{(3x-4)^2}$

20. $D_x\left[\frac{2x-3}{(x^2+4)^2}\right] = \frac{(x^2+4)^2 D_x(2x-3)-(2x-3)D_x(x^2+4)^2}{(x^2+4)^4}$

$= \frac{(x^2+4)^2(2)-(2x-3)(2)(x^2+4)(2x)}{(x^2+4)^4} = \frac{-6x^2+12x+8}{(x^2+4)^3}$

21. $y' = 2(x^2+4)(x^2+4)' = 2(x^2+4)(2x) = 4x(x^2+4)$

22. $y' = 2(x+\sin x)(x+\sin x)' = 2(x+\sin x)(1+\cos x)$

23. $D_t\left(\frac{3t-2}{t+5}\right)^3 = 3\left(\frac{3t-2}{t+5}\right)^2 \frac{(t+5)D_t(3t-2)-(3t-2)D_t(t+5)}{(t+5)^2}$

$= 3\left(\frac{3t-2}{t+5}\right)^2 \frac{(t+5)(3)-(3t-2)(1)}{(t+5)^2} = \frac{51(3t-2)^2}{(t+5)^4}$

24. $D_s\left(\frac{s^2-9}{s+4}\right) = \frac{(s+4)D_s(s^2-9)-(s^2-9)D_s(s+4)}{(s+4)^2} = \frac{(s+4)(2s)-(s^2-9)(1)}{(s+4)^2} = \frac{s^2+8s+9}{(s+4)^2}$

25. $\frac{d}{dt}\left(\frac{(3t-2)^3}{t+5}\right) = \frac{(t+5)\frac{d}{dt}(3t-2)^3-(3t-2)^3\frac{d}{dt}(t+5)}{(t+5)^2} = \frac{(t+5)(3)(3t-2)^2(3)-(3t-2)^3(1)}{(t+5)^2}$

$= \frac{(6t+47)(3t-2)^2}{(t+5)^2}$

26. $\frac{d}{d\theta}(\sin^3\theta) = 3\sin^2\theta\cos\theta$

27. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos 2x}\right)^3 = 3\left(\frac{\sin x}{\cos 2x}\right)^2 \cdot \frac{d}{dx}\frac{\sin x}{\cos 2x} = 3\left(\frac{\sin x}{\cos 2x}\right)^2 \cdot \frac{(\cos 2x)\frac{d}{dx}(\sin x)-(\sin x)\frac{d}{dx}(\cos 2x)}{\cos^2 2x}$

$= 3\left(\frac{\sin x}{\cos 2x}\right)^2 \frac{\cos x\cos 2x+2\sin x\sin 2x}{\cos^2 2x} = \frac{3\sin^2 x\cos x\cos 2x+6\sin^3 x\sin 2x}{\cos^4 2x}$

$= \frac{3(\sin^2 x)(\cos x\cos 2x+2\sin x\sin 2x)}{\cos^4 2x}$

28. $\frac{dy}{dt} = \frac{d}{dt}[\sin t \tan(t^2+1)] = \sin t \cdot \frac{d}{dt}[\tan(t^2+1)] + \tan(t^2+1)\cdot\frac{d}{dt}(\sin t)$

$= (\sin t)[\sec^2(t^2+1)](2t) + \tan(t^2+1)\cos t = 2t\sin t\sec^2(t^2+1) + \cos t\tan(t^2+1)$

29. $f'(x) = 3\left(\frac{x^2+1}{x+2}\right)^2 \frac{(x+2)D_x(x^2+1)-(x^2+1)D_x(x+2)}{(x+2)^2} = 3\left(\frac{x^2+1}{x+2}\right)^2 \frac{2x^2+4x-x^2-1}{(x+2)^2} = \frac{3(x^2+1)^2(x^2+4x-1)}{(x+2)^4}$

$f'(3) = 9.6$

30. $G'(t) = (t^2+9)^3 D_t(t^2-2)^4 + (t^2-2)^4 D_t(t^2+9)^3 = (t^2+9)^3(4)(t^2-2)^3(2t) + (t^2-2)^4(3)(t^2+9)^2(2t)$

$= 2t(7t^2+30)(t^2+9)^2(t^2-2)^3$

$G'(1) = -7400$

31. $F'(t) = [\cos(t^2+3t+1)](2t+3) = (2t+3)\cos(t^2+3t+1)$; $F'(1) = 5\cos 5 \approx 1.4183$

32. $g'(s) = (\cos \pi s)D_s(\sin^2 \pi s) + (\sin^2 \pi s)D_s(\cos \pi s) = (\cos \pi s)(2\sin \pi s)(\cos \pi s)(\pi) + (\sin^2 \pi s)(-\sin \pi s)(\pi)$

$= \pi \sin \pi s[2\cos^2 \pi s - \sin^2 \pi s]$

$g'\left(\frac{1}{2}\right) = -\pi$

33. $D_x[\sin^4(x^2+3x)] = 4\sin^3(x^2+3x)D_x \sin(x^2+3x) = 4\sin^3(x^2+3x)\cos(x^2+3x)D_x(x^2+3x)$

$= 4\sin^3(x^2+3x)\cos(x^2+3x)(2x+3) = 4(2x+3)\sin^3(x^2+3x)\cos(x^2+3x)$

34. $D_t[\cos^5(4t-19)] = 5\cos^4(4t-19)D_t \cos(4t-19) = 5\cos^4(4t-19)[-\sin(4t-19)]D_t(4t-19)$

$= -5\cos^4(4t-19)\sin(4t-19)(4) = -20\cos^4(4t-19)\sin(4t-19)$

35. $D_t[\sin^3(\cos t)] = 3\sin^2(\cos t)D_t \sin(\cos t) = 3\sin^2(\cos t)\cos(\cos t)D_t(\cos t)$

$= 3\sin^2(\cos t)\cos(\cos t)(-\sin t) = -3\sin t\sin^2(\cos t)\cos(\cos t)$

36 . $D_u\left[\cos^4\left(\frac{u+1}{u-1}\right)\right] = 4\cos^3\left(\frac{u+1}{u-1}\right)D_u \cos\left(\frac{u+1}{u-1}\right) = 4\cos^3\left(\frac{u+1}{u-1}\right)\left[-\sin\left(\frac{u+1}{u-1}\right)\right]D_u\left(\frac{u+1}{u-1}\right)$

$= -4\cos^3\left(\frac{u+1}{u-1}\right)\sin\left(\frac{u+1}{u-1}\right)\frac{(u-1)D_u(u+1)-(u+1)D_u(u-1)}{(u-1)^2} = \frac{8}{(u-1)^2}\cos^3\left(\frac{u+1}{u-1}\right)\sin\left(\frac{u+1}{u-1}\right)$

37. $D_\theta[\cos^4(\sin\theta^2)] = 4\cos^3(\sin\theta^2)D_\theta \cos(\sin\theta^2) = 4\cos^3(\sin\theta^2)[-\sin(\sin\theta^2)]D_\theta(\sin\theta^2)$

$= -4\cos^3(\sin\theta^2)\sin(\sin\theta^2)(\cos\theta^2)D_\theta(\theta^2) = -8\theta\cos^3(\sin\theta^2)\sin(\sin\theta^2)(\cos\theta^2)$

38. $D_x[x\sin^2(2x)] = x D_x \sin^2(2x) + \sin^2(2x)D_x x = x[2\sin(2x)D_x \sin(2x)] + \sin^2(2x)(1)$

$= x[2\sin(2x)\cos(2x)D_x(2x)] + \sin^2(2x) = x[4\sin(2x)\cos(2x)] + \sin^2(2x) = 2x\sin(4x) + \sin^2(2x)$

39. $D_x\{\sin[\cos(\sin 2x)]\} = \cos[\cos(\sin 2x)]D_x \cos(\sin 2x) = \cos[\cos(\sin 2x)][-\sin(\sin 2x)]D_x(\sin 2x)$

$= -\cos[\cos(\sin 2x)]\sin(\sin 2x)(\cos 2x)D_x(2x) = -2\cos[\cos(\sin 2x)]\sin(\sin 2x)(\cos 2x)$

40. $D_t\{\cos^2[\cos(\cos t)]\} = 2\cos[\cos(\cos t)]D_t \cos[\cos(\cos t)] = 2\cos[\cos(\cos t)]\{-\sin[\cos(\cos t)]\}D_t \cos(\cos t)$

$= -2\cos[\cos(\cos t)]\sin[\cos(\cos t)][-\sin(\cos t)]D_t(\cos t) = 2\cos[\cos(\cos t)]\sin[\cos(\cos t)]\sin(\cos t)(-\sin t)$

$= -2\sin t\cos[\cos(\cos t)]\sin[\cos(\cos t)]\sin(\cos t)$

41. $(f+g)'(4) = f'(4) + g'(4)$
$\approx \frac{1}{2} + \frac{3}{2} \approx 2$

42. $(f-2g)'(2) = f'(2) - (2g)'(2)$
$= f'(2) - 2g'(2)$
$= 1 - 2(0) = 1$

43. $(fg)'(2) = (fg' + gf')(2) = 2(0) + 1(1) = 1$

44. $(f/g)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g^2(2)}$
$\approx \frac{(1)(1)-(3)(0)}{(1)^2} = 1$

45. $(f \circ g)'(6) = f'(g(6))g'(6)$
$= f'(2)g'(6) \approx (1)(-1) = -1$

46. $(g \circ f)'(3) = g'(f(3))f'(3)$
$= g'(4)f'(3) \approx \left(\frac{3}{2}\right)(1) = \frac{3}{2}$

47. $D_x F(2x) = F'(2x) D_x(2x) = 2F'(2x)$

48. $D_x F(x^2+1) = F'(x^2+1) D_x(x^2+1)$
$= 2xF'(x^2+1)$

49. $D_t\left[(F(t))^{-2}\right] = -2(F(t))^{-3} F'(t)$

50. $\frac{d}{dz}\left[\frac{1}{(F(z))^2}\right] = -2(F(z))^{-3} F'(z)$

51. $\frac{d}{dz}\left[(1+F(2z))^2\right] = 2(1+F(2z))\frac{d}{dz}(1+F(2z))$
$= 2(1+F(2z))(2F'(2z)) = 4(1+F(2z))F'(2z)$

52. $\frac{d}{dy}\left[y^2 + \frac{1}{F(y^2)}\right] = 2y + \frac{d}{dy}\left[\left(F(y^2)\right)^{-1}\right]$
$= 2y - F'(y^2)\frac{d}{dy}y^2 = 2y - \frac{2yF'(y^2)}{\left(F(y^2)\right)^2}$
$= 2y\left(1 - \frac{F'(y^2)}{\left(F(y^2)\right)^2}\right)$

53. $\frac{d}{dx}F(\cos x) = F'(\cos x)\frac{d}{dx}(\cos x)$
$= -\sin x F'(\cos x)$

54. $\frac{d}{dx}\cos(F(x)) = -\sin(F(x))\frac{d}{dx}F(x)$
$= -F'(x)\sin(F(x))$

55. $D_x\left[\tan(F(2x))\right] = \sec^2(F(2x))D_x\left[F(2x)\right]$
$= \sec^2(F(2x)) \times F'(2x) \times D_x[2x]$
$= 2F'(2x)\sec^2(F(2x))$

56. $\frac{d}{dx}\left[g(\tan 2x)\right] = g'(\tan 2x) \cdot \frac{d}{dx}\tan 2x$
$= g'(\tan 2x)\left(\sec^2 2x\right) \cdot 2$
$= 2g'(\tan 2x)\sec^2 2x$

57. $D_x\left[F(x)\sin^2 F(x)\right]$
$= F(x) \times D_x\left[\sin^2 F(x)\right] + \sin^2 F(x) \times D_x F(x)$
$= F(x) \times 2\sin F(x) \times D_x\left[\sin F(x)\right]$
$+ F'(x)\sin^2 F(x)$
$= F(x) \times 2\sin F(x) \times \cos(F(x)) \times D_x\left[F(x)\right]$
$+ F'(x)\sin^2 F(x)$
$= 2F(x)F'(x)\sin F(x)\cos F(x)$
$+ F'(x)\sin^2 F(x)$

58. $D_x\left[\sec^3 F(x)\right] = 3\sec^2\left[F(x)\right]D_x\left[\sec F(x)\right]$
$= 3\sec^2\left[F(x)\right]\sec F(x)\tan F(x)D_x[x]$
$= 3F'(x)\sec^3 F(x)\tan F(x)$

59. $g'(x) = -\sin f(x) D_x f(x) = -f'(x)\sin f(x)$
$g'(0) = -f'(0)\sin f(0) = -2\sin 1 \approx -1.683$

60. $G'(x) = \frac{(1+\sec F(2x))\frac{d}{dx}x - x\frac{d}{dx}(1+\sec F(2x))}{(1+\sec F(2x))^2}$
$= \frac{(1+\sec F(2x)) - 2xF'(2x)\sec F(2x)\tan F(2x)}{(1+\sec F(2x))^2}$
$G'(0) = \frac{1+\sec F(0) - 0}{(1+\sec F(0))^2} = \frac{1+\sec F(0)}{(1+\sec F(0))^2}$
$= \frac{1}{1+\sec F(0)} = \frac{1}{1+\sec 2} \approx -0.713$

61. $F'(x) = -f(x)g'(x)\sin g(x) + f'(x)\cos g(x)$

$F'(1) = -f(1)g'(1)\sin g(1) + f'(1)\cos g(1)$

$= -2(1)\sin 0 + -1\cos 0 = -1$

62. $y = 1 + x\sin 3x;\quad y' = 3x\cos 3x + \sin 3x$

$y'(\pi/3) = 3\frac{\pi}{3}\cos 3\frac{\pi}{3} + \sin\frac{\pi}{3} = -\pi + 0 = -\pi$

$y - 1 = -\pi x - \pi/3$

$y = -\pi x - \pi/3 + 1$

The line crosses the x-axis at $x = \frac{3-\pi}{3}$.

63. $y = \sin^2 x;\quad y' = 2\sin x\cos x = \sin 2x = 1$

$x = \pi/4 + k\pi,\ k = 0, \pm 1, \pm 2, \ldots$

64. $y' = (x^2+1)^3\, 2(x^4+1)x^3 + 3(x^2+1)^2 x(x^4+1)^2$

$= 2x^3(x^4+1)(x^2+1)^3 + 3x(x^4+1)^2(x^2+1)^2$

$y'(1) = 2(2)(2)^3 + 3(1)(2)^2(2)^2 = 32 + 48 = 80$

$y - 32 = 80x - 1,\quad y = 80x + 31$

65. $y' = -2(x^2+1)^{-3}(2x) = -4x(x^2+1)^{-3}$

$y'(1) = -4(1)(1+1)^{-3} = -1/2$

$y - \frac{1}{4} = -\frac{1}{2}x + \frac{1}{2},\quad y = -\frac{1}{2}x + \frac{3}{4}$

66. $y' = 3(2x+1)^2(2) = 6(2x+1)^2$

$y'(0) = 6(1)^2 = 6$

$y - 1 = 6x - 0,\quad y = 6x + 1$

The line crosses the x-axis at $x = -1/6$.

67. $y' = -2(x^2+1)^{-3}(2x) = -4x(x^2+1)^{-3}$

$y'(1) = -4(2)^{-3} = -1/2$

$y - \frac{1}{4} = -\frac{1}{2}x + \frac{1}{2},\quad y = -\frac{1}{2}x + \frac{3}{4}$

Set $y = 0$ and solve for x. The line crosses the x-axis at $x = 3/2$.

68. **a.** $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{7}\right)^2 = \left(\frac{4\cos 2t}{4}\right)^2 + \left(\frac{7\sin 2t}{7}\right)^2$

$= \cos^2 2t + \sin^2 2t = 1$

b. $L = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$

$= \sqrt{(4\cos 2t)^2 + (7\sin 2t)^2}$

$= \sqrt{16\cos^2 2t + 49\sin^2 2t}$

c. $D_t L = \frac{1}{2\sqrt{16\cos^2 2t + 49\sin^2 2t}} D_t(16\cos^2 2t + 49\sin^2 2t)$

$= \frac{32\cos 2t\, D_t(\cos 2t) + 98\sin 2t\, D_t(\sin 2t)}{2\sqrt{16\cos^2 2t + 49\sin^2 2t}}$

$= \frac{-64\cos 2t\sin 2t + 196\sin 2t\cos 2t}{2\sqrt{16\cos^2 2t + 49\sin^2 2t}}$

$= \frac{-16\sin 4t + 49\sin 4t}{\sqrt{16\cos^2 2t + 49\sin^2 2t}}$

$= \frac{33\sin 4t}{\sqrt{16\cos^2 2t + 49\sin^2 2t}}$

At $t = \frac{\pi}{8}$: rate $= \frac{33}{\sqrt{16\cdot\frac{1}{2} + 49\cdot\frac{1}{2}}} \approx 5.8$ ft/sec.

69. **a.** $(10\cos 8\pi t, 10\sin 8\pi t)$

b. $D_t(10\sin 8\pi t) = 10\cos(8\pi t)D_t(8\pi t)$

$= 80\pi\cos(8\pi t)$

At $t = 1$: rate $= 80\pi \approx 251$ cm/s

P is rising at the rate of 251 cm/s.

70. **a.** $(\cos 2t, \sin 2t)$

b. $(0 - \cos 2t)^2 + (y - \sin 2t)^2 = 5^2$, so

$y = \sin 2t + \sqrt{25 - \cos^2 2t}$

c. $D_t\left(\sin 2t + \sqrt{25 - \cos^2 2t}\right)$

$= 2\cos 2t + \frac{1}{2\sqrt{25 - \cos^2 2t}} \cdot 4\cos 2t\sin 2t$

$= 2\cos 2t\left(1 + \frac{\sin 2t}{\sqrt{25 - \cos^2 2t}}\right)$

71. 60 revolutions per minute is 120π radians per minute or 2π radians per second.

a. $(\cos 2\pi t, \sin 2\pi t)$

b. $(0 - \cos 2\pi t)^2 + (y - \sin 2\pi t)^2 = 5^2$, so

$y = \sin 2\pi t + \sqrt{25 - \cos^2 2\pi t}$

c. $D_t\left(\sin 2\pi t + \sqrt{25 - \cos^2 2\pi t}\right)$

$= 2\pi\cos 2\pi t$

$+ \frac{1}{2\sqrt{25 - \cos^2 2\pi t}} \cdot 4\pi\cos 2\pi t\sin 2\pi t$

$= 2\pi\cos 2\pi t\left(1 + \frac{\sin 2\pi t}{\sqrt{25 - \cos^2 2\pi t}}\right)$

72. The minute hand makes 1 revolution every hour, so at t minutes after the hour, it makes an angle of $\frac{\pi t}{30}$ radians with the vertical. By the Law of Cosines, the length of the elastic string is

$$s = \sqrt{10^2 + 10^2 - 2(10)(10)\cos\frac{\pi t}{30}}$$
$$= 10\sqrt{2 - 2\cos\frac{\pi t}{30}}$$

$$\frac{ds}{dt} = 10 \cdot \frac{1}{2\sqrt{2 - 2\cos\frac{\pi t}{30}}} \cdot \frac{\pi}{15}\sin\frac{\pi t}{30}$$
$$= \frac{\pi \sin\frac{\pi t}{30}}{3\sqrt{2 - 2\cos\frac{\pi t}{30}}}$$

At 12:15, the string is stretching at the rate of

$$\frac{\pi \sin\frac{\pi}{2}}{3\sqrt{2 - 2\cos\frac{\pi}{2}}} = \frac{\pi}{3\sqrt{2}} \approx 0.74 \text{ cm/min}$$

73. The minute hand makes 1 revolution every hour, so at t minutes after noon it makes an angle of $\frac{\pi t}{30}$ radians with the vertical. Similarly, at t minutes after noon the hour hand makes an angle of $\frac{\pi t}{360}$ with the vertical. Thus, by the Law of Cosines, the distance between the tips of the hands is

$$s = \sqrt{6^2 + 8^2 - 2 \cdot 6 \cdot 8\cos\left(\frac{\pi t}{30} - \frac{\pi t}{360}\right)}$$
$$= \sqrt{100 - 96\cos\frac{11\pi t}{360}}$$
$$\frac{ds}{dt} = \frac{1}{2\sqrt{100 - 96\cos\frac{11\pi t}{360}}} \cdot \frac{44\pi}{15}\sin\frac{11\pi t}{360}$$
$$= \frac{22\pi \sin\frac{11\pi t}{360}}{15\sqrt{100 - 96\cos\frac{11\pi t}{360}}}$$

At 12:20,

$$\frac{ds}{dt} = \frac{22\pi \sin\frac{11\pi}{18}}{15\sqrt{100 - 96\cos\frac{11\pi}{18}}} \approx 0.38 \text{ in./min}$$

74. From Problem 73, $\frac{ds}{dt} = \frac{22\pi \sin\frac{11\pi t}{360}}{15\sqrt{100 - 96\cos\frac{11\pi t}{360}}}$.

Using a computer algebra system or graphing utility to view $\frac{ds}{dt}$ for $0 \le t \le 60$, $\frac{ds}{dt}$ is largest when $t \approx 7.5$. Thus, the distance between the tips of the hands is increasing most rapidly at about 12:08.

75.
$$\sin x_0 = \sin 2x_0$$
$$\sin x_0 = 2\sin x_0 \cos x_0$$
$$\cos x_0 = \frac{1}{2} \text{ [if } \sin x_0 \ne 0]$$
$$x_0 = \frac{\pi}{3}$$

$D_x(\sin x) = \cos x$, $D_x(\sin 2x) = 2\cos 2x$, so at x_0, the tangent lines to $y = \sin x$ and $y = \sin 2x$ have slopes of $m_1 = \frac{1}{2}$ and $m_2 = 2\left(-\frac{1}{2}\right) = -1$, respectively. From Problem 40 of Section 0.7, $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$ where θ is the angle between the tangent lines. $\tan\theta = \frac{-1 - \frac{1}{2}}{1 + \left(\frac{1}{2}\right)(-1)} = \frac{-\frac{3}{2}}{\frac{1}{2}} = -3$, so $\theta \approx -1.25$. The curves intersect at an angle of 1.25 radians.

76. $\frac{1}{2}\overline{AB} = \overline{OA}\sin\frac{t}{2}$

$$D = \frac{1}{2}\overline{OA}\cos\frac{t}{2} \cdot \overline{AB} = \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2}$$

$E = D$ + area (semi-circle)

$$= \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\pi\left(\frac{1}{2}\overline{AB}\right)^2$$
$$= \overline{OA}^2 \cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\pi\overline{OA}^2 \sin^2\frac{t}{2}$$
$$= \overline{OA}^2 \sin\frac{t}{2}\left(\cos\frac{t}{2} + \frac{1}{2}\pi\sin\frac{t}{2}\right)$$

$$\frac{D}{E} = \frac{\cos\frac{t}{2}}{\cos\frac{t}{2} + \frac{1}{2}\pi\sin\frac{t}{2}}$$

$$\lim_{t\to 0^+}\frac{D}{E} = \frac{1}{1+0} = 1$$

$$\lim_{t\to \pi^-}\frac{D}{E} = \lim_{t\to \pi^-}\frac{\cos(t/2)}{\cos(t/2) + \frac{\pi}{2}\sin(t/2)}$$
$$= \frac{0}{0 + \frac{\pi}{2}} = 0$$

77. $y = \sqrt{u}$ and $u = x^2$

$D_x y = D_u y \cdot D_x u$

$$= \frac{1}{2\sqrt{u}} \cdot 2x = \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|} = \frac{|x|}{x}$$

78. $D_x \left|x^2 - 1\right| = \frac{\left|x^2 - 1\right|}{x^2 - 1} D_x(x^2 - 1)$

$$= \frac{\left|x^2 - 1\right|}{x^2 - 1}(2x) = \frac{2x\left|x^2 - 1\right|}{x^2 - 1}$$

79. $D_x |\sin x| = \frac{|\sin x|}{\sin x} D_x(\sin x)$

$$= \frac{|\sin x|}{\sin x}\cos x = \cot x |\sin x|$$

80. a. $D_x L\left(x^2\right) = L'\left(x^2\right) D_x\left(x^2\right) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$

b. $D_x L(\cos^4 x) = \sec^4 x \, D_x(\cos^4 x)$

$= \sec^4 x(4\cos^3 x) D_x(\cos x)$

$= 4\sec^4 x \cos^3 x(-\sin x)$

$= 4 \cdot \frac{1}{\cos^4 x} \cdot \cos^3 x \cdot (-\sin x)$

$= -4\sec x \sin x = -4\tan x$

81. $[f(f(f(f(0))))]'$

$= f'(f(f(f(0)))) \cdot f'(f(f(0))) \cdot f'(f(0)) \cdot f'(0)$

$= 2 \cdot 2 \cdot 2 \cdot 2 = 16$

82. a. $\frac{d}{dx} f^{[2]} = f'(f(x)) \cdot f'(x)$

$$= f'(f^{[1]}) \cdot \frac{d}{dx} f^{[1]}(x)$$

b. $\frac{d}{dx} f^{[3]} = f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$

$$= f'(f^{[2]}(x)) \cdot f'(f^{[1]}(x)) \cdot \frac{d}{dx} f^{[1]}(x)$$

$$= f'(f^{[2]}(x)) \cdot \frac{d}{dx} f^{[2]}(x)$$

c. Conjecture:

$$\frac{d}{dx} f^{[n]}(x) = f'(f^{[n-1]}(x)) \cdot \frac{d}{dx} f^{[n-1]}(x)$$

83. $D_x\left(\frac{f(x)}{g(x)}\right) = D_x\left(f(x) \cdot \frac{1}{g(x)}\right) = D_x\left(f(x) \cdot (g(x))^{-1}\right) = f(x) D_x\left((g(x))^{-1}\right) + (g(x))^{-1} D_x f(x)$

$$= f(x) \cdot (-1)(g(x))^{-2} D_x g(x) + (g(x))^{-1} D_x f(x) = -f(x)(g(x))^{-2} D_x g(x) + (g(x))^{-1} D_x f(x)$$

$$= \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{D_x f(x)}{g(x)} = \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{g(x)}{g(x)} \cdot \frac{D_x f(x)}{g(x)} = \frac{-f(x) D_x g(x)}{g^2(x)} + \frac{g(x) D_x f(x)}{g^2(x)}$$

$$= \frac{g(x) D_x f(x) - f(x) D_x g(x)}{g^2(x)}$$

84. $g'(x) = f'\left(f\left(f\left(f(x)\right)\right)\right) f'\left(f\left(f(x)\right)\right) f'\left(f(x)\right) f'(x)$

$g'(x_1) = f'\left(f\left(f\left(f(x_1)\right)\right)\right) f'\left(f\left(f(x_1)\right)\right) f'\left(f(x_1)\right) f'(x_1)$

$= f'\left(f\left(f(x_2)\right)\right) f'\left(f(x_2)\right) f'(x_2) f'(x_1) = f'\left(f(x_1)\right) f'(x_1) f'(x_2) f'(x_1)$

$= \left[f'(x_1)\right]^2 \left[f'(x_2)\right]^2$

$g'(x_2) = f'\left(f\left(f\left(f(x_2)\right)\right)\right) f'\left(f\left(f(x_2)\right)\right) f'\left(f(x_2)\right) f'(x_2)$

$= f'\left(f\left(f(x_1)\right)\right) f'\left(f(x_1)\right) f'(x_1) f'(x_2) = f'\left(f(x_2)\right) f'(x_2) f'(x_1) f'(x_2)$

$= \left[f'(x_1)\right]^2 \left[f'(x_2)\right]^2 = g'(x_1)$

2.6 Concepts Review

1. $f'''(x), D_x^3 y, \dfrac{d^3y}{dx^3}, \; y'''$

2. $\dfrac{ds}{dt}; \left|\dfrac{ds}{dt}\right|; \dfrac{d^2s}{dt^2}$

3. $f'(t) > 0$

4. 0; < 0

Problem Set 2.6

1. $\dfrac{dy}{dx} = 3x^2 + 6x + 6$

$\dfrac{d^2y}{dx^2} = 6x + 6$

$\dfrac{d^3y}{dx^3} = 6$

2. $\dfrac{dy}{dx} = 5x^4 + 4x^3$

$\dfrac{d^2y}{dx^2} = 20x^3 + 12x^2$

$\dfrac{d^3y}{dx^3} = 60x^2 + 24x$

3. $\dfrac{dy}{dx} = 3(3x+5)^2(3) = 9(3x+5)^2$

$\dfrac{d^2y}{dx^2} = 18(3x+5)(3) = 162x + 270$

$\dfrac{d^3y}{dx^3} = 162$

4. $\dfrac{dy}{dx} = 5(3-5x)^4(-5) = -25(3-5x)^4$

$\dfrac{d^2y}{dx^2} = -100(3-5x)^3(-5) = 500(3-5x)^3$

$\dfrac{d^3y}{dx^3} = 1500(3-5x)^2(-5) = -7500(3-5x)^2$

5. $\dfrac{dy}{dx} = 7\cos(7x)$

$\dfrac{d^2y}{dx^2} = -7^2\sin(7x)$

$\dfrac{d^3y}{dx^3} = -7^3\cos(7x) = -343\cos(7x)$

6. $\dfrac{dy}{dx} = 3x^2\cos(x^3)$

$\dfrac{d^2y}{dx^2} = 3x^2[-3x^2\sin(x^3)] + 6x\cos(x^3) \quad = -9x^4\sin(x^3) + 6x\cos(x^3)$

$\dfrac{d^3y}{dx^3} = -9x^4\cos(x^3)(3x^2) + \sin(x^3)(-36x^3) + 6x[-\sin(x^3)(3x^2)] + 6\cos(x^3)$

$= -27x^6\cos(x^3) - 36x^3\sin(x^3) - 18x^3\sin(x^3) + 6\cos(x^3) \quad = (6-27x^6)\cos(x^3) - 54x^3\sin(x^3)$

7. $\dfrac{dy}{dx} = \dfrac{(x-1)(0)-(1)(1)}{(x-1)^2} = -\dfrac{1}{(x-1)^2}$

$\dfrac{d^2y}{dx^2} = -\dfrac{(x-1)^2(0)-2(x-1)}{(x-1)^4} = \dfrac{2}{(x-1)^3}$

$\dfrac{d^3y}{dx^3} = \dfrac{(x-1)^3(0)-2[3(x-1)^2]}{(x-1)^6}$

$= -\dfrac{6}{(x-1)^4}$

8. $\dfrac{dy}{dx} = \dfrac{(1-x)(3)-(3x)(-1)}{(1-x)^2} = \dfrac{3}{(x-1)^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{(x-1)^2(0)-3[2(x-1)]}{(x-1)^4} = -\dfrac{6}{(x-1)^3}$

$\dfrac{d^3y}{dx^3} = -\dfrac{(x-1)^3(0)-6(3)(x-1)^2}{(x-1)^6}$

$= \dfrac{18}{(x-1)^4}$

9. $f'(x) = 2x;\ f''(x) = 2;\ f''(2) = 2$

10. $f'(x) = 15x^2 + 4x + 1$

$f''(x) = 30x + 4$

$f''(2) = 64$

11. $f'(t) = -\frac{2}{t^2}$

$f''(t) = \frac{4}{t^3}$

$f''(2) = \frac{4}{8} = \frac{1}{2}$

12. $f'(u) = \frac{(5-u)(4u) - (2u^2)(-1)}{(5-u)^2} = \frac{20u - 2u^2}{(5-u)^2}$

$f''(u) = \frac{(5-u)^2(20-4u) - (20u - 2u^2)2(5-u)(-1)}{(5-u)^4}$

$= \frac{100}{(5-u)^3}$

$f''(2) = \frac{100}{3^3} = \frac{100}{27}$

13. $f'(\theta) = -2(\cos\theta\pi)^{-3}(-\sin\theta\pi)\pi \ = 2\pi(\cos\theta\pi)^{-3}(\sin\theta\pi)$

$f''(\theta) = 2\pi[(\cos\theta\pi)^{-3}(\pi)(\cos\theta\pi) + (\sin\theta\pi)(-3)(\cos\theta\pi)^{-4}(-\sin\theta\pi)(\pi)] \ = 2\pi^2[(\cos\theta\pi)^{-2} + 3\sin^2\theta\pi(\cos\theta\pi)^{-4}]$

$f''(2) = 2\pi^2[1 + 3(0)(1)] = 2\pi^2$

14. $f'(t) = t\cos\left(\frac{\pi}{t}\right)\left(-\frac{\pi}{t^2}\right) + \sin\left(\frac{\pi}{t}\right) \ = \left(-\frac{\pi}{t}\right)\cos\left(\frac{\pi}{t}\right) + \sin\left(\frac{\pi}{t}\right)$

$f''(t) = \left(-\frac{\pi}{t}\right)\left[-\sin\left(\frac{\pi}{t}\right)\left(-\frac{\pi}{t^2}\right)\right] + \left(\frac{\pi}{t^2}\right)\cos\left(\frac{\pi}{t}\right) + \left(-\frac{\pi}{t^2}\right)\cos\left(\frac{\pi}{t}\right) \ = -\frac{\pi^2}{t^3}\sin\left(\frac{\pi}{t}\right)$

$f''(2) = -\frac{\pi^2}{8}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{8} \approx -1.23$

15. $f'(s) = s(3)(1-s^2)^2(-2s) + (1-s^2)^3 \ = -6s^2(1-s^2)^2 + (1-s^2)^3 \ = -7s^6 + 15s^4 - 9s^2 + 1$

$f''(s) = -42s^5 + 60s^3 - 18s$

$f''(2) = -900$

16. $f'(x) = \frac{(x-1)2(x+1) - (x+1)^2}{(x-1)^2} \ = \frac{x^2 - 2x - 3}{(x-1)^2}$

$f''(x) = \frac{(x-1)^2(2x-2) - (x^2 - 2x - 3)2(x-1)}{(x-1)^4} \ = \frac{(x-1)(2x-2) - (x^2 - 2x - 3)(2)}{(x-1)^3} = \frac{8}{(x-1)^3}$

$f''(2) = \frac{8}{1^3} = 8$

17. $D_x(x^n) = nx^{n-1}$

$D_x^2(x^n) = n(n-1)x^{n-2}$

$D_x^3(x^n) = n(n-1)(n-2)x^{n-3}$

$D_x^4(x^n) = n(n-1)(n-2)(n-3)x^{n-4}$

$\vdots$

$D_x^{n-1}(x^n) = n(n-1)(n-2)(n-3)...(2)x$

$D_x^n(x^n) = n(n-1)(n-2)(n-3)...2(1)x^0 = n!$

18. Let $k < n$.

$D_x^n(x^k) = D_x^{n-k}[D_x^k(x^k)] = D_x(k!) = 0$

so $D_x^n[a_n x^{n-1} + \ldots + a_1 x + a_0] = 0$

19. **a.** $D_x^4(3x^3 + 2x - 19) = 0$

b. $D_x^{12}(100x^{11} - 79x^{10}) = 0$

c. $D_x^{11}(x^2 - 3)^5 = 0$

20. $D_x\left(\frac{1}{x}\right)=-\frac{1}{x^2}$

$D_x^2\left(\frac{1}{x}\right)=D_x(-x^{-2})=2x^{-3}=\frac{2}{x^3}$

$D_x^3\left(\frac{1}{x}\right)=D_x(2x^{-3})=-\frac{3(2)}{x^4}$

$D_x^4\left(\frac{1}{x}\right)=\frac{4(3)(2)}{x^5}$

$D_x^n\left(\frac{1}{x}\right)=\frac{(-1)^n n!}{x^{n+1}}$

21. $f'(x)=3x^2+6x-45=3(x+5)(x-3)$
$3(x+5)(x-3)=0$
$x=-5,\ x=3$
$f''(x)=6x+6$
$f''(-5)=-24$
$f''(3)=24$

22. $g'(t)=2at+b$
$g''(t)=2a$
$g''(1)=2a=-4$
$a=-2$
$g'(1)=2a+b=3$
$2(-2)+b=3$
$b=7$
$g(1)=a+b+c=5$
$(-2)+(7)+c=5$
$c=0$

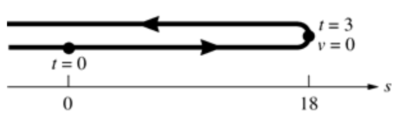
23. a. $v(t)=\frac{ds}{dt}=12-4t$

$a(t)=\frac{d^2s}{dt^2}=-4$

b. $12-4t>0$
$4t<12$
$t<3;\ (-\infty,3)$

c. $12-4t<0$
$t>3;\ (3,\infty)$

d. $a(t)=-4<0$ for all t

e. 

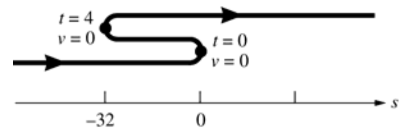
24. a. $v(t)=\frac{ds}{dt}=3t^2-12t$

$a(t)=\frac{d^2s}{dt^2}=6t-12$

b. $3t^2-12t>0$
$3t(t-4)>0;\ (-\infty,0)\cup(4,\infty)$

c. $3t^2-12t<0$
$(0, 4)$

d. $6t-12<0$
$6t<12$
$t<2;\ (-\infty,2)$

e. 

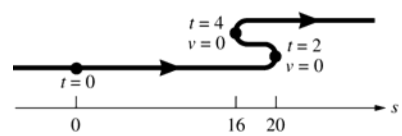
25. a. $v(t)=\frac{ds}{dt}=3t^2-18t+24$

$a(t)=\frac{d^2s}{dt^2}=6t-18$

b. $3t^2-18t+24>0$
$3(t-2)(t-4)>0$
$(-\infty,2)\cup(4,\infty)$

c. $3t^2-18t+24<0$
$(2, 4)$

d. $6t-18<0$
$6t<18$
$t<3;\ (-\infty,3)$

e. 

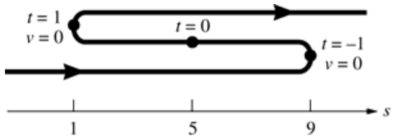
26. a. $v(t)=\frac{ds}{dt}=6t^2-6$

$a(t)=\frac{d^2s}{dt^2}=12t$

b. $6t^2-6>0$
$6(t+1)(t-1)>0$
$(-\infty,-1)\cup(1,\infty)$

c. $6t^2-6<0$
$(-1, 1)$

d. $12t<0$
$t<0$
The acceleration is negative for negative t.

e. 

27. a. $v(t) = \frac{ds}{dt} = 2t - \frac{16}{t^2}$

$a(t) = \frac{d^2s}{dt^2} = 2 + \frac{32}{t^3}$

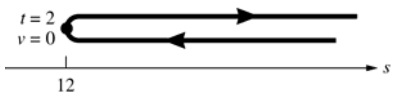
b. $2t - \frac{16}{t^2} > 0$

$\frac{2t^3 - 16}{t^2} > 0;\ (2, \infty)$

c. $2t - \frac{16}{t^2} < 0;\ (0, 2)$

d. $2 + \frac{32}{t^3} < 0$

$\frac{2t^3 + 32}{t^3} < 0$; The acceleration is not negative for any positive t.

e. 

28. a. $v(t) = \frac{ds}{dt} = 1 - \frac{4}{t^2}$

$a(t) = \frac{d^2s}{dt^2} = \frac{8}{t^3}$

b. $1 - \frac{4}{t^2} > 0$

$\frac{t^2 - 4}{t^2} > 0;\ (2, \infty)$

c. $1 - \frac{4}{t^2} < 0;\ (0, 2)$

d. $\frac{8}{t^3} < 0$; The acceleration is not negative for any positive t.

e. 

29. $v(t) = \frac{ds}{dt} = 2t^3 - 15t^2 + 24t$

$a(t) = \frac{d^2s}{dt^2} = 6t^2 - 30t + 24$

$6t^2 - 30t + 24 = 0$
$6(t - 4)(t - 1) = 0$
$t = 4, 1$
$v(4) = -16,\ v(1) = 11$

30. $v(t) = \frac{ds}{dt} = \frac{1}{10}(4t^3 - 42t^2 + 120t)$

$a(t) = \frac{d^2s}{dt^2} = \frac{1}{10}(12t^2 - 84t + 120)$

$\frac{1}{10}(12t^2 - 84t + 120) = 0$

$\frac{12}{10}(t - 2)(t - 5) = 0$

$t = 2,\ t = 5$
$v(2) = 10.4,\ v(5) = 5$

31. $v_1(t) = \frac{ds_1}{dt} = 4 - 6t$

$v_2(t) = \frac{ds_2}{dt} = 2t - 2$

a. $4 - 6t = 2t - 2$
$8t = 6$
$t = \frac{3}{4}$ sec

b. $|4 - 6t| = |2t - 2|;\ 4 - 6t = -2t + 2$

$t = \frac{1}{2}$ sec and $t = \frac{3}{4}$ sec

c. $4t - 3t^2 = t^2 - 2t$
$4t^2 - 6t = 0$
$2t(2t - 3) = 0$
$t = 0$ sec and $t = \frac{3}{2}$ sec

32. $v_1(t) = \frac{ds_1}{dt} = 9t^2 - 24t + 18$

$v_2(t) = \frac{ds_2}{dt} = -3t^2 + 18t - 12$

$9t^2 - 24t + 18 = -3t^2 + 18t - 12$
$12t^2 - 42t + 30 = 0$
$2t^2 - 7t + 5 = 0$
$(2t - 5)(t - 1) = 0$
$t = 1,\ \frac{5}{2}$

33. a. $v(t) = -32t + 48$
initial velocity = v_0 = 48 ft/sec

b. $-32t + 48 = 0$
$t = \frac{3}{2}$ sec

c. $s = -16(1.5)^2 + 48(1.5) + 256 = 292$ ft

d. $-16t^2 + 48t + 256 = 0$
$t = \frac{-48 \pm \sqrt{48^2 - 4(-16)(256)}}{-32} \approx -2.77, 5.77$
The object hits the ground at t = 5.77 sec.

e. $v(5.77) \approx -137$ ft/sec;
speed = $|-137| = 137$ ft/sec.

34. $v(t) = 48 - 32t$

a. $48 - 32t = 0$
$t = 1.5$
$s = 48(1.5) - 16(1.5)^2 = 36$ ft

b. $v(1)$ = 16 ft/sec upward

c. $48t - 16t^2 = 0$
$-16t(-3 + t) = 0$
t = 3 sec

35. $v(t) = v_0 - 32t$
$v_0 - 32t = 0$
$t = \frac{v_0}{32}$
$v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = 5280$
$\frac{{v_0}^2}{32} - \frac{{v_0}^2}{64} = 5280$
$\frac{{v_0}^2}{64} = 5280$
$v_0 = \sqrt{337,920} \approx 581$ ft/sec

36. $v(t) = v_0 + 32t$
$v_0 + 32t = 140$
$v_0 + 32(3) = 140$
$v_0 = 44$
$s = 44(3) + 16(3)^2 = 276$ ft

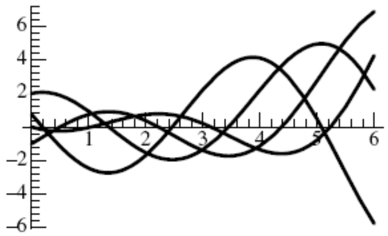
37. $v(t) = 3t^2 - 6t - 24$
$\frac{d}{dt}\left|3t^2 - 6t - 24\right| = \frac{\left|3t^2 - 6t - 24\right|}{3t^2 - 6t - 24}(6t - 6)$
$= \frac{|(t-4)(t+2)|}{(t-4)(t+2)}(6t - 6)$
$\frac{|(t-4)(t+2)|(6t-6)}{(t-4)(t+2)} < 0$
$t < -2,\ 1 < t < 4;\ (-\infty, -2) \cup (1, 4)$

38. Point slowing down when
$\frac{d}{dt}|v(t)| < 0$
$\frac{d}{dt}|v(t)| = \frac{|v(t)|a(t)}{v(t)}$
$\frac{|v(t)|a(t)}{v(t)} < 0$ when $a(t)$ and $v(t)$ have opposite signs.

39. $D_x(uv) = uv' + u'v$
$D_x^2(uv) = uv'' + u'v' + u'v' + u''v$
$= uv'' + 2u'v' + u''v$
$D_x^3(uv) = uv''' + u'v'' + 2(u'v'' + u''v') + u''v' + u'''v$
$= uv''' + 3u'v'' + 3u''v' + u'''v$
$D_x^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_x^{n-k}(u) D_x^k(v)$
where $\binom{n}{k}$ is the binomial coefficient
$\frac{n!}{(n-k)!k!}$.

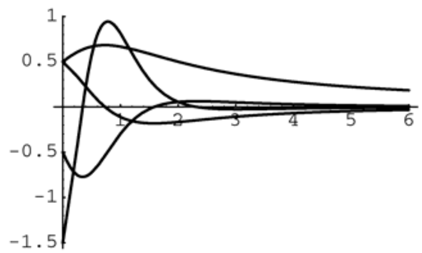
40. $D_x^4(x^4 \sin x) = \binom{4}{0} D_x^4(x^4) D_x^0(\sin x)$
$+ \binom{4}{1} D_x^3(x^4) D_x^1(\sin x) + \binom{4}{2} D_x^2(x^4) D_x^2(\sin x)$
$+ \binom{4}{3} D_x^1(x^4) D_x^3(\sin x) + \binom{4}{4} D_x^0(x^4) D_x^4(\sin x)$
$= 24 \sin x + 96x \cos x - 72x^2 \sin x$
$-16x^3 \cos x + x^4 \sin x$

41. a.



b. $f'''(2.13) \approx -1.2826$

42. a.



b. $f'''(2.13) \approx 0.0271$

2.7 Concepts Review

1. $\dfrac{9}{x^3-3}$

2. $3y^2\dfrac{dy}{dx}$

3. $x(2y)\dfrac{dy}{dx}+y^2+3y^2\dfrac{dy}{dx}-\dfrac{dy}{dx}=3x^2$

4. $\dfrac{p}{q}x^{p/q-1};\ \dfrac{5}{3}(x^2-5x)^{2/3}(2x-5)$

Problem Set 2.7

1. $2y\,D_xy-2x=0$

$D_xy=\dfrac{2x}{2y}=\dfrac{x}{y}$

2. $18x+8y\,D_xy=0$

$D_xy=\dfrac{-18x}{8y}=-\dfrac{9x}{4y}$

3. $x\,D_xy+y=0$

$D_xy=-\dfrac{y}{x}$

4. $2x+2\alpha^2yD_xy=0$

$D_xy=-\dfrac{2x}{2\alpha^2y}=-\dfrac{x}{\alpha^2y}$

5. $x(2y)D_xy+y^2=1$

$D_xy=\dfrac{1-y^2}{2xy}$

6. $2x+2x^2D_xy+4xy+3x\,D_xy+3y=0$

$D_xy(2x^2+3x)=-2x-4xy-3y$

$D_xy=\dfrac{-2x-4xy-3y}{2x^2+3x}$

7. $12x^2+7x(2y)D_xy+7y^2=6y^2D_xy$

$12x^2+7y^2=6y^2D_xy-14xyD_xy$

$D_xy=\dfrac{12x^2+7y^2}{6y^2-14xy}$

8. $x^2D_xy+2xy=y^2+x(2y)D_xy$

$x^2D_xy-2xyD_xy=y^2-2xy$

$D_xy=\dfrac{y^2-2xy}{x^2-2xy}$

9. $\dfrac{1}{2\sqrt{5xy}}\cdot(5x\,D_xy+5y)+2D_xy$

$=2y\,D_xy+x(3y^2)D_xy+y^3$

$\dfrac{5x}{2\sqrt{5xy}}D_xy+2D_xy-2y\,D_xy-3xy^2D_xy$

$=y^3-\dfrac{5y}{2\sqrt{5xy}}$

$D_xy=\dfrac{y^3-\frac{5y}{2\sqrt{5xy}}}{\frac{5x}{2\sqrt{5xy}}+2-2y-3xy^2}$

10. $x\dfrac{1}{2\sqrt{y+1}}D_xy+\sqrt{y+1}=x\,D_xy+y$

$\dfrac{x}{2\sqrt{y+1}}D_xy-x\,D_xy=y-\sqrt{y+1}$

$D_xy=\dfrac{y-\sqrt{y+1}}{\frac{x}{2\sqrt{y+1}}-x}$

11. $x D_x y + y + \cos(xy)(x D_x y + y) = 0$

$x D_x y + x\cos(xy) D_x y = -y - y\cos(xy)$

$D_x y = \dfrac{-y - y\cos(xy)}{x + x\cos(xy)} = -\dfrac{y}{x}$

12. $-\sin(xy^2)(2xy\, D_x y + y^2) = 2y D_x y + 1$

$-2xy\sin(xy^2) D_x y - 2y\, D_x y = 1 + y^2 \sin(xy^2)$

$D_x y = \dfrac{1 + y^2\sin(xy^2)}{-2xy\sin(xy^2) - 2y}$

13. $x^3 y' + 3x^2 y + y^3 + 3xy^2 y' = 0$

$y'(x^3 + 3xy^2) = -3x^2 y - y^3$

$y' = \dfrac{-3x^2 y - y^3}{x^3 + 3xy^2}$

At (1, 3), $y' = -\dfrac{36}{28} = -\dfrac{9}{7}$

Tangent line: $y - 3 = -\dfrac{9}{7}(x-1)$

14. $x^2(2y)y' + 2xy^2 + 4xy' + 4y = 12y'$

$y'(2x^2 y + 4x - 12) = -2xy^2 - 4y$

$y' = \dfrac{-2xy^2 - 4y}{2x^2 y + 4x - 12} = \dfrac{-xy^2 - 2y}{x^2 y + 2x - 6}$

At (2, 1), $y' = -2$

Tangent line: $y - 1 = -2(x-2)$

15. $\cos(xy)(xy' + y) = y'$

$y'[x\cos(xy) - 1] = -y\cos(xy)$

$y' = \dfrac{-y\cos(xy)}{x\cos(xy) - 1} = \dfrac{y\cos(xy)}{1 - x\cos(xy)}$

At $\left(\dfrac{\pi}{2}, 1\right)$, $y' = 0$

Tangent line: $y - 1 = 0\left(x - \dfrac{\pi}{2}\right)$

$y = 1$

16. $y' + [-\sin(xy^2)][2xyy' + y^2] + 6x = 0$

$y'[1 - 2xy\sin(xy^2)] = y^2\sin(xy^2) - 6x$

$y' = \dfrac{y^2\sin(xy^2) - 6x}{1 - 2xy\sin(xy^2)}$

At (1, 0), $y' = -\dfrac{6}{1} = -6$

Tangent line: $y - 0 = -6(x - 1)$

17. $\dfrac{2}{3}x^{-1/3} - \dfrac{2}{3}y^{-1/3}y' - 2y' = 0$

$\dfrac{2}{3}x^{-1/3} = y'\left(\dfrac{2}{3}y^{-1/3} + 2\right)$

$y' = \dfrac{\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3} + 2}$

At (1, –1), $y' = \dfrac{\frac{2}{3}}{\frac{4}{3}} = \dfrac{1}{2}$

Tangent line: $y + 1 = \dfrac{1}{2}(x-1)$

18. $\dfrac{1}{2\sqrt{y}}y' + 2xyy' + y^2 = 0$

$y'\left(\dfrac{1}{2\sqrt{y}} + 2xy\right) = -y^2$

$y' = \dfrac{-y^2}{\frac{1}{2\sqrt{y}} + 2xy}$

At (4, 1), $y' = \dfrac{-1}{\frac{17}{2}} = -\dfrac{2}{17}$

Tangent line: $y - 1 = -\dfrac{2}{17}(x-4)$

19. $\dfrac{dy}{dx} = 5x^{2/3} + \dfrac{1}{2\sqrt{x}}$

20. $\dfrac{dy}{dx} = \dfrac{1}{3}x^{-2/3} - 7x^{5/2} = \dfrac{1}{3\sqrt[3]{x^2}} - 7x^{5/2}$

21. $\dfrac{dy}{dx} = \dfrac{1}{3}x^{-2/3} - \dfrac{1}{3}x^{-4/3} = \dfrac{1}{3\sqrt[3]{x^2}} - \dfrac{1}{3\sqrt[3]{x^4}}$

22. $\dfrac{dy}{dx} = \dfrac{1}{4}(2x+1)^{-3/4}(2) = \dfrac{1}{2\sqrt[4]{(2x+1)^3}}$

23. $\dfrac{dy}{dx} = \dfrac{1}{4}(3x^2 - 4x)^{-3/4}(6x - 4)$

$= \dfrac{6x - 4}{4\sqrt[4]{(3x^2 - 4x)^3}} = \dfrac{3x - 2}{2\sqrt[4]{(3x^2 - 4x)^3}}$

24. $\dfrac{dy}{dx} = \dfrac{1}{3}(x^3 - 2x)^{-2/3}(3x^2 - 2)$

25. $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x^3 + 2x)^{-2/3}]$

$= -\dfrac{2}{3}(x^3 + 2x)^{-5/3}(3x^2 + 2) = -\dfrac{6x^2 + 4}{3\sqrt[3]{(x^3 + 2x)^5}}$

26. $\frac{dy}{dx} = -\frac{5}{3}(3x-9)^{-8/3}(3) = -5(3x-9)^{-8/3}$

27. $\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+\sin x}}(2x+\cos x)$

$= \frac{2x+\cos x}{2\sqrt{x^2+\sin x}}$

28. $\frac{dy}{dx} = \frac{1}{2\sqrt{x^2\cos x}}[x^2(-\sin x)+2x\cos x]$

$= \frac{2x\cos x - x^2\sin x}{2\sqrt{x^2\cos x}}$

29. $\frac{dy}{dx} = \frac{d}{dx}[(x^2\sin x)^{-1/3}]$

$= -\frac{1}{3}(x^2\sin x)^{-4/3}(x^2\cos x + 2x\sin x)$

$= -\frac{x^2\cos x + 2x\sin x}{3\sqrt[3]{(x^2\sin x)^4}}$

30. $\frac{dy}{dx} = \frac{1}{4}(1+\sin 5x)^{-3/4}(\cos 5x)(5)$

$= \frac{5\cos 5x}{4\sqrt[4]{(1+\sin 5x)^3}}$

31. $\frac{dy}{dx} = \frac{[1+\cos(x^2+2x)]^{-3/4}[-\sin(x^2+2x)(2x+2)]}{4}$

$= -\frac{(x+1)\sin(x^2+2x)}{2\sqrt[4]{[1+\cos(x^2+2x)]^3}}$

32. $\frac{dy}{dx} = \frac{(\tan^2 x+\sin^2 x)^{-1/2}(2\tan x\sec^2 x + 2\sin x\cos x)}{2}$

$= \frac{\tan x\sec^2 x + \sin x\cos x}{\sqrt{\tan^2 x+\sin^2 x}}$

33. $s^2 + 2st\frac{ds}{dt} + 3t^2 = 0$

$\frac{ds}{dt} = \frac{-s^2-3t^2}{2st} = -\frac{s^2+3t^2}{2st}$

$s^2\frac{dt}{ds} + 2st + 3t^2\frac{dt}{ds} = 0$

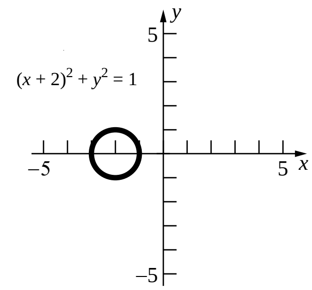
$\frac{dt}{ds}(s^2+3t^2) = -2st$

$\frac{dt}{ds} = -\frac{2st}{s^2+3t^2}$

34. $1 = \cos(x^2)(2x)\frac{dx}{dy} + 6x^2\frac{dx}{dy}$

$\frac{dx}{dy} = \frac{1}{2x\cos(x^2)+6x^2}$

35.



$2x + 4 + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x+4}{2y} = -\frac{x+2}{y}$

The tangent line at (x_0, y_0) has equation

$y - y_0 = -\frac{x_0+2}{y_0}(x - x_0)$ which simplifies to

$2x_0 - yy_0 - 2x - xx_0 + y_0^2 + x_0^2 = 0.$ Since

(x_0, y_0) is on the circle, $x_0^2 + y_0^2 = -3 - 4x_0,$

so the equation of the tangent line is

$-yy_0 - 2x_0 - 2x - xx_0 = 3.$

If (0, 0) is on the tangent line, then $x_0 = -\frac{3}{2}.$

Solve for y_0 in the equation of the circle to get

$y_0 = \pm\frac{\sqrt{3}}{2}.$ Put these values into the equation of the tangent line to get that the tangent lines are $\sqrt{3}y + x = 0$ and $\sqrt{3}y - x = 0.$

36. $16(x^2+y^2)(2x+2yy') = 100(2x-2yy')$

$32x^3 + 32x^2yy' + 32xy^2 + 32y^3y' = 200x - 200yy'$

$y'(4x^2y + 4y^3 + 25y) = 25x - 4x^3 - 4xy^2$

$y' = \frac{25x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 25y}$

The slope of the normal line $= -\frac{1}{y'}$

$= \frac{4x^2y + 4y^3 + 25y}{4x^3 + 4xy^2 - 25x}$

At (3, 1), slope $= \frac{65}{45} = \frac{13}{9}$

Normal line: $y - 1 = \frac{13}{9}(x-3)$

37. a. $xy' + y + 3y^2 y' = 0$

$y'(x + 3y^2) = -y$

$y' = -\frac{y}{x + 3y^2}$

b. $xy'' + \left(\frac{-y}{x+3y^2}\right) + \left(\frac{-y}{x+3y^2}\right) + 3y^2 y''$

$+6y\left(\frac{-y}{x+3y^2}\right)^2 = 0$

$xy'' + 3y^2 y'' - \frac{2y}{x+3y^2} + \frac{6y^3}{(x+3y^2)^2} = 0$

$y''(x+3y^2) = \frac{2y}{x+3y^2} - \frac{6y^3}{(x+3y^2)^2}$

$y''(x+3y^2) = \frac{2xy}{(x+3y^2)^2}$

$y'' = \frac{2xy}{(x+3y^2)^3}$

38. $3x^2 - 8yy' = 0$

$y' = \frac{3x^2}{8y}$

$6x - 8(yy'' + (y')^2) = 0$

$6x - 8yy'' - 8\left(\frac{3x^2}{8y}\right)^2 = 0$

$6x - 8yy'' - \frac{9x^4}{8y^2} = 0$

$\frac{48xy^2 - 9x^4}{8y^2} = 8yy''$

$y'' = \frac{48xy^2 - 9x^4}{64y^3}$

39. $2(x^2 y' + 2xy) - 12y^2 y' = 0$

$2x^2 y' - 12y^2 y' = -4xy$

$y' = \frac{2xy}{6y^2 - x^2}$

$2(x^2 y'' + 2xy' + 2xy' + 2y) - 12[y^2 y'' + 2y(y')^2] = 0$

$2x^2 y'' - 12y^2 y'' = -8xy' - 4y + 24y(y')^2$

$y''(2x^2 - 12y^2) = -\frac{16x^2 y}{6y^2 - x^2} - 4y + \frac{96x^2 y^3}{(6y^2 - x^2)^2}$

$y''(2x^2 - 12y^2) = \frac{12x^4 y + 48x^2 y^3 - 144y^5}{(6y^2 - x^2)^2}$

$y''(6y^2 - x^2) = \frac{72y^5 - 6x^4 y - 24x^2 y^3}{(6y^2 - x^2)^2}$

$y'' = \frac{72y^5 - 6x^4 y - 24x^2 y^3}{(6y^2 - x^2)^3}$

At (2, 1), $y'' = \frac{-120}{8} = -15$

40. $2x + 2yy' = 0$

$y' = -\frac{2x}{2y} = -\frac{x}{y}$

$2 + 2[yy'' + (y')^2] = 0$

$2 + 2yy'' + 2\left(-\frac{x}{y}\right)^2 = 0$

$2yy'' = -2 - \frac{2x^2}{y^2}$

$y'' = -\frac{1}{y} - \frac{x^2}{y^3} = -\frac{y^2 + x^2}{y^3}$

At (3, 4), $y'' = -\frac{25}{64}$

41. $3x^2 + 3y^2 y' = 3(xy' + y)$

$y'(3y^2 - 3x) = 3y - 3x^2$

$y' = \frac{y - x^2}{y^2 - x}$

At $\left(\frac{3}{2}, \frac{3}{2}\right)$, $y' = -1$

Slope of the normal line is 1.

Normal line: $y - \frac{3}{2} = 1\left(x - \frac{3}{2}\right)$; $y = x$

This line includes the point (0, 0).

42. $xy' + y = 0$

$y' = -\frac{y}{x}$

$2x - 2yy' = 0$

$y' = \frac{x}{y}$

The slopes of the tangents are negative reciprocals, so the hyperbolas intersect at right angles.

43. Implicitly differentiate the first equation.

$4x + 2yy' = 0$

$y' = -\frac{2x}{y}$

Implicitly differentiate the second equation.

$2yy' = 4$

$y' = \frac{2}{y}$

Solve for the points of intersection.

$2x^2 + 4x = 6$

$2(x^2 + 2x - 3) = 0$

$(x + 3)(x - 1) = 0$

$x = -3, x = 1$

$x = -3$ is extraneous, and $y = -2, 2$ when $x = 1$.

The graphs intersect at (1, –2) and (1, 2).

At (1, –2): $m_1 = 1, m_2 = -1$

At (1, 2): $m_1 = -1, m_2 = 1$

44. Find the intersection points:

$x^2 + y^2 = 1 \quad \to \quad y^2 = 1 - x^2$

$(x-1)^2 + y^2 = 1$

$(x-1)^2 + (1 - x^2) = 1$

$x^2 - 2x + 1 + 1 - x^2 = 1 \quad \Rightarrow \quad x = \frac{1}{2}$

Points of intersection: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Implicitly differentiate the first equation.

$2x + 2yy' = 0$

$y' = -\frac{x}{y}$

Implicitly differentiate the second equation.

$2(x-1) + 2yy' = 0$

$y' = \frac{1-x}{y}$

At $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right): m_1 = -\frac{1}{\sqrt{3}}, m_2 = \frac{1}{\sqrt{3}}$

$$\tan\theta = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \sqrt{3} \quad \to \quad \theta = \frac{\pi}{3}$$

At $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right): m_1 = \frac{1}{\sqrt{3}}, m_2 = -\frac{1}{\sqrt{3}}$

$$\tan\theta = \frac{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} = \frac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}} = -\sqrt{3}$$

$\theta = \frac{2\pi}{3}$

45. $x^2 - x(2x) + 2(2x)^2 = 28$

$7x^2 = 28$

$x^2 = 4$

$x = -2, 2$

Intersection point in first quadrant: (2, 4)

$y_1' = 2$

$2x - xy_2' - y + 4yy_2' = 0$

$y_2'(4y - x) = y - 2x$

$y_2' = \frac{y - 2x}{4y - x}$

At (2, 4): $m_1 = 2, m_2 = 0$

$$\tan\theta = \frac{0 - 2}{1 + (0)(2)} = -2; \theta = \pi + \tan^{-1}(-2) \approx 2.034$$

46. The equation is $mv^2 - mv_0^2 = kx_0^2 - kx^2$.

Differentiate implicitly with respect to t to get $2mv\frac{dv}{dt} = -2kx\frac{dx}{dt}$. Since $v = \frac{dx}{dt}$ this simplifies to $2mv\frac{dv}{dt} = -2kxv$ or $m\frac{dv}{dt} = -kx$.

47. $x^2 - xy + y^2 = 16$, when $y = 0$,

$x^2 = 16$

$x = -4, 4$

The ellipse intersects the x-axis at (–4, 0) and (4, 0).

$2x - xy' - y + 2yy' = 0$

$y'(2y - x) = y - 2x$

$y' = \frac{y - 2x}{2y - x}$

At (–4, 0), $y' = 2$

At (4, 0), $y' = 2$

Tangent lines: $y = 2(x + 4)$ and $y = 2(x - 4)$

48. $x^2 + 2xy\frac{dx}{dy} - 2xy - y^2\frac{dx}{dy} = 0$

$\frac{dx}{dy}(2xy - y^2) = 2xy - x^2;$

$\frac{dx}{dy} = \frac{2xy - x^2}{2xy - y^2}$

$\frac{2xy - x^2}{2xy - y^2} = 0$ if $x(2y - x) = 0$, which occurs when $x = 0$ or $y = \frac{x}{2}$. There are no points on $x^2y - xy^2 = 2$ where $x = 0$. If $y = \frac{x}{2}$, then

$2 = x^2\left(\frac{x}{2}\right) - x\left(\frac{x}{2}\right)^2 = \frac{x^3}{2} - \frac{x^3}{4} = \frac{x^3}{4}$ so $x = 2$,

$y = \frac{2}{2} = 1.$

The tangent line is vertical at (2, 1).

49. $2x + 2y\frac{dy}{dx} = 0; \frac{dy}{dx} = -\frac{x}{y}$

The tangent line at (x_0, y_0) has slope $-\frac{x_0}{y_0}$, hence the equation of the tangent line is $y - y_0 = -\frac{x_0}{y_0}(x - x_0)$ which simplifies to $yy_0 + xx_0 - (x_0^2 + y_0^2) = 0$ or $yy_0 + xx_0 = 1$ since (x_0, y_0) is on $x^2 + y^2 = 1$. If (1.25, 0) is on the tangent line through (x_0, y_0), $x_0 = 0.8$. Put this into $x^2 + y^2 = 1$ to get $y_0 = 0.6$, since $y_0 > 0$. The line is $6y + 8x = 10$. When $x = -2$, $y = \frac{13}{3}$, so the light bulb must be $\frac{13}{3}$ units high.

2.8 Concepts Review

1. $\frac{du}{dt}; t = 2$

2. 400 mi/hr

3. negative

4. negative; positive

Problem Set 2.8

1. $V = x^3; \frac{dx}{dt} = 3$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

When $x = 12$, $\frac{dV}{dt} = 3(12)^2(3) = 1296$ in.^{3}/s.

2. $V = \frac{4}{3}\pi r^3; \frac{dV}{dt} = 3$

$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

When $r = 3$, $3 = 4\pi(3)^2\frac{dr}{dt}$.

$\frac{dr}{dt} = \frac{1}{12\pi} \approx 0.027$ in./s

3. $y^2 = x^2 + 1^2; \frac{dx}{dt} = 400$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$ mi/hr

When $x = 5, y = \sqrt{26}, \frac{dy}{dt} = \frac{5}{\sqrt{26}}(400)$

$\approx$ 392 mi/h.

4. $V = \frac{1}{3}\pi r^2 h; \frac{r}{h} = \frac{3}{10}; r = \frac{3h}{10}$

$V = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h = \frac{3\pi h^3}{100}; \frac{dV}{dt} = 3, h = 5$

$\frac{dV}{dt} = \frac{9\pi h^2}{100}\frac{dh}{dt}$

When $h = 5$, $3 = \frac{9\pi(5)^2}{100}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{3\pi} \approx 0.42$ cm/s

5. $s^2 = (x + 300)^2 + y^2; \frac{dx}{dt} = 300, \frac{dy}{dt} = 400,$

$2s\frac{ds}{dt} = 2(x + 300)\frac{dx}{dt} + 2y\frac{dy}{dt}$

$s\frac{ds}{dt} = (x + 300)\frac{dx}{dt} + y\frac{dy}{dt}$

When $x = 300$, $y = 400$, $s = 200\sqrt{13}$, so

$200\sqrt{13}\frac{ds}{dt} = (300 + 300)(300) + 400(400)$

$\frac{ds}{dt} \approx 471$ mi/h

6. $y^2 = x^2 + (10)^2; \frac{dy}{dt} = 2$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

When $y = 25$, $x \approx 22.9$, so

$\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt} \approx \frac{25}{22.9}(2) \approx 2.18$ ft/s

7. $20^2 = x^2 + y^2; \frac{dx}{dt} = 1$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

When $x = 5$, $y = \sqrt{375} = 5\sqrt{15}$, so

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{5}{5\sqrt{15}}(1) \approx -0.258$ ft/s

The top of the ladder is moving down at 0.258 ft/s.

8. $\frac{dV}{dt} = -4$ ft^3/h; $V = \pi hr^2; \frac{dh}{dt} = -0.0005$ ft/h

$A = \pi r^2 = \frac{V}{h} = Vh^{-1}$, so $\frac{dA}{dt} = h^{-1}\frac{dV}{dt} - \frac{V}{h^2}\frac{dh}{dt}$.

When $h = 0.001$ ft, $V = \pi(0.001)(250)^2 = 62.5\pi$

and $\frac{dA}{dt} = 1000(-4) - 1{,}000{,}000(62.5\pi)(-0.0005)$

$= -4000 + 31{,}250\pi \approx 94{,}175$ ft^2/h.

(The height is decreasing due to the spreading of the oil rather than the bacteria.)

9. $V = \frac{1}{3}\pi r^2 h; h = \frac{d}{4} = \frac{r}{2}, r = 2h$

$V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3; \frac{dV}{dt} = 16$

$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$

When $h = 4$, $16 = 4\pi(4)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{4\pi} \approx 0.0796$ ft/s

10. $y^2 = x^2 + (90)^2; \frac{dx}{dt} = 5$

$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

When $y = 150$, $x = 120$, so

$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{120}{150}(5) = 4$ ft/s

11. $V = \frac{hx}{2}(20); \frac{40}{5} = \frac{x}{h}, x = 8h$

$V = 10h(8h) = 80h^2; \frac{dV}{dt} = 40$

$\frac{dV}{dt} = 160h\frac{dh}{dt}$

When $h = 3$, $40 = 160(3)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{12}$ ft/min

12. $y = \sqrt{x^2 - 4}; \frac{dx}{dt} = 5$

$\frac{dy}{dt} = \frac{1}{2\sqrt{x^2 - 4}}(2x)\frac{dx}{dt} = \frac{x}{\sqrt{x^2 - 4}}\frac{dx}{dt}$

When $x = 3$, $\frac{dy}{dt} = \frac{3}{\sqrt{3^2 - 4}}(5) = \frac{15}{\sqrt{5}} \approx 6.7$ units/s

13. $A = \pi r^2; \frac{dr}{dt} = 0.02$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

When $r = 8.1$, $\frac{dA}{dt} = 2\pi(0.02)(8.1) = 0.324\pi$

≈ 1.018 in.^{2}/s

14. $s^2 = x^2 + (y + 48)^2; \frac{dx}{dt} = 30, \frac{dy}{dt} = 24$

$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2(y + 48)\frac{dy}{dt}$

$s\frac{ds}{dt} = x\frac{dx}{dt} + (y + 48)\frac{dy}{dt}$

At 2:00 p.m., $x = 3(30) = 90$, $y = 3(24) = 72$, so $s = 150$.

$(150)\frac{ds}{dt} = 90(30) + (72 + 48)(24)$

$\frac{ds}{dt} = \frac{5580}{150} = 37.2$ knots/h

15. Let x be the distance from the beam to the point opposite the lighthouse and θ be the angle between the beam and the line from the lighthouse to the point opposite.

$\tan\theta = \frac{x}{1}; \frac{d\theta}{dt} = 2(2\pi) = 4\pi$ rad/min,

$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$

At $x = \frac{1}{2}, \theta = \tan^{-1}\frac{1}{2}$ and $\sec^2\theta = \frac{5}{4}$.

$\frac{dx}{dt} = \frac{5}{4}(4\pi) \approx 15.71$ km/min

16. $\tan\theta = \frac{4000}{x}$

$\sec^2\theta \frac{d\theta}{dt} = -\frac{4000}{x^2}\frac{dx}{dt}$

When $\theta = \frac{1}{2}, \frac{d\theta}{dt} = \frac{1}{10}$ and $x = \frac{4000}{\tan\frac{1}{2}} \approx 7322$.

$\frac{dx}{dt} \approx \sec^2\frac{1}{2}\left(\frac{1}{10}\right)\left[-\frac{(7322)^2}{4000}\right]$

≈ -1740 ft/s or -1186 mi/h

The plane's ground speed is 1186 mi/h.

17. a. Let x be the distance along the ground from the light pole to Chris, and let s be the distance from Chris to the tip of his shadow. By similar triangles, $\frac{6}{s} = \frac{30}{x+s}$, so $s = \frac{x}{4}$ and $\frac{ds}{dt} = \frac{1}{4}\frac{dx}{dt}$. $\frac{dx}{dt} = 2$ ft/s, hence $\frac{ds}{dt} = \frac{1}{2}$ ft/s no matter how far from the light pole Chris is.

b. Let $l = x + s$, then

$\frac{dl}{dt} = \frac{dx}{dt} + \frac{ds}{dt} = 2 + \frac{1}{2} = \frac{5}{2}$ ft/s.

c. The angular rate at which Chris must lift his head to follow his shadow is the same as the rate at which the angle that the light makes with the ground is decreasing. Let θ be the angle that the light makes with the ground at the tip of Chris' shadow.

$\tan\theta = \frac{6}{s}$ so $\sec^2\theta \frac{d\theta}{dt} = -\frac{6}{s^2}\frac{ds}{dt}$ and

$\frac{d\theta}{dt} = -\frac{6\cos^2\theta}{s^2}\frac{ds}{dt}$. $\frac{ds}{dt} = \frac{1}{2}$ ft/s

When $s = 6$, $\theta = \frac{\pi}{4}$, so

$$\frac{d\theta}{dt} = -\frac{6\left(\frac{1}{\sqrt{2}}\right)^2}{6^2}\left(\frac{1}{2}\right) = -\frac{1}{24}.$$

Chris must lift his head at the rate of $\frac{1}{24}$ rad/s.

18. Let θ be the measure of the vertex angle, a be the measure of the equal sides, and b be the measure of the base. Observe that $b = 2a\sin\frac{\theta}{2}$ and the height of the triangle is $a\cos\frac{\theta}{2}$.

$A = \frac{1}{2}\left(2a\sin\frac{\theta}{2}\right)\left(a\cos\frac{\theta}{2}\right) = \frac{1}{2}a^2\sin\theta$

$A = \frac{1}{2}(100)^2\sin\theta = 5000\sin\theta; \frac{d\theta}{dt} = \frac{1}{10}$

$\frac{dA}{dt} = 5000\cos\theta\frac{d\theta}{dt}$

When $\theta = \frac{\pi}{6}, \frac{dA}{dt} = 5000\left(\cos\frac{\pi}{6}\right)\left(\frac{1}{10}\right) = 250\sqrt{3}$

$\approx 433\ \text{cm}^2/\text{min}$.

19. Let p be the point on the bridge directly above the railroad tracks. If a is the distance between p and the automobile, then $\frac{da}{dt} = 66$ ft/s. If l is the distance between the train and the point directly below p, then $\frac{dl}{dt} = 88$ ft/s. The distance from the train to p is $\sqrt{100^2 + l^2}$, while the distance from p to the automobile is a. The distance between the train and automobile is

$D = \sqrt{a^2 + \left(\sqrt{100^2 + l^2}\right)^2} = \sqrt{a^2 + l^2 + 100^2}$.

$\frac{dD}{dt} = \frac{1}{2\sqrt{a^2 + l^2 + 100^2}} \cdot \left(2a\frac{da}{dt} + 2l\frac{dl}{dt}\right)$

$= \frac{a\frac{da}{dt} + l\frac{dl}{dt}}{\sqrt{a^2 + l^2 + 100^2}}$. After 10 seconds, $a = 660$ and $l = 880$, so

$\frac{dD}{dt} = \frac{660(66) + 880(88)}{\sqrt{660^2 + 880^2 + 100^2}} \approx 110$ ft/s.

20. $V=\frac{1}{3}\pi h\cdot(a^2+ab+b^2); a=20, b=\frac{h}{4}+20,$

$$V=\frac{1}{3}\pi h\left(400+5h+400+\frac{h^2}{16}+10h+400\right)$$

$$=\frac{1}{3}\pi\left(1200h+15h^2+\frac{h^3}{16}\right)$$

$$\frac{dV}{dt}=\frac{1}{3}\pi\left(1200+30h+\frac{3h^2}{16}\right)\frac{dh}{dt}$$

When $h = 30$ and $\frac{dV}{dt}=2000,$

$$2000=\frac{1}{3}\pi\left(1200+900+\frac{675}{4}\right)\frac{dh}{dt}=\frac{3025\pi}{4}\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{320}{121\pi}\approx 0.84 \text{ cm/min.}$$

21. $V=\pi h^2\left[r-\frac{h}{3}\right]; \frac{dV}{dt}=-2, r=8$

$$V=\pi rh^2-\frac{\pi h^3}{3}=8\pi h^2-\frac{\pi h^3}{3}$$

$$\frac{dV}{dt}=16\pi h\frac{dh}{dt}-\pi h^2\frac{dh}{dt}$$

When $h = 3$, $-2=\frac{dh}{dt}[16\pi(3)-\pi(3)^2]$

$$\frac{dh}{dt}=\frac{-2}{39\pi}\approx -0.016 \text{ ft/hr}$$

22. $s^2=a^2+b^2-2ab\cos\theta;$

$a = 5$, $b = 4$, $\frac{d\theta}{dt}=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$ rad/h

$$s^2=41-40\cos\theta$$

$$2s\frac{ds}{dt}=40\sin\theta\frac{d\theta}{dt}$$

At 3:00, $\theta=\frac{\pi}{2}$ and $s=\sqrt{41}$, so

$$2\sqrt{41}\frac{ds}{dt}=40\sin\left(\frac{\pi}{2}\right)\left(\frac{11\pi}{6}\right)=\frac{220\pi}{3}$$

$$\frac{ds}{dt}\approx 18 \text{ in./hr}$$

23. Let P be the point on the ground where the ball hits. Then the distance from P to the bottom of the light pole is 10 ft. Let s be the distance between P and the shadow of the ball. The height of the ball t seconds after it is dropped is $64-16t^2$.

By similar triangles, $\frac{48}{64-16t^2}=\frac{10+s}{s}$

(for $t > 1$), so $s=\frac{10t^2-40}{1-t^2}$.

$$\frac{ds}{dt}=\frac{20t(1-t^2)-(10t^2-40)(-2t)}{(1-t^2)^2}=-\frac{60t}{(1-t^2)^2}$$

The ball hits the ground when $t = 2$, $\frac{ds}{dt}=-\frac{120}{9}$.

The shadow is moving $\frac{120}{9}\approx 13.33$ ft/s.

24. $V=\pi h^2\left(r-\frac{h}{3}\right); r=20$

$$V=\pi h^2\left(20-\frac{h}{3}\right)=20\pi h^2-\frac{\pi}{3}h^3$$

$$\frac{dV}{dt}=(40\pi h-\pi h^2)\frac{dh}{dt}$$

At 7:00 a.m., $h = 15$, $\frac{dh}{dt}\approx -3$, so

$$\frac{dV}{dt}=(40\pi(15)-\pi(15)^2)(-3)\approx -1125\pi\approx -3534.$$

Webster City residents used water at the rate of 2400 + 3534 = 5934 ft^3/h.

25. Assuming that the tank is now in the shape of an upper hemisphere with radius r, we again let t be the number of hours past midnight and h be the height of the water at time t. The volume, V, of water in the tank at that time is given by

$$V=\frac{2}{3}\pi r^3-\frac{\pi}{3}(r-h)^2(2r+h)$$

and so $V=\frac{16000}{3}\pi-\frac{\pi}{3}(20-h)^2(40+h)$

from which

$$\frac{dV}{dt}=-\frac{\pi}{3}(20-h)^2\frac{dh}{dt}+\frac{2\pi}{3}(20-h)(40+h)\frac{dh}{dt}$$

At $t=7$, $\frac{dV}{dt}\approx -525\pi\approx -1649$

Thus Webster City residents were using water at the rate of $2400+1649=4049$ cubic feet per hour at 7:00 A.M.

26. The amount of water used by Webster City can be found by:

usage = beginning amount + added amount
− remaining amount

Thus the usage is

$$\approx \pi(20)^2(9)+2400(12)-\pi(20)^2(10.5)\approx 26{,}915 \text{ ft}^3$$

over the 12 hour period.

27. a. Let x be the distance from the bottom of the wall to the end of the ladder on the ground, so $\frac{dx}{dt} = 2$ ft/s. Let y be the height of the opposite end of the ladder. By similar triangles, $\frac{y}{12} = \frac{18}{\sqrt{144+x^2}}$, so $y = \frac{216}{\sqrt{144+x^2}}$.

$$\frac{dy}{dt} = -\frac{216}{2(144+x^2)^{3/2}} 2x \frac{dx}{dt} = -\frac{216x}{(144+x^2)^{3/2}} \frac{dx}{dt}$$

When the ladder makes an angle of 60° with the ground, $x = 4\sqrt{3}$ and $\frac{dy}{dt} = -\frac{216(4\sqrt{3})}{(144+48)^{3/2}} \cdot 2 = -1.125$ ft/s.

b. $$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(-\frac{216x}{(144+x^2)^{3/2}}\frac{dx}{dt}\right) = \frac{d}{dt}\left(-\frac{216x}{(144+x^2)^{3/2}}\right)\frac{dx}{dt} - \frac{216x}{(144+x^2)^{3/2}} \cdot \frac{d^2x}{dt^2}$$

Since $\frac{dx}{dt} = 2, \frac{d^2x}{dt^2} = 0,$ thus

$$\frac{d^2y}{dt^2} = \left[\frac{-216(144+x^2)^{3/2}\frac{dx}{dt} + 216x\left(\frac{3}{2}\right)\sqrt{144+x^2}\,(2x)\frac{dx}{dt}}{(144+x^2)^3}\right]\frac{dx}{dt}$$

$$= \frac{-216(144+x^2)+648x^2}{(144+x^2)^{5/2}}\left(\frac{dx}{dt}\right)^2 = \frac{432x^2 - 31{,}104}{(144+x^2)^{5/2}}\left(\frac{dx}{dt}\right)^2$$

When the ladder makes an angle of 60° with the ground,

$$\frac{d^2y}{dt^2} = \frac{432 \cdot 48 - 31{,}104}{(144+48)^{5/2}}(2)^2 \approx -0.08 \text{ ft/s}^2$$

28. a. If the ball has radius 6 in., the volume of the water in the tank is

$$V = 8\pi h^2 - \frac{\pi h^3}{3} - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3$$

$$= 8\pi h^2 - \frac{\pi h^3}{3} - \frac{\pi}{6}$$

$$\frac{dV}{dt} = 16\pi h\frac{dh}{dt} - \pi h^2 \frac{dh}{dt}$$

This is the same as in Problem 21, so $\frac{dh}{dt}$ is again –0.016 ft/hr.

b. If the ball has radius 2 ft, and the height of the water in the tank is h feet with $2 \le h \le 3$, the part of the ball in the water has volume

$$\frac{4}{3}\pi(2)^3 - \pi(4-h)^2\left[2 - \frac{4-h}{3}\right] = \frac{(6-h)h^2\pi}{3}.$$

The volume of water in the tank is

$$V = 8\pi h^2 - \frac{\pi h^3}{3} - \frac{(6-h)h^2\pi}{3} = 6h^2\pi$$

$$\frac{dV}{dt} = 12h\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{12h\pi}\frac{dV}{dt}$$

When $h = 3$, $\frac{dh}{dt} = \frac{1}{36\pi}(-2) \approx -0.018$ ft/hr.

29. $$\frac{dV}{dt} = k(4\pi r^2)$$

a. $$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$k(4\pi r^2) = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = k$$

b. If the original volume was V_0, the volume after 1 hour is $\frac{8}{27}V_0$. The original radius was $r_0 = \sqrt[3]{\frac{3}{4\pi}V_0}$ while the radius after 1 hour is $r_1 = \sqrt[3]{\frac{8}{27}V_0 \cdot \frac{3}{4\pi}} = \frac{2}{3}r_0$. Since $\frac{dr}{dt}$ is constant, $\frac{dr}{dt} = -\frac{1}{3}r_0$ unit/hr. The snowball will take 3 hours to melt completely.

30. $PV = k$

$$P\frac{dV}{dt} + V\frac{dP}{dt} = 0$$

At $t = 6.5$, $P \approx 67$, $\frac{dP}{dt} \approx -30$, $V = 300$

$$\frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt} = -\frac{300}{67}(-30) \approx 134 \text{ in.}^3\text{/min}$$

31. Let l be the distance along the ground from the brother to the tip of the shadow. The shadow is controlled by both siblings when $\frac{3}{l} = \frac{5}{l+4}$ or $l = 6$. Again using similar triangles, this occurs when $\frac{y}{20} = \frac{6}{3}$, so $y = 40$. Thus, the girl controls the tip of the shadow when $y \ge 40$ and the boy controls it when $y < 40$.

Let x be the distance along the ground from the light pole to the girl. $\frac{dx}{dt} = -4$

When $y \ge 40$, $\frac{20}{y} = \frac{5}{y-x}$ or $y = \frac{4}{3}x$.

When $y < 40$, $\frac{20}{y} = \frac{3}{y-(x+4)}$ or $y = \frac{20}{17}(x+4)$.

$x = 30$ when $y = 40$. Thus,

$$y = \begin{cases} \frac{4}{3}x & \text{if } x \ge 30 \\ \frac{20}{17}(x+4) & \text{if } x < 30 \end{cases}$$

and

$$\frac{dy}{dt} = \begin{cases} \frac{4}{3}\frac{dx}{dt} & \text{if } x \ge 30 \\ \frac{20}{17}\frac{dx}{dt} & \text{if } x < 30 \end{cases}$$

Hence, the tip of the shadow is moving at the rate of $\frac{4}{3}(4) = \frac{16}{3}$ ft/s when the girl is at least 30 feet from the light pole, and it is moving $\frac{20}{17}(4) = \frac{80}{17}$ ft/s when the girl is less than 30 ft from the light pole.

2.9 Concepts Review

1. $f'(x)dx$

2. Δy; dy

3. Δx is small.

4. larger ; smaller

Problem Set 2.9

1. $dy = (2x + 1)dx$

2. $dy = (21x^2 + 6x)dx$

3. $dy = -4(2x+3)^{-5}(2)dx = -8(2x+3)^{-5}dx$

4. $dy = -2(3x^2 + x + 1)^{-3}(6x+1)dx$

$= -2(6x+1)(3x^2 + x + 1)^{-3}dx$

5. $dy = 3(\sin x + \cos x)^2(\cos x - \sin x)dx$

6. $dy = 3(\tan x + 1)^2(\sec^2 x)dx$

$= 3\sec^2 x(\tan x + 1)^2 dx$

7. $dy = -\frac{3}{2}(7x^2 + 3x - 1)^{-5/2}(14x+3)dx$

$= -\frac{3}{2}(14x+3)(7x^2 + 3x - 1)^{-5/2}dx$

8. $dy = 2(x^{10} + \sqrt{\sin 2x})[10x^9 + \frac{1}{2\sqrt{\sin 2x}} \cdot (\cos 2x)(2)]dx$

$= 2\left(10x^9 + \frac{\cos 2x}{\sqrt{\sin 2x}}\right)(x^{10} + \sqrt{\sin 2x})dx$

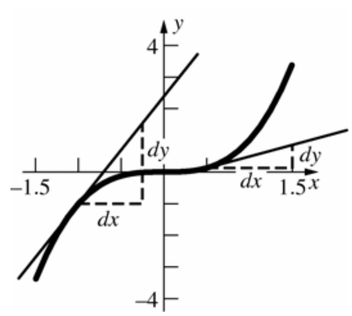
9. $ds = \frac{3}{2}(t^2 - \cot t + 2)^{1/2}(2t + \csc^2 t)dt$

$= \frac{3}{2}(2t + \csc^2 t)\sqrt{t^2 - \cot t + 2}dt$

10. a. $dy = 3x^2 dx = 3(0.5)^2(1) = 0.75$

b. $dy = 3x^2 dx = 3(-1)^2(0.75) = 2.25$

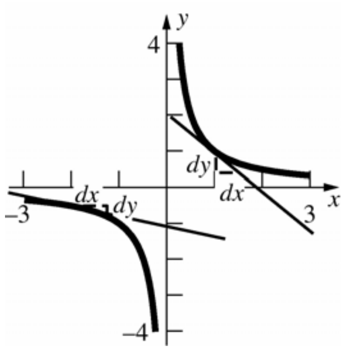
11.



12. a. $dy = -\frac{dx}{x^2} = -\frac{0.5}{(1)^2} = -0.5$

b. $dy = -\frac{dx}{x^2} = -\frac{0.75}{(-2)^2} = -0.1875$

13.



14. a. $\Delta y = (1.5)^3 - (0.5)^3 = 3.25$

b. $\Delta y = (-0.25)^3 - (-1)^3 = 0.984375$

15. a. $\Delta y = \frac{1}{1.5} - \frac{1}{1} = -\frac{1}{3}$

b. $\Delta y = \frac{1}{-1.25} + \frac{1}{2} = -0.3$

16. a. $\Delta y = [(2.5)^2 - 3] - [(2)^2 - 3] = 2.25$

$dy = 2x dx = 2(2)(0.5) = 2$

b. $\Delta y = [(2.88)^2 - 3] - [(3)^2 - 3] = -0.7056$

$dy = 2x dx = 2(3)(-0.12) = -0.72$

17. a. $\Delta y = [(3)^4 + 2(3)] - [(2)^4 + 2(2)] = 67$

$dy = (4x^3 + 2)dx = [4(2)^3 + 2](1) = 34$

b. $\Delta y = [(2.005)^4 + 2(2.005)] - [(2)^4 + 2(2)]$

≈ 0.1706

$dy = (4x^3 + 2)dx = [4(2)^3 + 2](0.005) = 0.17$

18. $y = \sqrt{x};\ dy = \frac{1}{2\sqrt{x}}dx;\ x = 400,\ dx = 2$

$dy = \frac{1}{2\sqrt{400}}(2) = 0.05$

$\sqrt{402} \approx \sqrt{400} + dy = 20 + 0.05 = 20.05$

19. $y = \sqrt{x};\ dy = \frac{1}{2\sqrt{x}}dx;\ x = 36,\ dx = -0.1$

$dy = \frac{1}{2\sqrt{36}}(-0.1) \approx -0.0083$

$\sqrt{35.9} \approx \sqrt{36} + dy = 6 - 0.0083 = 5.9917$

20. $y = \sqrt[3]{x};\ dy = \frac{1}{3}x^{-2/3}dx = \frac{1}{3\sqrt[3]{x^2}}dx;$

$x = 27,\ dx = -0.09$

$dy = \frac{1}{3\sqrt[3]{(27)^2}}(-0.09) \approx -0.0033$

$\sqrt[3]{26.91} \approx \sqrt[3]{27} + dy = 3 - 0.0033 = 2.9967$

21. $V = \frac{4}{3}\pi r^3;\ r = 5,\ dr = 0.125$

$dV = 4\pi r^2 dr = 4\pi(5)^2(0.125) \approx 39.27\ \text{cm}^3$

22. $V = x^3;\ x = \sqrt[3]{40},\ dx = 0.5$

$dV = 3x^2 dx = 3(\sqrt[3]{40})^2(0.5) \approx 17.54\ \text{in.}^3$

23. $V = \frac{4}{3}\pi r^3;\ r = 6\,\text{ft} = 72\,\text{in.},\ dr = -0.3$

$dV = 4\pi r^2 dr = 4\pi(72)^2(-0.3) \approx -19{,}543$

$V \approx \frac{4}{3}\pi(72)^3 - 19{,}543$

$\approx 1{,}543{,}915\ \text{in}^3 \approx 893\ \text{ft}^3$

24. $V = \pi r^2 h;\ r = 6\,\text{ft} = 72\,\text{in.},\ dr = -0.05,$

$h = 8\,\text{ft} = 96\,\text{in.}$

$dV = 2\pi rh\,dr = 2\pi(72)(96)(-0.05) \approx -2171\,\text{in.}^3$

About 9.4 gal of paint are needed.

25. $C = 2\pi r$; r = 4000 mi = 21,120,000 ft, $dr = 2$

$dC = 2\pi dr = 2\pi(2) = 4\pi \approx 12.6\ ft$

26. $T = 2\pi\sqrt{\frac{L}{32}};\ L = 4,\ dL = -0.03$

$dT = \frac{2\pi}{2\sqrt{\frac{L}{32}}} \cdot \frac{1}{32} \cdot dL = \frac{\pi}{\sqrt{32L}}dL$

$dT = \frac{\pi}{\sqrt{32(4)}}(-0.03) \approx -0.0083$

The time change in 24 hours is $(0.0083)(60)(60)(24) \approx 717$ sec

27. $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(10)^3 \approx 4189$

$dV = 4\pi r^2 dr = 4\pi(10)^2(0.05) \approx 62.8$ The volume is 4189 ± 62.8 cm^3.

The absolute error is ≈ 62.8 while the relative error is $62.8/4189 \approx 0.015$ or 1.5% .

28. $V = \pi r^2 h = \pi(3)^2(12) \approx 339$

$dV = 24\pi r dr = 24\pi(3)(0.0025) \approx 0.565$

The volume is 339 ± 0.565 in.^{3}

The absolute error is ≈ 0.565 while the relative error is $0.565/339 \approx 0.0017$ or 0.17%.

29. $s = \sqrt{a^2 + b^2 - 2ab\cos\theta}$

$= \sqrt{151^2 + 151^2 - 2(151)(151)\cos 0.53} \approx 79.097$

$s = \sqrt{45,602 - 45,602\cos\theta}$

$$ds = \frac{1}{2\sqrt{45,602 - 45,602\cos\theta}} \cdot 45,602\sin\theta\, d\theta$$

$$= \frac{22,801\sin\theta}{\sqrt{45,602 - 45,602\cos\theta}} d\theta$$

$$= \frac{22,801\sin 0.53}{\sqrt{45,602 - 45,602\cos 0.53}}(0.005) \approx 0.729$$

$s \approx 79.097 \pm 0.729$ cm

The absolute error is ≈ 0.729 while the relative error is $0.729/79.097 \approx 0.0092$ or 0.92%.

30. $A = \frac{1}{2}ab\sin\theta = \frac{1}{2}(151)(151)\sin 0.53 \approx 5763.33$

$A = \frac{22,801}{2}\sin\theta; \theta = 0.53, d\theta = 0.005$

$dA = \frac{22,801}{2}(\cos\theta)d\theta$

$= \frac{22,801}{2}(\cos 0.53)(0.005) \approx 49.18$

$A \approx 5763.33 \pm 49.18$ cm^2

The absolute error is ≈ 49.18 while the relative error is $49.18/5763.33 \approx 0.0085$ or 0.85%.

31. $y = 3x^2 - 2x + 11; x = 2, dx = 0.001$

$dy = (6x - 2)dx = [6(2) - 2](0.001) = 0.01$

$\frac{d^2y}{dx^2} = 6$, so with $\Delta x = 0.001$,

$|\Delta y - dy| \le \frac{1}{2}(6)(0.001)^2 = 0.000003$

32. Using the approximation

$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$

we let $x = 1.02$ and $\Delta x = -0.02$. We can rewrite the above form as

$f(x) \approx f(x + \Delta x) - f'(x)\Delta x$

which gives

$f(1.02) \approx f(1) - f'(1.02)(-0.02)$

$= 10 + 12(0.02) = 10.24$

33. Using the approximation

$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$

we let $x = 3.05$ and $\Delta x = -0.05$. We can rewrite the above form as

$f(x) \approx f(x + \Delta x) - f'(x)\Delta x$

which gives

$f(3.05) \approx f(3) - f'(3.05)(-0.05)$

$= 8 + \frac{1}{4}(0.05) = 8.0125$

34. From similar triangles, the radius at height h is $\frac{2}{5}h$. Thus, $V = \frac{1}{3}\pi r^2 h = \frac{4}{75}\pi h^3$, so

$dV = \frac{4}{25}\pi h^2 dh$. $h = 10$, $dh = -1$:

$dV = \frac{4}{25}\pi(100)(-1) \approx -50\,\text{cm}^3$

The ice cube has volume $3^3 = 27\,\text{cm}^3$, so there is room for the ice cube without the cup overflowing.

35. $V = \pi r^2 h + \frac{4}{3}\pi r^3$

$V = 100\pi r^2 + \frac{4}{3}\pi r^3; r = 10, dr = 0.1$

$dV = (200\pi r + 4\pi r^2)dr$

$= (2000\pi + 400\pi)(0.1) = 240\pi \approx 754$ cm^3

36. The percent increase in mass is $\frac{dm}{m}$.

$$dm = -\frac{m_0}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)dv$$

$$= \frac{m_0 v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} dv$$

$$\frac{dm}{m} = \frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1} dv = \frac{v}{c^2}\left(\frac{c^2}{c^2 - v^2}\right)dv$$

$$= \frac{v}{c^2 - v^2}dv$$

$v = 0.9c$, $dv = 0.02c$

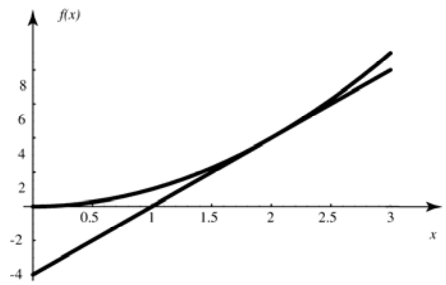
$$\frac{dm}{m} = \frac{0.9c}{c^2 - 0.81c^2}(0.02c) = \frac{0.018}{0.19} \approx 0.095$$

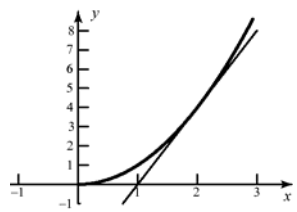
The percent increase in mass is about 9.5.

37. $f(x)=x^2;\ f'(x)=2x;\ a=2$

The linear approximation is then

$$L(x)=f(2)+f'(2)(x-2)$$
$$=4+4(x-2)=4x-4$$



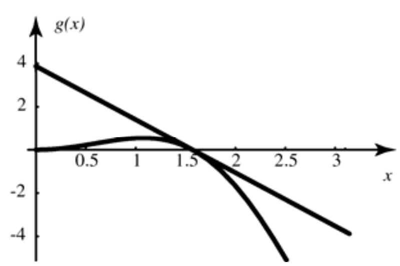


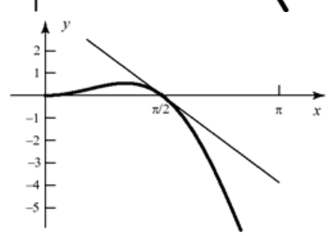
38. $g(x)=x^2\cos x;\ g'(x)=-x^2\sin x+2x\cos x$

$a=\pi/2$

The linear approximation is then

$$L(x)=0+-\left(\frac{\pi}{2}\right)^2\left(x-\frac{\pi}{2}\right)$$
$$=-\frac{\pi^2}{4}x+\frac{\pi^3}{8}$$
$$L(x)=0+-\frac{\pi^2}{4}\left(x-\frac{\pi}{2}\right)$$
$$=-\frac{\pi^2}{4}x+\frac{\pi^3}{8}$$

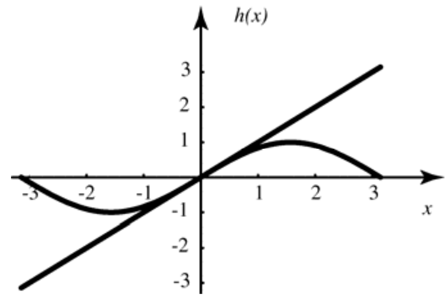


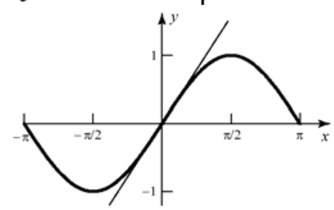


39. $h(x)=\sin x;\ h'(x)=\cos x;\ a=0$

The linear approximation is then

$$L(x)=0+1(x-0)=x$$

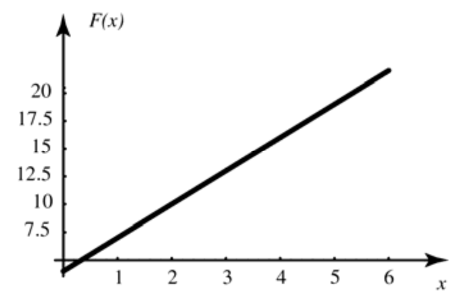


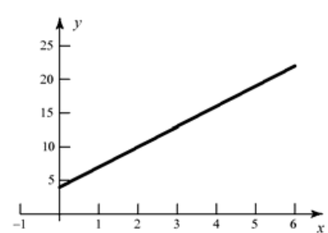


40. $F(x)=3x+4;\ F'(x)=3;\ a=3$

The linear approximation is then

$$L(x)=13+3(x-3)=13+3x-9$$
$$=3x+4$$





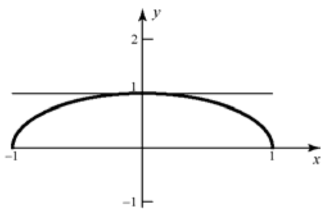
41. $f(x)=\sqrt{1-x^2}$;

$f'(x)=\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)$

$=\frac{-x}{\sqrt{1-x^2}}$, $a=0$

The linear approximation is then

$L(x)=1+0(x-0)=1$

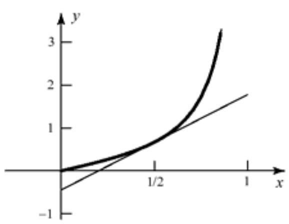


42. $g(x)=\frac{x}{1-x^2}$;

$g'(x)=\frac{\left(1-x^2\right)-x(-2x)}{\left(1-x^2\right)^2}=\frac{1+x^2}{\left(1-x^2\right)^2}, a=\frac{1}{2}$

The linear approximation is then

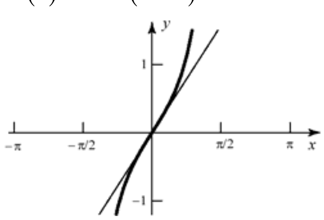
$L(x)=\frac{2}{3}+\frac{20}{9}\left(x-\frac{1}{2}\right)=\frac{20}{9}x-\frac{4}{9}$



43. $h(x)=x\sec x; h'(x)=\sec x+x\sec x\tan x, a=0$

The linear approximation is then

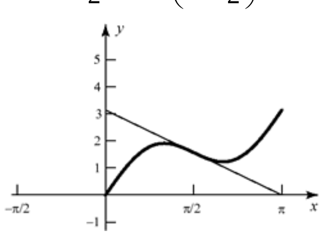
$L(x)=0+1(x-0)=x$



44. $G(x)=x+\sin 2x;\ G'(x)=1+2\cos 2x, a=\pi/2$

The linear approximation is then

$L(x)=\frac{\pi}{2}+(-1)\left(x-\frac{\pi}{2}\right)=-x+\pi$



45. $f(x)=mx+b;\ f'(x)=m$

The linear approximation is then

$L(x)=ma+b+m(x-a)=am+b+mx-ma$

$=mx+b \qquad f(x)=L(x)$

46. $L(x)-f(x)=\sqrt{a}+\frac{1}{2\sqrt{a}}(x-a)-\sqrt{x}$

$=\frac{x}{2\sqrt{a}}-\sqrt{x}+\frac{\sqrt{a}}{2}=\frac{x-2\sqrt{a}\sqrt{x}+a}{2\sqrt{a}}$

$=\frac{\left(\sqrt{x}-\sqrt{a}\right)^2}{2\sqrt{a}}\geq 0$

47. The linear approximation to $f(x)$ at a is

$L(x)=f(a)+f'(a)(x-a)$

$=a^2+2a(x-a)$

$=2ax-a^2$

Thus,

$f(x)-L(x)=x^2-\left(2ax-a^2\right)$

$=x^2-2ax+a^2$

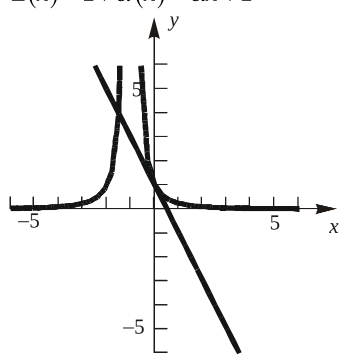
$=(x-a)^2$

≥ 0

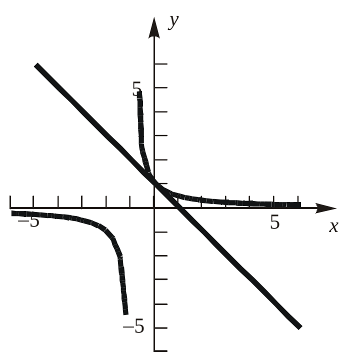
48. $f(x)=(1+x)^\alpha$, $f'(x)=\alpha(1+x)^{\alpha-1}$, $a=0$

The linear approximation is then

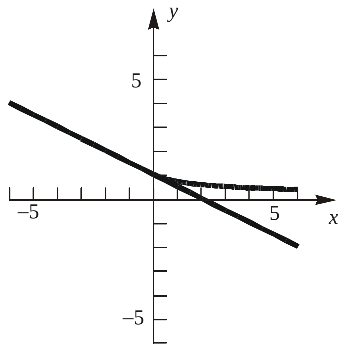
$L(x)=1+\alpha(x)=\alpha x+1$



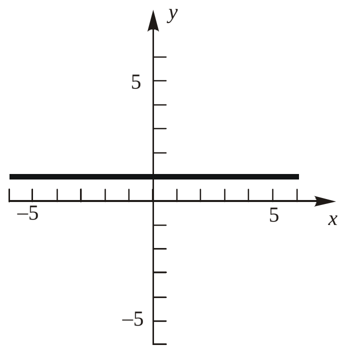
$\alpha=-2$



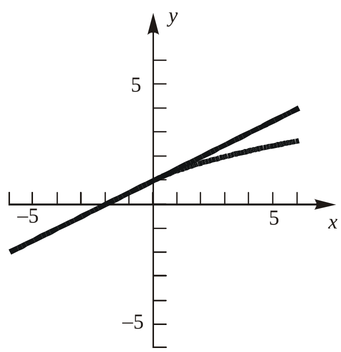
$\alpha=-1$



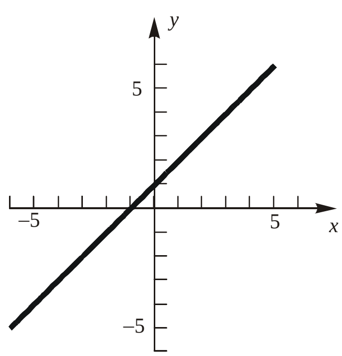
$\alpha = -0.5$



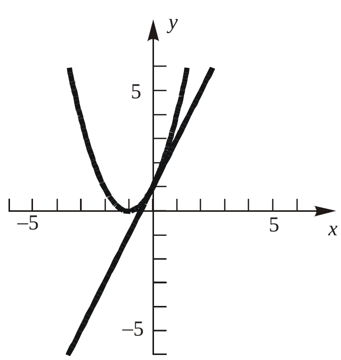
$\alpha = 0$



$\alpha = 0.5$



$\alpha = 1$



$\alpha = 2$

49. a. $\lim_{h\to 0}\varepsilon(h) = \lim_{h\to 0}\left(f(x+h) - f(x) - f'(x)h\right)$
$= f(x) - f(x) - f'(x)0 = 0$

b. $\lim_{h\to 0}\frac{\varepsilon(h)}{h} = \lim\left[\frac{f(x+h) - f(x)}{h} - f'(x)\right]$
$= f'(x) - f'(x) = 0$

2.10 Chapter Review

Concepts Test

1. False: If $f(x) = x^3$, $f'(x) = 3x^2$ and the tangent line $y = 0$ at $x = 0$ crosses the curve at the point of tangency.

2. False: The tangent line can touch the curve at infinitely many points.

3. True: $m_{\tan} = 4x^3$, which is unique for each value of x.

4. False: $m_{\tan} = -\sin x$, which is periodic.

5. True: If the velocity is negative and increasing, the speed is decreasing.

6. True: If the velocity is negative and decreasing, the speed is increasing.

7. True: If the tangent line is horizontal, the slope must be 0.

8. False: $f(x) = ax^2 + b$, $g(x) = ax^2 + c$, $b \neq c$. Then $f'(x) = 2ax = g'(x)$, but $f(x) \neq g(x)$.

9. True: $D_x f(g(x)) = f'(g(x))g'(x)$; since $g(x) = x$, $g'(x) = 1$, so $D_x f(g(x)) = f'(g(x))$.

10. False: $D_x y = 0$ because π is a constant, not a variable.

11. True: Theorem 3.2.A

12. True: The derivative does not exist when the tangent line is vertical.

13. False: $(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x)$

14. True: Negative acceleration indicates decreasing velocity.

15. True: If $f(x) = x^3 g(x)$, then
$D_x f(x) = x^3 g'(x) + 3x^2 g(x)$
$= x^2[xg'(x) + 3g(x)]$.

16. False: $D_x y = 3x^2$; At (1, 1):
$m_{\tan} = 3(1)^2 = 3$
Tangent line: $y - 1 = 3(x - 1)$

17. False: $D_x y = f(x)g'(x) + g(x)f'(x)$
$D_x^2 y = f(x)g''(x) + g'(x)f'(x)$
$+ g(x)f''(x) + f'(x)g'(x)$
$= f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$

18. True: The degree of $y = (x^3 + x)^8$ is 24, so
$D_x^{25} y = 0$.

19. True: $f(x) = ax^n$; $f'(x) = anx^{n-1}$

20. True: $D_x \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

21. True: $h'(x) = f(x)g'(x) + g(x)f'(x)$
$h'(c) = f(c)g'(c) + g(c)f'(c)$
$= f(c)(0) + g(c)(0) = 0$

22. True: $f'\left(\frac{\pi}{2}\right) = \lim_{x \to \frac{\pi}{2}} \frac{\sin x - \sin\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}}$
$= \lim_{x \to \frac{\pi}{2}} \frac{\sin x - 1}{x - \frac{\pi}{2}}$

23. True: $D^2(kf) = kD^2 f$ and
$D^2(f + g) = D^2 f + D^2 g$

24. True: $h'(x) = f'(g(x)) \cdot g'(x)$
$h'(c) = f'(g(c)) \cdot g'(c) = 0$

25. True: $(f \circ g)'(2) = f'(g(2)) \cdot g'(2)$
$= f'(2) \cdot g'(2) = 2 \cdot 2 = 4$

26. False: Consider $f(x) = \sqrt{x}$. The curve always lies below the tangent.

27. False: The rate of volume change depends on the radius of the sphere.

28. True: $c = 2\pi r$; $\frac{dr}{dt} = 4$
$\frac{dc}{dt} = 2\pi \frac{dr}{dt} = 2\pi(4) = 8\pi$

29. True: $D_x(\sin x) = \cos x$;
$D_x^2(\sin x) = -\sin x$;
$D_x^3(\sin x) = -\cos x$;
$D_x^4(\sin x) = \sin x$;
$D_x^5(\sin x) = \cos x$

30. False: $D_x(\cos x) = -\sin x$;
$D_x^2(\cos x) = -\cos x$;
$D_x^3(\cos x) = \sin x$;
$D_x^4(\cos x) = D_x[D_x^3(\cos x)] = D_x(\sin x)$
Since $D_x^{1+3}(\cos x) = D_x^1(\sin x)$,
$D_x^{n+3}(\cos x) = D_x^n(\sin x)$.

31. True: $\lim_{x \to 0} \frac{\tan x}{3x} = \frac{1}{3} \lim_{x \to 0} \frac{\sin x}{x \cos x}$
$= \frac{1}{3} \cdot 1 = \frac{1}{3}$

32. True: $v = \frac{ds}{dt} = 15t^2 + 6$ which is greater than 0 for all t.

33. True: $V = \frac{4}{3}\pi r^3$
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$
If $\frac{dV}{dt} = 3$, then $\frac{dr}{dt} = \frac{3}{4\pi r^2}$ so
$\frac{dr}{dt} > 0$.
$\frac{d^2 r}{dt^2} = -\frac{3}{2\pi r^3} \frac{dr}{dt}$ so $\frac{d^2 r}{dt^2} < 0$

34. True: When $h > r$, then $\frac{d^2 h}{dt^2} > 0$

35. True: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
$dV = 4\pi r^2 dr = S \cdot dr$
If $\Delta r = dr$, then $dV = S \cdot \Delta r$

36. False: $dy = 5x^4 dx$, so $dy > 0$ when $dx > 0$, but $dy < 0$ when $dx < 0$.

37. False: The slope of the linear approximation is equal to
$f'(a) = f'(0) = -\sin(0) = 0$.

Sample Test Problems

1. a. $f'(x)=\lim_{h\to 0}\frac{3(x+h)^3-3x^3}{h} = \lim_{h\to 0}\frac{9x^2h+9xh^2+3h^3}{h} = \lim_{h\to 0}(9x^2+9xh+3h^2)=9x^2$

b. $f'(x)=\lim_{h\to 0}\frac{[2(x+h)^5+3(x+h)]-(2x^5+3x)}{h} = \lim_{h\to 0}\frac{10x^4h+20x^3h^2+20x^2h^3+10xh^4+2h^5+3h}{h}$

$=\lim_{h\to 0}(10x^4+20x^3h+20x^2h^2+10xh^3+2h^4+3) = 10x^4+3$

c. $f'(x)=\lim_{h\to 0}\frac{\frac{1}{3(x+h)}-\frac{1}{3x}}{h}=\lim_{h\to 0}\left[-\frac{h}{3(x+h)x}\right]\frac{1}{h}=\lim_{h\to 0}-\left(\frac{1}{3x(x+h)}\right)=-\frac{1}{3x^2}$

d. $f'(x)=\lim_{h\to 0}\left[\left(\frac{1}{3(x+h)^2+2}-\frac{1}{3x^2+2}\right)\frac{1}{h}\right]=\lim_{h\to 0}\left[\frac{3x^2+2-3(x+h)^2-2}{(3(x+h)^2+2)(3x^2+2)}\cdot\frac{1}{h}\right]$

$=\lim_{h\to 0}\left[\frac{-6xh-3h^2}{(3(x+h)^2+2)(3x^2+2)}\cdot\frac{1}{h}\right]=\lim_{h\to 0}\frac{-6x-3h}{(3(x+h)^2+2)(3x^2+2)}=-\frac{6x}{(3x^2+2)^2}$

e. $f'(x)=\lim_{h\to 0}\frac{\sqrt{3(x+h)}-\sqrt{3x}}{h} = \lim_{h\to 0}\frac{(\sqrt{3x+3h}-\sqrt{3x})(\sqrt{3x+3h}+\sqrt{3x})}{h(\sqrt{3x+3h}+\sqrt{3x})}$

$=\lim_{h\to 0}\frac{3h}{h(\sqrt{3x+3h}+\sqrt{3x})}=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h}+\sqrt{3x}} = \frac{3}{2\sqrt{3x}}$

f. $f'(x)=\lim_{h\to 0}\frac{\sin[3(x+h)]-\sin 3x}{h}=\lim_{h\to 0}\frac{\sin(3x+3h)-\sin 3x}{h}$

$=\lim_{h\to 0}\frac{\sin 3x\cos 3h+\sin 3h\cos 3x-\sin 3x}{h}=\lim_{h\to 0}\frac{\sin 3x(\cos 3h-1)}{h}+\lim_{h\to 0}\frac{\sin 3h\cos 3x}{h}$

$=3\sin 3x\lim_{h\to 0}\frac{\cos 3h-1}{3h}+\cos 3x\lim_{h\to 0}\frac{\sin 3h}{h} = (3\sin 3x)(0)+(\cos 3x)3\lim_{h\to 0}\frac{\sin 3h}{3h} = (\cos 3x)(3)(1)=3\cos 3x$

g. $f'(x)=\lim_{h\to 0}\frac{\sqrt{(x+h)^2+5}-\sqrt{x^2+5}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{(x+h)^2+5}-\sqrt{x^2+5}\right)\left(\sqrt{(x+h)^2+5}+\sqrt{x^2+5}\right)}{h\left(\sqrt{(x+h)^2+5}+\sqrt{x^2+5}\right)}$

$=\lim_{h\to 0}\frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+5}+\sqrt{x^2+5}\right)} = \lim_{h\to 0}\frac{2x+h}{\sqrt{(x+h)^2+5}+\sqrt{x^2+5}}=\frac{2x}{2\sqrt{x^2+5}}=\frac{x}{\sqrt{x^2+5}}$

h. $f'(x)=\lim_{h\to 0}\frac{\cos[\pi(x+h)]-\cos\pi x}{h}=\lim_{h\to 0}\frac{\cos(\pi x+\pi h)-\cos\pi x}{h}=\lim_{h\to 0}\frac{\cos\pi x\cos\pi h-\sin\pi x\sin\pi h-\cos\pi x}{h}$

$=\lim_{h\to 0}\left(-\pi\cos\pi x\frac{1-\cos\pi h}{\pi h}\right)-\lim_{h\to 0}\left(\pi\sin\pi x\frac{\sin\pi h}{\pi h}\right)=(-\pi\cos\pi x)(0)-(\pi\sin\pi x)=-\pi\sin\pi x$

2. a. $g'(x)=\lim_{t\to x}\frac{2t^2-2x^2}{t-x}=\lim_{t\to x}\frac{2(t-x)(t+x)}{t-x}$

$=2\lim_{t\to x}(t+x)=2(2x)=4x$

b. $g'(x)=\lim_{t\to x}\frac{(t^3+t)-(x^3+x)}{t-x}$

$=\lim_{t\to x}\frac{(t-x)(t^2+tx+x^2)+(t-x)}{t-x}$

$=\lim_{t\to x}(t^2+tx+x^2+1)=3x^2+1$

c. $g'(x)=\lim_{t\to x}\frac{\frac{1}{t}-\frac{1}{x}}{t-x}=\lim_{t\to x}\frac{x-t}{tx(t-x)}$

$=\lim_{t\to x}\frac{-1}{tx}=-\frac{1}{x^2}$

d. $g'(x)=\lim_{t\to x}\left[\left(\frac{1}{t^2+1}-\frac{1}{x^2+1}\right)\left(\frac{1}{t-x}\right)\right]$

$=\lim_{t\to x}\frac{x^2-t^2}{(t^2+1)(x^2+1)(t-x)}$

$=\lim_{t\to x}\frac{-(x+t)(t-x)}{(t^2+1)(x^2+1)(t-x)}$

$=\lim_{t\to x}\frac{-(x+t)}{(t^2+1)(x^2+1)}=-\frac{2x}{(x^2+1)^2}$

e. $g'(x)=\lim_{t\to x}\frac{\sqrt{t}-\sqrt{x}}{t-x}$

$=\lim_{t\to x}\frac{(\sqrt{t}-\sqrt{x})(\sqrt{t}+\sqrt{x})}{(t-x)(\sqrt{t}+\sqrt{x})}$

$=\lim_{t\to x}\frac{t-x}{(t-x)(\sqrt{t}+\sqrt{x})}=\lim_{t\to x}\frac{1}{\sqrt{t}+\sqrt{x}}$

$=\frac{1}{2\sqrt{x}}$

f. $g'(x)=\lim_{t\to x}\frac{\sin\pi t-\sin\pi x}{t-x}$

Let $v = t - x$, then $t = v + x$ and as $t\to x, v\to 0$.

$\lim_{t\to x}\frac{\sin\pi t-\sin\pi x}{t-x}=\lim_{v\to 0}\frac{\sin\pi(v+x)-\sin\pi x}{v}$

$=\lim_{v\to 0}\frac{\sin\pi v\cos\pi x+\sin\pi x\cos\pi v-\sin\pi x}{v}$

$=\lim_{v\to 0}\left[\pi\cos\pi x\frac{\sin\pi v}{\pi v}+\pi\sin\pi x\frac{\cos\pi v-1}{\pi v}\right]$

$=\pi\cos\pi x\cdot 1+\pi\sin\pi x\cdot 0=\pi\cos\pi x$

Other method:
Use the subtraction formula

$\sin\pi t-\sin\pi x=2\cos\frac{\pi(t+x)}{2}\sin\frac{\pi(t-x)}{2}$

g. $g'(x)=\lim_{t\to x}\frac{\sqrt{t^3+C}-\sqrt{x^3+C}}{t-x}$

$=\lim_{t\to x}\frac{\left(\sqrt{t^3+C}-\sqrt{x^3+C}\right)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}{(t-x)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}$

$=\lim_{t\to x}\frac{t^3-x^3}{(t-x)\left(\sqrt{t^3+C}+\sqrt{x^3+C}\right)}$

$=\lim_{t\to x}\frac{t^2+tx+x^2}{\sqrt{t^3+C}+\sqrt{x^3+C}}=\frac{3x^2}{2\sqrt{x^3+C}}$

h. $g'(x)=\lim_{t\to x}\frac{\cos 2t-\cos 2x}{t-x}$

Let $v = t - x$, then $t = v + x$ and as $t\to x, v\to 0$.

$\lim_{t\to x}\frac{\cos 2t-\cos 2x}{t-x}=\lim_{v\to 0}\frac{\cos 2(v+x)-\cos 2x}{v}$

$=\lim_{v\to 0}\frac{\cos 2v\cos 2x-\sin 2v\sin 2x-\cos 2x}{v}$

$=\lim_{v\to 0}\left[2\cos 2x\frac{\cos 2v-1}{2v}-2\sin 2x\frac{\sin 2v}{2v}\right]$

$=2\cos 2x\cdot 0-2\sin 2x\cdot 1=-2\sin 2x$

Other method:
Use the subtraction formula

$\cos 2t-\cos 2x=-2\sin(t+x)\sin(t-x)$.

3. a. $f(x) = 3x$ at $x = 1$

b. $f(x)=4x^3$ at $x = 2$

c. $f(x)=\sqrt{x^3}$ at $x = 1$

d. $f(x) = \sin x$ at $x=\pi$

e. $f(x)=\frac{4}{x}$ at x

f. $f(x) = -\sin 3x$ at x

g. $f(x) = \tan x$ at $x=\frac{\pi}{4}$

h. $f(x)=\frac{1}{\sqrt{x}}$ at $x = 5$

4. a. $f'(2)\approx -\frac{3}{4}$

b. $f'(6)\approx\frac{3}{2}$

c. $V_{avg} = \dfrac{6-\frac{3}{2}}{7-3} = \dfrac{9}{8}$

d. $\dfrac{d}{dt} f(t^2) = f'(t^2)(2t)$

At $t = 2$, $4f'(4) \approx 4\left(\dfrac{2}{3}\right) = \dfrac{8}{3}$

e. $\dfrac{d}{dt}[f^2(t)] = 2f(t)f'(t)$

At $t = 2$,

$2f(2)f'(2) \approx 2(2)\left(-\dfrac{3}{4}\right) = -3$

f. $\dfrac{d}{dt}(f(f(t))) = f'(f(t))f'(t)$

At $t = 2$, $f'(f(2))f'(2) = f'(2)f'(2)$

$\approx \left(-\dfrac{3}{4}\right)\left(-\dfrac{3}{4}\right) = \dfrac{9}{16}$

5. $D_x(3x^5) = 15x^4$

6. $D_x(x^3 - 3x^2 + x^{-2}) = 3x^2 - 6x + (-2)x^{-3}$

$= 3x^2 - 6x - 2x^{-3}$

7. $D_z(z^3 + 4z^2 + 2z) = 3z^2 + 8z + 2$

8. $D_x\left(\dfrac{3x-5}{x^2+1}\right) = \dfrac{(x^2+1)(3) - (3x-5)(2x)}{(x^2+1)^2}$

$= \dfrac{-3x^2 + 10x + 3}{(x^2+1)^2}$

9. $D_t\left(\dfrac{4t-5}{6t^2+2t}\right) = \dfrac{(6t^2+2t)(4) - (4t-5)(12t+2)}{(6t^2+2t)^2}$

$= \dfrac{-24t^2 + 60t + 10}{(6t^2+2t)^2}$

10. $D_x(3x+2)^{2/3} = \dfrac{2}{3}(3x+2)^{-1/3}(3)$

$= 2(3x+2)^{-1/3}$

$D_x^2(3x+2)^{2/3} = -\dfrac{2}{3}(3x+2)^{-4/3}(3)$

$= -2(3x+2)^{-4/3}$

11. $\dfrac{d}{dx}\left(\dfrac{4x^2-2}{x^3+x}\right) = \dfrac{(x^3+x)(8x) - (4x^2-2)(3x^2+1)}{(x^3+x)^2}$

$= \dfrac{-4x^4 + 10x^2 + 2}{(x^3+x)^2}$

12. $D_t(t\sqrt{2t+6}) = t\dfrac{1}{2\sqrt{2t+6}}(2) + \sqrt{2t+6}$

$= \dfrac{t}{\sqrt{2t+6}} + \sqrt{2t+6}$

13. $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x^2+4}}\right) = \dfrac{d}{dx}(x^2+4)^{-1/2}$

$= -\dfrac{1}{2}(x^2+4)^{-3/2}(2x)$

$= -\dfrac{x}{\sqrt{(x^2+4)^3}}$

14. $\dfrac{d}{dx}\sqrt{\dfrac{x^2-1}{x^3-x}} = \dfrac{d}{dx}\dfrac{1}{\sqrt{x}} = \dfrac{d}{dx}x^{-1/2} = -\dfrac{1}{2x^{3/2}}$

15. $D_\theta(\sin\theta + \cos^3\theta) = \cos\theta + 3\cos^2\theta(-\sin\theta)$

$= \cos\theta - 3\sin\theta\cos^2\theta$

$D_\theta^2(\sin\theta + \cos^3\theta)$

$= -\sin\theta - 3[\sin\theta(2)(\cos\theta)(-\sin\theta) + \cos^3\theta]$

$= -\sin\theta + 6\sin^2\theta\cos\theta - 3\cos^3\theta$

16. $\dfrac{d}{dt}[\sin(t^2) - \sin^2(t)] = \cos(t^2)(2t) - (2\sin t)(\cos t)$

$= 2t\cos(t^2) - \sin(2t)$

17. $D_\theta[\sin(\theta^2)] = \cos(\theta^2)(2\theta) = 2\theta\cos(\theta^2)$

18. $\dfrac{d}{dx}(\cos^3 5x) = (3\cos^2 5x)(-\sin 5x)(5)$

$= -15\cos^2 5x \sin 5x$

19. $\frac{d}{d\theta}[\sin^2(\sin(\pi\theta))] = 2\sin(\sin(\pi\theta))\cos(\sin(\pi\theta))(\cos(\pi\theta))(\pi) = 2\pi\sin(\sin(\pi\theta))\cos(\sin(\pi\theta))\cos(\pi\theta)$

20. $\frac{d}{dt}[\sin^2(\cos 4t)] = 2\sin(\cos 4t)\left(\cos(\cos 4t)\right)(-\sin 4t)(4) = -8\sin(\cos 4t)\cos(\cos 4t)\sin 4t$

21. $D_\theta \tan 3\theta = (\sec^2 3\theta)(3) = 3\sec^2 3\theta$

22. $\frac{d}{dx}\left(\frac{\sin 3x}{\cos 5x^2}\right) = \frac{(\cos 5x^2)(\cos 3x)(3) - (\sin 3x)(-\sin 5x^2)(10x)}{\cos^2 5x^2} = \frac{3\cos 5x^2 \cos 3x + 10x\sin 3x \sin 5x^2}{\cos^2 5x^2}$

23. $f'(x) = (x^2-1)^2(9x^2-4) + (3x^3-4x)(2)(x^2-1)(2x) = (x^2-1)^2(9x^2-4) + 4x(x^2-1)(3x^3-4x)$

$f'(2) = 672$

24. $g'(x) = 3\cos 3x + 2(\sin 3x)(\cos 3x)(3) = 3\cos 3x + 3\sin 6x$

$g''(x) = -9\sin 3x + 18\cos 6x$

$g''(0) = 18$

25. $\frac{d}{dx}\left(\frac{\cot x}{\sec x^2}\right) = \frac{(\sec x^2)(-\csc^2 x) - (\cot x)(\sec x^2)(\tan x^2)(2x)}{\sec^2 x^2} = \frac{-\csc^2 x - 2x\cot x\tan x^2}{\sec x^2}$

26. $D_t\left(\frac{4t\sin t}{\cos t - \sin t}\right) = \frac{(\cos t - \sin t)(4t\cos t + 4\sin t) - (4t\sin t)(-\sin t - \cos t)}{(\cos t - \sin t)^2}$

$= \frac{4t\cos^2 t + 2\sin 2t - 4\sin^2 t + 4t\sin^2 t}{(\cos t - \sin t)^2} = \frac{4t + 2\sin 2t - 4\sin^2 t}{(\cos t - \sin t)^2}$

27. $f'(x) = (x-1)^3 2(\sin \pi x - x)(\pi\cos \pi x - 1) + (\sin \pi x - x)^2 3(x-1)^2$

$= 2(x-1)^3(\sin \pi x - x)(\pi\cos \pi x - 1) + 3(\sin \pi x - x)^2(x-1)^2$

$f'(2) = 16 - 4\pi \approx 3.43$

28. $h'(t) = 5(\sin(2t) + \cos(3t))^4(2\cos(2t) - 3\sin(3t))$

$h''(t) = 5(\sin(2t) + \cos(3t))^4(-4\sin(2t) - 9\cos(3t)) + 20(\sin(2t) + \cos(3t))^3(2\cos(2t) - 3\sin(3t))^2$

$h''(0) = 5\cdot 1^4 \cdot (-9) + 20\cdot 1^3 \cdot 2^2 = 35$

29. $g'(r) = 3(\cos^2 5r)(-\sin 5r)(5) = -15\cos^2 5r\sin 5r$

$g''(r) = -15[(\cos^2 5r)(\cos 5r)(5) + (\sin 5r)2(\cos 5r)(-\sin 5r)(5)] = -15[5\cos^3 5r - 10(\sin^2 5r)(\cos 5r)]$

$g'''(r) = -15[5(3)(\cos^2 5r)(-\sin 5r)(5) - (10\sin^2 5r)(-\sin 5r)(5) - (\cos 5r)(20\sin 5r)(\cos 5r)(5)]$

$= -15[-175(\cos^2 5r)(\sin 5r) + 50\sin^3 5r]$

$g'''(1) \approx 458.8$

30. $f'(t) = h'(g(t))g'(t) + 2g(t)g'(t)$

31. $G'(x) = F'(r(x) + s(x))(r'(x) + s'(x)) + s'(x)$

$G''(x) = F'(r(x) + s(x))(r''(x) + s''(x)) + (r'(x) + s'(x))F''(r(x) + s(x))(r'(x) + s'(x)) + s''(x)$

$= F'(r(x) + s(x))(r''(x) + s''(x)) + (r'(x) + s'(x))^2 F''(r(x) + s(x)) + s''(x)$

32. $F'(x) = Q'(R(x))R'(x) = 3[R(x)]^2(-\sin x)$
$= -3\cos^2 x \sin x$

33. $F'(z) = r'(s(z))s'(z) = [3\cos(3s(z))](9z^2)$
$= 27z^2 \cos(9z^3)$

34. $\frac{dy}{dx} = 2(x-2)$
$2x - y + 2 = 0$; $y = 2x + 2$; $m = 2$
$2(x-2) = -\frac{1}{2}$
$x = \frac{7}{4}$
$y = \left(\frac{7}{4} - 2\right)^2 = \frac{1}{16}; \left(\frac{7}{4}, \frac{1}{16}\right)$

35. $V = \frac{4}{3}\pi r^3$
$\frac{dV}{dr} = 4\pi r^2$
When $r = 5$, $\frac{dV}{dr} = 4\pi(5)^2 = 100\pi \approx 314$ m^3 per meter of increase in the radius.

36. $V = \frac{4}{3}\pi r^3; \frac{dV}{dt} = 10$
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$
When $r = 5$, $10 = 4\pi(5)^2 \frac{dr}{dt}$
$\frac{dr}{dt} = \frac{1}{10\pi} \approx 0.0318$ m/h

37. $V = \frac{1}{2}bh(12); \frac{6}{4} = \frac{b}{h}; b = \frac{3h}{2}$
$V = 6\left(\frac{3h}{2}\right)h = 9h^2; \frac{dV}{dt} = 9$
$\frac{dV}{dt} = 18h\frac{dh}{dt}$
When $h = 3$, $9 = 18(3)\frac{dh}{dt}$
$\frac{dh}{dt} = \frac{1}{6} \approx 0.167$ ft/min

38. **a.** $v = 128 - 32t$
$v = 0$, when $t = 4s$
$s = 128(4) - 16(4)^2 = 256$ ft

b. $128t - 16t^2 = 0$
$-16t(t-8) = 0$
The object hits the ground when $t = 8$s
$v = 128 - 32(8) = -128$ ft/s

39. $s = t^3 - 6t^2 + 9t$
$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$
$a(t) = \frac{d^2s}{dt^2} = 6t - 12$

a. $3t^2 - 12t + 9 < 0$
$3(t-3)(t-1) < 0$
$1 < t < 3$; $(1, 3)$

b. $3t^2 - 12t + 9 = 0$
$3(t-3)(t-1) = 0$
$t = 1, 3$
$a(1) = -6$, $a(3) = 6$

c. $6t - 12 > 0$
$t > 2$; $(2, \infty)$

40. **a.** $D_x^{20}(x^{19} + x^{12} + x^5 + 100) = 0$

b. $D_x^{20}(x^{20} + x^{19} + x^{18}) = 20!$

c. $D_x^{20}(7x^{21} + 3x^{20}) = (7 \cdot 21!)x + (3 \cdot 20!)$

d. $D_x^{20}(\sin x + \cos x) = D_x^4(\sin x + \cos x)$
$= \sin x + \cos x$

e. $D_x^{20}(\sin 2x) = 2^{20} \sin 2x$
$= 1{,}048{,}576 \sin 2x$

f. $D_x^{20}\left(\frac{1}{x}\right) = \frac{(-1)^{20}(20!)}{x^{21}} = \frac{20!}{x^{21}}$

41. **a.** $2(x-1) + 2y\frac{dy}{dx} = 0$
$\frac{dy}{dx} = \frac{-(x-1)}{y} = \frac{1-x}{y}$

b. $x(2y)\frac{dy}{dx} + y^2 + y(2x) + x^2\frac{dy}{dx} = 0$
$\frac{dy}{dx}(2xy + x^2) = -(y^2 + 2xy)$
$\frac{dy}{dx} = -\frac{y^2 + 2xy}{x^2 + 2xy}$

c. $3x^2 + 3y^2 \frac{dy}{dx} = x^3(3y^2)\frac{dy}{dx} + 3x^2y^3$

$\frac{dy}{dx}(3y^2 - 3x^3y^2) = 3x^2y^3 - 3x^2$

$\frac{dy}{dx} = \frac{3x^2y^3 - 3x^2}{3y^2 - 3x^3y^2} = \frac{x^2y^3 - x^2}{y^2 - x^3y^2}$

d. $x\cos(xy)\left[x\frac{dy}{dx} + y\right] + \sin(xy) = 2x$

$x^2\cos(xy)\frac{dy}{dx} = 2x - \sin(xy) - xy\cos(xy)$

$\frac{dy}{dx} = \frac{2x - \sin(xy) - xy\cos(xy)}{x^2\cos(xy)}$

e. $x\sec^2(xy)\left(x\frac{dy}{dx} + y\right) + \tan(xy) = 0$

$x^2\sec^2(xy)\frac{dy}{dx} = -[\tan(xy) + xy\sec^2(xy)]$

$\frac{dy}{dx} = -\frac{\tan(xy) + xy\sec^2(xy)}{x^2\sec^2(xy)}$

42. $2yy_1' = 12x^2$

$y_1' = \frac{6x^2}{y}$

At (1, 2): $y_1' = 3$

$4x + 6yy_2' = 0$

$y_2' = -\frac{2x}{3y}$

At (1, 2): $y_2' = -\frac{1}{3}$

Since $(y_1')(y_2') = -1$ at (1, 2), the tangents are perpendicular.

43. $dy = [\pi\cos(\pi x) + 2x]dx$; $x = 2$, $dx = 0.01$

$dy = [\pi\cos(2\pi) + 2(2)](0.01) = (4 + \pi)(0.01)$

≈ 0.0714

44. $x(2y)\frac{dy}{dx} + y^2 + 2y[2(x+2)] + (x+2)^2(2)\frac{dy}{dx} = 0$

$\frac{dy}{dx}[2xy + 2(x+2)^2] = -[y^2 + 2y(2x+4)]$

$\frac{dy}{dx} = \frac{-(y^2 + 4xy + 8y)}{2xy + 2(x+2)^2}$

$dy = -\frac{y^2 + 4xy + 8y}{2xy + 2(x+2)^2}dx$

When $x = -2$, $y = \pm 1$

a. $dy = -\frac{(1)^2 + 4(-2)(1) + 8(1)}{2(-2)(1) + 2(-2+2)^2}(-0.01)$

$= -0.0025$

b. $dy = -\frac{(-1)^2 + 4(-2)(-1) + 8(-1)}{2(-2)(-1) + 2(-2+2)^2}(-0.01)$

$= 0.0025$

45. a. $\frac{d}{dx}[f^2(x) + g^3(x)]$

$= 2f(x)f'(x) + 3g^2(x)g'(x)$

$2f(2)f'(2) + 3g^2(2)g'(2)$

$= 2(3)(4) + 3(2)^2(5) = 84$

b. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

$f(2)g'(2) + g(2)f'(2) = (3)(5) + (2)(4) = 23$

c. $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

$f'(g(2))g'(2) = f'(2)g'(2) = (4)(5) = 20$

d. $D_x[f^2(x)] = 2f(x)f'(x)$

$D_x^2[f^2(x)] = 2[f(x)f''(x) + f'(x)f'(x)]$

$= 2f(2)f''(2) + 2[f'(2)]^2$

$= 2(3)(-1) + 2(4)^2 = 26$

46. $(13)^2 = x^2 + y^2$; $\frac{dx}{dt} = 2$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$

When $y = 5$, $x = 12$, so

$\frac{dy}{dt} = -\frac{12}{5}(2) = -\frac{24}{5} = -4.8$ ft/s

47. $\sin 15° = \frac{y}{x}, \frac{dx}{dt} = 400$

$y = x\sin 15°$

$\frac{dy}{dt} = \sin 15° \frac{dx}{dt}$

$\frac{dy}{dt} = 400\sin 15° \approx 104$ mi/hr

48. a. $D_x(|x|^2) = 2|x| \cdot \frac{|x|}{x} = \frac{2(|x|^2)}{x} = \frac{2x^2}{x} = 2x$

b. $D_x^2|x| = D_x\left(\frac{|x|}{x}\right) = \frac{x\left(\frac{|x|}{x}\right) - |x|}{x^2} = \frac{|x| - |x|}{x^2} = 0$

c. $D_x^3|x| = D_x(D_x^2|x|) = D_x(0) = 0$

d. $D_x^2(|x|^2) = D_x(2x) = 2$

49. a. $D_\theta|\sin\theta| = \dfrac{|\sin\theta|}{\sin\theta}\cos\theta = \cot\theta|\sin\theta|$

b. $D_\theta|\cos\theta| = \dfrac{|\cos\theta|}{\cos\theta}(-\sin\theta) = -\tan\theta|\cos\theta|$

50. a. $f(x) = \sqrt{x+1};\ f'(x) = -\frac{1}{2}(x+1)^{-1/2};\ a = 3$

$$L(x) = f(3) + f'(3)(x-3)$$
$$= \sqrt{4} + -\frac{1}{2}(4)^{-1/2}(x-3)$$
$$= 2 - \frac{1}{4}x + \frac{3}{4} = -\frac{1}{4}x + \frac{11}{4}$$

b. $f(x) = x\cos x;\ f'(x) = -x\sin x + \cos x;\ a = 1$

$$L(x) = f(1) + f'(1)(x-1)$$
$$= \cos 1 + (-\sin 1 + \cos 1)(x-1)$$
$$= \cos 1 - (\sin 1)x + \sin 1 + (\cos 1)x - \cos 1$$
$$= (\cos 1 - \sin 1)x + \sin 1$$
$$\approx -0.3012x + 0.8415$$

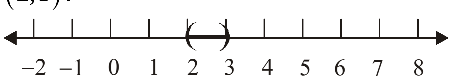
Review and Preview Problems

1. $(x-2)(x-3) < 0$

$(x-2)(x-3) = 0$

$x = 2$ or $x = 3$

The split points are 2 and 3. The expression on the left can only change signs at the split points. Check a point in the intervals $(-\infty, 2)$, $(2,3)$, and $(3,\infty)$. The solution set is $\{x \mid 2 < x < 3\}$ or $(2,3)$.



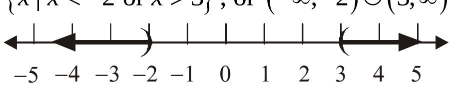
2. $x^2 - x - 6 > 0$

$(x-3)(x+2) > 0$

$(x-3)(x+2) = 0$

$x = 3$ or $x = -2$

The split points are 3 and -2. The expression on the left can only change signs at the split points. Check a point in the intervals $(-\infty, -2)$, $(-2, 3)$, and $(3,\infty)$. The solution set is $\{x \mid x < -2 \text{ or } x > 3\}$, or $(-\infty, -2) \cup (3, \infty)$.

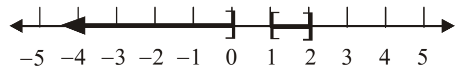


3. $x(x-1)(x-2) \le 0$

$x(x-1)(x-2) = 0$

$x = 0, x = 1$ or $x = 2$

The split points are 0, 1, and 2. The expression on the left can only change signs at the split points. Check a point in the intervals $(-\infty, 0)$, $(0,1)$, $(1,2)$, and $(2,\infty)$. The solution set is $\{x \mid x \le 0 \text{ or } 1 \le x \le 2\}$, or $(-\infty, 0] \cup [1, 2]$.



4. $x^3 + 3x^2 + 2x \ge 0$

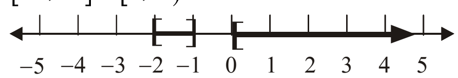
$x(x^2 + 3x + 2) \ge 0$

$x(x+1)(x+2) \ge 0$

$x(x+1)(x+2) = 0$

$x = 0, x = -1, x = -2$

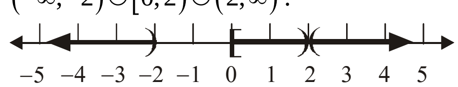
The split points are 0, -1, and -2. The expression on the left can only change signs at the split points. Check a point in the intervals $(-\infty, -2)$, $(-2,-1)$, $(-1, 0)$, and $(0,\infty)$. The solution set is $\{x \mid -2 \le x \le -1 \text{ or } x \ge 0\}$, or $[-2, -1] \cup [0, \infty)$.



5. $\dfrac{x(x-2)}{x^2-4} \ge 0$

$\dfrac{x(x-2)}{(x-2)(x+2)} \ge 0$

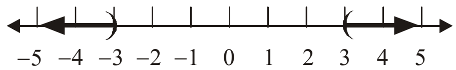
The expression on the left is equal to 0 or undefined at $x = 0$, $x = 2$, and $x = -2$. These are the split points. The expression on the left can only change signs at the split points. Check a point in the intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. The solution set is $\{x \mid x < -2 \text{ or } 0 \le x < 2 \text{ or } x > 2\}$, or $(-\infty, -2) \cup [0, 2) \cup (2, \infty)$.



6. $$\frac{x^2-9}{x^2+2}>0$$

$$\frac{(x-3)(x+3)}{x^2+2}>0$$

The expression on the left is equal to 0 at $x=3$, and $x=-3$. These are the split points. The expression on the left can only change signs at the split points. Check a point in the intervals: $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$. The solution set is $\{x \mid x<-3 \text{ or } x>3\}$, or $(-\infty,-3)\cup(3,\infty)$.



7. $f'(x)=4(2x+1)^3(2)=8(2x+1)^3$

8. $f'(x)=\cos(\pi x)\cdot\pi=\pi\cos(\pi x)$

9. $f'(x)=(x^2-1)\cdot -\sin(2x)\cdot 2+\cos(2x)\cdot(2x)$
$=-2(x^2-1)\sin(2x)+2x\cos(2x)$

10. $f'(x)=\dfrac{x\cdot\sec x\tan x-\sec x\cdot 1}{x^2}$
$=\dfrac{\sec x(x\tan x-1)}{x^2}$

11. $f'(x)=2(\tan 3x)\cdot\sec^2 3x\cdot 3$
$=6(\sec^2 3x)(\tan 3x)$

12. $f'(x)=\dfrac{1}{2}(1+\sin^2 x)^{-1/2}(2\sin x)(\cos x)$
$=\dfrac{\sin x\cos x}{\sqrt{1+\sin^2 x}}$

13. $f'(x)=\cos(\sqrt{x})\cdot\dfrac{1}{2}x^{-1/2}=\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$

(note: you cannot cancel the $\sqrt{x}$ here because it is not a factor of both the numerator and denominator. It is the argument for the cosine in the numerator.)

14. $f'(x)=\dfrac{1}{2}(\sin 2x)^{-1/2}\cdot\cos 2x\cdot 2=\dfrac{\cos 2x}{\sqrt{\sin 2x}}$

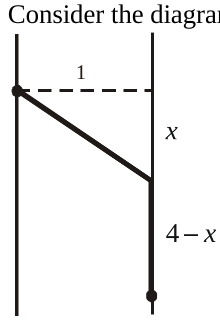
15. The tangent line is horizontal when the derivative is 0.
$y'=2\tan x\cdot\sec^2 x$
$2\tan x\sec x=0$
$\dfrac{2\sin x}{\cos^2 x}=0$
The tangent line is horizontal whenever $\sin x=0$. That is, for $x=k\pi$ where k is an integer.

16. The tangent line is horizontal when the derivative is 0.
$y'=1+\cos x$
The tangent line is horizontal whenever $\cos x=-1$. That is, for $x=(2k+1)\pi$ where k is an integer.

17. The line $y=2+x$ has slope 1, so any line parallel to this line will also have a slope of 1.
For the tangent line to $y=x+\sin x$ to be parallel to the given line, we need its derivative to equal 1.
$y'=1+\cos x=1$
$\cos x=0$
The tangent line will be parallel to $y=2+x$ whenever $x=(2k+1)\dfrac{\pi}{2}$.

18. Length: $24-2x$
Width: $9-2x$
Height: x
Volume: $l\cdot w\cdot h=(24-2x)(9-2x)x$
$=x(9-2x)(24-2x)$

19. Consider the diagram:



His distance swimming will be $\sqrt{1^2+x^2}=\sqrt{x^2+1}$ kilometers. His distance running will be $4-x$ kilometers.
Using the distance traveled formula, $d=r\cdot t$, we solve for t to get $t=\dfrac{d}{r}$. Andy can swim at 4 kilometers per hour and run 10 kilometers per hour. Therefore, the time to get from A to D will be $\dfrac{\sqrt{x^2+1}}{4}+\dfrac{4-x}{10}$ hours.

20. **a.** $f(0) = 0 - \cos(0) = 0 - 1 = -1$

$f(\pi) = \pi - \cos(\pi) = \pi - (-1) = \pi + 1$

Since $x - \cos x$ is continuous, $f(0) < 0$, and $f(\pi) > 0$, there is at least one point c .in the interval $(0, \pi)$ where $f(c) = 0$. (Intermediate Value Theorem)

b. $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

$f'(x) = 1 + \sin x$

$f'\left(\frac{\pi}{2}\right) = 1 + \sin\left(\frac{\pi}{2}\right) = 1 + 1 = 2$

The slope of the tangent line is $m = 2$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore,

$y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{2}\right)$ or $y = 2x - \frac{\pi}{2}$.

c. $2x - \frac{\pi}{2} = 0$.

$2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$

The tangent line will intersect the x-axis at $x = \frac{\pi}{4}$.

21. **a.** The derivative of x^2 is $2x$ and the derivative of a constant is 0. Therefore, one possible function is $f(x) = x^2 + 3$.

b. The derivative of $-\cos x$ is $\sin x$ and the derivative of a constant is 0. Therefore, one possible function is $f(x) = -(\cos x) + 8$.

c. The derivative of x^3 is $3x^2$, so the derivative of $\frac{1}{3}x^3$ is x^2. The derivative of x^2 is $2x$, so the derivative of $\frac{1}{2}x^2$ is x. The derivative of x is 1, and the derivative of a constant is 0. Therefore, one possible function is $\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 2$.

22. Yes. Adding 1 only changes the constant term in the function and the derivative of a constant is 0. Therefore, we would get the same derivative regardless of the value of the constant.