CHAPTER 4 The Definite Integral

4.1 Concepts Review

1. $2\cdot\frac{5(6)}{2}=30; 2(5)=10$

2. 3(9) – 2(7) = 13; 9 + 4(10) = 49

3. inscribed; circumscribed

4. $0+1+2+3=6$

Problem Set 4.1

1. $\sum_{k=1}^{6}(k-1)=\sum_{k=1}^{6}k-\sum_{k=1}^{6}1$
$=\frac{6(7)}{2}-6(1)$
$=15$

2. $\sum_{i=1}^{6}i^2=\frac{6(7)(13)}{6}=91$

3. $\sum_{k=1}^{7}\frac{1}{k+1}=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}$
$+\frac{1}{4+1}+\frac{1}{5+1}+\frac{1}{6+1}+\frac{1}{7+1}$
$=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$
$=\frac{1443}{840}$
$=\frac{481}{280}$

4. $\sum_{l=3}^{8}(l+1)^2=4^2+5^2+6^2+7^2+8^2+9^2=271$

5. $\sum_{m=1}^{8}(-1)^m 2^{m-2}$
$=(-1)^1 2^{-1}+(-1)^2 2^0+(-1)^3 2^1$
$+(-1)^4 2^2+(-1)^5 2^3+(-1)^6 2^4$
$+(-1)^7 2^5+(-1)^8 2^6$
$=-\frac{1}{2}+1-2+4-8+16-32+64$
$=\frac{85}{2}$

6. $\sum_{k=3}^{7}\frac{(-1)^k 2^k}{(k+1)}$
$=\frac{(-1)^3 2^3}{4}+\frac{(-1)^4 2^4}{5}$
$+\frac{(-1)^5 2^5}{6}+\frac{(-1)^6 2^6}{7}+\frac{(-1)^7 2^7}{8}$
$=-\frac{1154}{105}$

7. $\sum_{n=1}^{6}n\cos(n\pi)=\sum_{n=1}^{6}(-1)^n\cdot n$
$=-1+2-3+4-5+6$
$=3$

8. $\sum_{k=-1}^{6}k\sin\left(\frac{k\pi}{2}\right)$
$=-\sin\left(-\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)+2\sin(\pi)$
$+3\sin\left(\frac{3\pi}{2}\right)+4\sin(2\pi)\ +5\sin\left(\frac{5\pi}{2}\right)+6\sin(3\pi)$
$=1+1+0-3+0+5+0$
$=4$

9. $1+2+3+\cdots+41=\sum_{i=1}^{41}i$

10. $2+4+6+8+\cdots+50=\sum_{i=1}^{25}2i$

11. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}=\sum_{i=1}^{100}\frac{1}{i}$

12. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{100}=\sum_{i=1}^{100}\frac{(-1)^{i+1}}{i}$

13. $a_1+a_3+a_5+a_7+\cdots+a_{99}=\sum_{i=1}^{50}a_{2i-1}$

14. $f(w_1)\Delta x+f(w_2)\Delta x+\cdots+f(w_n)\Delta x$

$=\sum_{i=1}^{n}f(w_i)\Delta x$

15. $\sum_{i=1}^{10}(a_i+b_i)$

$=\sum_{i=1}^{10}a_i+\sum_{i=1}^{10}b_i$

$=40+50$

$=90$

16. $\sum_{n=1}^{10}(3a_n+2b_n)$

$=3\sum_{n=1}^{10}a_n+2\sum_{n=1}^{10}b_n$

$= 3(40) + 2(50)$

$= 220$

17. $\sum_{p=0}^{9}(a_{p+1}-b_{p+1})$

$=\sum_{p=1}^{10}a_p-\sum_{p=1}^{10}b_p$

$=40-50$

$=-10$

18. $\sum_{q=1}^{10}(a_q-b_q-q)$

$=\sum_{q=1}^{10}a_q-\sum_{q=1}^{10}b_q-\sum_{q=1}^{10}q$

$=40-50-\frac{10(11)}{2}$

$=-65$

19. $\sum_{i=1}^{100}(3i-2)$

$=3\sum_{i=1}^{100}i-\sum_{i=1}^{100}2$

$=3(5050)-2(100)$

$= 14,950$

20. $\sum_{i=1}^{10}\left[(i-1)(4i+3)\right]$

$=\sum_{i=1}^{10}(4i^2-i-3)$

$=4\sum_{i=1}^{10}i^2-\sum_{i=1}^{10}i-\sum_{i=1}^{10}3$

$= 4(385) – 55 – 3(10)$

$= 1455$

21. $\sum_{k=1}^{10}(k^3-k^2)=\sum_{k=1}^{10}k^3-\sum_{k=1}^{10}k^2$

$=3025-385$

$= 2640$

22. $\sum_{k=1}^{10}5k^2(k+4)=\sum_{k=1}^{10}(5k^3+20k^2)$

$=5\sum_{k=1}^{10}k^3+20\sum_{k=1}^{10}k^2$

$= 5(3025) + 20(385)$

$= 22,825$

23. $\sum_{i=1}^{n}(2i^2-3i+1)=2\sum_{i=1}^{n}i^2-3\sum_{i=1}^{n}i+\sum_{i=1}^{n}1$

$=\frac{2n(n+1)(2n+1)}{6}-\frac{3n(n+1)}{2}+n$

$=\frac{2n^3+3n^2+n}{3}-\frac{3n^2+3n}{2}+n$

$=\frac{4n^3-3n^2-n}{6}$

24. $\sum_{i=1}^{n}(2i-3)^2=\sum_{i=1}^{n}(4i^2-12i+9)$

$=4\sum_{i=1}^{n}i^2-12\sum_{i=1}^{n}i+\sum_{i=1}^{n}9$

$=\frac{4n(n+1)(2n+1)}{6}-\frac{12n(n+1)}{2}+9n$

$=\frac{4n^3-12n^2+11n}{3}$

25.
$$\begin{aligned} S &= 1+2+3+\cdots+(n-2)+(n-1)+n \\ +\,S &= n+(n-1)+(n-2)+\cdots+3+2+1 \\ \hline 2S &= (n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+1) \end{aligned}$$

$2S = n(n + 1)$

$S=\frac{n(n+1)}{2}$

26. $S - rS = a + ar + ar^2 + \cdots + ar^n$

$$- \underline{\quad (ar + ar^2 + \cdots + ar^n + ar^{n+1})}$$

$$= a - ar^{n+1}$$

$$= S(1-r);\ S = \frac{a - ar^{n+1}}{1-r}$$

27. a. $\sum_{k=0}^{10}\left(\frac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^{11}}{\frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{10}$, so

$$\sum_{k=1}^{10}\left(\frac{1}{2}\right)^k = 1 - \left(\frac{1}{2}\right)^{10} = \frac{1023}{1024}.$$

b. $\sum_{k=0}^{10} 2^k = \frac{1-2^{11}}{-1} = 2^{11} - 1$, so

$$\sum_{k=1}^{10} 2^k = 2^{11} - 2 = 2046.$$

28.

$$\begin{aligned}
&S = a + (a+d) + (a+2d) + \cdots\cdots + [a+(n-2)d] + [a+(n-1)d] + (a+nd)\\
&+\underline{S = (a+nd) + [a+(n-1)d] + [a+(n-2)d] + \cdots + (a+2d) + (a+d) + a}\\
&2S = (2a+nd) + (2a+nd) + (2a+nd) + \cdots + (2a+nd) + (2a+nd) + (2a+nd)\\
&2S = (n+1)(2a+nd)\\
&S = \frac{(n+1)(2a+nd)}{2}
\end{aligned}$$

29.

$$\begin{aligned}
(i+1)^3 - i^3 &= 3i^2 + 3i + 1\\
\sum_{i=1}^{n}\left[(i+1)^3 - i^3\right] &= \sum_{i=1}^{n}\left(3i^2 + 3i + 1\right)\\
(n+1)^3 - 1^3 &= 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\\
n^3 + 3n^2 + 3n &= 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n\\
2n^3 + 6n^2 + 6n &= 6\sum_{i=1}^{n} i^2 + 3n^2 + 3n + 2n\\
\frac{2n^3 + 3n^2 + n}{6} &= \sum_{i=1}^{n} i^2\\
\frac{n(n+1)(2n+1)}{6} &= \sum_{i=1}^{n} i^2
\end{aligned}$$

30. $$(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1$$

$$\sum_{i=1}^{n}\left[(i+1)^4 - i^4\right] = \sum_{i=1}^{n}\left(4i^3 + 6i^2 + 4i + 1\right)$$

$$(n+1)^4 - 1^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum_{i=1}^{n} i^3 + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n$$

Solving for $\sum_{i=1}^{n} i^3$ gives

$$4\sum_{i=1}^{n} i^3 = n^4 + 4n^3 + 6n^2 + 4n - \left(2n^3 + 3n^2 + n\right) - \left(2n^2 + 2n\right) - n$$

$$4\sum_{i=1}^{n} i^3 = n^4 + 2n^3 + n^2$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + 2n^3 + n^2}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

31. $$(i+1)^5 - i^5 = 5i^4 + 10i^3 + 10i^2 + 5i + 1$$

$$\sum_{i=1}^{n}\left[(i+1)^5 - i^5\right] = 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$(n+1)^5 - 1^5 = 5\sum_{i=1}^{n} i^4 + 10\frac{n^2(n+1)^2}{4} + 10\frac{n(n+1)(2n+1)}{6} + 5\frac{n(n+1)}{2} + n$$

$$n^5 + 5n^4 + 10n^3 + 10n^2 + 5n = 5\sum_{i=1}^{n} i^4 + \tfrac{5}{2}n^2(n+1)^2 + \tfrac{10}{6}n(n+1)(2n+1) + \frac{5}{2}n(n+1) + n$$

Solving for $\sum_{i=1}^{n} i^4$ yields

$$\sum_{i=1}^{n} i^4 = \tfrac{1}{5}\left[n^5 + \tfrac{5}{2}n^4 + \tfrac{5}{3}n^3 - \tfrac{1}{6}n\right] = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

32. Suppose we have a $(n+1)\times n$ grid. Shade in $n + 1 - k$ boxes in the kth column. There are n columns, and the shaded area is $1 + 2 + \cdots + n$. The shaded area is also half the area of the grid or $\frac{n(n+1)}{2}$. Thus, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Suppose we have a square grid with sides of length $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. From the diagram the area is $1^3 + 2^3 + \cdots + n^3$ or $\left[\frac{n(n+1)}{2}\right]^2$. Thus, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

33. $\bar{x} = \frac{1}{7}(2 + 5 + 7 + 8 + 9 + 10 + 14) = \frac{55}{7} \approx 7.86$

$$s^2 = \frac{1}{7}\left[\left(2 - \frac{55}{7}\right)^2 + \left(5 - \frac{55}{7}\right)^2 + \left(7 - \frac{55}{7}\right)^2 + \left(8 - \frac{55}{7}\right)^2 + \left(9 - \frac{55}{7}\right)^2 + \left(10 - \frac{55}{7}\right)^2 + \left(14 - \frac{55}{7}\right)^2\right] = \frac{608}{49} \approx 12.4$$

34. a. $\overline{x} = 1,\ s^2 = 0$

b. $\overline{x} = 1001,\ s^2 = 0$

c. $\overline{x} = 2$

$$s^2 = \tfrac{1}{3}\left[(1-2)^2 + (2-2)^2 + (3-2)^2\right] = \tfrac{1}{3}\left[(-1)^2 + 0^2 + 1^2\right] = \tfrac{1}{3}(2) = \tfrac{2}{3}$$

d. $\overline{x} = 1,000,002$

$$s^2 = \tfrac{1}{3}\left[(-1)^2 + 0^2 + 1^2\right] = \tfrac{2}{3}$$

35. a. $\sum_{i=1}^{n}(x_i - \overline{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\overline{x} = n\overline{x} - n\overline{x} = 0$

b. $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i^2 - 2\overline{x}\,x_i + \overline{x}^2)$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{2\overline{x}}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n}\overline{x}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{2\overline{x}}{n}(n\overline{x}) + \frac{1}{n}(n\overline{x}^2)$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - 2\overline{x}^2 + \overline{x}^2 = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \overline{x}^2$$

36. The variance of n identical numbers is 0. Let c be the constant. Then

$$s^2 = \tfrac{1}{n}\left[(c-c)^2 + (c-c)^2 + \cdots + (c-c)^2\right] = 0$$

37. Let $S(c) = \sum_{i=1}^{n}(x_i - c)^2$. Then

$$S'(c) = \frac{d}{dc}\sum_{i=1}^{n}(x_i - c)^2$$

$$= \sum_{i=1}^{n}\frac{d}{dc}(x_i - c)^2$$

$$= \sum_{i=1}^{n} 2(x_i - c)(-1)$$

$$= -2\sum_{i=1}^{n} x_i + 2nc$$

$S''(c) = 2n$

Set $S'(c) = 0$ and solve for c:

$$-2\sum_{i=1}^{n} x_i + 2nc = 0$$

$$c = \tfrac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

Since $S''(\overline{x}) = 2n > 0$ we know that $\overline{x}$ minimizes $S(c)$.

38. a. The number of gifts given on the nth day is $\sum_{m=1}^{i} m = \frac{i(i+1)}{2}$.

The total number of gifts is $\sum_{i=1}^{12} \frac{i(i+1)}{2} = 364$.

b. For n days, the total number of gifts is $\sum_{i=1}^{n} \frac{i(i+1)}{2}$.

$$\sum_{i=1}^{n} \frac{i(i+1)}{2} = \sum_{i=1}^{n} \frac{i^2}{2} + \sum_{i=1}^{n} \frac{i}{2} = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{2}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{1}{4}n(n+1)\left(\frac{2n+1}{3}+1\right) = \frac{1}{12}n(n+1)(2n+4) = \frac{1}{6}n(n+1)(n+2)$$

39. The bottom layer contains 10 · 16 = 160 oranges, the next layer contains 9 · 15 = 135 oranges, the third layer contains 8 · 14 = 112 oranges, and so on, up to the top layer, which contains 1 · 7 = 7 oranges. The stack contains

1 · 7 + 2 · 8+ ··· + 9· 15 + 10 · 16

$= \sum_{i=1}^{10} i(6+i) = 715$ oranges.

40. If the bottom layer is 50 oranges by 60 oranges, the stack contains $\sum_{i=1}^{50} i(10+i) = 55,675.$

41. For a general stack whose base is m rows of n oranges with $m \le n$, the stack contains

$$\sum_{i=1}^{m} i(n-m+i) = (n-m)\sum_{i=1}^{m} i + \sum_{i=1}^{m} i^2$$

$$= (n-m)\frac{m(m+1)}{2} + \frac{m(m+1)(2m+1)}{6}$$

$$= \frac{m(m+1)(3n-m+1)}{6}$$

42. $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

43. $A = \frac{1}{2}\left[1+\frac{3}{2}+2+\frac{5}{2}\right] = \frac{7}{2}$

44. $A = \frac{1}{4}\left[1+\frac{5}{4}+\frac{3}{2}+\frac{7}{4}+2+\frac{9}{4}+\frac{5}{2}+\frac{11}{4}\right] = \frac{15}{4}$

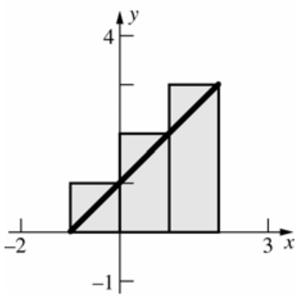
45. $A = \frac{1}{2}\left[\frac{3}{2}+2+\frac{5}{2}+3\right] = \frac{9}{2}$

46. $A = \frac{1}{4}\left[\frac{5}{4}+\frac{3}{2}+\frac{7}{4}+2+\frac{9}{4}+\frac{5}{2}+\frac{11}{4}+3\right] = \frac{17}{4}$

47. $A=\frac{1}{2}\left[\left(\frac{1}{2}\cdot 0^2+1\right)+\left(\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2+1\right)+\left(\frac{1}{2}\cdot 1^2+1\right)+\left(\frac{1}{2}\cdot\left(\frac{3}{2}\right)^2+1\right)\right]=\frac{1}{2}\left(1+\frac{9}{8}+\frac{3}{2}+\frac{17}{8}\right)=\frac{23}{8}$

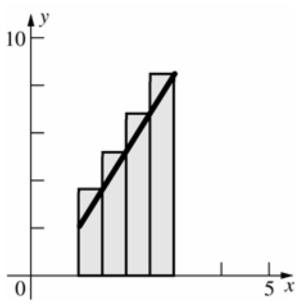
48. $A=\frac{1}{2}\left[\left(\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2+1\right)+\left(\frac{1}{2}\cdot 1^2+1\right)+\left(\frac{1}{2}\cdot\left(\frac{3}{2}\right)^2+1\right)+\left(\frac{1}{2}\cdot 2^2+1\right)\right]=\frac{1}{2}\left(\frac{9}{8}+\frac{3}{2}+\frac{17}{8}+3\right)=\frac{31}{8}$

49.



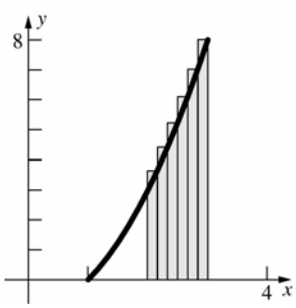
$A = 1(1 + 2 + 3) = 6$

50.



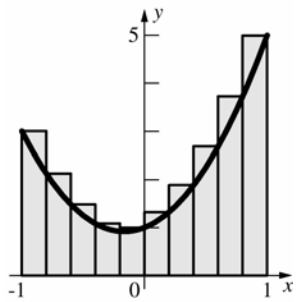
$A=\frac{1}{2}\left[\left(3\cdot\frac{3}{2}-1\right)+(3\cdot 2-1)+\left(3\cdot\frac{5}{2}-1\right)+(3\cdot 3-1)\right]=\frac{1}{2}\left(\frac{7}{2}+5+\frac{13}{2}+8\right)=\frac{23}{2}$

51.



$A=\frac{1}{6}\left[\left(\left(\frac{13}{6}\right)^2-1\right)+\left(\left(\frac{7}{3}\right)^2-1\right)+\left(\left(\frac{5}{2}\right)^2-1\right)+\left(\left(\frac{8}{3}\right)^2-1\right)+\left(\left(\frac{17}{6}\right)^2-1\right)+(3^2-1)\right]$

$=\frac{1}{6}\left(\frac{133}{36}+\frac{40}{9}+\frac{21}{4}+\frac{55}{9}+\frac{253}{36}+8\right)=\frac{1243}{216}$

52. 

$$A=\frac{1}{5}\left[(3(-1)^2+(-1)+1)+\left(3\left(-\frac{4}{5}\right)^2+\left(-\frac{4}{5}\right)+1\right)+\left(3\left(-\frac{3}{5}\right)^2+\left(-\frac{3}{5}\right)+1\right)+\left(3\left(-\frac{2}{5}\right)^2+\left(-\frac{2}{5}\right)+1\right)+(3(0)^2+0+1)\right.$$

$$\left.+\left(3\left(\frac{1}{5}\right)^2+\frac{1}{5}+1\right)+\left(3\left(\frac{2}{5}\right)^2+\frac{2}{5}+1\right)+\left(3\left(\frac{3}{5}\right)^2+\frac{3}{5}+1\right)\left(3\left(\frac{4}{5}\right)^2+\frac{4}{5}+1\right)+(3(1)^2+1+1)\right]$$

$$=\frac{1}{5}[3+2.12+1.48+1.08+1+1.32+1.88+2.68+3.72+5]=4.656$$

53. $\Delta x=\frac{1}{n},\ x_i=\frac{i}{n}$

$$f(x_i)\Delta x=\left(\frac{i}{n}+2\right)\left(\frac{1}{n}\right)=\frac{i}{n^2}+\frac{2}{n}$$

$$A(S_n)=\left[\left(\frac{1}{n^2}+\frac{2}{n}\right)+\left(\frac{2}{n^2}+\frac{2}{n}\right)+\cdots+\left(\frac{n}{n^2}+\frac{2}{n}\right)\right]=\frac{1}{n^2}(1+2+3+\cdots+n)+2\ =\frac{n(n+1)}{2n^2}+2\ =\frac{1}{2n}+\frac{5}{2}$$

$$\lim_{n\to\infty}A(S_n)=\lim_{n\to\infty}\left(\frac{1}{2n}+\frac{5}{2}\right)=\frac{5}{2}$$

54. $\Delta x=\frac{1}{n},\ x_i=\frac{i}{n}$

$$f(x_i)\Delta x=\left[\frac{1}{2}\cdot\left(\frac{i}{n}\right)^2+1\right]\left(\frac{1}{n}\right)=\frac{i^2}{2n^3}+\frac{1}{n}$$

$$A(S_n)=\left[\left(\frac{1^2}{2n^3}+\frac{1}{n}\right)+\left(\frac{2^2}{2n^3}+\frac{1}{n}\right)+\cdots+\left(\frac{n^2}{2n^3}+\frac{1}{n}\right)\right]=\frac{1}{2n^3}(1^2+2^2+3^2+\cdots+n^2)+1$$

$$=\frac{1}{2n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+1\ =\frac{1}{12}\left[\frac{2n^3+3n^2+n}{n^3}\right]+1\ =\frac{1}{12}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]+1$$

$$\lim_{n\to\infty}A(S_n)=\lim_{n\to\infty}\left[\frac{1}{12}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+1\right]=\frac{7}{6}$$

55. $\Delta x = \frac{2}{n},\ x_i = -1 + \frac{2i}{n}$

$$f(x_i)\Delta x = \left[2\left(-1+\frac{2i}{n}\right)+2\right]\left(\frac{2}{n}\right) = \frac{8i}{n^2}$$

$$A(S_n) = \left[\left(\frac{8}{n^2}\right)+\left(\frac{16}{n^2}\right)+\cdots+\left(\frac{8n}{n^2}\right)\right]$$

$$= \frac{8}{n^2}(1+2+3+\cdots+n) \quad = \frac{8}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 4\left[\frac{n^2+n}{n^2}\right] \quad = 4 + \frac{4}{n}$$

$$\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty}\left(4+\frac{4}{n}\right) = 4$$

56. First, consider $a = 0$ and $b = 2$.

$\Delta x = \frac{2}{n},\ x_i = \frac{2i}{n}$

$$f(x_i)\Delta x = \left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \frac{8i^2}{n^3}$$

$$A(S_n) = \left[\left(\frac{8}{n^3}\right)+\left(\frac{8(2^2)}{n^3}\right)+\cdots+\left(\frac{8n^2}{n^3}\right)\right]$$

$$= \frac{8}{n^3}(1^2+2^2+\cdots+n^2) = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{4}{3}\left[\frac{2n^3+3n^2+n}{n^3}\right] = \frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}$$

$$\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}\right) = \frac{8}{3}.$$

By symmetry, $A = 2\left(\frac{8}{3}\right) = \frac{16}{3}$.

57. $\Delta x = \frac{1}{n},\ x_i = \frac{i}{n}$

$$f(x_i)\Delta x = \left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \frac{i^3}{n^4}$$

$$A(S_n) = \left[\frac{1}{n^4}(1^3)+\frac{1}{n^4}(2^3)+\cdots+\frac{1}{n^4}(n^3)\right]$$

$$= \frac{1}{n^4}(1^3+2^3+\cdots+n^3) \quad = \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

$$= \frac{1}{n^4}\left[\frac{n^4+2n^3+n^2}{4}\right] \quad = \frac{1}{4}\left[1+\frac{2}{n}+\frac{1}{n^2}\right]$$

$$\lim_{n\to\infty} A(S_n) = \lim_{n\to\infty}\frac{1}{4}\left[1+\frac{2}{n}+\frac{1}{n^2}\right] = \frac{1}{4}$$

58. $\Delta x = \frac{1}{n},\ x_i = \frac{i}{n}$

$$f(x_i)\Delta x = \left[\left(\frac{i}{n}\right)^3+\frac{i}{n}\right]\left(\frac{1}{n}\right) = \frac{i^3}{n^4}+\frac{i}{n^2}$$

$$A(S_n) = \frac{1}{n^4}(1^3+2^3+\cdots+n^3)+\frac{1}{n^2}(1+2+\cdots+n)$$

$$= \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2+\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{n^2+2n+1}{4n^2}+\frac{n^2+n}{2n^2} = \frac{3n^2+4n+1}{4n^2} = \frac{3}{4}+\frac{1}{n}+\frac{1}{4n^2}$$

$$\lim_{n\to\infty} A(S_n) = \frac{3}{4}$$

59. $f(t_i)\Delta t = \left[\frac{i}{n}+2\right]\frac{1}{n} = \frac{i}{n^2}+\frac{2}{n}$

$$A(S_n) = \sum_{i=1}^{n}\left(\frac{i}{n^2}+\frac{2}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} i + \sum_{i=1}^{n}\frac{2}{n}$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]+2$$

$$= \left[\frac{n^2+n}{2n^2}\right]+2$$

$$= \left(\frac{1}{2}+\frac{1}{2n}\right)+2$$

$$\lim_{n\to\infty} A(S_n) = \frac{1}{2}+2 = \frac{5}{2}$$

The object traveled $2\frac{1}{2}$ ft.

60. $f(t_i)\Delta t = \left[\frac{1}{2}\left(\frac{i}{n}\right)^2+1\right]\frac{1}{n} = \frac{i^2}{2n^3}+\frac{1}{n}$

$$A(S_n) = \sum_{i=1}^{n}\left(\frac{1i^2}{2n^3}+\frac{1}{n}\right) = \frac{1}{2n^3}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\frac{1}{n}$$

$$= \frac{1}{2n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+1 \quad = \frac{1}{12}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]+1$$

$$\lim_{n\to\infty} A(S_n) = \frac{1}{12}(2)+1 = \frac{7}{6} \approx 1.17$$

The object traveled about 1.17 feet.

61. a. $f(x_i)\Delta x = \left(\frac{ib}{n}\right)^2\left(\frac{b}{n}\right) = \frac{b^3 i^2}{n^3}$

$A_0^b = \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 = \frac{b^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{b^3}{6}\left[2 + \frac{3}{n} + \frac{1}{n^2}\right]$

$\lim_{n\to\infty} A_0^b = \frac{2b^3}{6} = \frac{b^3}{3}$

b. Since $a \geq 0$, $A_0^b = A_0^a + A_a^b$, or

$A_a^b = A_0^b - A_0^a = \frac{b^3}{3} - \frac{a^3}{3}$.

62. $A_3^5 = \frac{5^3}{3} - \frac{3^3}{3} = \frac{98}{3} \approx 32.7$

The object traveled about 32.7 m.

63. a. $A_0^5 = \frac{5^3}{3} = \frac{125}{3}$

b. $A_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$

c. $A_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{117}{3} = 39$

64. a. $\Delta x = \frac{b}{n}, x_i = \frac{bi}{n}$

$f(x_i)\Delta x = \left(\frac{bi}{n}\right)^m\left(\frac{b}{n}\right) = \frac{b^{m+1}i^m}{n^{m+1}}$

$A(S_n) = \frac{b^{m+1}}{n^{m+1}}\sum_{i=1}^{n} i^m$

$= \frac{b^{m+1}}{n^{m+1}}\left[\frac{n^{m+1}}{m+1} + C_n\right]$

$= \frac{b^{m+1}}{m+1} + \frac{b^{m+1}C_n}{n^{m+1}}$

$A_0^b(x^m) = \lim_{n\to\infty} A(S_n) = \frac{b^{m+1}}{m+1}$

$\lim_{n\to\infty}\frac{C_n}{n^{m+1}} = 0$ since C_n is a polynomial in n of degree m.

b. Notice that $A_a^b(x^m) = A_0^b(x^m) - A_0^a(x^m)$.

Thus, using part a, $A_a^b(x^m) = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}$.

65. a. $A_0^2(x^3) = \frac{2^{3+1}}{3+1} = 4$

b. $A_1^2(x^3) = \frac{2^{3+1}}{3+1} - \frac{1^{3+1}}{3+1} = 4 - \frac{1}{4} = \frac{15}{4}$

c. $A_1^2(x^5) = \frac{2^{5+1}}{5+1} - \frac{1^{5+1}}{5+1} = \frac{32}{3} - \frac{1}{6} = \frac{63}{6}$

$= \frac{21}{2} = 10.5$

d. $A_0^2(x^9) = \frac{2^{9+1}}{9+1} = \frac{1024}{10} = 102.4$

66. Inscribed:
Consider an isosceles triangle formed by one side of the polygon and the center of the circle. The angle at the center is $\frac{2\pi}{n}$. The length of the base is $2r\sin\frac{\pi}{n}$. The height is $r\cos\frac{\pi}{n}$. Thus the area of the triangle is $r^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{1}{2}r^2\sin\frac{2\pi}{n}$.

$A_n = n\left(\frac{1}{2}r^2\sin\frac{2\pi}{n}\right) = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$

Circumscribed:
Consider an isosceles triangle formed by one side of the polygon and the center of the circle. The angle at the center is $\frac{2\pi}{n}$. The length of the base is $2r\tan\frac{\pi}{n}$. The height is r. Thus the area of the triangle is $r^2\tan\frac{\pi}{n}$.

$B_n = n\left(r^2\tan\frac{\pi}{n}\right) = nr^2\tan\frac{\pi}{n}$

$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\frac{1}{2}nr^2\sin\frac{2\pi}{n} = \lim_{n\to\infty}\pi r^2\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)$

$= \pi r^2$

$\lim_{n\to\infty} B_n = \lim_{n\to\infty} nr^2\tan\frac{\pi}{n} = \lim_{n\to\infty}\frac{\pi r^2}{\cos\frac{\pi}{n}}\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)$

$= \pi r^2$

4.2 Concepts Review

1. Riemann sum

2. definite integral; $\int_a^b f(x)dx$

3. $A_{\text{up}} - A_{\text{down}}$

4. $8 - \frac{1}{2} = \frac{15}{2}$

Problem Set 4.2

1. $R_P = f(2)(2.5-1) + f(3)(3.5-2.5) + f(4.5)(5-3.5)$ = 4(1.5) + 3(1) + (–2.25)(1.5) = 5.625

2. R_P = f(0.5)(0.7 – 0) + f(1.5)(1.7 – 0.7) + f(2)(2.7 – 1.7) + f(3.5)(4 – 2.7)
= 1.25(0.7) + (–0.75)(1) + (–1)(1) + 1.25(1.3) = 0.75

3. $R_P = \sum_{i=1}^{5} f(\overline{x}_i)\Delta x_i = f(3)(3.75-3) + f(4)(4.25-3.75) + f(4.75)(5.5-4.25) + f(6)(6-5.5) + f(6.5)(7-6)$
= 2(0.75) + 3(0.5) + 3.75(1.25) + 5(0.5) + 5.5(1) = 15.6875

4. $R_P = \sum_{i=1}^{4} f(\overline{x}_i)\Delta x_i = f(-2)(-1.3+3) + f(-0.5)(0+1.3) + f(0)(0.9-0) + f(2)(2-0.9)$
= 4(1.7) + 3.25(1.3) + 3(0.9) + 2(1.1) = 15.925

5. $R_P = \sum_{i=1}^{8} f(\overline{x}_i)\Delta x_i = [f(-1.75) + f(-1.25) + f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)](0.5)$
= [–0.21875 – 0.46875 – 0.46875 – 0.21875 + 0.28125 + 1.03125 + 2.03125 + 3.28125](0.5) = 2.625

6. $R_P = \sum_{i=1}^{6} f(\overline{x}_i)\Delta x_i = [f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)](0.5)$
= [1.5 + 5 + 14.5 + 33 + 63.5 + 109](0.5) = 113.25

7. $\int_1^3 x^3\, dx$

8. $\int_0^2 (x+1)^3\, dx$

9. $\int_{-1}^1 \frac{x^2}{1+x}\, dx$

10. $\int_0^\pi (\sin x)^2\, dx$

11. $\Delta x = \frac{2}{n}, \overline{x}_i = \frac{2i}{n}$

$f(\overline{x}_i) = \overline{x}_i + 1 = \frac{2i}{n} + 1$

$\sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\frac{2}{n}$

$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{4}{n^2}\sum_{i=1}^{n} i = \frac{2}{n}(n) + \frac{4}{n^2}\left[\frac{n(n+1)}{2}\right]$

$= 2 + 2\left(1 + \frac{1}{n}\right)$

$\int_0^2 (x+1)dx = \lim_{n\to\infty}\left[2 + 2\left(1 + \frac{1}{n}\right)\right] = 4$

12. $\Delta x = \frac{2}{n}, \bar{x}_i = \frac{2i}{n}$

$$f(\bar{x}_i) = \left(\frac{2i}{n}\right)^2 + 1 = \frac{4i^2}{n^2} + 1$$

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i^2\left(\frac{4}{n^2}\right)\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{2}{n}(n) + \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 2 + \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\int_0^2 (x^2 + 1)\,dx = \lim_{n\to\infty}\left[2 + \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = \frac{14}{3}$$

13. $\Delta x = \frac{3}{n}, \bar{x}_i = -2 + \frac{3i}{n}$

$$f(\bar{x}_i) = 2\left(-2 + \frac{3i}{n}\right) + \pi = \pi - 4 + \frac{6i}{n}$$

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \sum_{i=1}^{n}\left[\pi - 4 + \frac{6i}{n}\right]\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n}(\pi - 4) + \frac{18}{n^2}\sum_{i=1}^{n} i = 3(\pi - 4) + \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 3\pi - 12 + 9\left(1 + \frac{1}{n}\right)$$

$$\int_{-2}^{1} (2x + \pi)\,dx = \lim_{n\to\infty}\left[3\pi - 12 + 9\left(1 + \frac{1}{n}\right)\right]$$
$$= 3\pi - 3$$

14. $\Delta x = \frac{3}{n}, \bar{x}_i = -2 + \frac{3i}{n}$

$$f(\bar{x}_i) = 3\left(-2 + \frac{3i}{n}\right)^2 + 2 = 14 - \frac{36i}{n} + \frac{27i^2}{n^2}$$

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \sum_{i=1}^{n}\left[14 - \left(\frac{36}{n}\right)i + \left(\frac{27}{n^2}\right)i^2\right]\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n} 14 - \frac{108}{n^2}\sum_{i=1}^{n} i + \frac{81}{n^3}\sum_{i=1}^{n} i^2$$

$$= 42 - \frac{108}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{81}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 42 - 54\left(1 + \frac{1}{n}\right) + \frac{27}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\int_{-2}^{1} (3x^2 + 2)\,dx$$
$$= \lim_{n\to\infty}\left[42 - 54\left(1 + \frac{1}{n}\right) + \frac{27}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = 15$$

15. $\Delta x = \frac{5}{n}, \bar{x}_i = \frac{5i}{n}$

$$f(\bar{x}_i) = 1 + \frac{5i}{n}$$

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \sum_{i=1}^{n}\left[1 + i\left(\frac{5}{n}\right)\right]\frac{5}{n}$$

$$= \frac{5}{n}\sum_{i=1}^{n} 1 + \frac{25}{n^2}\sum_{i=1}^{n} i = 5 + \frac{25}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= 5 + \frac{25}{2}\left(1 + \frac{1}{n}\right)$$

$$\int_0^5 (x + 1)\,dx = \lim_{n\to\infty}\left[5 + \frac{25}{2}\left(1 + \frac{1}{n}\right)\right] = \frac{35}{2}$$

16. $\Delta x = \frac{20}{n}, \bar{x}_i = -10 + \frac{20i}{n}$

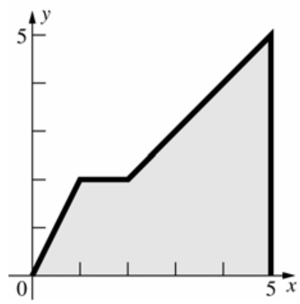
$$f(\bar{x}_i) = \left(-10 + \frac{20i}{n}\right)^2 + \left(-10 + \frac{20i}{n}\right) = 90 - \frac{380i}{n} + \frac{400i^2}{n^2}$$

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \sum_{i=1}^{n}\left[90 - i\left(\frac{380}{n}\right) + i^2\left(\frac{400}{n^2}\right)\right]\frac{20}{n} = \frac{20}{n}\sum_{i=1}^{n} 90 - \frac{7600}{n^2}\sum_{i=1}^{n} i + \frac{8000}{n^3}\sum_{i=1}^{n} i^2$$

$$= 1800 - \frac{7600}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{8000}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = 1800 - 3800\left(1 + \frac{1}{n}\right) + \frac{4000}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\int_{-10}^{10} (x^2 + x)\,dx = \lim_{n\to\infty}\left[1800 - 3800\left(1 + \frac{1}{n}\right) + \frac{4000}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = \frac{2000}{3}$$

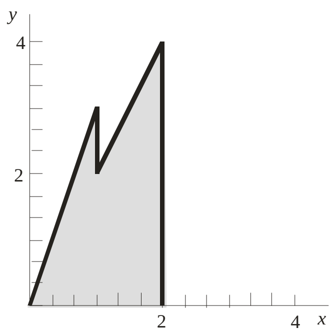
17.



$\int_0^5 f(x)\,dx$

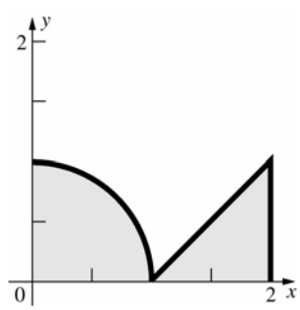
$= \frac{1}{2}(1)(2) + 1(2) + 3(2) + \frac{1}{2}(3)(3) = \frac{27}{2}$

18.



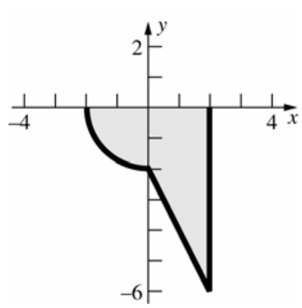
$\int_0^2 f(x)\,dx = \frac{1}{2}(1)(3) + (1)(2) + \frac{1}{2}(1)(2) = \frac{9}{2}$

19.



$\int_0^2 f(x)\,dx = \frac{1}{4}(\pi \cdot 1^2) + \frac{1}{2}(1)(1) = \frac{1}{2} + \frac{\pi}{4}$

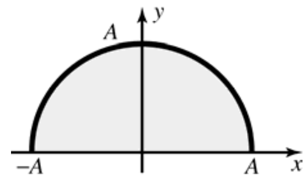
20.



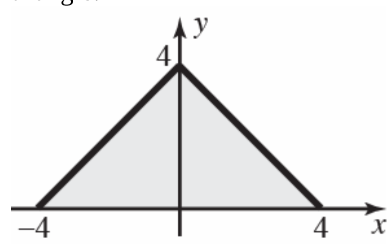
$\int_{-2}^2 f(x)\,dx = -\frac{1}{4}(\pi \cdot 2^2) - (2)(2) - \frac{1}{2}(2)(4)$

$= -\pi - 8$

21. The area under the curve is equal to the area of a semi-circle: $\int_{-A}^{A} \sqrt{A^2 - x^2}\,dx = \frac{1}{2}\pi A^2$.



22. The area under the curve is equal to the area of a triangle:



$\int_{-4}^{4} f(x)\,dx = 2\left(\frac{1}{2}\right)4 \cdot 4 = 16$

23. $s(4) = \int_0^4 v(t)\,dt = \frac{1}{2}4\left(\frac{4}{60}\right) = \frac{2}{15}$

24. $s(4) = \int_0^4 v(t)\,dt = 4 + \frac{1}{2}4(9-1) = 20$

25. $s(4) = \int_0^4 v(t)\,dt = \frac{1}{2}2(1) + 2(1) = 3$

26. $s(4) = \int_0^4 v(t)\,dt = \frac{1}{4}\pi(2)^2 + 0 = \pi$

27.

t	$s(t)$
20	40
40	80
60	120
80	160
100	200
120	240

28.

t	$s(t)$
20	10
40	40
60	90
80	160
100	250
120	360

29.

t	$s(t)$
20	20
40	80
60	160
80	240
100	320
120	400

30.

t	$s(t)$
20	20
40	60
60	80
80	60
100	0
120	-100

31. a. $\int_{-3}^{3}[\![x]\!]\,dx = (-3-2-1+0+1+2)(1) = -3$

b. $\int_{-3}^{3}[\![x]\!]^2\,dx = [(-3)^2+(-2)^2$

$+(-1)^2+0+1+4](1) = 19$

c. $\int_{-3}^{3}\left(x-[\![x]\!]\right)dx = 6\left[\frac{1}{2}(1)(1)\right] = 3$

d. $\int_{-3}^{3}\left(x-[\![x]\!]\right)^2 dx = 6\int_0^1 x^2 dx = 6\cdot\frac{1^3}{3} = 2$

e. $\int_{-3}^{3}|x|\,dx = \frac{1}{2}(3)(3)+\frac{1}{2}(3)(3) = 9$

f. $\int_{-3}^{3}x|x|\,dx = \frac{(-3)^3}{3}+\frac{(3)^3}{3} = 0$

g. $\int_{-1}^{2}|x|[\![x]\!]\,dx = -\int_{-1}^{0}|x|\,dx + 0\int_0^1|x|\,dx + \int_1^2|x|\,dx$

$= -\frac{1}{2}(1)(1)+1(1)+\frac{1}{2}(1)(1) = 1$

h. $\int_{-1}^{2}x^2[\![x]\!]\,dx = -\int_{-1}^{0}x^2\,dx + 0\int_0^1 x^2\,dx$

$+\int_1^2 x^2\,dx$

$= -\frac{1^3}{3}+\left(\frac{2^3}{3}-\frac{1^3}{3}\right) = 2$

32. a. $\int_{-1}^{1} f(x)\,dx = 0$ because this is an odd function.

b. $\int_{-1}^{1} g(x)\,dx = 3+3 = 6$

c. $\int_{-1}^{1}|f(x)|\,dx = 3+3 = 6$

d. $\int_{-1}^{1}[-g(x)]\,dx = -3+(-3) = -6$

e. $\int_{-1}^{1} xg(x)\,dx = 0$ because $xg(x)$ is an odd function.

f. $\int_{-1}^{1} f^3(x)g(x)\,dx = 0$ because $f^3(x)g(x)$ is an odd function.

33. $R_P = \frac{1}{2}\sum_{i=1}^{n}(x_i+x_{i-1})(x_i-x_{i-1})$

$= \frac{1}{2}\sum_{i=1}^{n}\left(x_i^2-x_{i-1}^2\right)$

$= \frac{1}{2}\left[(x_1^2-x_0^2)+(x_2^2-x_1^2)+(x_3^2-x_2^2)\right.$

$\left.+\cdots+(x_n^2-x_{n-1}^2)\right]$

$= \frac{1}{2}(x_n^2-x_0^2)$

$= \frac{1}{2}(b^2-a^2)$

$\lim_{n\to\infty}\frac{1}{2}(b^2-a^2) = \frac{1}{2}(b^2-a^2)$

34. Note that $\overline{x}_i = \left[\frac{1}{3}\left(x_{i-1}^2 + x_{i-1}x_i + x_i^2\right)\right]^{1/2}$

$\geq \left[\frac{1}{3}(x_{i-1}^2 + x_{i-1}^2 + x_{i-1}^2\right]^{1/2} = x_{i-1}$ and

$\overline{x}_i = \left[\frac{1}{3}(x_{i-1}^2 + x_{i-1}x_i + x_i^2\right]^{1/2}$

$\leq \left[\frac{1}{3}(x_i^2 + x_i^2 + x_i^2)\right]^{1/2} = x_i$.

$$R_p = \sum_{i=1}^{n} \overline{x}_i^2 \Delta x_i$$

$$= \sum_{i=1}^{n} \frac{1}{3}(x_i^2 + x_{i-1}\, x_i + x_{i-1}^2)(x_i - x_{i-1})$$

$$= \frac{1}{3}\sum_{i=1}^{n}(x_i^3 - x_{i-1}^3)$$

$$= \frac{1}{3}\Big[(x_1^3 - x_0^3) + (x_2^3 - x_1^3) + (x_3^3 - x_2^3) + \cdots + (x_n^3 - x_{n-1}^3)\Big]$$

$$= \frac{1}{3}(x_n^3 - x_0^3) = \frac{1}{3}(b^3 - a^3)$$

35. Left: $\int_0^2 (x^3 + 1)\, dx = 5.24$

Right: $\int_0^2 (x^3 + 1)\, dx = 6.84$

Midpoint: $\int_0^2 (x^3 + 1)\, dx = 5.98$

36. Left: $\int_0^1 \tan x\, dx \approx 0.5398$

Right: $\int_0^1 \tan x\, dx \approx 0.6955$

Midpoint: $\int_0^1 \tan x\, dx \approx 0.6146$

37. Left: $\int_0^1 \cos x\, dx \approx 0.8638$

Right: $\int_0^1 \cos x\, dx \approx 0.8178$

Midpoint: $\int_0^1 \cos x\, dx \approx 0.8418$

38. Left: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.1682$

Right: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.0349$

Midpoint: $\int_1^3 \left(\frac{1}{x}\right) dx \approx 1.0971$

39. Partition [0, 1] into n regular intervals, so $\|P\| = \frac{1}{n}$.

If $\overline{x}_i = \frac{i}{n} + \frac{1}{2n}$, $f(\overline{x}_i) = 1$.

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} = 1$$

If $\overline{x}_i = \frac{i}{n} + \frac{1}{\pi n}$, $f(\overline{x}_i) = 0$.

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} 0 = 0$$

Thus f is not integrable on [0, 1].

4.3 Concepts Review

1. $4(4-2) = 8$; $16(4-2) = 32$

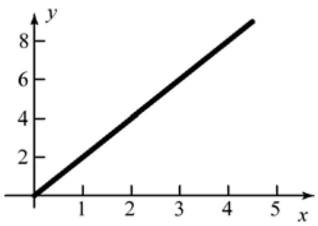
2. $\sin^3 x$

3. $\int_1^4 f(x)\,dx$; $\int_2^5 \sqrt{x}\,dx$

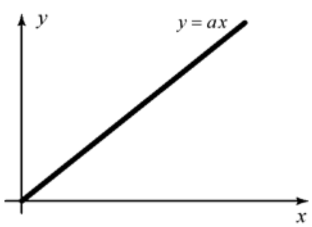
4. 5

Problem Set 4.3

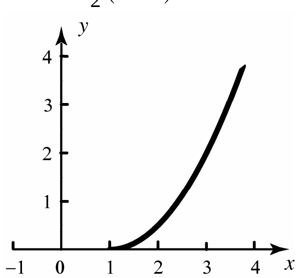
1. $A(x) = 2x$



2. $A(x) = ax$

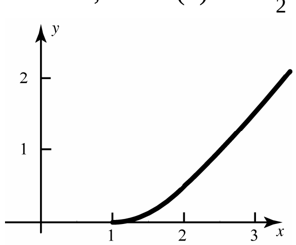


3. $A(x) = \frac{1}{2}(x-1)^2, \quad x \ge 1$

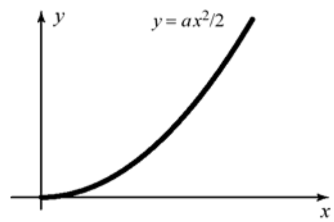


4. If $1 \le x \le 2$, then $A(x) = \frac{1}{2}(x-1)^2$.

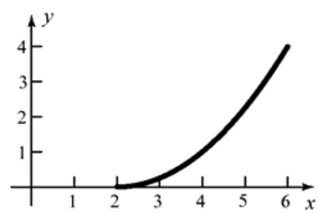
If $2 \le x$, then $A(x) = x - \frac{3}{2}$



5. $A(x) = \dfrac{1}{2}x(ax) = \dfrac{ax^2}{2}$

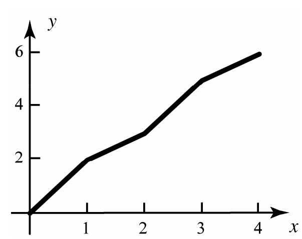


6. $A(x) = \dfrac{1}{2}(x-2)(-1+x/2) = \dfrac{1}{4}(x-2)^2, \quad x \ge 2$



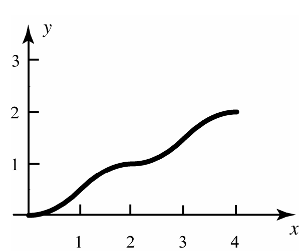
7.

$$A(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 2 + (x-1) & 1 < x \le 2 \\ 3 + 2(x-2) & 2 < x \le 3 \\ 5 + (x-3) & 3 < x \le 4 \\ \text{etc.} \end{cases}$$



8.

$$A(x) = \begin{cases} \frac{1}{2}x^2 & 0 \le x \le 1 \\ \frac{1}{2} + \frac{1}{2}(3-x)(x-1) & 1 < x \le 2 \\ 1 + \frac{1}{2}(x-2)^2 & 2 < x \le 3 \\ \frac{3}{2} + \frac{1}{2}(5-x)(x-3) & 3 < x \le 4 \\ 2 + \frac{1}{2}(x-4)^2 & 4 < x \le 5 \\ \text{etc.} \end{cases}$$



9. $\int_1^2 2f(x)\,dx = 2\int_1^2 f(x)\,dx = 2(3) = 6$

10. $\int_0^2 2f(x)\,dx = 2\int_0^2 f(x)\,dx$

$= 2\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)dx\right] = 2(2+3) = 10$

11. $\int_0^2 [2f(x)+g(x)]\,dx = 2\int_0^2 f(x)\,dx + \int_0^2 g(x)\,dx$

$= 2\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx\right] + \int_0^2 g(x)\,dx$

= 2(2 + 3) + 4 = 14

12. $\int_0^1 [2f(s)+g(s)]\,ds = 2\int_0^1 f(s)\,ds + \int_0^1 g(s)\,ds$

= 2(2) + (–1) = 3

13. $\int_2^1 [2f(s)+5g(s)]\,ds = -2\int_1^2 f(s)\,ds - 5\int_1^2 g(s)\,ds$

$= -2(3) - 5\left[\int_0^2 g(s)\,ds - \int_0^1 g(s)\,ds\right]$

= –6 – 5[4 + 1] = –31

14. $\int_1^1 [3f(x)+2g(x)]\,dx = 0$

15. $\int_0^2 [3f(t)+2g(t)]\,dt$

$= 3\left[\int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt\right] + 2\int_0^2 g(t)\,dt$

= 3(2 + 3) + 2(4) = 23

16. $\int_0^2 \left[\sqrt{3}\,f(t) + \sqrt{2}\,g(t) + \pi\right] dt$

$= \sqrt{3}\left[\int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt\right] + \sqrt{2}\int_0^2 g(t)\,dt$

$+\pi\int_0^2 dt$

$= \sqrt{3}(2+3) + \sqrt{2}(4) + 2\pi = 5\sqrt{3} + 4\sqrt{2} + 2\pi$

17. $G'(x) = D_x\left[\int_1^x 2t\,dt\right] = 2x$

18. $G'(x) = D_x\left[\int_x^1 2t\,dt\right] = D_x\left[-\int_1^x 2t\,dt\right] = -2x$

19. $G'(x) = D_x\left[\int_0^x \left(2t^2+\sqrt{t}\right)dt\right] = 2x^2 + \sqrt{x}$

20. $G'(x) = D_x\left[\int_1^x \cos^3(2t)\tan(t)\,dt\right]$

$= \cos^3(2x)\tan(x)$

21. $G'(x) = D_x\left[\int_x^{\pi/4} (s-2)\cot(2s)ds\right]$

$= D_x\left[-\int_{\pi/4}^x (s-2)\cot(2s)ds\right]$

$= -(x-2)\cot(2x)$

22. $G'(x) = D_x\left[\int_1^x xt\,dt\right] = D_x\left[x\int_1^x t\,dt\right]$

$= D_x\left[x\left[\frac{t^2}{2}\right]_1^x\right] = D_x\left[x\left(\frac{x^2-1}{2}\right)\right]$

$= D_x\left(\frac{x^3}{2} - \frac{x}{2}\right) = \frac{3}{2}x^2 - \frac{1}{2}$

23. $G'(x) = D_x\left[\int_1^{x^2} \sin t\,dt\right] = 2x\sin(x^2)$

24. $G'(x) = D_x\left[\int_1^{x^2+x} \sqrt{2z+\sin z}\,dz\right]$

$= (2x+1)\sqrt{2(x^2+x)+\sin(x^2+x)}$

25. $G(x) = \int_{-x^2}^x \frac{t^2}{1+t^2}\,dt$

$= \int_{-x^2}^0 \frac{t^2}{1+t^2}\,dt + \int_0^x \frac{t^2}{1+t^2}\,dt$

$= -\int_0^{-x^2} \frac{t^2}{1+t^2}\,dt + \int_0^x \frac{t^2}{1+t^2}\,dt$

$G'(x) = -\frac{\left(-x^2\right)^2}{1+\left(-x^2\right)^2}(-2x) + \frac{x^2}{1+x^2}$

$= \frac{2x^5}{1+x^4} + \frac{x^2}{1+x^2}$

26. $G(x) = D_x\left[\int_{\cos x}^{\sin x} t^5 dt\right]$

$= D_x\left[\int_0^{\sin x} t^5 dt + \int_{\cos x}^0 t^5 dt\right]$

$= D_x\left[\int_0^{\sin x} t^5 dt - \int_0^{\cos x} t^5 dt\right]$

$= \sin^5 x\cos x + \cos^5 x\sin x$

27. $f'(x) = \frac{x}{\sqrt{1+x^2}}$; $f''(x) = \frac{1}{\left(x^2+1\right)^{3/2}}$

So, $f(x)$ is increasing on $[0,\infty)$ and concave up on $(0,\infty)$.

28. $f'(x) = \dfrac{1+x}{1+x^2}$

$$f''(x) = \frac{(1+x^2)-(1+x)2x}{(x^2+1)^2} = -\frac{x^2+2x-1}{(x^2+1)^2}$$

So, $f(x)$ is increasing on $[0,\infty)$ and concave up on $\left(0,-1+\sqrt{2}\right)$.

29. $f'(x) = \cos x;\quad f''(x) = -\sin x$

So, $f(x)$ is increasing on $\left[0,\frac{\pi}{2}\right],\left[\frac{3\pi}{2},\frac{5\pi}{2}\right],\ldots$ and concave up on $(\pi,2\pi),(3\pi,4\pi),\ldots$.

30. $f'(x) = x+\sin x;\ f''(x) = 1+\cos x$

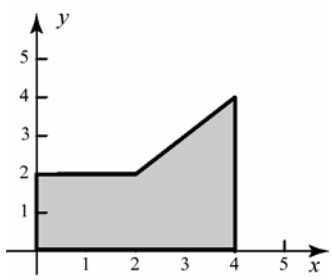
So, $f(x)$ is increasing on $(0,\infty)$ and concave up on $(0,\infty)$.

31. $f'(x) = \dfrac{1}{x};\quad f''(x) = -\dfrac{1}{x^2}$

So, $f(x)$ is increasing on $(0,\infty)$ and never concave up.

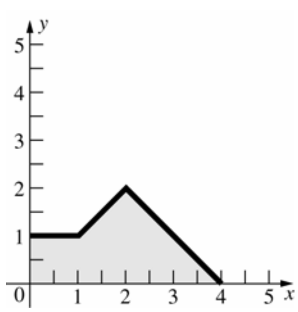
32. $f(x)$ is increasing on $x \ge 0$ and concave up on $(0,1),(2,3),\ldots$

33.



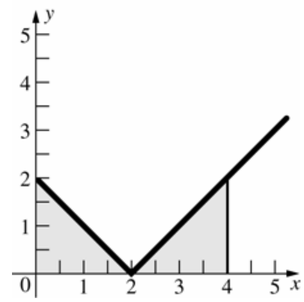
$$\int_0^4 f(x)\,dx = \int_0^2 2\,dx + \int_2^4 x\,dx = 4+6 = 10$$

34.



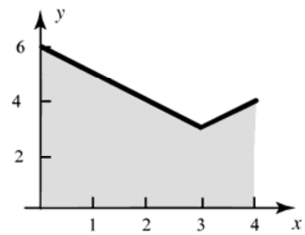
$$\int_0^4 f(x)\,dx = \int_0^1 dx + \int_1^2 x\,dx + \int_2^4 (4-x)\,dx$$
$$= 1+1.5+2.0 = 4.5$$

35.



$$\int_0^4 f(x)\,dx = \int_0^2 (2-x)dx + \int_2^4 (x-2)\,dx$$
$$= 2+2 = 4$$

36.



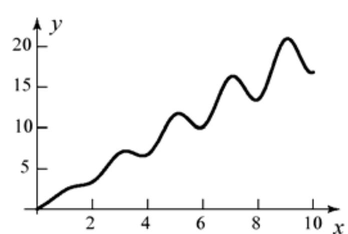
$$\int_0^4 (3+|x-3|)\,dx$$
$$= \int_0^3 (3+|x-3|)\,dx + \int_3^4 (3+|x-3|)\,dx$$
$$= \int_0^3 (6-x)\,dx + \int_3^4 x\,dx = \frac{27}{2}+\frac{7}{2} = 17$$

37. **a.** Local minima at 0, ≈ 3.8, ≈ 5.8, ≈ 7.9, ≈ 9.9;
local maxima at ≈ 3.1, ≈ 5, ≈ 7.1, ≈ 9, 10

b. Absolute minimum at 0, absolute maximum at ≈ 9

c. $\approx$ (0.7, 1.5), (2.5, 3.5), (4.5, 5.5), (6.5, 7.5), (8.5, 9.5)

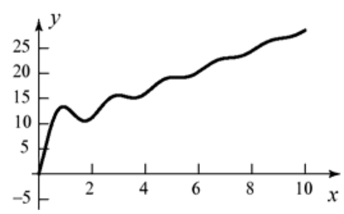
d.

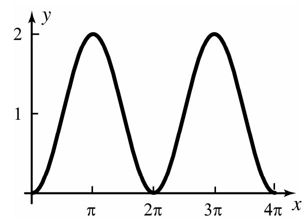


38. **a.** Local minima at 0, ≈ 1.8, ≈ 3.8, ≈ 5.8;
local maxima at ≈ 1, ≈ 2.9, ≈ 5.2, ≈ 10

b. Absolute minimum at 0, absolute maximum at 10

c. (0.5, 1.5), (2.2, 3.2), (4.2,5.2), (6.2,7.2), (8.2, 9.2)

d. 

f. 

39. **a.** $F(0) = \int_0^0 \left(t^4 + 1\right) dt = 0$

b. $y = F(x)$

$\frac{dy}{dx} = F'(x) = x^4 + 1$

$dy = \left(x^4 + 1\right) dx$

$y = \frac{1}{5}x^5 + x + C$

c. Now apply the initial condition $y(0) = 0$:

$0 = \frac{1}{5}0^5 + 0 + C$

$C = 0$

Thus $y = F(x) = \frac{1}{5}x^5 + x$

d. $\int_0^1 \left(x^4 + 1\right) dx = F(1) = \frac{1}{5}1^5 + 1 = \frac{6}{5}$.

40. **a.** $G(x) = \int_0^x \sin t\, dt$

$G(0) = \int_0^0 \sin t\, dt = 0$

$G(2\pi) = \int_0^{2\pi} \sin t\, dt = 0$

b. Let $y = G(x)$. Then

$\frac{dy}{dx} = G'(x) = \sin x$.

$dy = \sin x\, dx$

$y = -\cos x + C$

c. Apply the initial condition $0 = y(0) = -\cos 0 + C$. Thus, $C = 1$, and hence $y = G(x) = 1 - \cos x$.

d. $\int_0^\pi \sin x\, dx = G(\pi) = 1 - \cos\pi = 2$

e. G attains the maximum of 2 when $x = \pi, 3\pi$.

G attains the minimum of 0 when $x = 0, 2\pi, 4\pi$

Inflection points of G occur at

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

41. For $t \ge 1$, $\sqrt{t} \le t$. Since $1 + x^4 \ge 1$ for all x, $1 \le \sqrt{1+x^4} \le 1 + x^4$.

$\int_0^1 dx \le \int_0^1 \sqrt{1+x^4}\, dx \le \int_0^1 (1+x^4)\, dx$

By problem 39d, $1 \le \int_0^1 \sqrt{1+x^4}\, dx \le \frac{6}{5}$

42. On the interval [0,1], $2 \le \sqrt{4+x^4} \le 4 + x^4$. Thus

$\int_0^1 2\, dx \le \int_0^1 \sqrt{4+x^2}\, dx \le \int_0^1 \left(4+x^2\right) dx$

$2 \le \int_0^1 \sqrt{4+x^2}\, dx \le \frac{21}{5}$

Here, we have used the result from problem 39:

$\int_0^1 \left(4+x^4\right) dx = \int_0^1 \left(3+1+x^4\right) dx$

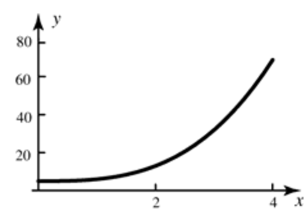
$= \int_0^1 3\, dx + \int_0^1 \left(1+x^4\right) dx$

$= 3 + \frac{6}{5} = \frac{21}{5}$

43. $5 \le f(x) \le 69$ so

$4 \cdot 5 \le \int_0^4 \left(5+x^3\right) dx \le 4 \cdot 69$

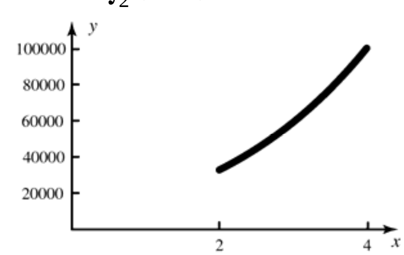
$20 \le \int_0^4 \left(5+x^3\right) dx \le 276$



44. On [2,4], $8^5 \le (x+6)^5 \le 10^5$. Thus,

$2 \cdot 8^5 \le \int_2^4 (x+6)^5\, dx \le 2 \cdot 10^5$

$65{,}536 \le \int_2^4 (x+6)^5\, dx \le 200{,}000$

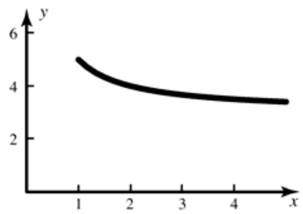


45. On [1,5],

$$3+\frac{2}{5}\le 3+\frac{2}{x}\le 3+\frac{2}{1}$$

$$4\left(\frac{17}{5}\right)\le \int_1^5\left(3+\frac{2}{x}\right)dx\le 4\cdot 5$$

$$\frac{68}{5}\le \int_1^5\left(3+\frac{2}{x}\right)dx\le 20$$



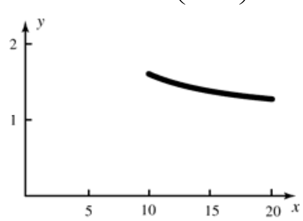
46. On [10, 20],

$$\left(1+\frac{1}{20}\right)^5\le\left(1+\frac{1}{x}\right)^5\le\left(1+\frac{1}{10}\right)^5$$

$$10\left(\frac{21}{20}\right)^5\le\int_{10}^{20}\left(1+\frac{1}{x}\right)^5dx\le 10\left(\frac{11}{10}\right)^5$$

$$\frac{4,084,101}{320,000}\le\int_{10}^{20}\left(1+\frac{1}{x}\right)^5dx\le\frac{161,051}{10,000}$$

$$12.7628\le\int_{10}^{20}\left(1+\frac{1}{x}\right)^5dx\le 16.1051$$

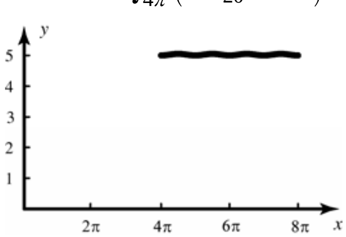


47. On $[4\pi, 8\pi]$

$$5\le 5+\tfrac{1}{20}\sin^2 x\le 5+\tfrac{1}{20}$$

$$(4\pi)(5)\le\int_{4\pi}^{8\pi}\left(5+\tfrac{1}{20}\sin^2 x\right)dx\le(4\pi)\left(5+\tfrac{1}{20}\right)$$

$$20\pi\le\int_{4\pi}^{8\pi}\left(5+\tfrac{1}{20}\sin^2 x\right)dx\le\frac{101}{5}\pi$$



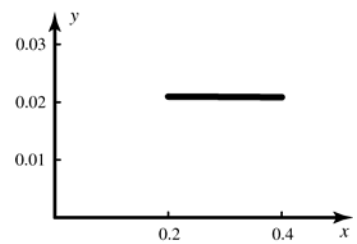
48. On [0.2,0.4],

$$0.002+0.0001\cos^2 0.4\le 0.002+0.0001\cos^2 x$$
$$\le 0.002+0.0001\cos^2 0.2$$

$$0.2\left(0.002+0.0001\cos^2 0.4\right)$$
$$\le\int_{0.2}^{0.4}\left(0.002+0.0001\cos^2 x\right)dx$$
$$\le 0.2\left(0.002+0.0001\cos^2 0.2\right)$$

Thus,

$$0.000417\le\int_{0.2}^{0.4}\left(0.002+0.0001\cos^2 x\right)dx$$
$$\le 0.000419$$



49. Let $F(x)=\int_0^x\frac{1+t}{2+t}dt$. Then

$$\lim_{x\to 0}\frac{1}{x}\int_0^x\frac{1+t}{2+t}dt=\lim_{x\to 0}\frac{F(x)-F(0)}{x-0}$$
$$=F'(0)=\frac{1+0}{2+0}=\frac{1}{2}$$

50. $$\lim_{x\to 1}\frac{1}{x-1}\int_1^x\frac{1+t}{2+t}dt$$
$$=\lim_{x\to 1}\frac{1}{x-1}\left[\int_0^x\frac{1+t}{2+t}dt-\int_0^1\frac{1+t}{2+t}dt\right]$$
$$=\lim_{x\to 1}\frac{F(x)-F(1)}{x-1}$$
$$=F'(1)=\frac{1+1}{2+1}=\frac{2}{3}$$

51. $$\int_1^x f(t)\,dt=2x-2$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\int_1^x f(t)\,dt=\frac{d}{dx}(2x-2)$$
$$f(x)=2$$

If such a function exists, it must satisfy $f(x)=2$, but both sides of the first equality may differ by a constant yet still have equal derivatives. When $x=1$ the left side is $\int_1^1 f(t)\,dt=0$ and the right side is $2\cdot 1-2=0$. Thus the function $f(x)=2$ satisfies

$$\int_1^x f(t)\,dt=2x-2.$$

52. $\int_0^x f(t)\,dt = x^2$

Differentiate both sides with respect to *x*:

$$\frac{d}{dx}\int_0^x f(t)\,dt = \frac{d}{dx}x^2$$
$$f(x) = 2x$$

53. $\int_0^{x^2} f(t)\,dt = \frac{1}{3}x^3$

Differentiate both sides with respect to *x*:

$$\frac{d}{dx}\int_0^{x^2} f(t)\,dt = \frac{d}{dx}\left(\frac{1}{3}x^3\right)$$
$$f\left(x^2\right)(2x) = x^2$$
$$f\left(x^2\right) = \frac{x}{2}$$
$$f(x) = \frac{\sqrt{x}}{2}$$

54. No such function exists. When $x = 0$ the left side is 0, whereas the right side is 1

55. True; by Theorem B (Comparison Property)

56. False. $a = -1$, $b = 2$, $f(x) = x$ is a counterexample.

57. False. $a = -1$, $b = 1$, $f(x) = x$ is a counterexample.

58. False; A counterexample is $f(x) = 0$ for all *x*, except $f(1) = 1$. Thus, $\int_0^2 f(x)\,dx = 0$, but *f* is not identically zero.

59. True. $\int_a^b f(x)dx - \int_a^b g(x)dx$
$= \int_a^b [f(x) - g(x)]dx$

60. False. $a = 0$, $b = 1$, $f(x) = 0$, $g(x) = -1$ is a counterexample.

61. $$v(t) = \begin{cases} 2+(t-2), & t \le 2 \\ 2-(t-2), & t > 2 \end{cases} = \begin{cases} t, & t \le 2 \\ 4-t, & t > 2 \end{cases}$$

$$s(t) = \int_0^t v(u)\,du$$
$$= \begin{cases} \int_0^t u\,du, & 0 \le t \le 2 \\ \int_0^2 u\,du + \int_2^t (4-u)\,du, & t > 2 \end{cases}$$
$$= \begin{cases} \frac{t^2}{2}, & 0 \le t \le 2 \\ 2 + \left[4t - \frac{t^2}{2}\right], & t > 2 \end{cases}$$
$$= \begin{cases} \frac{t^2}{2}, & 0 \le t \le 2 \\ -4 + 4t - \frac{t^2}{2} & t > 2 \end{cases}$$

$\frac{t^2}{2} - 4t + 4 = 0;\ t = 4 + 2\sqrt{2} \approx 6.83$

62. **a.** $$s(t) = \begin{cases} \int_0^t 5\,du, & 0 \le t \le 100 \\ \int_0^{100} 5\,du + \int_{100}^t \left(6 - \frac{u}{100}\right)du & 100 < t \le 700 \\ \int_0^{100} 5\,du + \int_{100}^{700} \left(6 - \frac{u}{100}\right)du + \int_{700}^t (-1)\,du, & t > 700 \end{cases}$$

$$= \begin{cases} 5t, & 0 \le t \le 100 \\ 500 + \left[6u - \frac{u^2}{200}\right]_{100}^t & 100 < t \le 700 \\ 500 + \left[6u - \frac{u^2}{200}\right]_{100}^{700} - (t-700) & t > 700 \end{cases}$$

$$= \begin{cases} 5t, & 0 \le t \le 100 \\ -50 + 6t - \frac{t^2}{200}, & 100 < t \le 700 \\ 2400 - t, & t > 700 \end{cases}$$

b. $v(t) > 0$ for $0 \le t < 600$ and $v(t) < 0$ for $t > 600$. So, $t = 600$ is the point at which the object is farthest to the right of the origin. At $t = 600$, $s(t) = 1750$.

c. $s(t) = 0 = 2400 - t;\ t = 2400$

63. $-|f(x)| \le f(x) \le |f(x)|$, so

$$\int_a^b -|f(x)|\,dx \le \int_a^b f(x)\,dx \Rightarrow$$

$$\int_a^b |f(x)|\,dx \ge -\int_a^b f(x)\,dx$$

and combining this with

$$\int_a^b |f(x)|\,dx \ge \int_a^b f(x)\,dx,$$

we can conclude that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

64. If $x > a$, $\int_a^x |f'(x)|\,dx \le M(x-a)$ by the Boundedness Property. If $x < a$,

$\int_x^a |f(x)|\,dx = -\int_a^x |f'(x)|\,dx \ge -M(x-a)$ by the Boundedness Property. Thus

$\int_a^x |f'(x)|\,dx \le M|x-a|$.

From Problem 63, $\int_a^x |f'(x)|\,dx \ge \left|\int_a^x f'(x)\,dx\right|$.

$$\left|\int_a^x f'(x)\,dx\right| = |f(x) - f(a)| \ge |f(x)| - |f(a)|$$

Therefore, $|f(x)| - |f(a)| \le M|x-a|$ or $|f(x)| \le |f(a)| + M|x-a|$.

4.4 Concepts Review

1. antiderivative; $F(b) - F(a)$

2. $F(b) - F(a)$

3. $F(d) - F(c)$

4. $\int_1^2 \frac{1}{3}u^4\,du$

Problem Set 4.4

1. $\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4 - 0 = 4$

2. $\int_{-1}^2 x^4\,dx = \left[\frac{x^5}{5}\right]_{-1}^2 = \frac{32}{5} + \frac{1}{5} = \frac{33}{5}$

3. $\int_{-1}^2 (3x^2 - 2x + 3)\,dx = \left[x^3 - x^2 + 3x\right]_{-1}^2$
$= (8 - 4 + 6) - (-1 - 1 - 3) = 15$

4. $\int_1^2 (4x^3 + 7)\,dx = \left[x^4 + 7x\right]_1^2$
$= (16 + 14) - (1 + 7) = 22$

5. $\int_1^4 \frac{1}{w^2}\,dw = \left[-\frac{1}{w}\right]_1^4 = \left(-\frac{1}{4}\right) - (-1) = \frac{3}{4}$

6. $\int_1^3 \frac{2}{t^3}\,dt = \left[-\frac{1}{t^2}\right]_1^3 = \left(-\frac{1}{9}\right) - (-1) = \frac{8}{9}$

7. $\int_0^4 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_0^4 = \left(\frac{2}{3}\cdot 8\right) - 0 = \frac{16}{3}$

8. $\int_1^8 \sqrt[3]{w}\,dw = \left[\frac{3}{4}w^{4/3}\right]_1^8 = \left(\frac{3}{4}\cdot 16\right) - \left(\frac{3}{4}\cdot 1\right) = \frac{45}{4}$

9. $\int_{-4}^{-2}\left(y^2 + \frac{1}{y^3}\right)dy = \left[\frac{y^3}{3} - \frac{1}{2y^2}\right]_{-4}^{-2}$
$= \left(-\frac{8}{3} - \frac{1}{8}\right) - \left(-\frac{64}{3} - \frac{1}{32}\right) = \frac{1783}{96}$

10. $\int_1^4 \frac{s^4 - 8}{s^2}\,ds = \int_1^4 (s^2 - 8s^{-2})\,ds = \left[\frac{s^3}{3} + \frac{8}{s}\right]_1^4$
$= \left(\frac{64}{3} + 2\right) - \left(\frac{1}{3} + 8\right) = 15$

11. $\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$

12. $\int_{\pi/6}^{\pi/2} 2\sin t\,dt = [-2\cos t]_{\pi/6}^{\pi/2} = 0 + \sqrt{3} = \sqrt{3}$

13. $\int_0^1 (2x^4 - 3x^2 + 5)\,dx = \left[\frac{2}{5}x^5 - x^3 + 5x\right]_0^1$
$= \left(\frac{2}{5} - 1 + 5\right) - 0 = \frac{22}{5}$

14. $\int_0^1 (x^{4/3} - 2x^{1/3})\,dx = \left[\frac{3}{7}x^{7/3} - \frac{3}{2}x^{4/3}\right]_0^1$
$= \left(\frac{3}{7} - \frac{3}{2}\right) - 0 = -\frac{15}{14}$

15. $u = 3x + 2$, $du = 3\,dx$

$$\int \sqrt{u}\cdot\frac{1}{3}du = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(3x+2)^{3/2} + C$$

16. $u = 2x - 4$, $du = 2\,dx$

$$\int u^{1/3}\cdot\frac{1}{2}du = \frac{3}{8}u^{4/3} + C = \frac{3}{8}(2x-4)^{4/3} + C$$

17. $u = 3x + 2$, $du = 3\,dx$

$$\int \cos(u)\cdot\frac{1}{3}du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3x+2) + C$$

18. $u = 2x - 4$, $du = 2\,dx$

$$\int \sin u\cdot\frac{1}{2}du = -\frac{1}{2}\cos u + C$$

$$= -\frac{1}{2}\cos(2x-4) + C$$

19. $u = 6x - 7$, $du = 6dx$

$$\int \sin u\cdot\frac{1}{6}du = -\frac{1}{6}\cos u + C$$

$$= -\frac{1}{6}\cos(6x-7) + C$$

20. $u = \pi v - \sqrt{7}$, $du = \pi\,dv$

$$\int \cos u\cdot\frac{1}{\pi}du = \frac{1}{\pi}\sin u + C = \frac{1}{\pi}\sin(\pi v - \sqrt{7}) + C$$

21. $u = x^2 + 4$, $du = 2x\,dx$

$$\int \sqrt{u}\cdot\frac{1}{2}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+4)^{3/2} + C$$

22. $u = x^3 + 5$, $du = 3x^2\,dx$

$$\int u^9\cdot\frac{1}{3}du = \frac{1}{30}u^{10} + C = \frac{1}{30}(x^3+5)^{10} + C$$

23. $u = x^2 + 3$, $du = 2x\,dx$

$$\int u^{-12/7}\cdot\frac{1}{2}du = -\frac{7}{10}u^{-5/7} + C$$

$$= -\frac{7}{10}(x^2+3)^{-5/7} + C$$

24. $u = \sqrt{3}v^2 + \pi$, $du = 2\sqrt{3}v\,dv$

$$\int u^{7/8}\cdot\frac{1}{2\sqrt{3}}du = \frac{4}{15\sqrt{3}}u^{15/8} + C$$

$$= \frac{4}{15\sqrt{3}}\left(\sqrt{3}v^2 + \pi\right)^{15/8} + C$$

25. $u = x^2 + 4$, $du = 2x\,dx$

$$\int \sin(u)\cdot\frac{1}{2}du = -\frac{1}{2}\cos u + C$$

$$= -\frac{1}{2}\cos(x^2+4) + C$$

26. $u = x^3 + 5$, $du = 3x^2\,dx$

$$\int \cos u\cdot\frac{1}{3}du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(x^3+5) + C$$

27. $u = \sqrt{x^2+4}$, $du = \frac{x}{\sqrt{x^2+4}}dx$

$$\int \sin u\,du = -\cos u + C = -\cos\sqrt{x^2+4} + C$$

28. $u = \sqrt[3]{z^2+3}$, $du = \frac{2z}{3\left(\sqrt[3]{z^2+3}\right)^2}dz$

$$\int \cos u\cdot\frac{3}{2}du = \frac{3}{2}\sin u + C = \frac{3}{2}\sin\sqrt[3]{z^2+3} + C$$

29. $u = (x^3+5)^9$,

$$du = 9(x^3+5)^8(3x^2)dx = 27x^2(x^3+5)^8\,dx$$

$$\int \cos u\cdot\frac{1}{27}du = \frac{1}{27}\sin u + C$$

$$= \frac{1}{27}\sin\left[(x^3+5)^9\right] + C$$

30. $u = (7x^7+\pi)^9$, $du = 441x^6(7x^7+\pi)^8\,dx$

$$\int \sin u\cdot\frac{1}{441}du = -\frac{1}{441}\cos u + C$$

$$= -\frac{1}{441}\cos(7x^7+\pi)^9 + C$$

31. $u = \sin(x^2+4)$, $du = 2x\cos(x^2+4)\,dx$

$$\int \sqrt{u}\cdot\frac{1}{2}du = \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}\left[\sin(x^2+4)\right]^{3/2} + C$$

32. $u = \cos(3x^7+9)$

$$du = -21x^6\sin(3x^7+9)\,dx$$

$$\int \sqrt[3]{u}\cdot\left(-\frac{1}{21}\right)du = -\frac{1}{28}u^{4/3} + C$$

$$= -\frac{1}{28}\left[\cos(3x^7+9)\right]^{4/3} + C$$

33. $u=\cos(x^3+5), du=-3x^2\sin(x^3+5)\,dx$

$$\int u^9\cdot\left(-\frac{1}{3}\right)du=-\frac{1}{30}u^{10}+C$$
$$=-\frac{1}{30}\cos^{10}(x^3+5)+C$$

34. $u=\tan(x^{-3}+1)$, $du=-3x^{-4}\sec^2(x^{-3}+1)\,dx$

$$\int\sqrt[5]{u}\cdot\left(-\frac{1}{3}\right)du=-\frac{5}{18}u^{6/5}+C$$
$$=-\frac{5}{18}\left[\tan(x^{-3}+1)\right]^{6/5}+C$$

35. $u=x^2+1, du=2x\,dx$

$$\int_0^1(x^2+1)^{10}(2x)dx=\int_1^2u^{10}du=\left[\frac{u^{11}}{11}\right]_1^2$$
$$=\left[\frac{1}{11}(2)^{11}\right]-\left[\frac{1}{11}(1)^{11}\right]=\frac{2047}{11}$$

36. $u=x^3+1, du=3x^2\,dx$

$$\int_{-1}^0\sqrt{x^3+1}(3x^2)\,dx=\int_0^1\sqrt{u}du=\left[\frac{2}{3}u^{3/2}\right]_0^1$$
$$=\left(\frac{2}{3}\cdot1^{3/2}\right)-\left(\frac{2}{3}\cdot0\right)=\frac{2}{3}$$

37. $u=t+2, du=dt$

$$\int_{-1}^3\frac{1}{(t+2)^2}dt=\int_1^5u^{-2}du=\left[-\frac{1}{u}\right]_1^5$$
$$=\left[-\frac{1}{5}\right]-[-1]=\frac{4}{5}$$

38. $u=y-1, du=dy$

$$\int_2^{10}\sqrt{y-1}\,dy=\int_1^9\sqrt{u}du=\left[\frac{2}{3}u^{3/2}\right]_1^9$$
$$=\left[\frac{2}{3}(27)\right]-\left[\frac{2}{3}(1)\right]=\frac{52}{3}$$

39. $u=3x+1, du=3\,dx$

$$\int_5^8\sqrt{3x+1}\,dx=\frac{1}{3}\int_5^8\sqrt{3x+1}\cdot3dx=\frac{1}{3}\int_{16}^{25}\sqrt{u}\,du$$
$$=\left[\frac{2}{9}u^{3/2}\right]_{16}^{25}=\left[\frac{2}{9}(125)\right]-\left[\frac{2}{9}(64)\right]=\frac{122}{9}$$

40. $u=2x+2, du=2\,dx$

$$\int_1^7\frac{1}{\sqrt{2x+2}}dx=\frac{1}{2}\int_1^7\frac{2}{\sqrt{2x+2}}dx$$
$$=\frac{1}{2}\int_4^{16}u^{-1/2}du=\left[\sqrt{u}\right]_4^{16}=4-2=2$$

41. $u=7+2t^2, du=4t\,dt$

$$\int_{-3}^3\sqrt{7+2t^2}\,(8t)\,dt=2\int_{-3}^3\sqrt{7+2t^2}\cdot(4t)\,dt$$
$$=2\int_{25}^{25}\sqrt{u}\,du=\left[\frac{4}{3}u^{3/2}\right]_{25}^{25}$$
$$=\left[\frac{4}{3}(125)\right]-\left[\frac{4}{3}(125)\right]=0$$

42. $u=x^3+3x, du=(3x^2+3)\,dx$

$$\int_1^3\frac{x^2+1}{\sqrt{x^3+3x}}dx=\frac{1}{3}\int_1^3\frac{3x^2+3}{\sqrt{x^3+3x}}dx$$
$$=\frac{1}{3}\int_4^{16}u^{-1/2}du=\left[\frac{2}{3}u^{1/2}\right]_4^{36}$$
$$=\left(\frac{2}{3}\cdot6\right)-\left(\frac{2}{3}\cdot2\right)=\frac{8}{3}$$

43. $u=\cos x, du=-\sin x\,dx$

$$\int_0^{\pi/2}\cos^2x\sin x\,dx=-\int_0^{\pi/2}\cos^2x(-\sin x)\,dx$$
$$=-\int_1^0u^2du=\left[-\frac{u^3}{3}\right]_1^0$$
$$=0-\left(-\frac{1}{3}\right)=\frac{1}{3}$$

44. $u=\sin3x, du=3\cos3x\,dx$

$$\int_0^{\pi/2}\sin^2 3x\cos3x\,dx$$
$$=\frac{1}{3}\int_0^{\pi/2}\sin^2 3x(3\cos3x)\,dx=\frac{1}{3}\int_0^{-1}u^2du$$
$$=\left[\frac{u^3}{9}\right]_0^{-1}=\left(-\frac{1}{9}\right)-0=-\frac{1}{9}$$

45. $u=x^2+2x,\ du=(2x+2)\,dx=2(x+1)\,dx$

$$\int_0^1(x+1)(x^2+2x)^2\,dx$$
$$=\int_0^1\frac{1}{2}(x^2+2x)^2\,2(x+1)\,dx$$
$$=\frac{1}{2}\int_0^3u^2\,du=\left[\frac{u^3}{6}\right]_0^3=\frac{9}{2}$$

46. $u=\sqrt{x}-1,\ du=\frac{1}{2\sqrt{x}}dx$

$$\int_1^4\frac{(\sqrt{x}-1)^3}{\sqrt{x}}dx=2\int_1^4\frac{(\sqrt{x}-1)^3}{2\sqrt{x}}dx$$
$$=2\int_0^1u^3\,du=2\left[\frac{u^4}{4}\right]_0^1=\frac{1}{2}$$

47. $u = \sin\theta, du = \cos\theta\, d\theta$

$$\int_0^{1/2} u^3 du = \left[\frac{u^4}{4}\right]_0^{1/2} = \frac{1}{64} - 0 = \frac{1}{64}$$

48. $u = \cos\theta, du = -\sin\theta\, d\theta$

$$-\int_1^{\sqrt{3}/2} u^{-3} du = \frac{1}{2}\left[u^{-2}\right]_1^{\sqrt{3}/2} = \frac{1}{2}\left(\frac{4}{3} - 1\right) = \frac{1}{6}$$

49. $u = 3x - 3, du = 3dx$

$$\frac{1}{3}\int_{-3}^{0} \cos u\, du = \frac{1}{3}[\sin u]_{-3}^{0} = \frac{1}{3}(0 - \sin(-3))$$

$$= \frac{\sin 3}{3}$$

50. $u = 2\pi x, du = 2\pi dx$

$$\frac{1}{2\pi}\int_0^{\pi} \sin u\, du = -\frac{1}{2\pi}[\cos u]_0^{\pi} = -\frac{1}{2\pi}(-1-1)$$

$$= \frac{1}{\pi}$$

51. $u = \pi x^2, du = 2\pi x\, dx$

$$\frac{1}{2\pi}\int_0^{\pi} \sin u\, du = -\frac{1}{2\pi}[\cos u]_0^{\pi} = -\frac{1}{2\pi}(-1-1)$$

$$= \frac{1}{\pi}$$

52. $u = 2x^5, du = 10x^4 dx$

$$\frac{1}{10}\int_0^{2\pi^5} \cos u\, du = \frac{1}{10}[\sin u]_0^{2\pi^5}$$

$$= \frac{1}{10}(\sin(2\pi^5) - 0) = \frac{1}{10}\sin(2\pi^5)$$

53. $u = 2x, du = 2dx$

$$\frac{1}{2}\int_0^{\pi/2} \cos u\, du + \frac{1}{2}\int_0^{\pi/2} \sin u\, du$$

$$= \frac{1}{2}[\sin u]_0^{\pi/2} - \frac{1}{2}[\cos u]_0^{\pi/2}$$

$$= \frac{1}{2}(1-0) - \frac{1}{2}(0-1) = 1$$

54. $u = 3x, du = 3dx; v = 5x, dv = 5dx$

$$\frac{1}{3}\int_{-3\pi/2}^{3\pi/2} \cos u\, du + \frac{1}{5}\int_{-5\pi/2}^{5\pi/2} \sin v\, dv$$

$$= \frac{1}{3}[\sin u]_{-3\pi/2}^{3\pi/2} - \frac{1}{5}[\cos v]_{-5\pi/2}^{5\pi/2}$$

$$= \frac{1}{3}[(-1) - 1] - \frac{1}{5}[0 - 0] = -\frac{2}{3}$$

55. $u = \cos x, du = -\sin x\, dx$

$$-\int_1^0 \sin u\, du = [\cos u]_1^0 = 1 - \cos 1$$

56. $u = \pi\sin\theta, du = \pi\cos\theta\, d\theta$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos u\, du = \frac{1}{\pi}[\sin u]_{-\pi}^{\pi} = 0$$

57. $u = \cos(x^2),\ du = -2x\sin(x^2)dx$

$$-\frac{1}{2}\int_1^{\cos 1} u^3 du = -\frac{1}{2}\left[\frac{u^4}{4}\right]_1^{\cos 1} = -\frac{\cos^4 1}{8} + \frac{1}{8}$$

$$= \frac{1 - \cos^4 1}{8}$$

58. $u = \sin(x^3), du = 3x^2\cos(x^3)dx$

$$\frac{1}{3}\int_{-\sin(\pi^3/8)}^{\sin(\pi^3/8)} u^2\, du = \frac{1}{9}\left[u^3\right]_{-\sin(\pi^3/8)}^{\sin(\pi^3/8)}$$

$$= \frac{2\sin^3\left(\frac{\pi^3}{8}\right)}{9}$$

59. **a.** Between 0 and 3, $f(x) > 0$. Thus, $\int_0^3 f(x)\, dx > 0$.

b. Since f is an antiderivative of f',

$$\int_0^3 f'(x)\, dx = f(3) - f(0) = 0 - 2 = -2 < 0$$

c. $$\int_0^3 f''(x)\, dx = f'(3) - f'(0) = -1 - 0 = -1 < 0$$

d. Since f is concave down at 0, $f''(0) < 0$.

$$\int_0^3 f'''(x)\, dx = f''(3) - f''(0) = 0 - (\text{negative number}) > 0$$

60. **a.** On $[0, 4]$, $f(x) > 0$. Thus, $\int_0^4 f(x)\, dx > 0$.

b. Since f is an antiderivative of f',

$$\int_0^4 f'(x)\, dx = f(4) - f(0) = 1 - 2 = -1 < 0$$

c. $$\int_0^4 f''(x)\, dx = f'(4) - f'(0) = \frac{1}{4} - (-2) = \frac{9}{4} > 0$$

d. $$\int_0^4 f'''(x)\, dx = f''(4) - f''(0) = (\text{negative}) - (\text{positive}) < 0$$

61. $V(t)=\int V'(t)=\int (20-t)\,dt=20t-\frac{1}{2}t^2+C$

$V(0)=C=0$ since no water has leaked out at time $t=0$. Thus, $V(t)=20t-\frac{1}{2}t^2$, so

$V(20)-V(10)=200-150=50$ gallons.

Time to drain: $20t-\frac{1}{2}t^2=200$; $t=20$ hours.

62. $V(1)-V(0)=\int_0^1 V'(t)\,dt=\left[t-\frac{t^2}{220}\right]_0^1=\frac{219}{220}$

$V(10)-V(9)=\int_9^{10}\left(1-\frac{t}{110}\right)dt=\frac{201}{220}$

$55=V(T)-V(0)=\int_0^T\left(1-\frac{t}{110}\right)dt=T-\frac{T^2}{220}$

$T\approx 110$ hrs

63. Use a midpoint Riemann sum with $n=12$ partitions.

$V=\sum_{i=1}^{12} f(x_i)\Delta x_i$

$\approx 1(5.4+6.3+6.4+6.5+6.9+7.5+8.4$
$+8.4+8.0+7.5+7.0+6.5)$

$=84.8$

64. Use a midpoint Riemann sum with $n=10$ partitions.

$V=\sum_{i=1}^{10} f(x_i)\Delta x_i$

$\approx 1\begin{pmatrix}6200+6300+6500+6500+6600\\+6700+6800+7000+7200+7200\end{pmatrix}$

$=67,000$

65. Use a midpoint Riemann sum with $n=12$ partitions.

$E=\sum_{i=0}^{12} P(t_i)\Delta t_i$

$\approx 2(3.0+3.0+3.8+5.8+7.8+6.9$
$+6.5+6.3+7.2+8.2+8.7+5.4)$

$=145.2$

66. $\delta(x)=m'(x)=1+\frac{x}{4}$

mass $=\int_0^2 \delta(x)\,dx=m(2)=\frac{5}{2}$

67. a. $\int_a^b x^n dx=B_n$; $\int_{a^n}^{b^n}\sqrt[n]{y}\,dy=A_n$

Using Figure 3 of the text,

$(a)(a^n)+A_n+B_n=(b)(b^n)$ or

$B_n+A_n=b^{n+1}-a^{n+1}$. Thus

$\int_a^b x^n dx+\int_{a^n}^{b^n}\sqrt[n]{y}\,dy=b^{n+1}-a^{n+1}$

b. $\int_a^b x^n\,dx+\int_{a^n}^{b^n}\sqrt[n]{y}\,dy$

$=\left[\frac{x^{n+1}}{n+1}\right]_a^b+\left[\frac{n}{n+1}y^{(n+1)/n}\right]_{a^n}^{b^n}$

$=\left(\frac{b^{n+1}}{n+1}-\frac{a^{n+1}}{n+1}\right)+\left(\frac{n}{n+1}b^{n+1}-\frac{n}{n+1}a^{n+1}\right)$

$=\frac{(n+1)b^{n+1}-(n+1)a^{n+1}}{n+1}=b^{n+1}-a^{n+1}$

c. $B_n=\int_a^b x^n\,dx=\frac{1}{n+1}\left[x^{n+1}\right]_a^b$

$=\frac{1}{n+1}(b^{n+1}-a^{n+1})$

$A_n=\int_{a^n}^{b^n}\sqrt[n]{y}\,dy=\left[\frac{n}{n+1}y^{(n+1)/n}\right]_{a^n}^{b^n}$

$=\frac{n}{n+1}\left(b^{n+1}-a^{n+1}\right)$

$nB_n=\frac{n}{n+1}(b^{n+1}-a^{n+1})=A_n$

68. Let $y=G(x)=\int_a^x f(t)\,dt$. Then

$\frac{dy}{dx}=G'(x)=f(x)$

$dy=f(x)\,dx$

Let F be any antiderivative of f. Then $G(x)=F(x)+C$. When $x=a$, we must have $G(a)=0$. Thus, $C=-F(a)$ and $G(x)=F(x)-F(a)$. Now choose $x=b$ to obtain

$\int_a^b f(t)\,dt=G(b)=F(b)-F(a)$

69. $\int_0^3 x^2 dx=\left[\frac{x^3}{3}\right]_0^3=9-0=9$

70. $\int_0^2 x^3\,dx=\left[\frac{x^4}{4}\right]_0^2=4-0=4$

71. $\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = 1+1 = 2$

72. $\int_0^2 (1+x+x^2)\,dx = \left[x + \frac{1}{2}x^2 + \frac{1}{3}x^3\right]_0^2$

$= \left(2+2+\frac{8}{3}\right) - 0 = \frac{20}{3}$

73. The right-endpoint Riemann sum is

$\sum_{i=1}^{n}\left(0+\frac{1-0}{n}i\right)^2\left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n} i^2$, which for

$n = 10$ equals $\frac{77}{200} = 0.385$.

$\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3} = 0.\overline{333}$

74 $\int_{-2}^{4} (2[\![x]\!] - 3|x|)\,dx = 2\int_{-2}^{4} [\![x]\!]\,dx - 3\int_{-2}^{4} |x|\,dx$

$= 2[(-2-1+0+1+2+3)(1)]$

$-3\left[\frac{1}{2}(2)(2) + \frac{1}{2}(4)(4)\right]$

$= -24$

75. $\frac{d}{dx}\left(\frac{1}{2}x|x|\right) = \frac{1}{2}x\left(\frac{|x|}{x}\right) + \frac{|x|}{2} = |x|$

$\int_a^b |x|\,dx = \left[\frac{1}{2}x|x|\right]_a^b = \frac{1}{2}(b|b| - a|a|)$

76. For $b > 0$, if b is an integer,

$\int_0^b [\![x]\!]\,dx = 0+1+2+\cdots+(b-1)$

$= \sum_{i=1}^{b-1} i = \frac{(b-1)b}{2}.$

If b is not an integer, let $n = [\![b]\!]$. Then

$\int_0^b [\![x]\!]\,dx = 0+1+2+\cdots+(n-1)+n(b-n)$

$= \frac{(n-1)n}{2} + n(b-n)$

$= \frac{([\![b]\!]-1)[\![b]\!]}{2} + [\![b]\!](b-[\![b]\!])$.

77. **a.** Let c be in (a,b). Then $G'(c) = f(c)$ by the First Fundamental Theorem of Calculus. Since G is differentiable at c, G is continuous there. Now suppose $c = a$.

Then $\lim_{x\to c} G(x) = \lim_{x\to a}\int_a^x f(t)\,dt$. Since f is continuous on $[a,b]$, there exist (by the Min-Max Existence Theorem) m and M such that $f(m) \le f(x) \le f(M)$ for all x in $[a,b]$.

Then

$\int_a^x f(m)\,dt \le \int_a^x f(t)\,dt \le \int_a^x f(M)\,dt$

$(x-a)f(m) \le G(x) \le (x-a)f(M)$

By the Squeeze Theorem

$\lim_{x\to a^+}(x-a)f(m) \le \lim_{x\to a^+} G(x)$

$\le \lim_{x\to a^+}(x-a)f(M)$

Thus,

$\lim_{x\to a^+} G(x) = 0 = \int_a^a f(t)\,dt = G(a)$

Therefore G is right-continuous at $x = a$.

Now, suppose $c = b$. Then

$\lim_{x\to b^-} G(x) = \lim_{x\to b^-}\int_x^b f(t)\,dt$

As before,

$(b-x)f(m) \le G(x) \le (b-x)f(M)$ so we can apply the Squeeze Theorem again to obtain

$\lim_{x\to b^-}(b-x)f(m) \le \lim_{x\to b^-} G(x)$

$\le \lim_{x\to b^-}(b-x)f(M)$

Thus

$\lim_{x\to b^-} G(x) = 0 = \int_b^b f(t)\,dt = G(b)$

Therefore, G is left-continuous at $x = b$.

b. Let F be any antiderivative of f. Note that G is also an antiderivative of f. Thus, $F(x) = G(x) + C$. We know from part (a) that $G(x)$ is continuous on $[a,b]$. Thus $F(x)$, being equal to $G(x)$ plus a constant, is also continuous on $[a,b]$.

78. Let $f(x)=\begin{cases}1, & x>0\\ 0, & x\le 0\end{cases}$ and $F(x)=\int_{-1}^{x} f(t)\,dt$.

If $x<0$, then $F(x)=0$. If $x\ge 0$, then

$$F(x)=\int_{-1}^{x} f(t)\,dt$$
$$=\int_{-1}^{0} 0\,dt+\int_{0}^{x} 1\,dt$$
$$=0+x=x$$

Thus,

$$F(x)=\begin{cases}x, & x\ge 0\\ 0, & x<0\end{cases}$$

which is continuous everywhere even though $f(x)$ is not continuous everywhere.

4.5 Concepts Review

1. $\frac{1}{b-a}\int_a^b f(x)\,dx$

2. $f(c)$

3. 0; $2\int_0^2 f(x)\,dx$

4. $f(x+p)=f(x)$; period

Problem Set 4.5

1. $\frac{1}{3-1}\int_1^3 4x^3 dx=\frac{1}{2}\left[x^4\right]_1^3=40$

2. $\frac{1}{4-1}\int_1^4 5x^2 dx=\frac{1}{3}\left[\frac{5}{3}x^3\right]_1^4=35$

3. $\frac{1}{3-0}\int_0^3 \frac{x}{\sqrt{x^2+16}}dx=\frac{1}{3}\left[\sqrt{x^2+16}\right]_0^3=\frac{1}{3}$

4. $\frac{1}{2-0}\int_0^2 \frac{x^2}{\sqrt{x^3+16}}dx=\frac{1}{2}\left[\frac{2}{3}\sqrt{x^3+16}\right]_0^2$

$=\frac{1}{3}\left(\sqrt{24}-4\right)=\frac{2}{3}\left(\sqrt{6}-2\right)$

5. $\frac{1}{1+2}\int_{-2}^{1}\left(2+|x|\right)dx$

$=\frac{1}{3}\left[\int_{-2}^{0}(2-x)\,dx+\int_0^1(2+x)\,dx\right]$

$=\frac{1}{3}\left\{\left[2x-\frac{1}{2}x^2\right]_{-2}^{0}+\left[2x+\frac{1}{2}x^2\right]_0^1\right\}$

$=\frac{1}{3}\left(-2(-2)+\frac{1}{2}(-2)^2+2+\frac{1}{2}\right)=\frac{17}{6}$

6. $\frac{1}{2+3}\int_{-3}^{2}\left(x+|x|\right)dx$

$=\frac{1}{5}\left(\int_{-3}^{0}(-x+x)\,dx+\int_0^2 2x\,dx\right)$

$=\frac{1}{5}\left[x^2\right]_0^2=\frac{4}{5}$

7. $\frac{1}{\pi}\int_0^{\pi}\cos x\,dx=\frac{1}{\pi}\left[\sin x\right]_0^{\pi}$

$=\frac{1}{\pi}\left[\sin\pi-\sin 0\right]=0$

8. $\frac{1}{\pi-0}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}\left(-\cos x\right)_0^{\pi}$

$=-\frac{1}{\pi}(-1-1)=\frac{2}{\pi}$

9. $\frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} x\cos x^2\,dx=\frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\sin x^2\right)_0^{\sqrt{\pi}}$

$=\frac{1}{\sqrt{\pi}}(0-0)=0$

10. $\frac{1}{\pi/2-0}\int_0^{\pi/2}\sin^2 x\cos x\,dx$

$=\frac{2}{\pi}\left[\frac{1}{3}\sin^3 x\right]_0^{\pi/2}=\frac{2}{3\pi}$

11. $\frac{1}{2-1}\int_1^2 y\left(1+y^2\right)^3 dy=\left[\frac{1}{8}\left(1+y^2\right)^4\right]_1^2$

$=\frac{625}{8}-2=\frac{609}{8}=76.125$

12. $\frac{1}{\pi/4-1}\int_0^{\pi/4}\tan x\sec^2 x=\frac{1}{\pi/4-1}\left[\frac{1}{2}\tan^2 x\right]_0^{\pi/4}$

$=\frac{2}{\pi-4}(1-0)=\frac{2}{\pi-4}$

13. $\frac{1}{\pi/4}\int_{\pi/4}^{\pi/2}\frac{\sin\sqrt{z}}{\sqrt{z}}dz=\frac{4}{\pi}\left[-2\cos\sqrt{z}\right]_{\pi/4}^{\pi/2}$

$=\frac{8}{\pi}\left(\cos\sqrt{\pi/4}-\cos\sqrt{\pi/2}\right)\approx 0.815$

14. $\frac{1}{\pi/2}\int_0^{\pi/2}\frac{\sin v\,\cos v}{\sqrt{1+\cos^2 v}}dv$

$=\frac{2}{\pi}\left[-\sqrt{1+\cos^2 v}\right]_0^{\pi/2}$

$=\frac{2}{\pi}\left(-1+\sqrt{2}\right)$

15. $\int_0^3 \sqrt{x+1}\,dx = \sqrt{c+1}(3-0)$

$\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = 3\sqrt{c+1}$

$14/3 = 3\sqrt{c+1};\ c = \frac{115}{81} \approx 1.42$

16. $\int_{-1}^1 x^2\,dx = c^2(1-(-1))$

$\left[\frac{1}{3}x^3\right]_{-1}^1 = 2c^2;\ c = \pm\frac{\sqrt{3}}{3} \approx \pm 0.58$

17. $\int_{-4}^3 (1-x^2)\,dx = (1-c^2)(3+4)$

$\left[x - \frac{1}{3}x^3\right]_{-4}^3 = 7 - 7c^2$

$c = \pm\frac{\sqrt{39}}{3} \approx \pm 2.08$

18. $\int_0^1 x(1-x)\,dx = c(1-c)(1-0)$

$\left[\frac{-x^2(2x-3)}{6}\right]_0^1 = c - c^2$

$c = \frac{3\pm\sqrt{3}}{6} \approx 0.21$ or 0.79

19. $\int_0^2 |x|\,dx = |c|(2-0);\ \left[\frac{x|x|}{2}\right]_0^2 = 2|c|;\ c = 1$

20. $\int_{-2}^2 |x|\,dx = |c|(2+2);\ \left[\frac{x|x|}{2}\right]_{-2}^2 = 4|c|;\ c = -1, 1$

21. $\int_{-\pi}^{\pi} \sin z\,dz = \sin c(\pi+\pi)$

$[-\cos z]_{-\pi}^{\pi} = 2\pi\sin c;\quad c = 0$

22. $\int_0^{\pi} \cos 2y\,dy = (\cos 2c)(\pi - 0)$

$\left[\frac{\sin 2y}{2}\right]_0^{\pi} = \pi\cos 2c;\quad c = \frac{\pi}{4}, \frac{3\pi}{4}$

23. $\int_0^2 (v^2 - v)\,dv = (c^2 - c)(2-0)$

$\left[\frac{1}{3}v^3 - \frac{1}{2}v^2\right]_0^2 = 2c^2 - 2c$

$c = \frac{\sqrt{21}+3}{6} \approx 1.26$

24. $\int_0^2 x^3 dx = c^3(2-0);\ \left[\frac{1}{4}x^4\right]_0^2 = 2c^3$

$c = \sqrt[3]{2} \approx 1.26$

25. $\int_1^4 (ax+b)\,dx = (ac+b)(4-1)$

$\left[\frac{a}{2}x^2 + bx\right]_1^4 = 3ac + 3b;\ c = \frac{5}{2}$

26. $\int_0^b y^2 dy = c^2(b-0);\ \left[\frac{1}{3}y^3\right]_0^b = bc^2$

$c = \frac{b}{\sqrt{3}}$

27. $\frac{\int_A^B (ax+b)\,dx}{B-A} = f(c)$

$\frac{\left[\frac{a}{2}x^2 + bx\right]_A^B}{B-A} = ac + b$

$\frac{\frac{a}{2}(B-A)(B+A) + b(B-A)}{B-A} = ac + b$

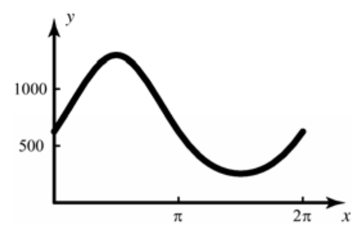
$\frac{a}{2}B + \frac{a}{2}A + b = ac + b;$

$c = \frac{1}{2}B + \frac{1}{2}A = (A+B)/2$

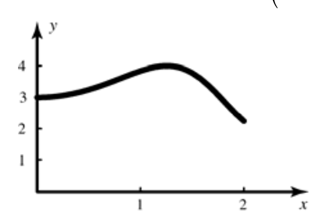
28. $\int_0^b ay^2 dy = ac^2(b-0);\ \left[\frac{1}{3}ay^3\right]_0^b = abc^2$

$c = \frac{b\sqrt{3}}{3}$

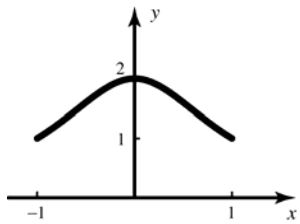
29.. Using $c = \pi$ yields $2\pi(5)^4 = 1250\pi \approx 3927$



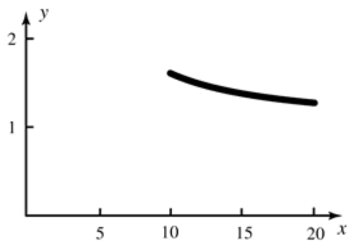
30. Using $c = 0.8$ yields $2\left(3 + \sin 0.8^2\right) \approx 7.19$



31. Using $c = 0.5$ yields $2\dfrac{2}{1+0.5^2} = 3.2$



32. Using c = 15 yields $\left(\dfrac{16}{15}\right)^5 (20-10) \approx 13.8$.



33. A rectangle with height 25 and width 7 has approximately the same area as that under the curve. Thus

$$\frac{1}{7}\int_0^7 H(t)\,dt \approx 25$$

34. **a.** A rectangle with height 28 and width 24 has approximately the same area as that under the curve. Thus,

$$\frac{1}{24-0}\int_0^{24} T(t)\,dt \approx 28$$

b. Yes. The Mean Value Theorem for Integrals guarantees the existence of a c such that

$$\frac{1}{24-0}\int_0^{24} T(t)\,dt = T(c)$$

The figure indicates that there are actually two such values of c, roughly, $c = 11$ and $c = 16$.

35. $\int_{-\pi}^{\pi} (\sin x + \cos x)\,dx = \int_{-\pi}^{\pi} \sin x\,dx + 2\int_0^{\pi} \cos x\,dx$

$= 0 + 2\left[\sin x\right]_0^{\pi} = 0$

36. $\int_{-1}^{1} \dfrac{x^3}{(1+x^2)^4}\,dx = 0$, since the integrand is odd.

37. $\int_{-\pi/2}^{\pi/2} \dfrac{\sin x}{1+\cos x}\,dx = 0$, since the integrand is odd.

38. $\int_{-\sqrt{3}\pi}^{\sqrt{3}\pi} x^2\cos(x^3)\,dx = 2\int_0^{\sqrt{3}\pi} x^2\cos(x^3)\,dx$

$= \dfrac{2}{3}\left[\sin(x^3)\right]_0^{\sqrt{3}\pi} = \dfrac{2}{3}\sin\left(3\sqrt{3}\,\pi^3\right)$

39. $\int_{-\pi}^{\pi} (\sin x + \cos x)^2\,dx$

$= \int_{-\pi}^{\pi} (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,dx$

$= \int_{-\pi}^{\pi} (1 + 2\sin x\cos x)\,dx = \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \sin 2x\,dx$

$= 2\int_0^{\pi} dx + 0 = 2[x]_0^{\pi} = 2\pi$

40. $\int_{-\pi/2}^{\pi/2} z\sin^2(z^3)\cos(z^3)dz = 0$, since

$(-z)\sin^2[(-z)^3]\cos[(-z)^3]$

$= -z\sin^2(-z^3)\cos(-z^3)$

$= -z[-\sin(z^3)]^2\cos(z^3)$

$= -z\sin^2(z^3)\cos(z^3)$

41. $\int_{-1}^{1} (1 + x + x^2 + x^3)\,dx$

$= \int_{-1}^{1} dx + \int_{-1}^{1} x\,dx + \int_{-1}^{1} x^2 dx + \int_{-1}^{1} x^3 dx$

$= 2\left[x\right]_0^1 + 0 + 2\left[\dfrac{x^3}{3}\right]_0^1 + 0 = \dfrac{8}{3}$

42. $\int_{-100}^{100} (v + \sin v + v\cos v + \sin^3 v)^5\,dv = 0$

since $(-v + \sin(-v) - v\cos(-v) + \sin^3(-v))^5$

$= (-v - \sin v - v\cos v - \sin^3 v)^5$

$= -(v + \sin v + v\cos v + \sin^3 v)^5$

43. $\int_{-1}^{1} \left(\left|x^3\right| + x^3\right)dx = 2\int_0^1 \left|x^3\right|dx + \int_{-1}^{1} x^3 dx$

$= 2\left[\dfrac{x^4}{4}\right]_0^1 + 0 = \dfrac{1}{2}$

44. $\int_{-\pi/4}^{\pi/4} \left(|x|\sin^5 x + |x|^2\tan x\right)dx = 0$

since $|-x|\sin^5(-x) + |-x|^2\tan(-x)$

$= -|x|\sin^5 x - |x|^2\tan x$

45. $\int_{-b}^{-a} f(x)\,dx = \int_a^b f(x)\,dx$ when f is even.

$\int_{-b}^{-a} f(x)\,dx = -\int_a^b f(x)\,dx$ when f is odd.

46. $u = -x, du = -dx$

$$\int_a^b f(-x)\,dx = -\int_{-a}^{-b} f(u)\,du$$

$= \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$ since the variable used in the integration is not important.

47. $$\int_0^{4\pi} |\cos x|\,dx = 8\int_0^{\pi/2} |\cos x|\,dx$$

$$= 8[\sin x]_0^{\pi/2} = 8$$

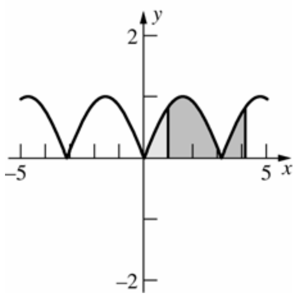
48. Since sin *x* is periodic with period 2π, sin 2*x* is periodic with period π.

$$\int_0^{4\pi} |\sin 2x|\,dx = 8\int_0^{\pi/2} \sin 2x\,dx$$

$$= 8\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} = -4(-1 - 1) = 8$$

49. $$\int_1^{1+\pi} |\sin x|\,dx = \int_0^{\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx$$

$$= [-\cos x]_0^{\pi} = 2$$



50. $$\int_2^{2+\pi/2} |\sin 2x|\,dx = \int_0^{\pi/2} |\sin 2x|\,dx$$

$$= \frac{1}{2}[-\cos 2x]_0^{\pi/2} = 1$$

51. $$\int_1^{1+\pi} |\cos x|\,dx = \int_0^{\pi} |\cos x|\,dx = 2\int_0^{\pi/2} \cos x\,dx$$

$$= 2[\sin x]_0^{\pi/2} = 2(1-0) = 2$$

52. The statement is true. Recall that

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$\int_a^b \overline{f}dx = \overline{f}\int_a^b dx = \frac{1}{b-a}\int_a^b f(x)dx \cdot \int_a^b dx$$

$$= \frac{1}{b-a}\int_a^b f(x)dx \cdot (b-a) = \int_a^b f(x)\,dx$$

53. All the statements are true.

a. $$\overline{u} + \overline{v} = \frac{1}{b-a}\int_a^b u\,dx + \frac{1}{b-a}\int_a^b v\,dx$$

$$= \frac{1}{b-a}\int_a^b (u+v)\,dx = \overline{u+v}$$

b. $$k\overline{u} = \frac{k}{b-a}\int_a^b u\,dx = \frac{1}{b-a}\int_a^b ku\,dx = \overline{ku}$$

c. Note that

$\overline{u} = \frac{1}{b-a}\int_a^b u(x)\,dx = \frac{1}{a-b}\int_b^a u(x)\,dx$, so we can assume $a < b$.

$$\overline{u} = \frac{1}{b-a}\int_a^b u\,dx \le \frac{1}{b-a}\int_a^b v\,dx = \overline{v}$$

54. **a.** $\overline{V} = 0$ by periodicity.

b. $\overline{V} = 0$ by periodicity.

c. $$V_{rms}^2 = \int_{\phi}^{\phi+1} \hat{V}^2 \sin^2(120\pi t + \phi)\,dt$$

$$= \int_0^1 \hat{V}^2 \sin^2(120\pi t)\,dt$$

by periodicity.

$u = 120\pi t$, $\quad du = 120\pi\,dt$

$$V_{rms}^2 = \frac{1}{120\pi}\int_0^{120\pi} \hat{V}^2 \sin^2 u\,du$$

$$= \frac{\hat{V}^2}{120\pi}\left[-\frac{1}{2}\cos u \sin u + \frac{1}{2}u\right]_0^{120\pi}$$

$$= \frac{1}{2}\hat{V}^2$$

d. $$120 = \frac{\hat{V}\sqrt{2}}{2}$$

$\hat{V} = 120\sqrt{2} \approx 169.71$ Volts

55. Since *f* is continuous on a closed interval $[a,b]$ there exist (by the Min-Max Existence Theorem) an *m* and *M* in $[a,b]$ such that

$f(m) \le f(x) \le f(M)$ for all x in $[a,b]$. Thus

$$\int_a^b f(m)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(M)\,dx$$

$$(b-a)\,f(m) \le \int_a^b f(x)\,dx \le (b-a)\,f(M)$$

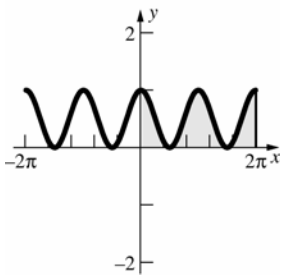
$$f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M)$$

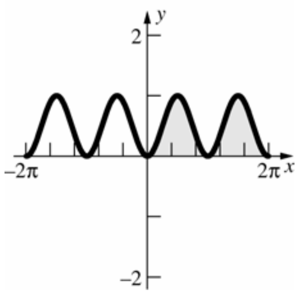
Since *f* is continuous, we can apply the Intermediate Value Theorem and say that *f* takes on every value between $f(m)$ and $f(M)$. Since $\frac{1}{b-a}\int_a^b f(x)\,dx$ is between $f(m)$ and $f(M)$, there exists a *c* in $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

56. **a.** $$\int_0^{2\pi} (\sin^2 x + \cos^2 x)\,dx = \int_0^{2\pi} dx = [x]_0^{2\pi} = 2\pi$$

b.





c. $2\pi = \int_0^{2\pi} \cos^2 x\,dx + \int_0^{2\pi} \sin^2 x\,dx$

$= 2\int_0^{2\pi} \cos^2 x\,dx$, thus $\int_0^{2\pi} \cos^2 x\,dx$

$= \int_0^{2\pi} \sin^2 x\,dx = \pi$

57. **a.** Even

b. 2π

c. On $[0,\pi]$, $|\sin x| = \sin x$.

$u = \cos x$, $du = -\sin x\,dx$

$$\int f(x)\,dx = \int \sin x \cdot \sin(\cos x)\,dx$$

$$= -\int \sin u\,du = \cos u + C$$

$$= \cos(\cos x) + C$$

Likewise, on $[\pi, 2\pi]$,

$$\int f(x)\,dx = -\cos(\cos x) + C$$

$$\int_0^{\pi/2} f(x)\,dx = 1 - \cos 1 \approx 0.46$$

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = 2\int_0^{\pi/2} f(x)\,dx$$

$$= 2(1 - \cos 1) \approx 0.92$$

$$\int_0^{3\pi/2} f(x)\,dx = \int_0^{\pi} f(x)\,dx + \int_{\pi}^{3\pi/2} f(x)\,dx$$

$$= \cos 1 - 1 \approx -0.46$$

$$\int_{-3\pi/2}^{3\pi/2} f(x)\,dx = 2\int_0^{3\pi/2} f(x)\,dx$$

$$= 2(\cos 1 - 1) \approx -0.92$$

$$\int_0^{2\pi} f(x)\,dx = 0$$

$$\int_{\pi/6}^{4\pi/3} f(x)\,dx = 2\cos 1 - \cos\left(\frac{\sqrt{3}}{2}\right) + \cos\left(\frac{1}{2}\right)$$

$$\approx -0.44$$

$$\int_{13\pi/6}^{10\pi/3} f(x)\,dx = \int_{\pi/6}^{4\pi/3} f(x)\,dx \approx -0.44$$

58. **a.** Odd

b. 2π

c. This function cannot be integrated in closed form. We can only simplify the integrals using symmetry and periodicity, and approximate them numerically.

Note that $\int_{-a}^{a} f(x)\,dx = 0$ since f is odd, and

$\int_{\pi-a}^{\pi+a} f(x)\,dx = 0$ since

$f(\pi + x) = -f(\pi - x)$.

$\int_0^{\pi/2} f(x)\,dx = \frac{\pi}{2} J_1(1) \approx 0.69$ (Bessel function)

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0$$

$$\int_0^{3\pi/2} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx \approx 0.69$$

$$\int_{-3\pi/2}^{3\pi/2} f(x)\,dx = 0\,;\ \int_0^{2\pi} f(x)\,dx = 0$$

$$\int_{\pi/6}^{13\pi/6} f(x)\,dx = \int_0^{2\pi} f(x)\,dx = 0$$

$\int_{\pi/6}^{4\pi/3} f(x)\,dx \approx 1.055$ (numeric integration)

$$\int_{13\pi/6}^{10\pi/3} f(x)\,dx = \int_{\pi/6}^{4\pi/3} f(x)\,dx \approx 1.055$$

59. **a.** Written response.

b. $A = \int_0^a g(x)\,dx = \int_0^a \frac{a}{c} f\left(\frac{c}{a}x\right)dx$

$= \int_0^c \frac{a}{c} f(x) \frac{a}{c}\,dx = \frac{a^2}{c^2}\int_0^c f(x)\,dx$

$B = \int_0^b h(x)\,dx = \int_0^b \frac{b}{c} f\left(\frac{c}{b}x\right)dx$

$= \int_0^c \frac{b}{c} f(x)\frac{b}{c}\,dx = \frac{b^2}{c^2}\int_0^c f(x)\,dx$

Thus, $\int_0^a g(x)\,dx + \int_0^b h(x)\,dx$

$= \frac{a^2}{c^2}\int_0^c f(x)\,dx + \frac{b^2}{c^2}\int_0^c f(x)\,dx$

$= \frac{a^2 + b^2}{c^2}\int_0^c f(x)\,dx = \int_0^c f(x)\,dx$ since

$a^2 + b^2 = c^2$ from the triangle.

60. If f is odd, then $f(-x) = -f(x)$ and we can write

$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} [-f(-x)]\,dx = \int_a^0 f(u)\,du$

$= -\int_0^a f(u)\,du = -\int_0^a f(x)\,dx$

On the second line, we have made the substitution $u = -x$.

4.6 Concepts Review

1. 1, 2, 2, 2, …, 2, 1

2. 1, 4, 2, 4, 2, …, 4, 1

3. n^4

4. large

Problem Set 4.6

1. $f(x)=\frac{1}{x^2}; h=\frac{3-1}{8}=0.25$

$x_0=1.00$	$f(x_0)=1$	$x_5=2.25$	$f(x_5)\approx 0.1975$
$x_1=1.25$	$f(x_1)=0.64$	$x_6=2.50$	$f(x_6)=0.16$
$x_2=1.50$	$f(x_2)\approx 0.4444$	$x_7=2.75$	$f(x_7)\approx 0.1322$
$x_3=1.75$	$f(x_3)\approx 0.3265$	$x_8=3.00$	$f(x_8)\approx 0.1111$
$x_4=2.00$	$f(x_4)=0.25$		

Left Riemann Sum: $\int_1^3 \frac{1}{x^2}dx \approx 0.25[f(x_0)+f(x_1)+\ldots+f(x_7)]\approx 0.7877$

Right Riemann Sum: $\int_1^3 \frac{1}{x^2}dx \approx 0.25[f(x_1)+f\left(x_2\right)+...+f(x_8)]\approx 0.5655$

Trapezoidal Rule: $\int_1^3 \frac{1}{x^2}dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)]\approx 0.6766$

Parabolic Rule: $\int_1^3 \frac{1}{x^2}dx \approx \frac{0.25}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_7)+f(x_8)]\approx 0.6671$

Fundamental Theorem of Calculus: $\int_1^3 \frac{1}{x^2}dx=\left[-\frac{1}{x}\right]_1^3=-\frac{1}{3}+1=\frac{2}{3}\approx 0.6667$

2. $f(x)=\frac{1}{x^3}; h=\frac{3-1}{8}=0.25$

$x_0=1.00$	$f(x_0)=1$	$x_5=2.25$	$f(x_5)\approx 0.0878$
$x_1=1.25$	$f(x_1)=0.5120$	$x_6=2.50$	$f(x_6)=0.0640$
$x_2=1.50$	$f(x_2)\approx 0.2963$	$x_7=2.75$	$f(x_7)\approx 0.0481$
$x_3=1.75$	$f(x_3)\approx 0.1866$	$x_8=3.00$	$f(x_8)\approx 0.0370$
$x_4=2.00$	$f(x_4)=0.1250$		

Left Riemann Sum: $\int_1^3 \frac{1}{x^3}dx \approx 0.25[f(x_0)+f(x_1)+\ldots+f(x_7)]\approx 0.5799$

Right Riemann Sum: $\int_1^3 \frac{1}{x^3}dx \approx 0.25[f(x_1)+f\left(x_2\right)+...+f(x_8)]\approx 0.3392$

Trapezoidal Rule: $\int_1^3 \frac{1}{x^3}dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)]\approx 0.4596$

Parabolic Rule: $\int_1^3 \frac{1}{x^3}dx \approx \frac{0.25}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_7)+f(x_8)]\approx 0.4455$

Fundamental Theorem of Calculus: $\int_1^3 \frac{1}{x^3}dx=\left[-\frac{1}{2x^2}\right]_1^3=\frac{4}{9}\approx 0.4444$

3. $f(x)=\sqrt{x};\, h=\frac{2-0}{8}=0.25$

$x_0=0.00$	$f(x_0)=0$	$x_5=1.25$	$f(x_5)\approx 1.1180$
$x_1=0.25$	$f(x_1)=0.5$	$x_6=1.50$	$f(x_6)\approx 1.2247$
$x_2=0.50$	$f(x_2)\approx 0.7071$	$x_7=1.75$	$f(x_7)\approx 1.3229$
$x_3=0.75$	$f(x_3)\approx 0.8660$	$x_8=2.00$	$f(x_8)\approx 1.4142$
$x_4=1.00$	$f(x_4)=1$		

Left Riemann Sum: $\int_0^2 \sqrt{x}\,dx \approx 0.25[f(x_0)+f(x_1)+\ldots+f(x_7)]\approx 1.6847$

Right Riemann Sum: $\int_0^2 \sqrt{x}\,dx \approx 0.25[f(x_1)+f\left(x_2\right)+...+f(x_8)]\approx 2.0383$

Trapezoidal Rule: $\int_0^2 \sqrt{x}dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)]\approx 1.8615$

Parabolic Rule: $\int_0^2 \sqrt{x}dx \approx \frac{0.25}{3}[f(x_1)+4f(x_2)+2f(x_3)+\ldots+4f(x_7)+f(x_8)]\approx 1.8755$

Fundamental Theorem of Calculus: $\int_0^2 \sqrt{x}dx=\left[\frac{2}{3}x^{3/2}\right]_0^2=\frac{4\sqrt{2}}{3}\approx 1.8856$

4. $f(x)=x\sqrt{x^2+1};\, h=\frac{3-1}{8}=0.25$

$x_0=1.00$	$f(x_0)\approx 1.4142$	$x_5=2.25$	$f(x_5)\approx 5.5400$
$x_1=1.25$	$f(x_1)\approx 2.0010$	$x_6=2.50$	$f(x_6)\approx 6.7315$
$x_2=1.50$	$f(x_2)\approx 2.7042$	$x_7=2.75$	$f(x_7)\approx 8.0470$
$x_3=1.75$	$f(x_3)\approx 3.5272$	$x_8=3.00$	$f(x_8)\approx 9.4868$
$x_4=2.00$	$f(x_4)\approx 4.4721$		

Left Riemann Sum: $\int_1^3 x\sqrt{x^2+1}\,dx \approx 0.25[f(x_0)+f(x_1)+\cdots+f(x_7)]\approx 8.6093$

Right Riemann Sum: $\int_1^3 x\sqrt{x^2+1}\,dx \approx 0.25[f(x_1)+f\left(x_2\right)+...+f(x_8)]\approx 10.6274$

Trapezoidal Rule: $\int_1^3 x\sqrt{x^2+1}\,dx \approx \frac{0.25}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_7)+f(x_8)]\approx 9.6184$

Parabolic Rule: $\int_1^3 x\sqrt{x^2+1}dx \approx \frac{0.25}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_7)+f(x_8)]\approx 9.5981$

Fundamental Theorem of Calculus: $\int_1^3 x\sqrt{x^2+1}dx=\left[\frac{1}{3}(x^2+1)^{3/2}\right]_1^3=\frac{1}{3}\left(10\sqrt{10}-2\sqrt{2}\right)\approx 9.5981$

5. $f(x)=x\left(x^2+1\right)^5;\, h=\frac{1-0}{8}=0.125$

$x_0=0.00$	$f(x_0)=0$	$x_5=0.625$	$f(x_5)\approx 3.2504$
$x_1=0.125$	$f(x_1)\approx 0.1351$	$x_6=0.750$	$f(x_6)\approx 6.9849$
$x_2=0.250$	$f(x_2)\approx 0.3385$	$x_7=0.875$	$f(x_7)\approx 15.0414$
$x_3=0.375$	$f(x_3)\approx 0.7240$	$x_8=1.000$	$f(x_8)=32$
$x_4=0.500$	$f(x_4)\approx 1.5259$		

Left Riemann Sum: $\int_0^1 x\left(x^2+1\right)^5 dx \approx 0.125[f(x_0)+f(x_1)+\cdots+f(x_7)] \approx 3.4966$

Right Riemann Sum: $\int_0^1 x\left(x^2+1\right)^5 dx \approx 0.125[f(x_1)+f(x_2)+...+f(x_8)] \approx 7.4966$

Trapezoidal Rule: $\int_0^1 x\left(x^2+1\right)^5 dx \approx \frac{0.125}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_7)+f(x_8)] \approx 5.4966$

Parabolic Rule: $\int_0^1 x\left(x^2+1\right)^5 dx \approx \frac{0.125}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_7)+f(x_8)] \approx 5.2580$

Fundamental Theorem of Calculus: $\int_0^1 x\left(x^2+1\right)^5 dx = \left[\frac{1}{12}\left(x^2+1\right)^6\right]_0^1 = 5.25$

6. $f(x)=(x+1)^{3/2}; h=\frac{4-1}{8}=0.375$

$x_0 = 1.000$	$f(x_0) \approx 2.8284$	$x_5 = 2.875$	$f(x_5) \approx 7.6279$
$x_1 = 1.375$	$f(x_1) \approx 3.6601$	$x_6 = 3.250$	$f(x_6) \approx 8.7616$
$x_2 = 1.750$	$f(x_2) \approx 4.5604$	$x_7 = 3.625$	$f(x_7) \approx 9.9464$
$x_3 = 2.125$	$f(x_3) \approx 5.5243$	$x_8 = 4.000$	$f(x_8) \approx 11.1803$
$x_4 = 2.500$	$f(x_4) \approx 6.5479$		

Left Riemann Sum: $\int_1^4 (x+1)^{3/2} dx \approx 0.375[f(x_0)+f(x_1)+\cdots+f(x_7)] \approx 18.5464$

Right Riemann Sum: $\int_1^4 (x+1)^{3/2} dx \approx 0.375[f(x_1)+f(x_2)+...+f(x_8)] \approx 21.6784$

Trapezoidal Rule: $\int_1^4 (x+1)^{3/2} dx \approx \frac{0.375}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_7)+f(x_8)] \approx 20.1124$

Parabolic Rule: $\int_1^4 (x+1)^{3/2} dx \approx \frac{0.375}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_7)+f(x_8)] \approx 20.0979$

Fundamental Theorem of Calculus: $\int_1^4 (x+1)^{3/2} dx = \left[\frac{2}{5}(x+1)^{5/2}\right]_1^4 \approx 20.0979$

7.

	LRS	RRS	MRS	Trap	Parabolic
n = 4	0.5728	0.3728	0.4590	0.4728	0.4637
n = 8	0.5159	0.4159	0.4625	0.4659	0.4636
n = 16	0.4892	0.4392	0.4634	0.4642	0.4636

8.

	LRS	RRS	MRS	Trap	Parabolic
n = 4	1.2833	0.9500	1.0898	1.1167	1.1000
n = 8	1.1865	1.0199	1.0963	1.1032	1.0987
n = 16	1.1414	1.0581	1.0980	1.0998	1.0986

9.

	LRS	RRS	MRS	Trap	Parabolic
$n = 4$	2.6675	3.2855	2.9486	2.9765	2.9580
$n = 8$	2.8080	3.1171	2.9556	2.9625	2.9579
$n = 16$	2.8818	3.0363	2.9573	2.9591	2.9579

10.

	LRS	RRS	MRS	Trap	Parabolic
$n = 4$	10.3726	17.6027	13.6601	13.9876	13.7687
$n = 8$	12.0163	15.6314	13.7421	13.8239	13.7693
$n = 16$	12.8792	14.6867	13.7625	13.7830	13.7693

11. $f'(x) = -\frac{1}{x^2}$; $f''(x) = \frac{2}{x^3}$

The largest that $|f''(c)|$ can be on $[1,3]$ occurs when $c = 1$, and $|f''(1)| = 2$

$\frac{(3-1)^3}{12n^2}(2) \le 0.01$; $n \ge \sqrt{\frac{400}{3}}$ Round up: $n = 12$

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{0.167}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{11}) + f(x_{12})]$$
≈ 1.1007

12. $f'(x) = -\frac{1}{(1+x)^2}$; $f''(x) = \frac{2}{(1+x)^3}$

The largest that $|f''(c)|$ can be on $[1,3]$ occurs when $c = 1$, and $|f''(1)| = \frac{1}{4}$.

$\frac{(3-1)^3}{12n^2}\left(\frac{1}{4}\right) \le 0.01$; $n \ge \sqrt{\frac{100}{6}}$ Round up: $n = 5$

$$\int_1^3 \frac{1}{1+x}\,dx \approx \frac{0.4}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_4) + f(x_5)]$$
≈ 0.6956

13. $f'(x) = \frac{1}{2\sqrt{x}}$; $f''(x) = -\frac{1}{4x^{3/2}}$

The largest that $|f''(c)|$ can be on $[1,4]$ occurs when $c = 1$, and $|f''(1)| = \frac{1}{4}$.

$\frac{(4-1)^3}{12n^2}\left(\frac{1}{4}\right) \le 0.01$; $n \ge \sqrt{\frac{900}{16}}$ Round up: $n = 8$

$$\int_1^4 \sqrt{x}\,dx \approx \frac{0.375}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_7) + f(x_8)]$$
≈ 4.6637

14. $f'(x)=\frac{1}{2\sqrt{x+1}};\ f''(x)=-\frac{1}{4(x+1)^{3/2}}$

The largest that $\left|f''(c)\right|$ can be on $[1,3]$ occurs when $c=1$, and $\left|f''(1)\right|=\frac{1}{4\times 2^{3/2}}$.

$\frac{(3-1)^3}{12n^2}\left(\frac{1}{4\times 2^{3/2}}\right)\le 0.01;\quad n\ge\sqrt{\frac{100}{12\sqrt{2}}}$ Round up: $n = 3$

$$\int_1^3 \sqrt{x+1}\,dx \approx \frac{0.667}{2}[f(x_0)+2f(x_1)+2f(x_2)+f(x_3)]$$
$$\approx 3.4439$$

15. $f'(x)=-\frac{1}{x^2};\ f''(x)=\frac{2}{x^3};\ f'''(x)=-\frac{6}{x^4};$

$f^{(4)}(x)=\frac{24}{x^5}$

The largest that $\left|f^{(4)}(c)\right|$ can be on $[1,3]$ occurs when $c=1$, and $\left|f^{(4)}(1)\right|=24$.

$\frac{(4-1)^5}{180n^4}(24)\le 0.01;\quad n\approx 4.545$ Round up to even: $n = 6$

$$\int_1^3 \frac{1}{x}dx \approx \frac{0.333}{3}[f(x_0)+4f(x_1)+\ldots+4f(x_5)+f(x_6)]$$
$$\approx 1.0989$$

16. $f'(x)=\frac{1}{2\sqrt{x+1}};\ f''(x)=-\frac{1}{4(x+1)^{3/2}};$

$f'''(x)=\frac{3}{8(x+1)^{5/2}};\ f^{(4)}(x)=-\frac{15}{16(x+1)^{7/2}}$

The largest that $\left|f^{(4)}(c)\right|$ can be on $[4,8]$ occurs when $c=4$, and $\left|f^{(4)}(4)\right|=\frac{3}{400\sqrt{5}}$.

$\frac{(8-4)^5}{180n^4}\left(\frac{3}{400\sqrt{5}}\right)\le 0.01;\quad n\approx 1.1753$ Round up to even: $n = 2$

$$\int_4^8 \sqrt{x+1}\,dx \approx \frac{2}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]\approx 10.5464$$

17. $\int_{m-h}^{m+h}(ax^2+bx+c)dx=\left[\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right]_{m-h}^{m+h}$

$=\frac{a}{3}(m+h)^3+\frac{b}{2}(m+h)^2+c(m+h)-\frac{a}{3}(m-h)^3-\frac{b}{2}(m-h)^2-c(m-h)$

$=\frac{a}{3}(6m^2h+2h^3)+\frac{b}{2}(4mh)+c(2h)\ =\frac{h}{3}[a(6m^2+2h^2)+b(6m)+6c]$

$\frac{h}{3}[f(m-h)+4f(m)+f(m+h)]$

$=\frac{h}{3}[a(m-h)^2+b(m-h)+c+4am^2+4bm+4c+a(m+h)^2+b(m+h)+c]$

$=\frac{h}{3}[a(6m^2+2h^2)+b(6m)+6c]$

18. **a.** To show that the Parabolic Rule is exact, examine it on the interval $[m-h, m+h]$.

Let $f(x)=ax^3+bx^2+cx+d$, then

$\int_{m-h}^{m+h}f(x)\,dx$

$=\frac{a}{4}\left[(m+h)^4-(m-h)^4\right]+\frac{b}{3}\left[(m+h)^3-(m-h)^3\right]+\frac{c}{2}\left[(m+h)^2-(m-h)^2\right]+d[(m+h)-(m-h)]$

$=\frac{a}{4}(8m^3h+8h^3m)+\frac{b}{3}(6m^2h+2h^3)+\frac{c}{2}(4mh)+d(2h).$

The Parabolic Rule with $n = 2$ gives

$\int_{m-h}^{m+h}f(x)\,dx=\frac{h}{3}[f(m-h)+4f(m)+f(m+h)]=2am^3h+2amh^3+2bm^2h+\frac{2}{3}bh^3+2chm+2dh$

$=\frac{a}{4}(8m^3h+8mh^3)+\frac{b}{3}(6m^2h+2h^3)+\frac{c}{2}(4mh)+d(2h)$

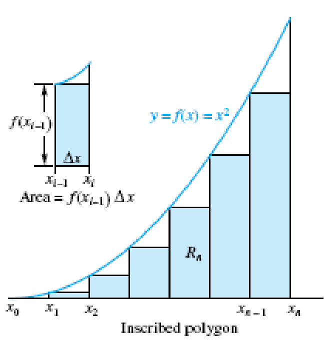
which agrees with the direct computation. Thus, the Parabolic Rule is exact for any cubic polynomial.

b. The error in using the Parabolic Rule is given by $E_n=-\frac{(l-k)^5}{180n^4}f^{(4)}(m)$ for some m between l and k.

However, $f'(x)=3ax^2+2bx+c,\ f''(x)=6ax+2b,\ f^{(3)}(x)=6a,$ and $f^{(4)}(x)=0$, so $E_n=0$.

19. The left Riemann sum will be smaller than $\int_a^b f(x)\,dx$.

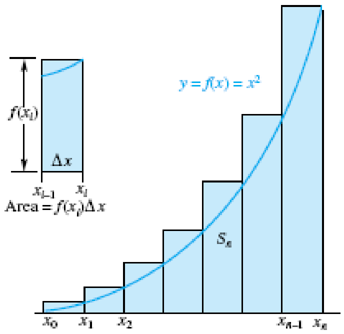
If the function is increasing, then $f(x_i)<f(x_{i+1})$ on the interval $[x_i,x_{i+1}]$. Therefore, the left Riemann sum will underestimate the value of the definite integral. The following example illustrates this behavior:



If f is increasing, then $f'(c)>0$ for any $c\in(a,b)$. Thus, the error $E_n=\frac{(b-a)^2}{2n}f'(c)>0$. Since the error is positive, then the Riemann sum must be less than the integral.

20. The right Riemann sum will be larger than $\int_a^b f(x)\,dx$.

If the function is increasing, then $f(x_i) < f(x_{i+1})$ on the interval $[x_i, x_{i+1}]$. Therefore, the right Riemann sum will overestimate the value of the definite integral. The following example illustrates this behavior:



Circumscribed polygon

If *f* is increasing, then $f'(c) > 0$ for any $c \in (a,b)$. Thus, the error $E_n = -\frac{(b-a)^2}{2n} f'(c) < 0$. Since the error is negative, then the Riemann sum must be greater than the integral.

21. The midpoint Riemann sum will be larger than $\int_a^b f(x)\,dx$.

If *f* is concave down, then $f''(c) < 0$ for any $c \in (a,b)$. Thus, the error $E_n = \frac{(b-a)^3}{24n^2} f''(c) < 0$. Since the error is negative, then the Riemann sum must be greater than the integral.

22. The Trapezoidal Rule approximation will be smaller than $\int_a^b f(x)\,dx$.

If *f* is concave down, then $f''(c) < 0$ for any $c \in (a,b)$. Thus, the error $E_n = -\frac{(b-a)^3}{12n^2} f''(c) > 0$. Since the error is positive, then the Trapezoidal Rule approximation must be less than the integral.

23. Let $n = 2$.

$f(x) = x^k;\ h = a$

$x_0 = -a \qquad f(x_0) = -a^k$

$x_1 = 0 \qquad f(x_1) = 0$

$x_2 = a \qquad f(x_2) = a^k$

$$\int_{-a}^{a} x^k\,dx \approx \frac{a}{2}[-a^k + 2\cdot 0 + a^k] = 0$$

$$\int_{-a}^{a} x^k\,dx = \left[\frac{1}{k+1}x^{k+1}\right]_{-a}^{a} = \frac{1}{k+1}[a^{k+1} - (-a)^{k+1}] = \frac{1}{k+1}[a^{k+1} - a^{k+1}] = 0$$

A corresponding argument works for all *n*.

24. **a.** $T \approx 48.9414;\ f'(x) = 4x^3$

$$T - \frac{[4(3)^3 - 4(1)^3](0.25)^2}{12} \approx 48.9414 - 0.5417 = 48.3997$$

The correct value is 48.4.

b. $T \approx 1.9886;\ f'(x) = \cos x$

$$T - \frac{[\cos\pi - \cos 0]\left(\frac{\pi}{12}\right)^2}{12} \approx 1.999987$$

The correct value is 2.

25. The integrand is increasing and concave down. By problems 19-22, LRS < TRAP < MRS < RRS.

26. The integrand is increasing and concave up. By problems 19-22, LRS < MRS < TRAP < RRS

27. $A \approx \frac{10}{2}[75+2\cdot71+2\cdot60+2\cdot45+2\cdot45+2\cdot52+2\cdot57+2\cdot60+59] = 4570\text{ ft}^2$

28. $A \approx \frac{3}{3}[23+4\cdot24+2\cdot23+4\cdot21+2\cdot18+4\cdot15+2\cdot12+4\cdot11+2\cdot10+4\cdot8+0] = 465\text{ ft}^2$

$V = A\cdot 6 \approx 2790\ \text{ft}^3$

29. $A \approx \frac{20}{3}[0+4\cdot7+2\cdot12+4\cdot18+2\cdot20+4\cdot20+2\cdot17+4\cdot10+0] = 2120\text{ ft}^2$

4 mi/h = 21,120 ft/h

(2120)(21,120)(24) = 1,074,585,600 ft^3

30. Using a right-Riemann sum,

$$\text{Distance} = \int_0^{24} v(t)\,dt \approx \sum_{i=1}^{8} v(t_i)\,\Delta t$$

$$= (31+54+53+52+35+31+28)\frac{3}{60}$$

$$= \frac{852}{60} = 14.2\ \text{miles}$$

31. Using a right-Riemann sum,

$$\text{Water Usage} = \int_0^{120} F(t)\,dt$$

$$\approx \sum_{i=1}^{10} F(t_i)\,\Delta t = 12(71+68+\cdots+148)$$

$$= 13{,}740\ \text{gallons}$$

4.7 Chapter Review

Concepts Test

1. True: Theorem 4.3.D

2. True: Obtained by integrating both sides of the Product Rule

3. True: If $F(x) = \int f(x)\,dx,\ f(x)$ is a derivative of $F(x)$.

4. False: $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 + 7x - 5$ are a counterexample.

5. False: The two sides will in general differ by a constant term.

6. True: At any given height, speed on the downward trip is the negative of speed on the upward.

7. True:

$$a_1 + a_0 + a_2 + a_1 + a_3 + a_2 + \cdots + a_{n-1} + a_{n-2} + a_n + a_{n-1} = a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-1} + a_n$$

8. True:

$$\sum_{i=1}^{100}(2i-1) = 2\sum_{i=1}^{100} i - \sum_{i=1}^{100} 1 = \frac{2(100)(100+1)}{2} - 100 = 10{,}000$$

9. True:

$$\sum_{i=1}^{10}(a_i+1)^2 = \sum_{i=1}^{10} a_i^2 + 2\sum_{i=1}^{10} a_i + \sum_{i=1}^{100} 1$$

= 100 + 2(20) + 10 = 150

10. False: f must also be continuous except at a finite number of points on $[a, b]$.

11. True: The area of a vertical line segment is 0.

12. False: $\int_{-1}^{1} x\,dx$ is a counterexample.

13. False: A counterexample is

$$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

with $\int_{-1}^{1} [f(x)]^2\,dx = 0$.

If $f(x)$ is continuous, then $[f(x)]^2 \geq 0$, and if $[f(x)]^2$ is greater than 0 on [*a*, *b*], the integral will be also.

14. False: $D_x\left[\int_a^x f(z)\,dz\right] = f(x)$

15. True: $\sin x + \cos x$ has period 2π, so $\int_x^{x+2\pi} (\sin x + \cos x)\,dx$ is independent of *x*.

16. True: $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$ and $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ when all the limits exist.

17. True: $\sin^{13} x$ is an odd function.

18. True: Theorem 4.2.B

19. False: The statement is not true if $c > d$.

20. False: $D_x\left[\int_0^{x^2} \frac{1}{1+t^2}\,dt\right] = \frac{2x}{1+x^2}$

21. True: Both sides equal 4.

22. True: Both sides equal 4.

23. True: If *f* is odd, then the accumulation function $F(x) = \int_0^x f(t)\,dt$ is even, and so is $F(x) + C$ for any *C*.

24. False: $f(x) = x^2$ is a counterexample.

25. False: $f(x) = x^2$ is a counterexample.

26. False: $f(x) = x^2$ is a counterexample.

27. False: $f(x) = x^2$, $v(x) = 2x + 1$ is a counterexample.

28. False: $f(x) = x^3$ is a counterexample.

29. False: $f(x) = \sqrt{x}$ is a counterexample.

30. True: All rectangles have height 4, regardless of $\overline{x}_i$.

31. True: $F(b) - F(a) = \int_a^b F'(x)\,dx = \int_a^b G'(x)\,dx = G(b) - G(a)$

32. False: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ because *f* is even.

33. False: $z(t) = t^2$ is a counterexample.

34. False: $\int_0^b f(x)\,dx = F(b) - F(0)$

35. True: Odd-exponent terms cancel themselves out over the interval, since they are odd.

36. False: $a = 0$, $b = 1$, $f(x) = -1$, $g(x) = 0$ is a counterexample.

37. False: $a = 0$, $b = 1$, $f(x) = -1$, $g(x) = 0$ is a counterexample.

38. True: $|a_1 + a_2 + a_3 + \cdots + a_n| \leq |a_1| + |a_2| + |a_3| + \cdots + |a_n|$ because any negative values of a_i make the left side smaller than the right side.

39. True: Note that $-|f(x)| \leq f(x) \leq |f(x)|$ and use Theorem 4.3.B.

40. True: Definition of Definite Integral

41. True: Definition of Definite Integral

42. False: Consider $\int \cos(x^2)\,dx$

43. True. Right Riemann sum always bigger.

44. True. Midpoint of *x* coordinate is midpoint of *y* coordinate.

45. False. Trapeziod rule overestimates integral.

46. True. Parabolic Rule gives exact value for quadratic and cubic functions.

Sample Test Problems

1. $\left[\frac{1}{4}x^4 - x^3 + 2x^{3/2}\right]_0^1 = \frac{5}{4}$

2. $\left[\frac{2}{3}x^3 - 3x - \frac{1}{x}\right]_1^2 = \frac{13}{6}$

3. $\left[\frac{1}{3}y^3 + 9\cos y - \frac{26}{y}\right]_1^{\pi} = \frac{50}{3} - \frac{26}{\pi} + \frac{\pi^3}{3} - 9\cos 1$

4. $\left[\frac{1}{3}(y^2 - 4)^{3/2}\right]_4^9 = -8\sqrt{3} + \frac{77\sqrt{77}}{3}$

5. $\left[\frac{3}{16}(2z^2 - 3)^{4/3}\right]_2^8 = \frac{-15\left(-125 + \sqrt[3]{5}\right)}{16}$

6. $\left[-\frac{1}{5}\cos^5 x\right]_0^{\pi/2} = \frac{1}{5}$

7. $u = \tan(3x^2 + 6x), du = (6x + 6)\sec^2(3x^2 + 6x)$

$\frac{1}{6}\int u^2 du = \frac{1}{18}u^3 + C$

$\frac{1}{18}\left[\tan^3(3x^2 + 6x)\right]_0^{\pi} = \frac{1}{18}\tan^3(3\pi^2 + 6\pi)$

8. $u = t^4 + 9, du = 4t^3 dt$

$\frac{1}{4}\int_9^{25} u^{-1/2} du = \frac{1}{2}\left[u^{1/2}\right]_9^{25} = 1$

9. $\frac{1}{5}\left[\frac{3}{5}(t^5 + 5)^{5/3}\right]_1^2 = \frac{3}{25}\left[37^{5/3} - 6^{5/3}\right] \approx 46.9$

10. $\left[\frac{1}{9y - 3y^3}\right]_2^3 = \frac{4}{27}$

11. $\int (x+1)\sin\left(x^2 + 2x + 3\right)dx$

$= \frac{1}{2}\int \sin\left(x^2 + 2x + 3\right)\left(2x + 2\right)dx$

$= \frac{1}{2}\int \sin u\, du$

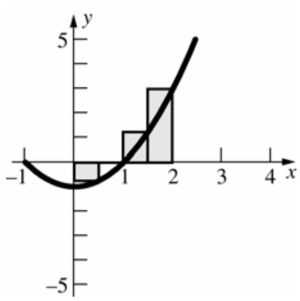
$= -\frac{1}{2}\cos\left(x^2 + 2x + 3\right) + C$

12. $u = 2y^3 + 3y^2 + 6y, du = (6y^2 + 6y + 6)\,dy$

$\int_1^5 \frac{\left(y^2 + y + 1\right)}{\sqrt[5]{2y^3 + 3y^2 + 6y}}dy = \frac{1}{6}\int_{11}^{355} u^{-1/5} du$

$= \frac{1}{6}\left[\frac{5}{4}u^{4/5}\right]_{11}^{355} = \frac{5}{24}\left(355^{4/5} - 11^{4/5}\right)$

13. $\sum_{i=1}^{4}\left[\left(\frac{i}{2}\right)^2 - 1\right]\left(\frac{1}{2}\right) = \frac{7}{4}$



14. $f'(x) = \frac{1}{x+3}, f'(7) = \frac{1}{10}$

15. $\int_0^3 (2 - \sqrt{x+1})^2 dx$

$= \int_0^3 \left(x + 5 - 4\sqrt{x+1}\right)dx$

$= \left[\frac{1}{2}x^2 + 5x - \frac{8}{3}(x+1)^{3/2}\right]_0^3 = \frac{5}{6}$

16. $\frac{1}{5-2}\int_2^5 3x^2\sqrt{x^3 - 4}\,dx = \frac{1}{3}\left[\frac{2}{3}(x^3 - 4)^{3/2}\right]_2^5$

$= 294$

17. $\int_2^4 \left(5 - \frac{1}{x^2}\right)dx = \left[5x + \frac{1}{x}\right]_2^4 = \frac{39}{4}$

18. $\sum_{i=1}^{n}(3^i - 3^{i-1})$

$= (3-1) + (3^2 - 3) + (3^3 - 3^2) + \cdots + (3^n - 3^{n-1})$

$= 3^n - 1$

19. $\sum_{i=1}^{10}(6i^2 - 8i) = 6\sum_{i=1}^{10} i^2 - 8\sum_{i=1}^{10} i$

$= 6\left[\frac{10(11)(21)}{6}\right] - 8\left[\frac{10(11)}{2}\right] = 1870$

20. a. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}$

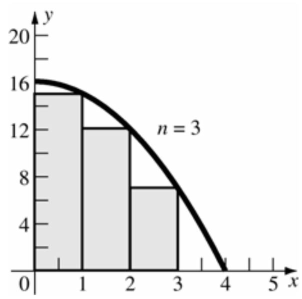
b. 1 + 0 + (–1) + (–2) + (–3) + (–4) = –9

c. $1+\frac{\sqrt{2}}{2}+0-\frac{\sqrt{2}}{2}-1=0$

21. a. $\sum_{n=2}^{78}\frac{1}{n}$

b. $\sum_{n=1}^{50}nx^{2n}$

22.



$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left[16-\left(\frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$
$$=\lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left[\frac{48}{n}-\frac{27}{n^3}i^2\right]\right\}$$
$$=\lim_{n\to\infty}\left\{\frac{48}{n}\sum_{i=1}^{n}1-\frac{27}{n^3}\sum_{i=1}^{n}i^2\right\}$$
$$=\lim_{n\to\infty}\left\{48-\frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right\}$$
$$=\lim_{n\to\infty}\left\{48-\frac{9}{2}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]\right\}$$
= 48 – 9 = 39

23. a. $\int_1^2 f(x)\,dx=\int_1^0 f(x)\,dx+\int_0^2 f(x)\,dx$
= –4 + 2 = –2

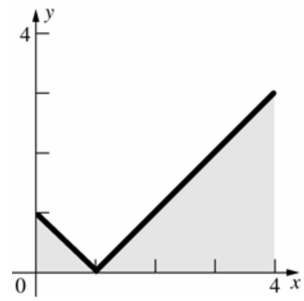
b. $\int_1^0 f(x)\,dx=-\int_0^1 f(x)\,dx=-4$

c. $\int_0^2 3f(u)\,du=3\int_0^2 f(u)\,du=3(2)=6$

d. $\int_0^2 [2g(x)-3f(x)]\,dx$
$=2\int_0^2 g(x)-3\int_0^2 f(x)\,dx$
= 2(–3) – 3(2) = –12

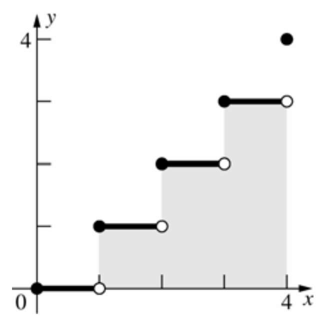
e. $\int_0^{-2} f(-x)\,dx=-\int_0^2 f(x)\,dx=-2$

24. a.



$\int_0^4 |x-1|\,dx=\frac{1}{2}(1)(1)+\frac{1}{2}(3)(3)=5$

b.



$\int_0^4 [\![x]\!]\,dx=1+2+3=6$

c. $\int_0^4 (x-[\![x]\!])\,dx=\int_0^4 x\,dx-\int_0^4 [\![x]\!]\,dx$
$\left[\frac{1}{2}x^2\right]_0^4-6=8-6=2$

25. a. $\int_{-2}^{2} f(x)\,dx=2\int_0^2 f(x)\,dx=2(-4)=-8$

b. Since $f(x)\le 0$, $|f(x)|=-f(x)$ and
$\int_{-2}^{2}|f(x)|\,dx=-\int_{-2}^{2}f(x)\,dx$
$=-2\int_0^2 f(x)\,dx=8$

c. $\int_{-2}^{2} g(x)\,dx=0$

d. $\int_{-2}^{2}[f(x)+f(-x)]\,dx$
$=2\int_0^2 f(x)\,dx+2\int_0^2 f(x)\,dx$
= 4(–4) = –16

e. $\int_0^2 [2g(x)+3f(x)]\,dx$
$= 2\int_0^2 g(x)\,dx + 3\int_0^2 f(x)\,dx$
$= 2(5) + 3(-4) = -2$

f. $\int_{-2}^0 g(x)\,dx = -\int_0^2 g(x)\,dx = -5$

26. $\int_{-100}^{100} (x^3+\sin^5 x)\,dx = 0$

27. $\int_{-4}^{-1} 3x^2 dx = 3c^2(-1+4)$
$\left[x^3\right]_{-4}^{-1} = 9c^2$
$c^2 = 7$
$c = -\sqrt{7} \approx -2.65$

28. **a.** $G'(x) = \dfrac{1}{x^2+1}$

b. $G'(x) = \dfrac{2x}{x^4+1}$

c. $G'(x) = \dfrac{3x^2}{x^6+1} - \dfrac{1}{x^2+1}$

29. **a.** $G'(x) = \sin^2 x$

b. $G'(x) = f(x+1) - f(x)$

c. $G'(x) = -\dfrac{1}{x^2}\int_0^x f(z)\,dz + \dfrac{1}{x}f(x)$

d. $G'(x) = \int_0^x f(t)\,dt$

e. $G(x) = \int_0^{g(x)} \dfrac{dg(u)}{du}du = [g(u)]_0^{g(x)}$
$= g(g(x)) - g(0)$
$G'(x) = g'(g(x))g'(x)$

f. $G(x) = \int_0^{-x} f(-t)\,dt = \int_0^x f(u)(-du)$
$= -\int_0^x f(u)\,du$
$G'(x) = -f(x)$

30. **a.** $\int_0^4 \sqrt{x}\,dx = \dfrac{2}{3}\left[x^{3/2}\right]_0^4 = \dfrac{16}{3}$

b. $\int_1^3 x^2 dx = \dfrac{1}{3}\left[x^3\right]_1^3 = \dfrac{26}{3}$

31. $f(x) = \int_{2x}^{5x}\dfrac{1}{t}dt = \int_1^{5x}\dfrac{1}{t}dt - \int_1^{2x}\dfrac{1}{t}dt$
$f'(x) = \dfrac{1}{5x}\cdot 5 - \dfrac{1}{2x}\cdot 2 = 0$

32. Left Riemann Sum: $\int_1^2 \dfrac{1}{1+x^4}dx \approx 0.125[f(x_0)+f(x_1)+\ldots+f(x_7)] \approx 0.2319$

Right Riemann Sum: $\int_1^2 \dfrac{1}{1+x^4}dx \approx 0.125[f(x_1)+f(x_2)+\ldots+f(x_8)] \approx 0.1767$

Midpoint Riemann Sum: $\int_1^2 \dfrac{1}{1+x^4}dx \approx 0.125[f(x_{0.5})+f(x_{1.5})+\ldots+f(x_{7.5})] \approx 0.2026$

33. $\int_1^2 \dfrac{1}{1+x^4}dx \approx \dfrac{0.125}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)] \approx 0.2043$

$$|f''(c)| = \left|\frac{4c^2(5c^4-3)}{(1+c^4)^3}\right| \le \frac{(4)(2^2)\left((5)(2^4)-3\right)}{\left(1+1^4\right)^3} = 154$$

$$|E_n| = \left|-\frac{(2-1)^3}{(12)8^2}f''(c)\right| = \frac{1}{(12)(64)}|f''(c)| \le \frac{154}{768} \approx 0.2005$$

Remark: A plot of f'' shows that in fact $|f''(c)| < 1.5$, so $|E_n| < 0.002$.

34. $\int_0^4 \frac{1}{1+2x}dx \approx \frac{0.5}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_7)+f(x_8)] \approx 1.1050$

$\left|f^{(4)}(c)\right| = \left|\frac{384}{(1+2c)^5}\right| \le 384$

$|E_n| = \left|-\frac{(4-0)^5}{180\cdot 8^4}\cdot f^{(4)}(c)\right| \le \frac{4^5\cdot 384}{180\cdot 8^4} = \frac{8}{15}$

35. $|f''(c)| = \left|\frac{4c^2(5c^4-3)}{(1+c^4)^3}\right| \le \frac{(4)(2^2)\left((5)(2^4)+3\right)}{\left(1+1^4\right)^3} = 166$

$|E_n| = \left|-\frac{(2-1)^3}{12n^2}f''(c)\right| = \frac{1}{12n^2}|f''(c)| \le \frac{166}{12n^2} < 0.0001$

$n^2 > \frac{166}{(12)(0.0001)} \approx 138{,}333$ so $n > \sqrt{138{,}333} \approx 371.9$ Round up to $n = 372$.

Remark: A plot of f'' shows that in fact $|f''(c)| < 1.5$ which leads to $n = 36$.

36. $\left|f^{(4)}(c)\right| = \left|\frac{384}{(1+2c)^5}\right| \le 384$

$|E_n| = \left|-\frac{(4-0)^5}{180\cdot n^4}\cdot f^{(4)}(c)\right| \le \frac{4^5\cdot 384}{180\cdot n^4} < 0.0001$

$n^4 > \frac{4^5\cdot 384}{180(0.0001)} \approx 21{,}845{,}333$, so $n \approx 68.4$. Round up to $n = 69$.

37. The integrand is decreasing and concave up. Therefore, we get:
Midpoint Rule, Trapezoidal rule, Left Riemann Sum

Review and Preview Problems

1. $\frac{1}{2}-\left(\frac{1}{2}\right)^2 = \frac{1}{2}-\frac{1}{4} = \frac{1}{4}$

2. $x - x^2$

3. the distance between $(1,4)$ and $(\sqrt[3]{4},4)$ is $\sqrt[3]{4}$ -1

4. the distance between $\left(\frac{y}{4},y\right)$ and $\left(\sqrt[3]{y},y\right)$ is $\sqrt[3]{y}-\frac{y}{4}$

5. the distance between (2,4) and (1,1) is $\sqrt{(2-1)^2+(4-1)^2} = \sqrt{10}$

6. $\sqrt{(x+h-x)^2+\left((x+h)^2-x^2\right)^2}$
$= \sqrt{h^2+\left(2xh+h^2\right)^2}$

7. $V = (\pi\cdot 2^2)0.4 = 1.6\pi$

8. $V = [\pi(4^2-1^2)]1 = 15\pi$

9. $V = [\pi(r_2^2 - r_1^2)]\Delta x$

10. $V = [\pi(5^2-4.5^2)]6 = 28.5\pi$

11. $\int_{-1}^{2}\left(x^4-2x^3+2\right)dx=\left[\frac{x^5}{5}-\frac{x^4}{2}+2x\right]_{-1}^{2}$

$=\frac{12}{5}-\left(-\frac{27}{10}\right)=\frac{51}{10}$

12. $\int_0^3 y^{2/3}dy=\frac{3}{5}\cdot y^{5/3}\Big|_0^3=\frac{3}{5}\cdot 3^{5/3}\approx 3.74$

13. $\int_0^2\left(1-\frac{x^2}{2}+\frac{x^4}{16}\right)dx=\left[x-\frac{x^3}{6}+\frac{x^5}{80}\right]_0^2=\frac{16}{15}$

14. Let $u=1+\frac{9}{4}x;$ then $du=\frac{9}{4}dx$ and

$\int\sqrt{1+\frac{9}{4}x}\,dx=\frac{4}{9}\int\sqrt{u}\,du=\frac{4}{9}\frac{2}{3}u^{3/2}+C$

$=\frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}+C$

Thus, $\int_1^4\sqrt{1+\frac{9}{4}x}\,dx=\left[\frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\right]_1^4$

$=\frac{8}{27}\left(10^{3/2}-\frac{13^{3/2}}{8}\right)\approx 7.63$