CHAPTER 7 Techniques of Integration

7.1 Concepts Review

1. elementary function

2. $\int u^5 du$

3. e^x

4. $\int_1^2 u^3 du$

Problem Set 7.1

1. $\int (x-2)^5 dx = \frac{1}{6}(x-2)^6 + C$

2. $\int \sqrt{3x}\, dx = \frac{1}{3}\int \sqrt{3x} \cdot 3dx = \frac{2}{9}(3x)^{3/2} + C$

3. $u = x^2 + 1, du = 2x\,dx$
When $x = 0, u = 1$ and when $x = 2, u = 5$.
$\int_0^2 x(x^2+1)^5 dx = \frac{1}{2}\int_0^2 (x^2+1)^5 (2x\,dx)$
$= \frac{1}{2}\int_1^5 u^5\, du$
$= \left[\frac{u^6}{12}\right]_1^5 = \frac{5^6 - 1^6}{12}$
$= \frac{15624}{12} = 1302$

4. $u = 1 - x^2, du = -2x\,dx$
When $x = 0, u = 1$ and when $x = 1, u = 0$.
$\int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int_0^1 \sqrt{1-x^2}\,(-2x\,dx)$
$= -\frac{1}{2}\int_1^0 u^{1/2}\, du$
$= \frac{1}{2}\int_0^1 u^{1/2}\, du$
$= \left[\frac{1}{3}u^{3/2}\right]_0^1 = \frac{1}{3}$

5. $\int \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

6. $u = 2 + e^x, du = e^x dx$
$\int \frac{e^x}{2+e^x} dx = \int \frac{du}{u}$
$= \ln|u| + C$
$= \ln\left|2+e^x\right| + C$
$= \ln(2+e^x) + C$

7. $u = x^2 + 4,\ du = 2x\, dx$
$\int \frac{x}{x^2+4} dx = \frac{1}{2}\int \frac{du}{u}$
$= \frac{1}{2}\ln|u| + C$
$= \frac{1}{2}\ln\left|x^2+4\right| + C$
$= \frac{1}{2}\ln(x^2+4) + C$

8. $\int \frac{2t^2}{2t^2+1} dt = \int \frac{2t^2+1-1}{2t^2+1} dt$
$= \int dt - \int \frac{1}{2t^2+1} dt$
$u = \sqrt{2}t, du = \sqrt{2}dt$
$t - \int \frac{1}{2t^2+1} dt = t - \frac{1}{\sqrt{2}}\int \frac{du}{1+u^2}$
$= t - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}t) + C$

9. $u = 4 + z^2,\ du = 2z\, dz$
$\int 6z\sqrt{4+z^2}dz = 3\int \sqrt{u}\, du$
$= 2u^{3/2} + C$
$= 2(4+z^2)^{3/2} + C$

10. $u = 2t + 1, du = 2dt$

$$\int \frac{5}{\sqrt{2t+1}} dt = \frac{5}{2} \int \frac{du}{\sqrt{u}}$$
$$= 5\sqrt{u} + C$$
$$= 5\sqrt{2t+1} + C$$

11. $\int \frac{\tan z}{\cos^2 z} dz = \int \tan z \sec^2 z \, dz$

$u = \tan z, \; du = \sec^2 z \, dz$

$$\int \tan z \sec^2 z \, dz = \int u \, du$$
$$= \frac{1}{2} u^2 + C$$
$$= \frac{1}{2} \tan^2 z + C$$

12. $u = \cos z, du = -\sin z \, dz$

$$\int e^{\cos z} \sin z \, dz = -\int e^{\cos z} (-\sin z \, dz)$$
$$= -\int e^u du = -e^u + C$$
$$= -e^{\cos z} + C$$

13. $u = \sqrt{t}, du = \frac{1}{2\sqrt{t}} dt$

$$\int \frac{\sin \sqrt{t}}{\sqrt{t}} dt = 2 \int \sin u \, du$$
$$= -2 \cos u + C$$
$$= -2 \cos \sqrt{t} + C$$

14. $u = x^2, \; du = 2x \, dx$

$$\int \frac{2x \, dx}{\sqrt{1-x^4}} = \int \frac{du}{\sqrt{1-u^2}}$$
$$= \sin^{-1} u + C$$
$$= \sin^{-1}(x^2) + C$$

15. $u = \sin x, du = \cos x \, dx$

$$\int_0^{\pi/4} \frac{\cos x}{1 + \sin^2 x} dx = \int_0^{\sqrt{2}/2} \frac{du}{1+u^2}$$
$$= [\tan^{-1} u]_0^{\sqrt{2}/2}$$
$$= \tan^{-1} \frac{\sqrt{2}}{2}$$
$$\approx 0.6155$$

16. $u = \sqrt{1-x}, du = -\frac{1}{2\sqrt{1-x}} dx$

$$\int_0^{3/4} \frac{\sin \sqrt{1-x}}{\sqrt{1-x}} dx = -2 \int_1^{1/2} \sin u \, du$$
$$= 2 \int_{1/2}^{1} \sin u \, du$$
$$= [-2 \cos u]_{1/2}^{1}$$
$$= -2\left(\cos 1 - \cos \frac{1}{2} \right)$$
$$\approx 0.6746$$

17. $\int \frac{3x^2 + 2x}{x+1} dx = \int (3x-1) dx + \int \frac{1}{x+1} dx$

$$= \frac{3}{2} x^2 - x + \ln|x+1| + C$$

18. $\int \frac{x^3 + 7x}{x-1} dx = \int (x^2 + x + 8) dx + 8 \int \frac{1}{x-1} dx$

$$= \frac{1}{3} x^3 + \frac{1}{2} x^2 + 8x + 8 \ln|x-1| + C$$

19. $u = \ln 4x^2, du = \frac{2}{x} dx$

$$\int \frac{\sin(\ln 4x^2)}{x} dx = \frac{1}{2} \int \sin u \, du$$
$$= -\frac{1}{2} \cos u + C$$
$$= -\frac{1}{2} \cos(\ln 4x^2) + C$$

20. $u = \ln x, \; du = \frac{1}{x} dx$

$$\int \frac{\sec^2 (\ln x)}{2x} dx = \frac{1}{2} \int \sec^2 u \, du$$
$$= \frac{1}{2} \tan u + C$$
$$= \frac{1}{2} \tan(\ln x) + C$$

21. $u = e^x, \quad du = e^x dx$

$$\int \frac{6e^x}{\sqrt{1-e^{2x}}} dx = 6 \int \frac{du}{\sqrt{1-u^2}} du$$
$$= 6 \sin^{-1} u + C$$
$$= 6 \sin^{-1}(e^x) + C$$

22. $u = x^2,\ du = 2x\,dx$

$$\int \frac{x}{x^4+4}dx = \frac{1}{2}\int \frac{du}{4+u^2}$$
$$= \frac{1}{4}\tan^{-1}\frac{u}{2} + C$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{x^2}{2}\right) + C$$

23. $u = 1 - e^{2x},\ du = -2e^{2x}dx$

$$\int \frac{3e^{2x}}{\sqrt{1-e^{2x}}}dx = -\frac{3}{2}\int \frac{du}{\sqrt{u}}$$
$$= -3\sqrt{u} + C$$
$$= -3\sqrt{1-e^{2x}} + C$$

24. $\int \frac{x^3}{x^4+4}dx = \frac{1}{4}\int \frac{4x^3}{x^4+4}dx$

$$= \frac{1}{4}\ln\left|x^4+4\right| + C$$
$$= \frac{1}{4}\ln(x^4+4) + C$$

25. $\int_0^1 t\,3^{t^2}\,dt = \frac{1}{2}\int_0^1 2t\,3^{t^2}\,dt$

$$= \left[\frac{3^{t^2}}{2\ln 3}\right]_0^1 = \frac{3}{2\ln 3} - \frac{1}{2\ln 3}$$
$$= \frac{1}{\ln 3} \approx 0.9102$$

26. $\int_0^{\pi/6} 2^{\cos x}\sin x\,dx = -\int_0^{\pi/6} 2^{\cos x}(-\sin x\,dx)$

$$= \left[-\frac{2^{\cos x}}{\ln 2}\right]_0^{\pi/6}$$
$$= -\frac{1}{\ln 2}(2^{\sqrt{3}/2} - 2)$$
$$= \frac{2 - 2^{\sqrt{3}/2}}{\ln 2}$$
$$\approx 0.2559$$

27. $\int \frac{\sin x - \cos x}{\sin x}dx = \int \left(1 - \frac{\cos x}{\sin x}\right)dx$

$u = \sin x,\ du = \cos x\,dx$

$$\int \frac{\sin x - \cos x}{\sin x}dx = x - \int \frac{du}{u}$$
$$= x - \ln|u| + C$$
$$= x - \ln|\sin x| + C$$

28. $u = \cos(4t - 1),\ du = -4\sin(4t - 1)dt$

$$\int \frac{\sin(4t-1)}{1-\sin^2(4t-1)}dt = \int \frac{\sin(4t-1)}{\cos^2(4t-1)}dt$$
$$= -\frac{1}{4}\int \frac{1}{u^2}du$$
$$= \frac{1}{4}u^{-1} + C = \frac{1}{4}\sec(4t-1) + C$$

29. $u = e^x,\ du = e^x dx$

$$\int e^x \sec e^x dx = \int \sec u\,du$$
$$= \ln\left|\sec u + \tan u\right| + C$$
$$= \ln\left|\sec e^x + \tan e^x\right| + C$$

30. $u = e^x,\ du = e^x dx$

$$\int e^x \sec^2(e^x)dx = \int \sec^2 u\,du = \tan u + C$$
$$= \tan(e^x) + C$$

31. $\int \frac{\sec^3 x + e^{\sin x}}{\sec x}dx = \int (\sec^2 x + e^{\sin x}\cos x)\,dx$

$$= \tan x + \int e^{\sin x}\cos x\,dx$$

$u = \sin x,\ du = \cos x\,dx$

$$\tan x + \int e^{\sin x}\cos x\,dx = \tan x + \int e^u\,du$$
$$= \tan x + e^u + C = \tan x + e^{\sin x} + C$$

32. $u = \sqrt{3t^2 - t - 1}$,

$$du = \frac{1}{2}(3t^2 - t - 1)^{-1/2}(6t-1)dt$$
$$\int \frac{(6t-1)\sin\sqrt{3t^2-t-1}}{\sqrt{3t^2-t-1}}dt = 2\int \sin u\,du$$
$$= -2\cos u + C$$
$$= -2\cos\sqrt{3t^2-t-1} + C$$

33. $u = t^3 - 2$, $du = 3t^2 dt$

$$\int \frac{t^2 \cos(t^3-2)}{\sin^2(t^3-2)} dt = \frac{1}{3}\int \frac{\cos u}{\sin^2 u} du$$

$v = \sin u$, $dv = \cos u\, du$

$$\frac{1}{3}\int \frac{\cos u}{\sin^2 u} du = \frac{1}{3}\int v^{-2} dv = -\frac{1}{3} v^{-1} + C$$

$$= -\frac{1}{3\sin u} + C$$

$$= -\frac{1}{3\sin(t^3-2)} + C.$$

34. $\int \frac{1+\cos 2x}{\sin^2 2x} dx = \int \frac{1}{\sin^2 2x} dx + \int \frac{\cos 2x}{\sin^2 2x} dx$

$$= \int \csc^2 2x\, dx + \int \cot 2x \csc 2x\, dx$$

$$= -\frac{1}{2}\cot 2x - \frac{1}{2}\csc 2x + C$$

35. $u = t^3 - 2$, $du = 3t^2 dt$

$$\int \frac{t^2 \cos^2(t^3-2)}{\sin^2(t^3-2)} dt = \frac{1}{3}\int \frac{\cos^2 u}{\sin^2 u} du$$

$$= \frac{1}{3}\int \cot^2 u\, du = \frac{1}{3}\int (\csc^2 u - 1) du$$

$$= \frac{1}{3}[-\cot u - u] + C_1$$

$$= \frac{1}{3}[-\cot(t^3-2) - (t^3-2)] + C_1$$

$$= -\frac{1}{3}[\cot(t^3-2) + t^3] + C$$

36. $u = 1 + \cot 2t$, $du = -2\csc^2 2t$

$$\int \frac{\csc^2 2t}{\sqrt{1+\cot 2t}} dt = -\frac{1}{2}\int \frac{1}{\sqrt{u}} du$$

$$= -\sqrt{u} + C$$

$$= -\sqrt{1+\cot 2t} + C$$

37. $u = \tan^{-1} 2t$, $du = \frac{2}{1+4t^2} dt$

$$\int \frac{e^{\tan^{-1} 2t}}{1+4t^2} dt = \frac{1}{2}\int e^u du$$

$$= \frac{1}{2} e^u + C = \frac{1}{2} e^{\tan^{-1} 2t} + C$$

38. $u = -t^2 - 2t - 5$,
$du = (-2t-2)dt = -2(t+1)dt$

$$\int (t+1) e^{-t^2-2t-5} = -\frac{1}{2}\int e^u du$$

$$= -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-t^2-2t-5} + C$$

39. $u = 3y^2$, $du = 6y\, dy$

$$\int \frac{y}{\sqrt{16-9y^4}} dy = \frac{1}{6}\int \frac{1}{\sqrt{4^2-u^2}} du$$

$$= \frac{1}{6}\sin^{-1}\left(\frac{u}{4}\right) + C$$

$$= \frac{1}{6}\sin^{-1}\left(\frac{3y^2}{4}\right) + C$$

40. $u = 3x$, $du = 3\,dx$

$$\int \cosh 3x\, dx$$

$$= \frac{1}{3}\int (\cosh u) du = \frac{1}{3}\sinh u + C$$

$$= \frac{1}{3}\sinh 3x + C$$

41. $u = x^3$, $du = 3x^2 dx$

$$\int x^2 \sinh x^3 dx = \frac{1}{3}\int \sinh u\, du$$

$$= \frac{1}{3}\cosh u + C$$

$$= \frac{1}{3}\cosh x^3 + C$$

42. $u = 2x$, $du = 2\, dx$

$$\int \frac{5}{\sqrt{9-4x^2}} dx = \frac{5}{2}\int \frac{1}{\sqrt{3^2-u^2}} du$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

43. $u = e^{3t}$, $du = 3e^{3t} dt$

$$\int \frac{e^{3t}}{\sqrt{4-e^{6t}}} dt = \frac{1}{3}\int \frac{1}{\sqrt{2^2-u^2}} du$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{e^{3t}}{2}\right) + C$$

44. $u = 2t$, $du = 2dt$

$$\int \frac{dt}{2t\sqrt{4t^2-1}} = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}} du$$

$$= \frac{1}{2}\left[\sec^{-1}|u|\right] + C$$

$$= \frac{1}{2}\sec^{-1}|2t| + C$$

45. $u = \cos x$, $du = -\sin x\,dx$

$$\int_0^{\pi/2} \frac{\sin x}{16+\cos^2 x}dx = -\int_1^0 \frac{1}{16+u^2}du$$

$$= \int_0^1 \frac{1}{16+u^2}du$$

$$= \left[\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right)\right]_0^1 = \left[\frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) - \frac{1}{4}\tan^{-1} 0\right]$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{1}{4}\right) \approx 0.0612$$

46. $u = e^{2x} + e^{-2x}$, $du = (2e^{2x} - 2e^{-2x})dx$

$= 2(e^{2x} - e^{-2x})dx$

$$\int_0^1 \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx = \frac{1}{2}\int_2^{e^2+e^{-2}} \frac{1}{u}du$$

$$= \frac{1}{2}\left[\ln|u|\right]_2^{e^2+e^{-2}} = \frac{1}{2}\ln\left|e^2+e^{-2}\right| - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\ln\left|\frac{e^4+1}{e^2}\right| - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\ln(e^4+1) - \frac{1}{2}\ln(e^2) - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\left(\ln\left(\frac{e^4+1}{2}\right) - 2\right) \approx 0.6625$$

47. $$\int \frac{1}{x^2+2x+5}dx = \int \frac{1}{x^2+2x+1+4}dx$$

$$= \int \frac{1}{(x+1)^2+2^2}d(x+1)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

48. $$\int \frac{1}{x^2-4x+9}dx = \int \frac{1}{x^2-4x+4+5}dx$$

$$= \int \frac{1}{(x-2)^2+(\sqrt{5})^2}d(x-2)$$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x-2}{\sqrt{5}}\right) + C$$

49. $$\int \frac{dx}{9x^2+18x+10} = \int \frac{dx}{9x^2+18x+9+1}$$

$$= \int \frac{dx}{(3x+3)^2+1^2}$$

$u = 3x + 3$, $du = 3\,dx$

$$\int \frac{dx}{(3x+3)^2+1^2} = \frac{1}{3}\int \frac{du}{u^2+1^2}$$

$$= \frac{1}{3}\tan^{-1}(3x+3) + C$$

50. $$\int \frac{dx}{\sqrt{16+6x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-6x+9-25)}}$$

$$= \int \frac{dx}{\sqrt{-(x-3)^2+5^2}} = \int \frac{dx}{\sqrt{5^2-(x-3)^2}}$$

$$= \sin^{-1}\left(\frac{x-3}{5}\right) + C$$

51. $$\int \frac{x+1}{9x^2+18x+10}dx = \frac{1}{18}\int \frac{18x+18}{9x^2+18x+10}dx$$

$$= \frac{1}{18}\ln\left|9x^2+18x+10\right| + C$$

$$= \frac{1}{18}\ln\left(9x^2+18x+10\right) + C$$

52. $$\int \frac{3-x}{\sqrt{16+6x-x^2}}dx = \frac{1}{2}\int \frac{6-2x}{\sqrt{16+6x-x^2}}dx$$

$$= \sqrt{16+6x-x^2} + C$$

53. $u = \sqrt{2}t$, $du = \sqrt{2}dt$

$$\int \frac{dt}{t\sqrt{2t^2-9}} = \int \frac{du}{u\sqrt{u^2-3^2}}$$

$$= \frac{1}{3}\sec^{-1}\left(\frac{\left|\sqrt{2}t\right|}{3}\right) + C$$

54. $$\int \frac{\tan x}{\sqrt{\sec^2 x-4}}dx = \int \frac{\cos x}{\cos x}\frac{\tan x}{\sqrt{\sec^2 x-4}}dx$$

$$= \int \frac{\sin x}{\sqrt{1-4\cos^2 x}}dx$$

$u = 2\cos x$, $du = -2\sin x\,dx$

$$\int \frac{\sin x}{\sqrt{1-4\cos^2 x}}dx = -\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$$

$$= -\frac{1}{2}\sin^{-1} u + C = -\frac{1}{2}\sin^{-1}(2\cos x) + C$$

55. The length is given by

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^{\pi/4} \sqrt{1+\left[\frac{1}{\cos x}(-\sin x)\right]^2}\,dx$$

$$= \int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx$$

$$= \int_0^{\pi/4} \sec x\,dx = \left[\ln\left|\sec x + \tan x\right|\right]_0^{\pi/4}$$

$$= \ln\left|\sqrt{2}+1\right| - \ln|1| = \ln\left|\sqrt{2}+1\right| \approx 0.881$$

56. $\sec x = \frac{1}{\cos x} = \frac{1+\sin x}{\cos x(1+\sin x)}$

$= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1+\sin x)} = \frac{\sin x(1+\sin x) + \cos^2 x}{\cos x(1+\sin x)}$

$= \frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$

$\int \sec x = \int \left(\frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}\right) dx$

$= \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1+\sin x} dx$

For the first integral use $u = \cos x$, $du = -\sin x\, dx$, and for the second integral use $v = 1 + \sin x$, $dv = \cos x\, dx$.

$\int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{1+\sin x} dx = -\int \frac{du}{u} + \int \frac{dv}{v}$

$= -\ln|u| + \ln|v| + C$

$= -\ln|\cos x| + \ln|1+\sin x| + C$

$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C$

$= \ln|\sec x + \tan x| + C$

57. $u = x - \pi$, $du = dx$

$\int_0^{2\pi} \frac{x|\sin x|}{1+\cos^2 x} dx = \int_{-\pi}^{\pi} \frac{(u+\pi)|\sin(u+\pi)|}{1+\cos^2(u+\pi)} du$

$= \int_{-\pi}^{\pi} \frac{(u+\pi)|\sin u|}{1+\cos^2 u} du$

$= \int_{-\pi}^{\pi} \frac{u|\sin u|}{1+\cos^2 u} du + \int_{-\pi}^{\pi} \frac{\pi|\sin u|}{1+\cos^2 u} du$

$\int_{-\pi}^{\pi} \frac{u|\sin u|}{1+\cos^2 u} du = 0$ by symmetry.

$\int_{-\pi}^{\pi} \frac{\pi|\sin u|}{1+\cos^2 u} du = 2\int_0^{\pi} \frac{\pi \sin u}{1+\cos^2 u} du$

$v = \cos u$, $dv = -\sin u\, du$

$-2\int_1^{-1} \frac{\pi}{1+v^2} dv = 2\pi\int_{-1}^{1} \frac{1}{1+v^2} dv$

$= 2\pi[\tan^{-1} v]_{-1}^{1} = 2\pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$

$= 2\pi\left(\frac{\pi}{2}\right) = \pi^2$

58. $V = 2\pi\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(x + \frac{\pi}{4}\right)|\sin x - \cos x|\, dx$

$u = x - \frac{\pi}{4}, du = dx$

$V = 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2}\right)\left|\sin\left(u + \frac{\pi}{4}\right) - \cos\left(u + \frac{\pi}{4}\right)\right| du$

$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2}\right)\left|\frac{\sqrt{2}}{2}\sin u + \frac{\sqrt{2}}{2}\cos u - \frac{\sqrt{2}}{2}\cos u + \frac{\sqrt{2}}{2}\sin u\right| du$

$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(u + \frac{\pi}{2}\right)\left|\sqrt{2}\sin u\right| du = 2\sqrt{2}\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u|\sin u|\, du + \sqrt{2}\pi^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin u|\, du$

$2\sqrt{2}\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u|\sin u|\, du = 0$ by symmetry. Therefore,

$V = \sqrt{2}\pi^2 2\int_0^{\frac{\pi}{2}} \sin u\, du = 2\sqrt{2}\pi^2[-\cos u]_0^{\frac{\pi}{2}} = 2\sqrt{2}\pi^2$

7.2 Concepts Review

1. $uv - \int v\,du$

2. x; $\sin x\,dx$

3. 1

4. reduction

Problem Set 7.2

1. $u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

2. $u = x \quad dv = e^{3x} dx$

$du = dx \quad v = \frac{1}{3}e^{3x}$

$\int xe^{3x} dx = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x} dx$

$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

3. $u = t \quad dv = e^{5t+\pi} dt$

$du = dt \quad v = \frac{1}{5}e^{5t+\pi}$

$\int te^{5t+\pi} dt = \frac{1}{5}te^{5t+\pi} - \int \frac{1}{5}e^{5t+\pi} dt$

$= \frac{1}{5}te^{5t+\pi} - \frac{1}{25}e^{5t+\pi} + C$

4. $u = t + 7 \quad dv = e^{2t+3} dt$

$du = dt \quad v = \frac{1}{2}e^{2t+3}$

$\int (t+7)e^{2t+3} dt = \frac{1}{2}(t+7)e^{2t+3} - \int \frac{1}{2}e^{2t+3} dt$

$= \frac{1}{2}(t+7)e^{2t+3} - \frac{1}{4}e^{2t+3} + C$

$= \frac{t}{2}e^{2t+3} + \frac{13}{4}e^{2t+3} + C$

5. $u = x \quad dv = \cos x\,dx$

$du = dx \quad v = \sin x$

$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C$

6. $u = x \quad dv = \sin 2x\,dx$

$du = dx \quad v = -\frac{1}{2}\cos 2x$

$\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\,dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

7. $u = t - 3 \quad dv = \cos(t-3)dt$

$du = dt \quad v = \sin(t-3)$

$\int (t-3)\cos(t-3)dt = (t-3)\sin(t-3) - \int \sin(t-3)dt$

$= (t-3)\sin(t-3) + \cos(t-3) + C$

8. $u = x - \pi \quad dv = \sin(x)dx$

$du = dx \quad v = -\cos x$

$\int (x-\pi)\sin(x)dx = -(x-\pi)\cos x + \int \cos x\,dx$

$= (\pi - x)\cos x + \sin x + C$

9. $u = t \quad dv = \sqrt{t+1}\,dt$

$du = dt \quad v = \frac{2}{3}(t+1)^{3/2}$

$\int t\sqrt{t+1}\,dt = \frac{2}{3}t(t+1)^{3/2} - \int \frac{2}{3}(t+1)^{3/2} dt$

$= \frac{2}{3}t(t+1)^{3/2} - \frac{4}{15}(t+1)^{5/2} + C$

10. $u = t \quad dv = \sqrt[3]{2t+7}\,dt$

$du = dt \quad v = \frac{3}{8}(2t+7)^{4/3}$

$\int t\sqrt[3]{2t+7}\,dt = \frac{3}{8}t(2t+7)^{4/3} - \int \frac{3}{8}(2t+7)^{4/3} dt$

$= \frac{3}{8}t(2t+7)^{4/3} - \frac{9}{112}(2t+7)^{7/3} + C$

11. $u = \ln 3x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$\int \ln 3x\,dx = x\ln 3x - \int x\frac{1}{x}dx = x\ln 3x - x + C$

12. $u = \ln(7x^5) \quad dv = dx$

$du = \frac{5}{x}dx \quad v = x$

$\int \ln(7x^5)dx = x\ln(7x^5) - \int x\frac{5}{x}dx$

$= x\ln(7x^5) - 5x + C$

13. $u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2}dx \qquad v = x$

$$\int \arctan x = x\arctan x - \int \frac{x}{1+x^2}dx$$

$$= x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

14. $u = \arctan 5x \qquad dv = dx$

$du = \frac{5}{1+25x^2}dx \qquad v = x$

$$\int \arctan 5x\,dx = x\arctan 5x - \int \frac{5x}{1+25x^2}dx$$

$$= x\arctan 5x - \frac{1}{10}\int \frac{50x\,dx}{1+25x^2}$$

$$= x\arctan 5x - \frac{1}{10}\ln(1+25x^2) + C$$

15. $u = \ln x \qquad dv = \frac{dx}{x^2}$

$du = \frac{1}{x}dx \qquad v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \int -\frac{1}{x}\left(\frac{1}{x}\right)dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

16. $u = \ln 2x^5 \qquad dv = \frac{1}{x^2}dx$

$du = \frac{5}{x}dx \qquad v = -\frac{1}{x}$

$$\int_2^3 \frac{\ln 2x^5}{x^2}dx = \left[-\frac{1}{x}\ln 2x^5\right]_2^3 + 5\int_2^3 \frac{1}{x^2}dx$$

$$= \left[-\frac{1}{x}\ln 2x^5 - \frac{5}{x}\right]_2^3$$

$$= \left(-\frac{1}{3}\ln(2\cdot 3^5) - \frac{5}{3}\right) - \left(-\frac{1}{2}\ln(2\cdot 2^5) - \frac{5}{2}\right)$$

$$= -\frac{1}{3}\ln 2 - \frac{5}{3}\ln 3 - \frac{5}{3} + 3\ln 2 + \frac{5}{2}$$

$$= \frac{8}{3}\ln 2 - \frac{5}{3}\ln 3 + \frac{5}{6} \approx 0.8507$$

17. $u = \ln t \qquad dv = \sqrt{t}\,dt$

$du = \frac{1}{t}dt \qquad v = \frac{2}{3}t^{3/2}$

$$\int_1^e \sqrt{t}\ln t\,dt = \left[\frac{2}{3}t^{3/2}\ln t\right]_1^e - \int_1^e \frac{2}{3}t^{1/2}dt$$

$$= \frac{2}{3}e^{3/2}\ln e - \frac{2}{3}\cdot 1\ln 1 - \left[\frac{4}{9}t^{3/2}\right]_1^e$$

$$= \frac{2}{3}e^{3/2} - 0 - \frac{4}{9}e^{3/2} + \frac{4}{9} = \frac{2}{9}e^{3/2} + \frac{4}{9} \approx 1.4404$$

18. $u = \ln x^3 \qquad dv = \sqrt{2x}dx$

$du = \frac{3}{x}dx \qquad v = \frac{1}{3}(2x)^{3/2}$

$$\int_1^5 \sqrt{2x}\ln x^3\,dx = \left[\frac{1}{3}(2x)^{3/2}\ln x^3\right]_1^5 - \int_1^5 2^{3/2}\sqrt{x}\,dx$$

$$= \left[\frac{1}{3}(2x)^{3/2}\ln x^3 - \frac{2^{5/2}}{3}x^{3/2}\right]_1^5$$

$$= \frac{1}{3}(10)^{3/2}\ln 5^3 - \frac{2^{5/2}}{3}5^{3/2} - \left(\frac{1}{3}(2)^{3/2}\ln 1^3 - \frac{2^{5/2}}{3}\right)$$

$$= -\frac{4\sqrt{2}}{3}5^{3/2} + \frac{4\sqrt{2}}{3} + 10^{3/2}\ln 5 \approx 31.699$$

19. $u = \ln z \qquad dv = z^3 dz$

$du = \frac{1}{z}dz \qquad v = \frac{1}{4}z^4$

$$\int z^3\ln z\,dz = \frac{1}{4}z^4\ln z - \int \frac{1}{4}z^4\cdot\frac{1}{z}dz$$

$$= \frac{1}{4}z^4\ln z - \frac{1}{4}\int z^3 dz$$

$$= \frac{1}{4}z^4\ln z - \frac{1}{16}z^4 + C$$

20. $u = \arctan t \qquad dv = t\,dt$

$du = \frac{1}{1+t^2}dt \qquad v = \frac{1}{2}t^2$

$$\int t\arctan t\,dt = \frac{1}{2}t^2\arctan t - \frac{1}{2}\int \frac{t^2}{1+t^2}dt$$

$$= \frac{1}{2}t^2\arctan t - \frac{1}{2}\int \frac{1+t^2-1}{1+t^2}dt$$

$$= \frac{1}{2}t^2\arctan t - \frac{1}{2}\int dt + \frac{1}{2}\int \frac{1}{1+t^2}dt$$

$$= \frac{1}{2}t^2\arctan t - \frac{1}{2}t + \frac{1}{2}\arctan t + C$$

21. $u = \arctan\left(\frac{1}{t}\right) \qquad dv = dt$

$du = -\frac{1}{1+t^2}dt \qquad v = t$

$$\int \arctan\left(\frac{1}{t}\right)dt = t\arctan\left(\frac{1}{t}\right) + \int \frac{t}{1+t^2}dt$$

$$= t\arctan\left(\frac{1}{t}\right) + \frac{1}{2}\ln(1+t^2) + C$$

22. $u = \ln(t^7) \qquad dv = t^5 dt$

$du = \frac{7}{t}dt \qquad v = \frac{1}{6}t^6$

$$\int t^5 \ln(t^7)dt = \frac{1}{6}t^6 \ln(t^7) - \frac{7}{6}\int t^5 dt$$

$$= \frac{1}{6}t^6 \ln(t^7) - \frac{7}{36}t^6 + C$$

23. $u = x \qquad dv = \csc^2 x\, dx$

$du = dx \qquad v = -\cot x$

$$\int_{\pi/6}^{\pi/2} x\csc^2 x\, dx = \left[-x\cot x\right]_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2} \cot x\, dx = \left[-x\cot x + \ln|\sin x|\right]_{\pi/6}^{\pi/2}$$

$$= -\frac{\pi}{2}\cdot 0 + \ln 1 + \frac{\pi}{6}\sqrt{3} - \ln\frac{1}{2} = \frac{\pi}{2\sqrt{3}} + \ln 2 \approx 1.60$$

24. $u = x \qquad dv = \sec^2 x\, dx$

$du = dx \qquad v = \tan x$

$$\int_{\pi/6}^{\pi/4} x\sec^2 x\, dx = \left[x\tan x\right]_{\pi/6}^{\pi/4} - \int_{\pi/6}^{\pi/4} \tan x\, dx = \left[x\tan x + \ln|\cos x|\right]_{\pi/6}^{\pi/4} = \frac{\pi}{4} + \ln\frac{\sqrt{2}}{2} - \left(\frac{\pi}{6\sqrt{3}} + \ln\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6\sqrt{3}} + \frac{1}{2}\ln\frac{2}{3} \approx 0.28$$

25. $u = x^3 \qquad dv = x^2\sqrt{x^3+4}dx$

$du = 3x^2 dx \qquad v = \frac{2}{9}(x^3+4)^{3/2}$

$$\int x^5\sqrt{x^3+4}dx = \frac{2}{9}x^3(x^3+4)^{3/2} - \int \frac{2}{3}x^2(x^3+4)^{3/2}dx = \frac{2}{9}x^3(x^3+4)^{3/2} - \frac{4}{45}(x^3+4)^{5/2} + C$$

26. $u = x^7 \qquad dv = x^6\sqrt{x^7+1}\,dx$

$du = 7x^6 dx \qquad v = \frac{2}{21}(x^7+1)^{3/2}$

$$\int x^{13}\sqrt{x^7+1}dx = \frac{2}{21}x^7(x^7+1)^{3/2} - \int \frac{2}{3}x^6(x^7+1)^{3/2}dx = \frac{2}{21}x^7(x^7+1)^{3/2} - \frac{4}{105}(x^7+1)^{5/2} + C$$

27. $u = t^4 \qquad dv = \frac{t^3}{(7-3t^4)^{3/2}}dt$

$du = 4t^3\, dt \qquad v = \frac{1}{6(7-3t^4)^{1/2}}$

$$\int \frac{t^7}{(7-3t^4)^{3/2}}dt = \frac{t^4}{6(7-3t^4)^{1/2}} - \frac{2}{3}\int \frac{t^3}{(7-3t^4)^{1/2}}dt = \frac{t^4}{6(7-3t^4)^{1/2}} + \frac{1}{9}(7-3t^4)^{1/2} + C$$

28. $u = x^2 \qquad dv = x\sqrt{4-x^2}\,dx$

$du = 2x\,dx \qquad v = -\frac{1}{3}(4-x^2)^{3/2}$

$$\int x^3\sqrt{4-x^2}\,dx = -\frac{1}{3}x^2(4-x^2)^{3/2} + \frac{2}{3}\int x(4-x^2)^{3/2}\,dx = -\frac{1}{3}x^2(4-x^2)^{3/2} - \frac{2}{15}(4-x^2)^{5/2} + C$$

29. $u = z^4 \qquad dv = \frac{z^3}{(4-z^4)^2}\,dz$

$du = 4z^3dz \qquad v = \frac{1}{4(4-z^4)}$

$$\int \frac{z^7}{(4-z^4)^2}\,dz = \frac{z^4}{4(4-z^4)} - \int \frac{z^3}{4-z^4}dz = \frac{z^4}{4(4-z^4)} + \frac{1}{4}\ln\left|4-z^4\right| + C$$

30. $u = x \qquad dv = \cosh x\,dx$

$du = dx \qquad v = \sinh x$

$$\int x\cosh x\,dx = x\sinh x - \int \sinh x\,dx = x\sinh x - \cosh x + C$$

31. $u = x \qquad dv = \sinh x\,dx$

$du = dx \qquad v = \cosh x$

$$\int x\sinh x\,dx = x\cosh x - \int \cosh x\,dx = x\cosh x - \sinh x + C$$

32. $u = \ln x \qquad dv = x^{-1/2}dx$

$du = \frac{1}{x}dx \qquad v = 2x^{1/2}$

$$\int \frac{\ln x}{\sqrt{x}}dx = 2\sqrt{x}\ln x - 2\int \frac{1}{x^{1/2}}dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

33. $u = x \qquad dv = (3x+10)^{49}dx$

$du = dx \qquad v = \frac{1}{150}(3x+10)^{50}$

$$\int x(3x+10)^{49}dx = \frac{x}{150}(3x+10)^{50} - \frac{1}{150}\int (3x+10)^{50}dx = \frac{x}{150}(3x+10)^{50} - \frac{1}{22,950}(3x+10)^{51} + C$$

34. $u = t \qquad dv = (t-1)^{12}\,dt$

$du = dt \qquad v = \frac{1}{13}(t-1)^{13}$

$$\int_0^1 t(t-1)^{12}\,dt = \left[\frac{t}{13}(t-1)^{13}\right]_0^1 - \frac{1}{13}\int_0^1 (t-1)^{13}\,dt$$

$$= \left[\frac{t}{13}(t-1)^{13} - \frac{1}{182}(t-1)^{14}\right]_0^1 = \frac{1}{182}$$

35. $u = x \qquad dv = 2^x dx$

$du = dx \qquad v = \frac{1}{\ln 2}2^x$

$$\int x2^x dx = \frac{x}{\ln 2}2^x - \frac{1}{\ln 2}\int 2^x dx$$

$$= \frac{x}{\ln 2}2^x - \frac{1}{(\ln 2)^2}2^x + C$$

36. $u = z \qquad dv = a^z dz$

$du = dz \qquad v = \frac{1}{\ln a}a^z$

$$\int za^z dz = \frac{z}{\ln a}a^z - \frac{1}{\ln a}\int a^z dz$$

$$= \frac{z}{\ln a}a^z - \frac{1}{(\ln a)^2}a^z + C$$

37. $u = x^2 \qquad dv = e^x dx$

$du = 2x\,dx \qquad v = e^x$

$$\int x^2e^x dx = x^2e^x - \int 2xe^x dx$$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$\int x^2e^x\,dx = x^2e^x - 2\left(xe^x - \int e^x\,dx\right)$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

38. $u = x^4 \qquad dv = xe^{x^2}dx$

$du = 4x^3dx \qquad v = \frac{1}{2}e^{x^2}$

$\int x^5e^{x^2}dx = \frac{1}{2}x^4e^{x^2} - \int 2x^3e^{x^2}dx$

$u = x^2 \qquad dv = 2xe^{x^2}dx$

$du = 2x\,dx \qquad v = e^{x^2}$

$\int x^5e^{x^2}dx = \frac{1}{2}x^4e^{x^2} - \left(x^2e^{x^2} - \int 2xe^{x^2}dx\right)$

$= \frac{1}{2}x^4e^{x^2} - x^2e^{x^2} + e^{x^2} + C$

39. $u = \ln^2 z \qquad dv = dz$

$du = \frac{2\ln z}{z}dz \qquad v = z$

$\int \ln^2 z\,dz = z\ln^2 z - 2\int \ln z\,dz$

$u = \ln z \qquad dv = dz$

$du = \frac{1}{z}dz \qquad v = z$

$\int \ln^2 z\,dz = z\ln^2 z - 2\left(z\ln z - \int dz\right)$

$= z\ln^2 z - 2z\ln z + 2z + C$

40. $u = \ln^2 x^{20} \qquad dv = dx$

$du = \frac{40\ln x^{20}}{x}dx \qquad v = x$

$\int \ln^2 x^{20}dx = x\ln^2 x^{20} - 40\int \ln x^{20}dx$

$u = \ln x^{20} \qquad dv = dx$

$du = \frac{20}{x}dx \qquad v = x$

$\int \ln^2 x^{20}dx = x\ln^2 x^{20} - 40\left(x\ln x^{20} - 20\int dx\right)$

$= x\ln^2 x^{20} - 40x\ln x^{20} + 800x + C$

41. $u = e^t \qquad dv = \cos t\,dt$

$du = e^t dt \qquad v = \sin t$

$\int e^t \cos t\,dt = e^t \sin t - \int e^t \sin t\,dt$

$u = e^t \qquad dv = \sin t\,dt$

$du = e^t dt \qquad v = -\cos t$

$\int e^t \cos t\,dt = e^t \sin t - \left[-e^t \cos t + \int e^t \cos t\,dt\right]$

$\int e^t \cos t\,dt = e^t \sin t + e^t \cos t - \int e^t \cos t\,dt$

$2\int e^t \cos t\,dt = e^t \sin t + e^t \cos t + C$

$\int e^t \cos t\,dt = \frac{1}{2}e^t(\sin t + \cos t) + C$

42. $u = e^{at} \qquad dv = \sin t\,dt$

$du = ae^{at}dt \qquad v = -\cos t$

$\int e^{at}\sin t\,dt = -e^{at}\cos t + a\int e^{at}\cos t\,dt$

$u = e^{at} \qquad dv = \cos t\,dt$

$du = ae^{at}dt \qquad v = \sin t$

$\int e^{at}\sin t\,dt = -e^{at}\cos t + a\left(e^{at}\sin t - a\int e^{at}\sin t\,dt\right)$

$\int e^{at}\sin t\,dt = -e^{at}\cos t + ae^{at}\sin t - a^2\int e^{at}\sin t\,dt$

$(1+a^2)\int e^{at}\sin t\,dt = -e^{at}\cos t + ae^{at}\sin t + C$

$\int e^{at}\sin t\,dt = \frac{-e^{at}\cos t}{a^2+1} + \frac{ae^{at}\sin t}{a^2+1} + C$

43. $u = x^2 \qquad dv = \cos x\,dx$

$du = 2x\,dx \qquad v = \sin x$

$\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx$

$u = 2x \qquad dv = \sin x\,dx$

$du = 2dx \qquad v = -\cos x$

$\int x^2\cos x\,dx = x^2\sin x - \left(-2x\cos x + \int 2\cos x\,dx\right)$

$= x^2\sin x + 2x\cos x - 2\sin x + C$

44. $u = r^2 \qquad dv = \sin r\,dr$

$du = 2r\,dr \qquad v = -\cos r$

$\int r^2\sin r\,dr = -r^2\cos r + 2\int r\cos r\,dr$

$u = r \qquad dv = \cos r\,dr$

$du = dr \qquad v = \sin r$

$\int r^2\sin r\,dr = -r^2\cos r + 2\left(r\sin r - \int \sin r\,dr\right) = -r^2\cos r + 2r\sin r + 2\cos r + C$

45. $u = \sin(\ln x)$ $\quad dv = dx$

$du = \cos(\ln x) \cdot \frac{1}{x} dx$ $\quad v = x$

$$\int \sin(\ln x) dx = x \sin(\ln x) - \int \cos(\ln x)\, dx$$

$u = \cos(\ln x)$ $\quad dv = dx$

$du = -\sin(\ln x) \cdot \frac{1}{x} dx$ $\quad v = x$

$$\int \sin(\ln x) dx = x \sin(\ln x) - \left[x \cos(\ln x) - \int -\sin(\ln x) dx \right]$$

$$\int \sin(\ln x) dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx$$

$$2\int \sin(\ln x) dx = x \sin(\ln x) - x \cos(\ln x) + C$$

$$\int \sin(\ln x) dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C$$

46. $u = \cos(\ln x)$ $\quad dv = dx$

$du = -\sin(\ln x) \frac{1}{x} dx$ $\quad v = x$

$$\int \cos(\ln x) dx = x \cos(\ln x) + \int \sin(\ln x) dx$$

$u = \sin(\ln x)$ $\quad dv = dx$

$du = \cos(\ln x) \frac{1}{x} dx$ $\quad v = x$

$$\int \cos(\ln x) dx = x \cos(\ln x) + \left[x \sin(\ln x) - \int \cos(\ln x) dx \right]$$

$$2\int \cos(\ln x) dx = x[\cos(\ln x) + \sin(\ln x)] + C$$

$$\int \cos(\ln x) dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C$$

47. $u = (\ln x)^3$ $\quad dv = dx$

$du = \frac{3\ln^2 x}{x} dx$ $\quad v = x$

$$\int (\ln x)^3 dx = x(\ln x)^3 - 3\int \ln^2 x\, dx$$

$$= x \ln^3 x - 3(x \ln^2 x - 2x \ln x + 2x + C)$$

$$= x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + C$$

48. $u = (\ln x)^4$ $\quad dv = dx$

$du = \frac{4\ln^3 x}{x} dx$ $\quad v = x$

$$\int (\ln x)^4 dx = x(\ln x)^4 - 4\int \ln^3 x\, dx = x \ln^4 x - 4(x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + C)$$

$$= x \ln^4 x - 4x \ln^3 x + 12x \ln^2 x - 24x \ln x + 24x + C$$

49. $u = \sin x$ $\quad dv = \sin(3x) dx$

$du = \cos x\, dx$ $\quad v = -\frac{1}{3}\cos(3x)$

$$\int \sin x \sin(3x) dx = -\frac{1}{3} \sin x \cos(3x) + \frac{1}{3}\int \cos x \cos(3x) dx$$

$u = \cos x$ $\quad dv = \cos(3x) dx$

$du = -\sin x\, dx$ $\quad v = \frac{1}{3}\sin(3x)$

$$\int \sin x \sin(3x)dx = -\frac{1}{3}\sin x\cos(3x) + \frac{1}{3}\left[\frac{1}{3}\cos x\sin(3x) + \frac{1}{3}\int \sin x\sin(3x)dx\right]$$

$$= -\frac{1}{3}\sin x\cos(3x) + \frac{1}{9}\cos x\sin(3x) + \frac{1}{9}\int \sin x\sin(3x)dx$$

$$\frac{8}{9}\int \sin x\sin(3x)dx = -\frac{1}{3}\sin x\cos(3x) + \frac{1}{9}\cos x\sin(3x) + C$$

$$\int \sin x\sin(3x)dx = -\frac{3}{8}\sin x\cos(3x) + \frac{1}{8}\cos x\sin(3x) + C$$

50. $u = \cos(5x)$ $\quad dv = \sin(7x)dx$

$du = -5\sin(5x)dx$ $\quad v = -\frac{1}{7}\cos(7x)$

$$\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{7}\int \sin(5x)\cos(7x)dx$$

$u = \sin(5x)$ $\quad dv = \cos(7x)dx$

$du = 5\cos(5x)dx$ $\quad v = \frac{1}{7}\sin(7x)$

$$\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{7}\left[\frac{1}{7}\sin(5x)\sin(7x) - \frac{5}{7}\int \cos(5x)\sin(7x)dx\right]$$

$$= -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{49}\sin(5x)\sin(7x) + \frac{25}{49}\int \cos(5x)\sin(7x)dx$$

$$\frac{24}{49}\int \cos(5x)\sin(7x)dx = -\frac{1}{7}\cos(5x)\cos(7x) - \frac{5}{49}\sin(5x)\sin(7x) + C$$

$$\int \cos(5x)\sin(7x)dx = -\frac{7}{24}\cos(5x)\cos(7x) - \frac{5}{24}\sin(5x)\sin(7x) + C$$

51. $u = e^{\alpha z}$ $\quad dv = \sin \beta z\, dz$

$du = \alpha e^{\alpha z} dz$ $\quad v = -\frac{1}{\beta}\cos \beta z$

$$\int e^{\alpha z}\sin \beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos \beta z + \frac{\alpha}{\beta}\int e^{\alpha z}\cos \beta z\, dz$$

$u = e^{\alpha z}$ $\quad dv = \cos \beta z\, dz$

$du = \alpha e^{\alpha z} dz$ $\quad v = \frac{1}{\beta}\sin \beta z$

$$\int e^{\alpha z}\sin \beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos \beta z + \frac{\alpha}{\beta}\left[\frac{1}{\beta}e^{\alpha z}\sin \beta z - \frac{\alpha}{\beta}\int e^{\alpha z}\sin \beta z\, dz\right]$$

$$= -\frac{1}{\beta}e^{\alpha z}\cos \beta z + \frac{\alpha}{\beta^2}e^{\alpha z}\sin \beta z - \frac{\alpha^2}{\beta^2}\int e^{\alpha z}\sin \beta z\, dz$$

$$\frac{\beta^2 + \alpha^2}{\beta^2}\int e^{\alpha z}\sin \beta z\, dz = -\frac{1}{\beta}e^{\alpha z}\cos \beta z + \frac{\alpha}{\beta^2}e^{\alpha z}\sin \beta z + C$$

$$\int e^{\alpha z}\sin \beta z\, dz = \frac{-\beta}{\alpha^2 + \beta^2}e^{\alpha z}\cos \beta z + \frac{\alpha}{\alpha^2 + \beta^2}e^{\alpha z}\sin \beta z + C = \frac{e^{\alpha z}(\alpha \sin \beta z - \beta \cos \beta z)}{\alpha^2 + \beta^2} + C$$

52. $u = e^{\alpha z}$ $\quad dv = \cos\beta z\ dz$

$du = \alpha e^{\alpha z} dz \quad v = \frac{1}{\beta}\sin\beta z$

$$\int e^{\alpha z}\cos\beta z\,dz = \frac{1}{\beta}e^{\alpha z}\sin\beta z - \frac{\alpha}{\beta}\int e^{\alpha z}\sin\beta z\,dz$$

$u = e^{\alpha z}$ $\quad dv = \sin\beta z\ dz$

$du = \alpha e^{\alpha z} dz \quad v = -\frac{1}{\beta}\cos\beta z$

$$\int e^{\alpha z}\cos\beta z\,dz = \frac{1}{\beta}e^{\alpha z}\sin\beta z - \frac{\alpha}{\beta}\left[-\frac{1}{\beta}e^{\alpha z}\cos\beta z + \frac{\alpha}{\beta}\int e^{\alpha z}\cos\beta z\,dz\right]$$

$$= \frac{1}{\beta}e^{\alpha z}\sin\beta z + \frac{\alpha}{\beta^2}e^{\alpha z}\cos\beta z - \frac{\alpha^2}{\beta^2}\int e^{\alpha z}\cos\beta z\,dz$$

$$\frac{\alpha^2+\beta^2}{\beta^2}\int e^{\alpha z}\cos\beta z\,dz = \frac{\alpha}{\beta^2}e^{\alpha z}\cos\beta z + \frac{1}{\beta}e^{\alpha z}\sin\beta z + C$$

$$\int e^{\alpha z}\cos\beta z\,dz = \frac{e^{\alpha z}(\alpha\cos\beta z + \beta\sin\beta z)}{\alpha^2+\beta^2} + C$$

53. $u = \ln x$ $\quad dv = x^{\alpha} dx$

$du = \frac{1}{x}dx \quad v = \frac{x^{\alpha+1}}{\alpha+1},\ \alpha \neq -1$

$$\int x^{\alpha}\ln x\,dx = \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{1}{\alpha+1}\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} + C,\ \alpha \neq -1$$

54. $u = (\ln x)^2$ $\quad dv = x^{\alpha} dx$

$du = \frac{2\ln x}{x}dx \quad v = \frac{x^{\alpha+1}}{\alpha+1},\ \alpha \neq -1$

$$\int x^{\alpha}(\ln x)^2 dx = \frac{x^{\alpha+1}}{\alpha+1}(\ln x)^2 - \frac{2}{\alpha+1}\int x^{\alpha}\ln x\,dx = \frac{x^{\alpha+1}}{\alpha+1}(\ln x)^2 - \frac{2}{\alpha+1}\left[\frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2}\right] + C$$

$$= \frac{x^{\alpha+1}}{\alpha+1}(\ln x)^2 - 2\frac{x^{\alpha+1}}{(\alpha+1)^2}\ln x + 2\frac{x^{\alpha+1}}{(\alpha+1)^3} + C,\ \alpha \neq -1$$

Problem 53 was used for $\int x^{\alpha}\ln x\,dx$.

55. $u = x^{\alpha}$ $\quad dv = e^{\beta x}\,dx$

$du = \alpha x^{\alpha-1}dx \quad v = \frac{1}{\beta}e^{\beta x}$

$$\int x^{\alpha}e^{\beta x}dx = \frac{x^{\alpha}e^{\beta x}}{\beta} - \frac{\alpha}{\beta}\int x^{\alpha-1}e^{\beta x}dx$$

56. $u = x^{\alpha}$ $\quad dv = \sin\beta x\ dx$

$du = \alpha x^{\alpha-1}dx \quad v = -\frac{1}{\beta}\cos\beta x$

$$\int x^{\alpha}\sin\beta x\,dx = -\frac{x^{\alpha}\cos\beta x}{\beta} + \frac{\alpha}{\beta}\int x^{\alpha-1}\cos\beta x\,dx$$

57. $u = x^{\alpha}$ $\qquad dv = \cos \beta x\, dx$

$du = \alpha x^{\alpha-1} dx \qquad v = \frac{1}{\beta}\sin \beta x$

$$\int x^{\alpha} \cos \beta x\, dx = \frac{x^{\alpha} \sin \beta x}{\beta} - \frac{\alpha}{\beta}\int x^{\alpha-1} \sin \beta x\, dx$$

58. $u = (\ln x)^{\alpha}$ $\qquad dv = dx$

$du = \frac{\alpha(\ln x)^{\alpha-1}}{x} dx \qquad v = x$

$$\int (\ln x)^{\alpha} dx = x(\ln x)^{\alpha} - \alpha \int (\ln x)^{\alpha-1} dx$$

59. $u = (a^2 - x^2)^{\alpha}$ $\qquad dv = dx$

$du = -2\alpha x(a^2 - x^2)^{\alpha-1} dx \qquad v = x$

$$\int (a^2 - x^2)^{\alpha} dx = x(a^2 - x^2)^{\alpha} + 2\alpha \int x^2 (a^2 - x^2)^{\alpha-1} dx$$

60. $u = \cos^{\alpha-1} x$ $\qquad dv = \cos x\, dx$

$du = -(\alpha - 1)\cos^{\alpha-2} x \sin x\, dx \qquad v = \sin x$

$$\int \cos^{\alpha} x\, dx = \cos^{\alpha-1} x \sin x + (\alpha-1)\int \cos^{\alpha-2} x \sin^2 x\, dx$$

$$= \cos^{\alpha-1} x \sin x + (\alpha-1)\int \cos^{\alpha-2} x(1-\cos^2 x)\, dx \quad = \cos^{\alpha-1} x \sin x + (\alpha-1)\int \cos^{\alpha-2} x\, dx - (\alpha-1)\int \cos^{\alpha} x\, dx$$

$$\alpha \int \cos^{\alpha} x\, dx = \cos^{\alpha-1} x \sin x + (\alpha-1)\int \cos^{\alpha-2} x\, dx$$

$$\int \cos^{\alpha} x\, dx = \frac{\cos^{\alpha-1} x \sin x}{\alpha} + \frac{\alpha-1}{\alpha}\int \cos^{\alpha-2} x\, dx$$

61. $u = \cos^{\alpha-1} \beta x$ $\qquad dv = \cos \beta x\, dx$

$du = -\beta(\alpha-1)\cos^{\alpha-2} \beta x \sin \beta x\, dx \qquad v = \frac{1}{\beta}\sin \beta x$

$$\int \cos^{\alpha} \beta x\, dx = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha-1)\int \cos^{\alpha-2} \beta x \sin^2 \beta x\, dx$$

$$= \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha-1)\int \cos^{\alpha-2} \beta x(1-\cos^2 \beta x)\, dx$$

$$= \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha-1)\int \cos^{\alpha-2} \beta x\, dx - (\alpha-1)\int \cos^{\alpha} \beta x\, dx$$

$$\alpha \int \cos^{\alpha} \beta x = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\beta} + (\alpha-1)\int \cos^{\alpha-2} \beta x\, dx$$

$$\int \cos^{\alpha} \beta x\, dx = \frac{\cos^{\alpha-1} \beta x \sin \beta x}{\alpha\beta} + \frac{\alpha-1}{\alpha}\int \cos^{\alpha-2} \beta x\, dx$$

62. $\int x^4 e^{3x} dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{3}\int x^3 e^{3x} dx = \frac{1}{3}x^4 e^{3x} - \frac{4}{3}\left[\frac{1}{3}x^3 e^{3x} - \int x^2 e^{3x} dx\right]$

$$= \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{3}\left[\frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} dx\right] \quad = \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{9}\left[\frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x} dx\right]$$

$$= \frac{1}{3}x^4 e^{3x} - \frac{4}{9}x^3 e^{3x} + \frac{4}{9}x^2 e^{3x} - \frac{8}{27}x e^{3x} + \frac{8}{81}e^{3x} + C$$

63. $\int x^4 \cos 3x\,dx = \frac{1}{3}x^4 \sin 3x - \frac{4}{3}\int x^3 \sin 3x\,dx \;= \frac{1}{3}x^4 \sin 3x - \frac{4}{3}\left[-\frac{1}{3}x^3 \cos 3x + \int x^2 \cos 3x\,dx\right]$

$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{3}\left[\frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x\,dx\right]$

$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{9}x^2 \sin 3x + \frac{8}{9}\left[-\frac{1}{3}x \cos 3x + \frac{1}{3}\int \cos 3x\,dx\right]$

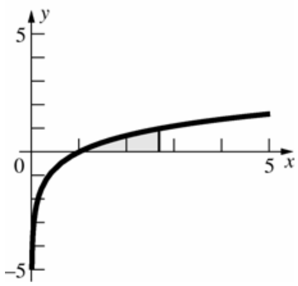
$= \frac{1}{3}x^4 \sin 3x + \frac{4}{9}x^3 \cos 3x - \frac{4}{9}x^2 \sin 3x - \frac{8}{27}x \cos 3x + \frac{8}{81}\sin 3x + C$

64. $\int \cos^6 3x\,dx = \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{6}\int \cos^4 3x\,dx \;= \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{6}\left[\frac{1}{12}\cos^3 3x \sin 3x + \frac{3}{4}\int \cos^2 3x\,dx\right]$

$= \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{72}\cos^3 3x \sin 3x + \frac{5}{8}\left[\frac{1}{6}\cos 3x \sin 3x + \frac{1}{2}\int dx\right]$

$= \frac{1}{18}\cos^5 3x \sin 3x + \frac{5}{72}\cos^3 3x \sin 3x + \frac{5}{48}\cos 3x \sin 3x + \frac{5x}{16} + C$

65. First make a sketch.



From the sketch, the area is given by

$\int_1^e \ln x\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$\int_1^e \ln x\,dx = [x \ln x]_1^e - \int_1^e dx \;= [x \ln x - x]_1^e = (e - e) - (1 \cdot 0 - 1) = 1$

66. $V = \int_1^e \pi(\ln x)^2\,dx$

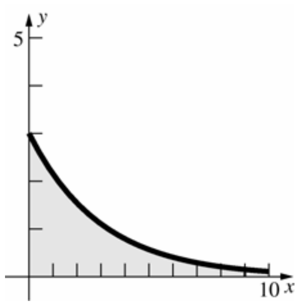
$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x}dx \qquad v = x$

$\pi\int_1^e (\ln x)^2\,dx = \pi\left(\left[x(\ln x)^2\right]_1^e - 2\int_1^e \ln x\,dx\right) = \pi\left[x(\ln x)^2 - 2(x\ln x - x)\right]_1^e = \pi[x(\ln x)^2 - 2x\ln x + 2x]_1^e$

$= \pi[(e - 2e + 2e) - (0 - 0 + 2)] = \pi(e - 2) \approx 2.26$

67.



$$\int_0^9 3e^{-x/3}dx = -9\int_0^9 e^{-x/3}\left(-\frac{1}{3}dx\right) = -9[e^{-x/3}]_0^9 = -\frac{9}{e^3}+9 \approx 8.55$$

68. $V = \int_0^9 \pi(3e^{-x/3})^2 dx = 9\pi\int_0^9 e^{-2x/3}dx$

$$= 9\pi\left(-\frac{3}{2}\right)\int_0^9 e^{-2x/3}\left(-\frac{2}{3}dx\right) = -\frac{27\pi}{2}[e^{-2x/3}]_0^9 = -\frac{27\pi}{2e^6}+\frac{27\pi}{2} \approx 42.31$$

69. $\int_0^{\pi/4}(x\cos x - x\sin x)dx = \int_0^{\pi/4} x\cos x\,dx - \int_0^{\pi/4} x\sin x\,dx$

$$= \left(\left[x\sin x\right]_0^{\pi/4} - \int_0^{\pi/4}\sin x\,dx\right) - \left([-x\cos x]_0^{\pi/4} + \int_0^{\pi/4}\cos x\,dx\right)$$

$$= [x\sin x + \cos x + x\cos x - \sin x]_0^{\pi/4} = \frac{\sqrt{2}\pi}{4} - 1 \approx 0.11$$

Use Problems 60 and 61 for $\int x\sin x\,dx$ and $\int x\cos x\,dx$.

70. $V = 2\pi\int_0^{2\pi} x\sin\left(\frac{x}{2}\right)dx$

$u = x \qquad dv = \sin\frac{x}{2}dx$

$du = dx \quad v = -2\cos\frac{x}{2}$

$$V = 2\pi\left(\left[-2x\cos\frac{x}{2}\right]_0^{2\pi} + \int_0^{2\pi} 2\cos\frac{x}{2}dx\right) = 2\pi\left(4\pi + \left[4\sin\frac{x}{2}\right]_0^{2\pi}\right) = 8\pi^2$$

71. $\int_1^e \ln x^2 dx = 2\int_1^e \ln x\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$2\int_1^e \ln x\,dx = 2\left([x\ln x]_1^e - \int_1^e dx\right) = 2\left(e - [x]_1^e\right) = 2$$

$\int_1^e x\ln x^2 dx = 2\int_1^e x\ln x\,dx$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$2\int_1^e x\ln x\,dx = 2\left(\left[\frac{1}{2}x^2\ln x\right]_1^e - \int_1^e \frac{1}{2}x\,dx\right) = 2\left(\frac{1}{2}e^2 - \left[\frac{1}{4}x^2\right]_1^e\right) = \frac{1}{2}(e^2+1)$$

$\frac{1}{2}\int_1^e (\ln x)^2 \, dx$

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x} dx \qquad v = x$

$\frac{1}{2}\int_1^e (\ln x)^2 \, dx = \frac{1}{2}\left([x(\ln x)^2]_1^e - 2\int_1^e \ln x \, dx \right) = \frac{1}{2}(e-2)$

$\overline{x} = \frac{\frac{1}{2}(e^2+1)}{2} = \frac{e^2+1}{4}$

$\overline{y} = \frac{\frac{1}{2}(e-2)}{2} = \frac{e-2}{4}$

72. **a.** $u = \cot x \qquad dv = \csc^2 x \, dx$

$du = -\csc^2 x \, dx \qquad v = -\cot x$

$\int \cot x \csc^2 x \, dx = -\cot^2 x - \int \cot x \csc^2 x \, dx$

$2\int \cot x \csc^2 x \, dx = -\cot^2 x + C$

$\int \cot x \csc^2 x \, dx = -\frac{1}{2}\cot^2 x + C$

b. $u = \csc x \qquad dv = \cot x \csc x \, dx$

$du = -\cot x \csc x \, dx \qquad v = -\csc x$

$\int \cot x \csc^2 x \, dx = -\csc^2 x - \int \cot x \csc^2 x \, dx$

$2\int \cot x \csc^2 x \, dx = -\csc^2 x + C$

$\int \cot x \csc^2 x \, dx = -\frac{1}{2}\csc^2 x + C$

c. $-\frac{1}{2}\cot^2 x = -\frac{1}{2}(\csc^2 x - 1) = -\frac{1}{2}\csc^2 x + \frac{1}{2}$

73. **a.** $p(x) = x^3 - 2x$

$g(x) = e^x$

All antiderivatives of $g(x) = e^x$

$\int (x^3 - 2x)e^x dx = (x^3 - 2x)e^x - (3x^2 - 2)e^x + 6xe^x - 6e^x + C$

b. $p(x) = x^2 - 3x + 1$

$g(x) = \sin x$

$G_1(x) = -\cos x$

$G_2(x) = -\sin x$

$G_3(x) = \cos x$

$\int (x^2 - 3x + 1)\sin x \, dx = (x^2 - 3x + 1)(-\cos x) - (2x - 3)(-\sin x) + 2\cos x + C$

74. **a.** We note that the nth arch extends from $x = 2\pi(n-1)$ to $x = \pi(2n-1)$, so the area of the nth arch is $A(n) = \int_{2\pi(n-1)}^{\pi(2n-1)} x \sin x\, dx$. Using integration by parts:

$u = x \qquad dv = \sin x\, dx$

$du = dx \qquad v = -\cos x$

$$A(n) = \int_{2\pi(n-1)}^{\pi(2n-1)} x \sin x\, dx = \left[-x\cos x\right]_{2\pi(n-1)}^{\pi(2n-1)} - \int_{2\pi(n-1)}^{\pi(2n-1)} -\cos x\, dx = \left[-x\cos x\right]_{2\pi(n-1)}^{\pi(2n-1)} + \left[\sin x\right]_{2\pi(n-1)}^{\pi(2n-1)}$$

$$= \left[-\pi(2n-1)\cos(\pi(2n-1)) + 2\pi(n-1)\cos(2\pi(n-1))\right] + \left[\sin(\pi(2n-1)) - \sin(2\pi(n-1))\right]$$

$$= -\pi(2n-1)(-1) + 2\pi(n-1)(1) + 0 - 0 = \pi\left[(2n-1) + (2n-2)\right]. \quad \text{So} \quad A(n) = (4n-3)\pi$$

b. $V = 2\pi \int_{2\pi}^{3\pi} x^2 \sin x\, dx$

$u = x^2 \qquad dv = \sin x\ dx$

$du = 2x\ dx \qquad v = -\cos x$

$$V = 2\pi\left(\left[-x^2 \cos x\right]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} 2x \cos x\, dx\right) = 2\pi\left(9\pi^2 + 4\pi^2 + \int_{2\pi}^{3\pi} 2x\cos x\, dx\right)$$

$u = 2x \qquad dv = \cos x\ dx$

$du = 2\ dx \qquad v = \sin x$

$$V = 2\pi\left(13\pi^2 + [2x\sin x]_{2\pi}^{3\pi} - \int_{2\pi}^{3\pi} 2\sin x\right)$$

$$= 2\pi\left(13\pi^2 + [2\cos x]_{2\pi}^{3\pi}\right) = 2\pi(13\pi^2 - 4) \approx 781$$

75. $u = f(x) \qquad dv = \sin nx\, dx$

$du = f'(x)dx \qquad v = -\frac{1}{n}\cos nx$

$$a_n = \frac{1}{\pi}\Bigg[\underbrace{\left[-\frac{1}{n}\cos(nx)\, f(x)\right]_{-\pi}^{\pi}}_{\text{Term 1}} + \underbrace{\frac{1}{n}\int_{-\pi}^{\pi} \cos(nx)\, f'(x)dx}_{\text{Term 2}}\Bigg]$$

Term 1 = $\frac{1}{n}\cos(n\pi)(f(-\pi) - f(\pi)) = \pm\frac{1}{n}(f(-\pi) - f(\pi))$

Since $f'(x)$ is continuous on $[-\infty, \infty]$, it is bounded. Thus, $\int_{-\pi}^{\pi} \cos(nx)\, f'(x)dx$ is bounded so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\pi n}\left[\pm(f(-\pi) - f(\pi)) + \int_{-\pi}^{\pi} \cos(nx)\, f'(x)\, dx\right] = 0.$$

76. $$\frac{G_n}{n} = \frac{[(n+1)(n+2)\cdots(n+n)]^{1/n}}{[n^n]^{1/n}} = \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

$$\ln\left(\frac{G_n}{n}\right) = \frac{1}{n}\ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]$$

$$= \frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + \cdots + \ln\left(1+\frac{n}{n}\right)\right]$$

$$\lim_{n\to\infty} \ln\left(\frac{G_n}{n}\right) = \int_1^2 \ln x\, dx = 2\ln 2 - 1$$

$$\lim_{n\to\infty} \left(\frac{G_n}{n}\right) = e^{2\ln 2 - 1} = 4e^{-1} = \frac{4}{e}$$

77. The proof fails to consider the constants when integrating $\frac{1}{t}$.

The symbol $\int(1/t)\,dt$ is a *family* of functions, all of who whom have derivative $\frac{1}{t}$. We know that any two of these functions will differ by a constant, so it is perfectly correct (notationally) to write $\int(1/t)\,dt = \int(1/t)\,dt + 1$

78. $\frac{d}{dx}[e^{5x}(C_1\cos 7x + C_2\sin 7x) + C_3] = 5e^{5x}(C_1\cos 7x + C_2\sin 7x) + e^{5x}(-7C_1\sin 7x + 7C_2\cos 7x)$

$= e^{5x}[(5C_1 + 7C_2)\cos 7x + (5C_2 - 7C_1)\sin 7x]$

Thus, $5C_1 + 7C_2 = 4$ and $5C_2 - 7C_1 = 6$.

Solving, $C_1 = -\frac{11}{37};\ C_2 = \frac{29}{37}$

79. $u = f(x)$ $\qquad dv = dx$

$du = f'(x)dx$ $\qquad v = x$

$$\int_a^b f(x)dx = [xf(x)]_a^b - \int_a^b xf'(x)dx$$

Starting with the same integral,

$u = f(x)$ $\qquad dv = dx$

$du = f'(x)dx$ $\qquad v = x - a$

$$\int_a^b f(x)\,dx = [(x-a)f(x)]_a^b - \int_a^b (x-a)f'(x)dx$$

80. $u = f'(x)$ $\qquad dv = dx$

$du = f''(x)dx$ $\qquad v = x - a$

$$f(b) - f(a) = \int_a^b f'(x)dx = [(x-a)f'(x)]_a^b - \int_a^b (x-a)f''(x)dx = f'(b)(b-a) - \int_a^b (x-a)f''(x)dx$$

Starting with the same integral,

$u = f'(x)$ $\qquad dv = dx$

$du = f''(x)dx$ $\qquad v = x - b$

$$f(b) - f(a) = \int_a^b f'(x)dx = [(x-b)f'(x)]_a^b - \int_a^b (x-b)f''(x)dx = f'(a)(b-a) - \int_a^b (x-b)f''(x)dx$$

81. Use proof by induction.

$n = 1$: $f(a) + f'(a)(t-a) + \int_a^t (t-x)f''(x)dx = f(a) + f'(a)(t-a) + [f'(x)(t-x)]_a^t + \int_a^t f'(x)dx$

$= f(a) + f'(a)(t-a) - f'(a)(t-a) + [f(x)]_a^t = f(t)$

Thus, the statement is true for $n = 1$. Note that integration by parts was used with $u = (t-x)$, $dv = f''(x)dx$.

Suppose the statement is true for n.

$$f(t) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(t-a)^i + \int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx$$

Integrate $\int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx$ by parts.

$u = f^{(n+1)}(x)$ $\qquad dv = \frac{(t-x)^n}{n!}dx$

$du = f^{(n+2)}(x)$ $\qquad v = -\frac{(t-x)^{n+1}}{(n+1)!}$

$$\int_a^t \frac{(t-x)^n}{n!} f^{(n+1)}(x)dx = \left[-\frac{(t-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\right]_a^t + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

$$= \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

Thus $f(t) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(t-a)^i + \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$

$$= f(a) + \sum_{i=1}^{n+1} \frac{f^{(i)}(a)}{i!}(t-a)^i + \int_a^t \frac{(t-x)^{n+1}}{(n+1)!} f^{(n+2)}(x)dx$$

Thus, the statement is true for $n + 1$.

82. **a.** $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ where $\alpha \geq 1, \beta \geq 1$

$x = 1 - u$, $dx = -du$

$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \int_1^0 (1-u)^{\alpha-1}(u)^{\beta-1}(-du) = \int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du = B(\beta, \alpha)$

Thus, $B(\alpha, \beta) = B(\beta, \alpha)$.

b. $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$

$u = x^{\alpha-1}$ $\qquad dv = (1-x)^{\beta-1}dx$

$du = (\alpha-1)x^{\alpha-2}dx$ $\qquad v = -\frac{1}{\beta}(1-x)^{\beta}$

$$B(\alpha, \beta) = \left[-\frac{1}{\beta}x^{\alpha-1}(1-x)^{\beta}\right]_0^1 + \frac{\alpha-1}{\beta}\int_0^1 x^{\alpha-2}(1-x)^{\beta}dx = \frac{\alpha-1}{\beta}\int_0^1 x^{\alpha-2}(1-x)^{\beta}dx \qquad (*)$$

$$= \frac{\alpha-1}{\beta}B(\alpha-1, \beta+1)$$

Similarly,

$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$

$u = (1-x)^{\beta-1}$ $\qquad dv = x^{\alpha-1}dx$

$du = -(\beta-1)(1-x)^{\beta-2}dx$ $\qquad v = \frac{1}{\alpha}x^{\alpha}$

$$B(\alpha, \beta) = \left[\frac{1}{\alpha}x^{\alpha}(1-x)^{\beta-1}\right]_0^1 + \frac{\beta-1}{\alpha}\int_0^1 x^{\alpha}(1-x)^{\beta-2}dx = \frac{\beta-1}{\alpha}\int_0^1 x^{\alpha}(1-x)^{\beta-2}dx = \frac{\beta-1}{\alpha}B(\alpha+1, \beta-1)$$

c. Assume that $n \leq m$. Using part (b) n times,

$$B(n, m) = \frac{n-1}{m}B(n-1, m+1) = \frac{(n-1)(n-2)}{m(m+1)}B(n-2, m+2)$$

$$= \ldots = \frac{(n-1)(n-2)(n-3)\ldots\cdot 2\cdot 1}{m(m+1)(m+2)\ldots(m+n-2)}B(1, m+n-1).$$

$$B(1, m+n-1) = \int_0^1 (1-x)^{m+n-2}dx = -\frac{1}{m+n-1}[(1-x)^{m+n-1}]_0^1 = \frac{1}{m+n-1}$$

Thus, $B(n, m) = \frac{(n-1)(n-2)(n-3)\ldots\cdot 2\cdot 1}{m(m+1)(m+2)\ldots(m+n-2)(m+n-1)} = \frac{(n-1)!(m-1)!}{(m+n-1)!} = \frac{(n-1)!(m-1)!}{(n+m-1)!}$

If $n > m$, then $B(n, m) = B(m, n) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$ by the above reasoning.

83. $u = f(t)$ $\quad dv = f''(t)dt$

$du = f'(t)dt \quad v = f'(t)$

$$\int_a^b f''(t) f(t)dt = \left[f(t) f'(t) \right]_a^b - \int_a^b [f'(t)]^2 dt$$

$$= f(b) f'(b) - f(a) f'(a) - \int_a^b [f'(t)]^2 dt = -\int_a^b [f'(t)]^2 dt$$

$[f'(t)]^2 \geq 0,$ so $-\int_a^b [f'(t)]^2 \leq 0$.

84. $\int_0^x \left(\int_0^t f(z)dz \right) dt$

$u = \int_0^t f(z)dz \quad dv = dt$

$du = f(t)dt \quad v = t$

$$\int_0^x \left(\int_0^t f(z)dz \right) dt = \left[t\int_0^t f(z)dz \right]_0^x - \int_0^x t\, f(t)dt = \int_0^x x\, f(z)dz - \int_0^x t\, f(t)dt$$

By letting $z = t$, $\int_0^x x\, f(z)dz = \int_0^x x\, f(t)dt$, so

$$\int_0^x \left(\int_0^t f(z)dz \right) dt = \int_0^x x\, f(t)dt - \int_0^x t\, f(t)dt = \int_0^x (x-t) f(t)dt$$

85. Let $I = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n)\, dt_n ... dt_2 dt_1$ be the iterated integral. Note that for $i \geq 2$, the limits of integration of the integral with respect to t_i are 0 to t_{i-1} so that any change of variables in an outer integral affects the limits, and hence the variables in all interior integrals. We use induction on *n*, noting that the case $n = 2$ is solved in the previous problem.

Assume we know the formula for $n-1$, and we want to show it for *n*.

$$I = \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(t_n)\, dt_n ... dt_3 dt_2\, dt_1 = \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-2}} F(t_{n-1})\, dt_{n-1} ... dt_3 dt_2 dt_1$$

where $F(t_{n-1}) = \int_0^{t_{n-1}} f(t_n)\, dn$.

By induction,

$$I = \frac{1}{(n-2)!} \int_0^x F(t_1)(x-t_1)^{n-2}\, dt_1$$

$u = F(t_1) = \int_0^{t_1} f(t_n)\, dt_n$, $\quad dv = (x-t_1)^{n-2}$

$du = f(t_1)\, dt_1$, $\quad v = -\frac{1}{n-1}(x-t_1)^{n-1}$

$$I = \frac{1}{(n-2)!} \left\{ \left[-\frac{1}{n-1}(x-t_1)^{n-1} \int_0^{t_1} f(t_n)\, dt_n \right]_{t_1=0}^{t_1=x} + \frac{1}{n-1} \int_0^x f(t_1)(x-t_1)^{n-1}\, dt_1 \right\}.$$

$$= \frac{1}{(n-1)!} \int_0^x f(t_1)(x-t_1)^{n-1} dt_1$$

(note: that the quantity in square brackets equals 0 when evaluated at the given limits)

86. Proof by induction.
$n = 1$:

$u = P_1(x) \qquad dv = e^x dx$

$du = \frac{dP_1(x)}{dx}dx \qquad v = e^x$

$$\int e^x P_1(x)dx = e^x P_1(x) - \int e^x \frac{dP_1(x)}{dx}dx = e^x P_1(x) - \frac{dP_1(x)}{dx}\int e^x dx = e^x P_1(x) - e^x \frac{dP_1(x)}{dx}$$

Note that $\frac{dP_1(x)}{dx}$ is a constant.

Suppose the formula is true for n. By using integration by parts with $u = P_{n+1}(x)$ and $dv = e^x dx$,

$$\int e^x P_{n+1}(x)dx = e^x P_{n+1}(x) - \int e^x \frac{dP_{n+1}(x)}{dx}dx$$

Note that $\frac{dP_{n+1}(x)}{dx}$ is a polynomial of degree n, so

$$\int e^x P_{n+1}(x)dx = e^x P_{n+1}(x) - \left[e^x \sum_{j=0}^{n}(-1)^j \frac{d^j}{dx^j}\left(\frac{dP_{n+1}(x)}{dx}\right)\right] = e^x P_{n+1}(x) - e^x \sum_{j=0}^{n}(-1)^j \frac{d^{j+1}P_{n+1}(x)}{dx^{j+1}}$$

$$= e^x P_{n+1}(x) + e^x \sum_{j=1}^{n+1}(-1)^j \frac{d^j P_{n+1}(x)}{dx^j} = e^x \sum_{j=0}^{n+1}(-1)^j \frac{d^j P_{n+1}(x)}{dx^j}$$

87. $\int (3x^4 + 2x^2)e^x dx = e^x \sum_{j=0}^{4}(-1)^j \frac{d^j(3x^4 + 2x^2)}{dx^j}$

$= e^x[3x^4 + 2x^2 - 12x^3 - 4x + 36x^2 + 4 - 72x + 72]$

$= e^x(3x^4 - 12x^3 + 38x^2 - 76x + 76)$

7.3 Concepts Review

1. $\int \frac{1+\cos 2x}{2}dx$

2. $\int (1-\sin^2 x)\cos x\, dx$

3. $\int \sin^2 x(1-\sin^2 x)\cos x\, dx$

4. $\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$

Problem Set 7.3

1. $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

2. $u = 6x$, $du = 6\, dx$

$\int \sin^4 6x\, dx = \frac{1}{6}\int \sin^4 u\, du$

$= \frac{1}{6}\int \left(\frac{1-\cos 2u}{2}\right)^2 du$

$= \frac{1}{24}\int (1 - 2\cos 2u + \cos^2 2u)du$

$= \frac{1}{24}\int du - \frac{1}{24}\int 2\cos 2u\, du + \frac{1}{48}\int (1+\cos 4u)du$

$= \frac{3}{48}\int du - \frac{1}{24}\int 2\cos 2u\, du + \frac{1}{192}\int 4\cos 4u\, du$

$= \frac{3}{48}(6x) - \frac{1}{24}\sin 12x + \frac{1}{192}\sin 24x + C$

$= \frac{3}{8}x - \frac{1}{24}\sin 12x + \frac{1}{192}\sin 24x + C$

3. $\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)dx$

$= \int \sin x\,dx - \int \sin x \cos^2 x\,dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

4. $\int \cos^3 x\,dx =$

$= \int \cos x(1-\sin^2 x)dx$

$= \int \cos x\,dx - \int \cos x \sin^2 x\,dx$

$= \sin x - \frac{1}{3}\sin^3 x + C$

5. $\int_0^{\pi/2} \cos^5 \theta\,d\theta = \int_0^{\pi/2} (1-\sin^2 \theta)^2 \cos\theta\,d\theta$

$= \int_0^{\pi/2} (1-2\sin^2 \theta + \sin^4 \theta)\cos\theta\,d\theta$

$= \left[\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta\right]_0^{\pi/2}$

$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$

6. $\int_0^{\pi/2} \sin^6 \theta\,d\theta = \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right)^3 d\theta$

$= \frac{1}{8}\int_0^{\pi/2} (1 - 3\cos 2\theta + 3\cos^2 2\theta - \cos^3 2\theta)d\theta$

$= \frac{1}{8}\int_0^{\pi/2} d\theta - \frac{3}{16}\int_0^{\pi/2} 2\cos 2\theta\,d\theta + \frac{3}{8}\int_0^{\pi/2} \cos^2 2\theta - \frac{1}{8}\int_0^{\pi/2} \cos^3 2\theta\,d\theta$

$= \frac{1}{8}[\theta]_0^{\pi/2} - \frac{3}{16}[\sin 2\theta]_0^{\pi/2} + \frac{3}{8}\int_0^{\pi/2} \left(\frac{1+\cos 4\theta}{2}\right) d\theta - \frac{1}{8}\int_0^{\pi/2} (1-\sin^2 2\theta)\cos 2\theta\,d\theta$

$= \frac{1}{8}\cdot\frac{\pi}{2} + \frac{3}{16}\int_0^{\pi/2} d\theta + \frac{3}{64}\int_0^{\pi/2} 4\cos 4\theta\,d\theta - \frac{1}{16}\int_0^{\pi/2} 2\cos 2\theta\,d\theta + \frac{1}{16}\int_0^{\pi/2} \sin^2 2\theta \cdot 2\cos 2\theta\,d\theta$

$= \frac{\pi}{16} + \frac{3\pi}{32} + \frac{3}{64}[\sin 4\theta]_0^{\pi/2} - \frac{1}{16}[\sin 2\theta]_0^{\pi/2} + \frac{1}{48}[\sin^3 2\theta]_0^{\pi/2} = \frac{5\pi}{32}$

7. $\int \sin^5 4x \cos^2 4x\,dx = \int (1-\cos^2 4x)^2 \cos^2 4x \sin 4x\,dx = \int (1 - 2\cos^2 4x + \cos^4 4x)\cos^2 4x \sin 4x\,dx$

$= -\frac{1}{4}\int (\cos^2 4x - 2\cos^4 4x + \cos^6 4x)(-4\sin 4x)dx = -\frac{1}{12}\cos^3 4x + \frac{1}{10}\cos^5 4x - \frac{1}{28}\cos^7 4x + C$

8. $\int (\sin^3 2t)\sqrt{\cos 2t}\,dt = \int (1-\cos^2 2t)(\cos 2t)^{1/2} \sin 2t\,dt = -\frac{1}{2}\int [(\cos 2t)^{1/2} - (\cos 2t)^{5/2}](-2\sin 2t)dt$

$= -\frac{1}{3}(\cos 2t)^{3/2} + \frac{1}{7}(\cos 2t)^{7/2} + C$

9. $\int \cos^3 3\theta \sin^{-2} 3\theta\,d\theta = \int (1-\sin^2 3\theta)\sin^{-2} 3\theta \cos 3\theta\,d\theta = \frac{1}{3}\int (\sin^{-2} 3\theta - 1)3\cos 3\theta\,d\theta$

$= -\frac{1}{3}\csc 3\theta - \frac{1}{3}\sin 3\theta + C$

10. $\int \sin^{1/2} 2z \cos^3 2z\,dz = \int (1-\sin^2 2z)\sin^{1/2} 2z \cos 2z\,dz$

$= \frac{1}{2}\int (\sin^{1/2} 2z - \sin^{5/2} 2z)2\cos 2z\,dz = \frac{1}{3}\sin^{3/2} 2z - \frac{1}{7}\sin^{7/2} 2z + C$

11. $\int \sin^4 3t \cos^4 3t\, dt = \int \left(\frac{1-\cos 6t}{2}\right)^2 \left(\frac{1+\cos 6t}{2}\right)^2 dt \;= \frac{1}{16}\int (1-2\cos^2 6t+\cos^4 6t)dt$

$= \frac{1}{16}\int \left[1-(1+\cos 12t)+\frac{1}{4}(1+\cos 12t)^2\right]dt \;= -\frac{1}{16}\int \cos 12t\, dt + \frac{1}{64}\int (1+2\cos 12t+\cos^2 12t)dt$

$= -\frac{1}{192}\int 12\cos 12t\, dt + \frac{1}{64}\int dt + \frac{1}{384}\int 12\cos 12t\, dt + \frac{1}{128}\int (1+\cos 24t)dt$

$= -\frac{1}{192}\sin 12t + \frac{1}{64}t + \frac{1}{384}\sin 12t + \frac{1}{128}t + \frac{1}{3072}\sin 24t + C \;= \frac{3}{128}t - \frac{1}{384}\sin 12t + \frac{1}{3072}\sin 24t + C$

12. $\int \cos^6\theta \sin^2\theta\, d\theta = \int \left(\frac{1+\cos 2\theta}{2}\right)^3 \left(\frac{1-\cos 2\theta}{2}\right) d\theta \;= \frac{1}{16}\int (1+2\cos 2\theta - 2\cos^3 2\theta - \cos^4 2\theta)d\theta$

$= \frac{1}{16}\int d\theta + \frac{1}{16}\int 2\cos 2\theta\, d\theta - \frac{1}{8}\int (1-\sin^2 2\theta)\cos 2\theta\, d\theta - \frac{1}{64}\int (1+\cos 4\theta)^2\, d\theta$

$= \frac{1}{16}\int d\theta + \frac{1}{16}\int 2\cos 2\theta\, d\theta - \frac{1}{16}\int 2\cos 2\theta\, d\theta + \frac{1}{16}\int 2\sin^2 2\theta \cos 2\theta\, d\theta - \frac{1}{64}\int (1+2\cos 4\theta + \cos^2 4\theta)d\theta$

$= \frac{1}{16}\int d\theta + \frac{1}{16}\int \sin^2 2\theta \cdot 2\cos 2\theta\, d\theta - \frac{1}{64}\int d\theta - \frac{1}{128}\int 4\cos 4\theta\, d\theta - \frac{1}{128}\int (1+\cos 8\theta)d\theta$

$= \frac{1}{16}\theta + \frac{1}{48}\sin^3 2\theta - \frac{1}{64}\theta - \frac{1}{128}\sin 4\theta - \frac{1}{128}\theta - \frac{1}{1024}\sin 8\theta + C$

$= \frac{5}{128}\theta + \frac{1}{48}\sin^3 2\theta - \frac{1}{128}\sin 4\theta - \frac{1}{1024}\sin 8\theta \;+ C$

13. $\int \sin 4y \cos 5y\, dy = \frac{1}{2}\int \left[\sin 9y + \sin(-y)\right]dy = \frac{1}{2}\int (\sin 9y - \sin y)dy$

$= \frac{1}{2}\left(-\frac{1}{9}\cos 9y + \cos y\right) + C = \frac{1}{2}\cos y - \frac{1}{18}\cos 9y + C$

14. $\int \cos y \cos 4y\, dy = \frac{1}{2}\int [\cos 5y + \cos(-3y)]dy \;= \frac{1}{10}\sin 5y - \frac{1}{6}\sin(-3y) + C \;= \frac{1}{10}\sin 5y + \frac{1}{6}\sin 3y + C$

15. $\int \sin^4\left(\frac{w}{2}\right)\cos^2\left(\frac{w}{2}\right)dw = \int \left(\frac{1-\cos w}{2}\right)^2 \left(\frac{1+\cos w}{2}\right)dw \;= \frac{1}{8}\int (1-\cos w - \cos^2 w + \cos^3 w)dw$

$= \frac{1}{8}\int \left[1-\cos w - \frac{1}{2}(1+\cos 2w) + (1-\sin^2 w)\cos w\right]dw \;= \frac{1}{8}\int \left[\frac{1}{2} - \frac{1}{2}\cos 2w - \sin^2 w \cos w\right]dw$

$= \frac{1}{16}w - \frac{1}{32}\sin 2w - \frac{1}{24}\sin^3 w + C$

16. $\int \sin 3t \sin t\; dt = \int -\frac{1}{2}\left[\cos 4t - \cos 2t\right]dt$

$= -\frac{1}{2}\left(\int \cos 4t dt - \int \cos 2t dt\right)$

$= -\frac{1}{2}\left(\frac{1}{4}\sin 4t - \frac{1}{2}\sin 2t\right) + C$

$= -\frac{1}{8}\sin 4t + \frac{1}{4}\sin 2t + C$

17. $\int x\cos^2 x\sin x\,dx$

$u = x \qquad du = 1\,dx$

$dv = \cos^2 x\sin x\ dx$

$v = -\underset{t=\cos x}{\int (\cos x)^2(-\sin x)\,dx} = -\frac{1}{3}\cos^3 x$

Thus

$\int x\cos^2 x\sin x\,dx =$

$x(-\frac{1}{3}\cos^3 x) - \int (1)(-\frac{1}{3}\cos^3 x)\,dx =$

$\frac{1}{3}\left[-x\cos^3 x + \int \cos^3 x\,dx\right] =$

$\frac{1}{3}\left[-x\cos^3 x + \int \cos x(1-\sin^2 x)\,dx\right] =$

$\frac{1}{3}\left[-x\cos^3 x + \underset{t=\sin x}{\int (\cos x - \cos x\sin^2 x)\ dx}\right] =$

$\frac{1}{3}\left[-x\cos^3 x + \sin x - \frac{1}{3}\sin^3 x\right] + C$

18. $\int x\sin^3 x\cos x\,dx$

$u = x \qquad du = 1\,dx$

$dv = \sin^3 x\cos x\ dx$

$v = \underset{t=\sin x}{\int (\sin x)^3(\cos x)\,dx} = \frac{1}{4}\sin^4 x$

Thus

$\int x\sin^3 x\cos x\,dx =$

$x(\frac{1}{4}\sin^4 x) - \int (1)(\frac{1}{4}\sin^4 x)\,dx =$

$\frac{1}{4}\left[x\sin^4 x - \int (\sin^2 x)^2\ dx\right] =$

$\frac{1}{4}\left[x\sin^4 x - \frac{1}{4}\int (1-\cos 2x)^2\ dx\right] =$

$\frac{1}{4}\left[x\sin^4 x - \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\,dx\right] =$

$\frac{1}{4}\left[x\sin^4 x - \frac{1}{4}x + \frac{1}{4}\sin 2x - \frac{1}{8}\int (1+\cos 4x)\,dx\right] =$

$\frac{1}{4}\left[x\sin^4 x - \frac{3}{8}x + \frac{1}{4}\sin 2x - \frac{1}{32}\sin 4x\right] + C$

19. $\int \tan^4 x\ dx = \int \left(\tan^2 x\right)\left(\tan^2 x\right)\ dx$

$= \int \left(\tan^2 x\right)(\sec^2 x - 1)\ dx$

$= \int \left(\tan^2 x\sec^2 x - \tan^2 x\right)dx$

$= \int \tan^2 x\sec^2 x\ dx - \int (\sec^2 x - 1)dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

20. $\int \cot^4 x\ dx = \int \left(\cot^2 x\right)\left(\cot^2 x\right)\ dx$

$= \int \left(\cot^2 x\right)(\csc^2 x - 1)\ dx$

$= \int \left(\cot^2 x\csc^2 x - \cot^2 x\right)dx$

$= \int \cot^2 x\csc^2 x\ dx - \int (\csc^2 x - 1)dx$

$= -\frac{1}{3}\cot^3 x + \cot x + x + C$

21. $\tan^3 \text{x} = \int \left(\tan x\right)\left(\tan^2 x\right)dx$

$= \int \left(\tan x\right)\left(\sec^2 x - 1\right)dx$

$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$

22. $\int \cot^3 2t\,dt = \int \left(\cot 2t\right)\left(\cot^2 2t\right)dt$

$= \int \left(\cot 2t\right)\left(\csc^2 2t - 1\right)dt$

$= \int \cot 2t\csc^2 2t\,dt - \int \cot 2t\,dt$

$= -\frac{1}{4}\cot^2 2t - \frac{1}{2}\ln|\sin 2t| + C$

23. $\int \tan^5\left(\frac{\theta}{2}\right)d\theta$

$u=\left(\frac{\theta}{2}\right);\ du=\frac{d\theta}{2}$

$$\int \tan^5\left(\frac{\theta}{2}\right)d\theta = 2\int \tan^5 u\ du$$

$$= 2\int\left(\tan^3 u\right)\left(\sec^2 u - 1\right)du$$

$$= 2\int \tan^3 u \sec^2 u\ du - 2\int \tan^3 u\ du$$

$$= 2\int \tan^3 u \sec^2 u\ du - 2\int \tan u\left(\sec^2 u - 1\right)du$$

$$= 2\int \tan^3 u \sec^2 u\ du - 2\int \tan u \sec^2 u\ du + 2\int \tan u\ \mathrm{du}$$

$$= \frac{1}{2}\tan^4\left(\frac{\theta}{2}\right) - \tan^2\left(\frac{\theta}{2}\right) - 2\ln\left|\cos\frac{\theta}{2}\right| + C$$

24. $\int \cot^5 2t\ dt$

$u = 2t; du = 2dt$

$$\int \cot^5 2t\ dt = \frac{1}{2}\int \cot^5 u\ du$$

$$= \frac{1}{2}\int\left(\cot^3 u\right)\left(\cot^2 u\right)du = \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 - 1\right)du$$

$$= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du - \frac{1}{2}\int \cot^3 u\ du$$

$$= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du - \frac{1}{2}\int\left(\cot u\right)\left(\csc^2 u - 1\right)du$$

$$= \frac{1}{2}\int\left(\cot^3 u\right)\left(\csc^2 u\right)du - \frac{1}{2}\int\left(\cot u\right)\left(\csc^2 u\right)du + \frac{1}{2}\int \cot u$$

$$= -\frac{1}{8}\cot^4 u + \frac{1}{4}\cot^2 u + \frac{1}{2}\ln\left|\sin u\right| + C$$

$$= -\frac{1}{8}\cot^4 2t + \frac{1}{4}\cot^2 2t + \frac{1}{2}\ln\left|\sin 2t\right| + C$$

25. $\int \tan^{-3} x \sec^4 x dx = \int\left(\tan^{-3} x\right)\left(\sec^2 x\right)\left(\sec^2 x\right)dx$

$$= \int\left(\tan^{-3} x\right)\left(1 + \tan^2 x\right)\left(\sec^2 x\right)dx$$

$$= \int \tan^{-3} x \sec^2 x\ \mathrm{dx} + \int\left(\tan x\right)^{-1}\sec^2 x\ dx$$

$$= -\frac{1}{2}\tan^{-2} x + \ln\left|\tan x\right| + C$$

26. $\int \tan^{-3/2} x \sec^4 x\ dx = \int\left(\tan^{-3/2} x\right)\left(\sec^2 x\right)\left(\sec^2 x\right)$

$$= \int\left(\tan^{-3/2} x\right)\left(1 + \tan^2 x\right)\left(\sec^2 x\right)dx$$

$$= \int \tan^{-3/2} x \sec^2 x\ dx + \int \tan^{1/2} x \sec^2 x\ dx$$

$$= -2\tan^{-1/2} x + \frac{2}{3}\tan^{3/2} x + C$$

27. $\int \tan^3 x \sec^2 x\, dx$

Let $u = \tan x$. Then $du = \sec^2 x\, dx$.

$$\int \tan^3 x \sec^2 x\, dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\tan^4 x + C$$

28. $\int \tan^3 x \sec^{-1/2} x\, dx = \int \tan^2 x \sec^{-3/2} x(\sec x \tan x) dx$

$$= \int \left(\sec^2 x - 1\right)\left(\sec^{-3/2} x\right)\left(\sec x \tan x\right) dx$$

$$= \int \sec^{1/2} x\left(\sec x \tan x\right) dx - \int \sec^{-3/2} x\left(\sec x \tan x\right) dx$$

$$= \frac{2}{3}\sec^{3/2} x + 2\sec^{-1/2} x + C$$

29. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos[(m+n)x] + \cos[(m-n)x])dx = \frac{1}{2}\left[\frac{1}{m+n}\sin[(m+n)x] + \frac{1}{m-n}\sin[(m-n)x]\right]_{-\pi}^{\pi}$

= 0 for $m \neq n$, since $\sin k\pi = 0$ for all integers k.

30. If we let $u = \frac{\pi x}{L}$ then $du = \frac{\pi}{L}dx$. Making the substitution and changing the limits as necessary, we get

$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\, dx = \frac{L}{\pi}\int_{-\pi}^{\pi}\cos mu \cos nu\, du = 0$ (See Problem 29

31. $\int_0^{\pi} \pi(x + \sin x)^2 dx = \pi\int_0^{\pi}(x^2 + 2x\sin x + \sin^2 x)\, dx = \pi\int_0^{\pi} x^2\, dx + 2\pi\int_0^{\pi} x\sin x\, dx + \frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)dx$

$$= \pi\left[\frac{1}{3}x^3\right]_0^{\pi} + 2\pi\left[\sin x - x\cos x\right]_0^{\pi} + \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{3}\pi^4 + 2\pi(0+\pi-0) + \frac{\pi}{2}(\pi-0-0) = \frac{1}{3}\pi^4 + \frac{5}{2}\pi^2 \approx 57.1437$$

Use Formula 40 with $u = x$ for $\int x\sin x\, dx$

32. $V = 2\pi\int_0^{\sqrt{\pi/2}} x\sin^2(x^2)dx$

$u = x^2$, $du = 2x\, dx$

$$V = \pi\int_0^{\pi/2}\sin^2 u\, du = \pi\int_0^{\pi/2}\frac{1-\cos 2u}{2}du = \pi\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^{\pi/2} = \frac{\pi^2}{4} \approx 2.4674$$

33. a. $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N} a_n\sin(nx)\right)\sin(mx)dx = \frac{1}{\pi}\sum_{n=1}^{N} a_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx$

From Example 6,

$\int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } n = m \end{cases}$ so every term in the sum is 0 except for when $n = m$.

If $m > N$, there is no term where $n = m$, while if $m \leq N$, then $n = m$ occurs. When $n = m$

$a_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\, dx = a_m\pi$ so when $m \leq N$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\, dx = \frac{1}{\pi}\cdot a_m\cdot\pi = a_m .$$

b. $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N} a_n \sin(nx)\right)\left(\sum_{m=1}^{N} a_m \sin(mx)\right)dx = \frac{1}{\pi}\sum_{n=1}^{N} a_n \sum_{m=1}^{N} a_m \int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx$

From Example 6, the integral is 0 except when $m = n$. When $m = n$, we obtain

$\frac{1}{\pi}\sum_{n=1}^{N} a_n(a_n\pi) = \sum_{n=1}^{N} a_n^2$.

34. a. Proof by induction

$n = 1$: $\cos\frac{x}{2} = \cos\frac{x}{2}$

Assume true for $k \le n$.

$$\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\cdot\cos\frac{x}{2^{n+1}} = \left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\frac{1}{2^{n-1}}\cos\frac{x}{2^{n+1}}$$

Note that

$$\left(\cos\frac{k}{2^n}x\right)\left(\cos\frac{1}{2^{n+1}}x\right) = \frac{1}{2}\left[\cos\frac{2k+1}{2^{n+1}}x + \cos\frac{2k-1}{2^{n+1}}x\right],\text{ so}$$

$$\left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\left(\cos\frac{1}{2^{n+1}}x\right)\frac{1}{2^{n-1}} = \left[\cos\frac{1}{2^{n+1}}x + \cos\frac{3}{2^{n+1}}x + \cdots + \cos\frac{2^{n+1}-1}{2^{n+1}}x\right]\frac{1}{2^n}$$

b. $$\lim_{n\to\infty}\left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\frac{1}{2^{n-1}} = \frac{1}{x}\lim_{n\to\infty}\left[\cos\frac{1}{2^n}x + \cos\frac{3}{2^n}x + \cdots + \cos\frac{2^n-1}{2^n}x\right]\frac{x}{2^{n-1}}$$

$$= \frac{1}{x}\int_0^x \cos t\,dt$$

c. $\frac{1}{x}\int_0^x \cos t\,dt = \frac{1}{x}[\sin t]_0^x = \frac{\sin x}{x}$

35. Using the half-angle identity $\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$, we see that since

$$\cos\frac{\pi}{4} = \cos\frac{\frac{\pi}{2}}{2} = \frac{\sqrt{2}}{2}$$

$$\cos\frac{\pi}{8} = \cos\frac{\frac{\pi}{2}}{4} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\cos\frac{\pi}{16} = \cos\frac{\frac{\pi}{2}}{8} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2},\text{ etc.}$$

Thus, $\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots = \cos\left(\frac{\frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{2}}{4}\right)\cos\left(\frac{\frac{\pi}{2}}{8}\right)\cdots$

$$= \lim_{n\to\infty}\cos\left(\frac{\frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{2}}{4}\right)\cdots\cos\left(\frac{\frac{\pi}{2}}{2^n}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$$

36. Since $(k-\sin x)^2 = (\sin x - k)^2$, the volume of S is $\int_0^\pi \pi(k-\sin x)^2 = \pi\int_0^\pi (k^2 - 2k\sin x + \sin^2 x)\,dx$

$$= \pi k^2 \int_0^\pi dx - 2k\pi\int_0^\pi \sin x\,dx + \frac{\pi}{2}\int_0^\pi (1-\cos 2x)\,dx = \pi k^2 \left[x\right]_0^\pi + 2k\pi\left[\cos x\right]_0^\pi + \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi$$

$$= \pi^2 k^2 + 2k\pi(-1-1) + \frac{\pi}{2}(\pi - 0) = \pi^2 k^2 - 4k\pi + \frac{\pi^2}{2}$$

Let $f(k) = \pi^2 k^2 - 4k\pi + \frac{\pi^2}{2}$, then $f'(k) = 2\pi^2 k - 4\pi$ and $f'(k) = 0$ when $k = \frac{2}{\pi}$.

The critical points of $f(k)$ on $0 \le k \le 1$ are 0, $\frac{2}{\pi}$, 1.

$$f(0) = \frac{\pi^2}{2} \approx 4.93,\ f\left(\frac{2}{\pi}\right) = 4 - 8 + \frac{\pi^2}{2} \approx 0.93,\ f(1) = \pi^2 - 4\pi + \frac{\pi^2}{2} \approx 2.24$$

a. S has minimum volume when $k = \frac{2}{\pi}$.

b. S has maximum volume when $k = 0$.

7.4 Concepts Review

1. $\sqrt{x-3}$
2. $2\sin t$
3. $2\tan t$
4. $2\sec t$

Problem Set 7.4

1. $u = \sqrt{x+1}, u^2 = x+1, 2u\,du = dx$

$$\int x\sqrt{x+1}dx = \int (u^2-1)u(2u\,du)$$
$$= \int (2u^4 - 2u^2)du = \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$$
$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

2. $u = \sqrt[3]{x+\pi}, u^3 = x+\pi, 3u^2 du = dx$

$$\int x\sqrt[3]{x+\pi}dx = \int (u^3-\pi)u(3u^2 du)$$
$$= \int (3u^6 - 3\pi u^3)du = \frac{3}{7}u^7 - \frac{3\pi}{4}u^4 + C$$
$$= \frac{3}{7}(x+\pi)^{7/3} - \frac{3\pi}{4}(x+\pi)^{4/3} + C$$

3. $u = \sqrt{3t+4}, u^2 = 3t+4,\ 2u\,du = 3\,dt$

$$\int \frac{t\,dt}{\sqrt{3t+4}} = \int \frac{\frac{1}{3}(u^2-4)\frac{2}{3}u\,du}{u} = \frac{2}{9}\int (u^2-4)du$$
$$= \frac{2}{27}u^3 - \frac{8}{9}u + C$$
$$= \frac{2}{27}(3t+4)^{3/2} - \frac{8}{9}(3t+4)^{1/2} + C$$

4. $u = \sqrt{x+4}, u^2 = x+4,\ 2u\,du = dx$

$$\int \frac{x^2+3x}{\sqrt{x+4}}dx = \int \frac{(u^2-4)^2 + 3(u^2-4)}{u}2u\,du$$
$$= 2\int (u^4 - 5u^2 + 4)du = \frac{2}{5}u^5 - \frac{10}{3}u^3 + 8u + C$$
$$= \frac{2}{5}(x+4)^{5/2} - \frac{10}{3}(x+4)^{3/2} + 8(x+4)^{1/2} + C$$

5. $u = \sqrt{t}, u^2 = t,\ 2u\,du = dt$

$$\int_1^2 \frac{dt}{\sqrt{t}+e} = \int_1^{\sqrt{2}} \frac{2u\,du}{u+e} = 2\int_1^{\sqrt{2}} \frac{u+e-e}{u+e}du$$
$$= 2\int_1^{\sqrt{2}} du - 2\int_1^{\sqrt{2}} \frac{e}{u+e}du$$
$$= 2[u]_1^{\sqrt{2}} - 2e\left[\ln|u+e|\right]_1^{\sqrt{2}}$$
$$= 2(\sqrt{2}-1) - 2e[\ln(\sqrt{2}+e) - \ln(1+e)]$$
$$= 2\sqrt{2} - 2 - 2e\ln\left(\frac{\sqrt{2}+e}{1+e}\right)$$

6. $u = \sqrt{t}, u^2 = t,\ 2u\,du = dt$

$$\int_0^1 \frac{\sqrt{t}}{t+1}dt = \int_0^1 \frac{u}{u^2+1}(2u\,du)$$
$$= 2\int_0^1 \frac{u^2}{u^2+1}du = 2\int_0^1 \frac{u^2+1-1}{u^2+1}du$$
$$= 2\int_0^1 du - 2\int_0^1 \frac{1}{u^2+1}du = 2[u]_0^1 - 2[\tan^{-1} u]_0^1$$
$$= 2 - 2\tan^{-1} 1 = 2 - \frac{\pi}{2} \approx 0.4292$$

7. $u=(3t+2)^{1/2}, u^2=3t+2, 2u\,du=3dt$

$$\int t(3t+2)^{3/2}\,dt=\int \frac{1}{3}(u^2-2)u^3\left(\frac{2}{3}u\,du\right)$$

$$=\frac{2}{9}\int (u^6-2u^4)du=\frac{2}{63}u^7-\frac{4}{45}u^5+C$$

$$=\frac{2}{63}(3t+2)^{7/2}-\frac{4}{45}(3t+2)^{5/2}+C$$

8. $u=(1-x)^{1/3}, u^3=1-x, 3u^2du=-dx$

$$\int x(1-x)^{2/3}dx=\int (1-u^3)u^2(-3u^2)du$$

$$=3\int (u^7-u^4)du=\frac{3}{8}u^8-\frac{3}{5}u^5+C$$

$$=\frac{3}{8}(1-x)^{8/3}-\frac{3}{5}(1-x)^{5/3}+C$$

9. $x = 2 \sin t$, $dx = 2 \cos t\, dt$

$$\int \frac{\sqrt{4-x^2}}{x}dx=\int \frac{2\cos t}{2\sin t}(2\cos t\,dt)$$

$$=2\int \frac{1-\sin^2 t}{\sin t}dt\ =2\int \csc t\,dt-2\int \sin t\,dt$$

$$=2\ln|\csc t-\cot t|+2\cos t+C$$

$$=2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right|+\sqrt{4-x^2}+C$$

10. $x=4\sin t, dx=4\cos t\,dt$

$$\int \frac{x^2dx}{\sqrt{16-x^2}}=16\int \frac{\sin^2 t\cos t}{\cos t}dt$$

$$=16\int \sin^2 t\,dt\ =8\int (1-\cos 2t)dt$$

$$=8t-4\sin 2t+C=8t-8\sin t\cos t+C$$

$$=8\sin^{-1}\left(\frac{x}{4}\right)-\frac{x\sqrt{16-x^2}}{2}+C$$

11. $x=2\tan t, dx=2\sec^2 t\,dt$

$$\int \frac{dx}{(x^2+4)^{3/2}}=\int \frac{2\sec^2 t\,dt}{(4\sec^2 t)^{3/2}}=\frac{1}{4}\int \cos t\,dt$$

$$=\frac{1}{4}\sin t+C\ =\frac{x}{4\sqrt{x^2+4}}+C$$

12. $t = \sec x$, $dt = \sec x \tan x\, dx$

Note that $0\le x<\frac{\pi}{2}.$

$$\sqrt{t^2-1}=|\tan x|=\tan x$$

$$\int_2^3 \frac{dt}{t^2\sqrt{t^2-1}}=\int_{\pi/3}^{\sec^{-1}(3)}\frac{\sec x\tan x}{\sec^2 x\tan x}dx$$

$$=\int_{\pi/3}^{\sec^{-1}(3)}\cos x\,dx$$

$$=[\sin x]_{\pi/3}^{\sec^{-1}(3)}=\sin[\sec^{-1}(3)]-\sin\frac{\pi}{3}$$

$$=\sin\left[\cos^{-1}\left(\frac{1}{3}\right)\right]-\frac{\sqrt{3}}{2}=\frac{2\sqrt{2}}{3}-\frac{\sqrt{3}}{2}\approx 0.0768$$

13. $t = \sec x$, $dt = \sec x \tan x\, dx$

Note that $\frac{\pi}{2}<x\le\pi.$

$$\sqrt{t^2-1}=|\tan x|=-\tan x$$

$$\int_{-2}^{-3}\frac{\sqrt{t^2-1}}{t^3}dt=\int_{2\pi/3}^{\sec^{-1}(-3)}\frac{-\tan x}{\sec^3 x}\sec x\tan x\,dx$$

$$=\int_{2\pi/3}^{\sec^{-1}(-3)}-\sin^2 x\,dx=\int_{2\pi/3}^{\sec^{-1}(-3)}\left(\frac{1}{2}\cos 2x-\frac{1}{2}\right)dx$$

$$=\left[\frac{1}{4}\sin 2x-\frac{1}{2}x\right]_{2\pi/3}^{\sec^{-1}(-3)}$$

$$=\left[\frac{1}{2}\sin x\cos x-\frac{1}{2}x\right]_{2\pi/3}^{\sec^{-1}(-3)}$$

$$=-\frac{\sqrt{2}}{9}-\frac{1}{2}\sec^{-1}(-3)+\frac{\sqrt{3}}{8}+\frac{\pi}{3}\approx 0.151252$$

14. $t = \sin x$, $dt = \cos x\, dx$

$$\int \frac{t}{\sqrt{1-t^2}}dt=\int \sin x\,dx\ = -\cos x + C$$

$$=-\sqrt{1-t^2}+C$$

15. $z = \sin t$, $dz = \cos t\, dt$

$$\int \frac{2z-3}{\sqrt{1-z^2}}dz=\int (2\sin t-3)dt$$

$$= -2 \cos t - 3t + C$$

$$=-2\sqrt{1-z^2}-3\sin^{-1}z+C$$

16. $x = \pi \tan t,\ dx = \pi \sec^2 t\, dt$

$$\int \frac{\pi x - 1}{\sqrt{x^2+\pi^2}}dx = \int (\pi^2 \tan t - 1)\sec t\, dt$$
$$= \pi^2 \int \tan t \sec t\, dt - \int \sec t\, dt$$
$$= \pi^2 \sec t - \ln\left|\sec t + \tan t\right| + C$$
$$= \pi\sqrt{x^2+\pi^2} - \ln\left|\frac{1}{\pi}\sqrt{x^2+\pi^2} + \frac{x}{\pi}\right| + C$$
$$\int_0^{\pi} \frac{\pi x - 1}{\sqrt{x^2+\pi^2}}dx$$
$$= \left[\pi\sqrt{x^2+\pi^2} - \ln\left|\frac{\sqrt{x^2+\pi^2}}{\pi} + \frac{x}{\pi}\right|\right]_0^{\pi}$$
$$= [\sqrt{2}\pi^2 - \ln(\sqrt{2}+1)] - [\pi^2 - \ln 1]$$
$$= (\sqrt{2}-1)\pi^2 - \ln(\sqrt{2}+1) \approx 3.207$$

17. $x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 4$

$u = x + 1,\ du = dx$
$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{du}{\sqrt{u^2+4}}$$
$u = 2\tan t,\ du = 2\sec^2 t\, dt$
$$\int \frac{du}{\sqrt{u^2+4}} = \int \sec t\, dt = \ln\left|\sec t + \tan t\right| + C_1$$
$$= \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C_1$$
$$= \ln\left|\frac{\sqrt{x^2+2x+5}+x+1}{2}\right| + C_1$$
$$= \ln\left|\sqrt{x^2+2x+5}+x+1\right| + C$$

18. $x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x+2)^2 + 1$

$u = x + 2,\ du = dx$
$$\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{du}{\sqrt{u^2+1}}$$
$u = \tan t,\ du = \sec^2 t\, dt$
$$\int \frac{du}{\sqrt{u^2+1}} = \int \sec t\, dt = \ln\left|\sec t + \tan t\right| + C$$
$$\int \frac{dx}{\sqrt{x^2+4x+5}} = \ln\left|\sqrt{u^2+1}+u\right| + C$$
$$= \ln\left|\sqrt{x^2+4x+5}+x+2\right| + C$$

19. $x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 4$

$u = x + 1,\ du = dx$
$$\int \frac{3x}{\sqrt{x^2+2x+5}}dx = \int \frac{3u-3}{\sqrt{u^2+4}}du$$
$$= 3\int \frac{u}{\sqrt{u^2+4}}du - 3\int \frac{du}{\sqrt{u^2+4}}$$
(Use the result of Problem 17.)
$$= 3\sqrt{u^2+4} - 3\ln\left|\sqrt{u^2+4}+u\right| + C$$
$$= 3\sqrt{x^2+2x+5} - 3\ln\left|\sqrt{x^2+2x+5}+x+1\right| + C$$

20. $x^2 + 4x + 5 = x^2 + 4x + 4 + 1 = (x+2)^2 + 1$

$u = x + 2,\ du = dx$
$$\int \frac{2x-1}{\sqrt{x^2+4x+5}}dx = \int \frac{2u-5}{\sqrt{u^2+1}}du$$
$$= \int \frac{2u\, du}{\sqrt{u^2+1}} - 5\int \frac{du}{\sqrt{u^2+1}}$$
(Use the result of Problem 18.)
$$= 2\sqrt{u^2+1} - 5\ln\left|\sqrt{u^2+1}+u\right| + C$$
$$= 2\sqrt{x^2+4x+5} - 5\ln\left|\sqrt{x^2+4x+5}+x+2\right| + C$$

21. $5 - 4x - x^2 = 9 - (4 + 4x + x^2) = 9 - (x+2)^2$

$u = x + 2,\ du = dx$
$$\int \sqrt{5-4x-x^2}\,dx = \int \sqrt{9-u^2}\,du$$
$u = 3\sin t,\ du = 3\cos t\, dt$
$$\int \sqrt{9-u^2}\,du = 9\int \cos^2 t\, dt = \frac{9}{2}\int (1+\cos 2t)dt$$
$$= \frac{9}{2}\left(t + \frac{1}{2}\sin 2t\right) + C = \frac{9}{2}(t + \sin t \cos t) + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{u}{3}\right) + \frac{1}{2}u\sqrt{9-u^2} + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + \frac{x+2}{2}\sqrt{5-4x-x^2} + C$$

22. $16 + 6x - x^2 = 25 - (9 - 6x + x^2) = 25 - (x-3)^2$

$u = x - 3,\ du = dx$
$$\int \frac{dx}{\sqrt{16+6x-x^2}} = \int \frac{du}{\sqrt{25-u^2}}$$
$u = 5\sin t,\ du = 5\cos t$
$$\int \frac{du}{\sqrt{25-u^2}} = \int dt = t + C = \sin^{-1}\left(\frac{u}{5}\right) + C$$
$$= \sin^{-1}\left(\frac{x-3}{5}\right) + C$$

23. $4x - x^2 = 4 - (4 - 4x + x^2) = 4 - (x-2)^2$

$u = x - 2$, $du = dx$

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{du}{\sqrt{4 - u^2}}$$

$u = 2 \sin t$, $du = 2 \cos t\, dt$

$$\int \frac{du}{\sqrt{4-u^2}} = \int dt = t + C = \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

24. $4x - x^2 = 4 - (4 - 4x + x^2) = 4 - (x-2)^2$

$u = x - 2$, $du = dx$

$$\int \frac{x}{\sqrt{4x - x^2}}\, dx = \int \frac{u+2}{\sqrt{4-u^2}}\, du$$

$$= -\int \frac{-u\, du}{\sqrt{4-u^2}} + 2\int \frac{du}{\sqrt{4-u^2}}$$

(Use the result of Problem 23.)

$$= -\sqrt{4-u^2} + 2\sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4x - x^2} + 2\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

25. $x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x+1)^2 + 1$

$u = x + 1$, $du = dx$

$$\int \frac{2x+1}{x^2+2x+2}\, dx = \int \frac{2u-1}{u^2+1}\, du$$

$$= \int \frac{2u}{u^2+1}\, du - \int \frac{du}{u^2+1}$$

$$= \ln\left|u^2+1\right| - \tan^{-1} u + C$$

$$= \ln\left(x^2+2x+2\right) - \tan^{-1}(x+1) + C$$

26. $x^2 - 6x + 18 = x^2 - 6x + 9 + 9 = (x-3)^2 + 9$

$u = x - 3$, $du = dx$

$$\int \frac{2x-1}{x^2-6x+18}\, dx = \int \frac{2u+5}{u^2+9}\, du$$

$$= \int \frac{2u\, du}{u^2+9} + 5\int \frac{du}{u^2+9}$$

$$= \ln\left(u^2+9\right) + \frac{5}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \ln\left(x^2-6x+18\right) + \frac{5}{3}\tan^{-1}\left(\frac{x-3}{3}\right) + C$$

27. $V = \pi\int_0^1 \left(\frac{1}{x^2+2x+5}\right)^2 dx$

$$= \pi\int_0^1 \left[\frac{1}{(x+1)^2+4}\right]^2 dx$$

$x + 1 = 2 \tan t$, $dx = 2\sec^2 t\, dt$

$$V = \pi\int_{\tan^{-1}(1/2)}^{\pi/4} \left(\frac{1}{4\sec^2 t}\right)^2 2\sec^2 t\, dt$$

$$= \frac{\pi}{8}\int_{\tan^{-1}(1/2)}^{\pi/4} \frac{1}{\sec^2 t}\, dt = \frac{\pi}{8}\int_{\tan^{-1}(1/2)}^{\pi/4} \cos^2 t\, dt$$

$$= \frac{\pi}{8}\int_{\tan^{-1}(1/2)}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt$$

$$= \frac{\pi}{8}\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_{\tan^{-1}(1/2)}^{\pi/4}$$

$$= \frac{\pi}{8}\left[\frac{1}{2}t + \frac{1}{2}\sin t\cos t\right]_{\tan^{-1}(1/2)}^{\pi/4}$$

$$= \frac{\pi}{8}\left[\left(\frac{\pi}{8} + \frac{1}{4}\right) - \left(\frac{1}{2}\tan^{-1}\frac{1}{2} + \frac{1}{5}\right)\right]$$

$$= \frac{\pi}{16}\left(\frac{1}{10} + \frac{\pi}{4} - \tan^{-1}\frac{1}{2}\right) \approx 0.082811$$

28. $V = 2\pi\int_0^1 \frac{1}{x^2+2x+5}\, x\, dx$

$$= 2\pi\int_0^1 \frac{x}{(x+1)^2+4}\, dx$$

$$= 2\pi\int_0^1 \frac{x+1}{(x+1)^2+4}\, dx - 2\pi\int_0^1 \frac{1}{(x+1)^2+4}\, dx$$

$$= 2\pi\left[\frac{1}{2}\ln[(x+1)^2+4]\right]_0^1 - 2\pi\left[\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\right]_0^1$$

$$= \pi[\ln 8 - \ln 5] - \pi\left[\tan^{-1} 1 - \tan^{-1}\frac{1}{2}\right]$$

$$= \pi\left(\ln\frac{8}{5} - \frac{\pi}{4} + \tan^{-1}\frac{1}{2}\right) \approx 0.465751$$

29. **a.** $u = x^2 + 9$, $du = 2x\, dx$

$$\int \frac{x\, dx}{x^2+9} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln\left|x^2+9\right| + C = \frac{1}{2}\ln\left(x^2+9\right) + C$$

b. $x = 3 \tan t$, $dx = 3\sec^2 t\, dt$

$$\int \frac{x\, dx}{x^2+9} = \int \tan t\, dt = -\ln|\cos t| + C$$

$$= -\ln\left|\frac{3}{\sqrt{x^2+9}}\right| + C_1 = -\ln\left(\frac{3}{\sqrt{x^2+9}}\right) + C_1$$

$$= \ln\left(\sqrt{x^2+9}\right) - \ln 3 + C_1$$

$$= \ln\left((x^2+9)^{1/2}\right) + C = \frac{1}{2}\ln\left(x^2+9\right) + C$$

30. $u = \sqrt{9+x^2}, u^2 = 9 + x^2,\ 2u\,du = 2x\,dx$

$$\int_0^3 \frac{x^3 dx}{\sqrt{9+x^2}} = \int_0^3 \frac{x^2}{\sqrt{9+x^2}} x\,dx = \int_3^{3\sqrt{2}} \frac{u^2-9}{u} u\,du$$

$$= \int_3^{3\sqrt{2}} (u^2-9)\,du = \left[\frac{u^3}{3} - 9u\right]_3^{3\sqrt{2}} = 18 - 9\sqrt{2}$$

≈ 5.272

31. a. $u = \sqrt{4-x^2}, u^2 = 4 - x^2,\ 2u\,du = -2x\,dx$

$$\int \frac{\sqrt{4-x^2}}{x} dx = \int \frac{\sqrt{4-x^2}}{x^2} x\,dx = -\int \frac{u^2 du}{4-u^2}$$

$$= \int \frac{-4+4-u^2}{4-u^2} du = -4\int \frac{1}{4-u^2} du + \int du$$

$$= -4 \cdot \frac{1}{4} \ln\left|\frac{u+2}{u-2}\right| + u + C$$

$$= -\ln\left|\frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2}\right| + \sqrt{4-x^2} + C$$

b. $x = 2\sin t$, $dx = 2\cos t\,dt$

$$\int \frac{\sqrt{4-x^2}}{x} dx = 2\int \frac{\cos^2 t}{\sin t} dt$$

$$= 2\int \frac{(1-\sin^2 t)}{\sin t} dt$$

$$= 2\int \csc t\,dt - 2\int \sin t\,dt$$

$$= 2\ln|\csc t - \cot t| + 2\cos t + C$$

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C$$

$$= 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C$$

To reconcile the answers, note that

$$-\ln\left|\frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2}\right| = \ln\left|\frac{\sqrt{4-x^2}-2}{\sqrt{4-x^2}+2}\right|$$

$$= \ln\left|\frac{(\sqrt{4-x^2}-2)^2}{(\sqrt{4-x^2}+2)(\sqrt{4-x^2}-2)}\right|$$

$$= \ln\left|\frac{(2-\sqrt{4-x^2})^2}{4-x^2-4}\right| = \ln\left|\frac{(2-\sqrt{4-x^2})^2}{-x^2}\right|$$

$$= \ln\left|\left(\frac{2-\sqrt{4-x^2}}{x}\right)^2\right| = 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right|$$

32. The equation of the circle with center (–a, 0) is $(x+a)^2 + y^2 = b^2$, so $y = \pm\sqrt{b^2-(x+a)^2}$. By symmetry, the area of the overlap is four times the area of the region bounded by $x = 0$, $y = 0$, and $y = \sqrt{b^2-(x+a)^2}dx$.

$$A = 4\int_0^{b-a} \sqrt{b^2-(x+a)^2}dx$$

$x + a = b\sin t$, $dx = b\cos t\,dt$

$$A = 4\int_{\sin^{-1}(a/b)}^{\pi/2} b^2\cos^2 t\,dt$$

$$= 2b^2\int_{\sin^{-1}(a/b)}^{\pi/2} (1+\cos 2t)dt$$

$$= 2b^2\left[t + \frac{1}{2}\sin 2t\right]_{\sin^{-1}(a/b)}^{\pi/2}$$

$$= 2b^2[t + \sin t\cos t]_{\sin^{-1}(a/b)}^{\pi/2}$$

$$= 2b^2\left[\frac{\pi}{2} - \left(\sin^{-1}\left(\frac{a}{b}\right) + \frac{a}{b}\frac{\sqrt{b^2-a^2}}{b}\right)\right]$$

$$= \pi b^2 - 2b^2\sin^{-1}\left(\frac{a}{b}\right) - 2a\sqrt{b^2-a^2}$$

33. a. The coordinate of C is (0, –a). The lower arc of the lune lies on the circle given by the equation $x^2 + (y+a)^2 = 2a^2$ or $y = \pm\sqrt{2a^2-x^2} - a.$ The upper arc of the lune lies on the circle given by the equation $x^2 + y^2 = a^2$ or $y = \pm\sqrt{a^2-x^2}$.

$$A = \int_{-a}^{a} \sqrt{a^2-x^2}dx - \int_{-a}^{a}\left(\sqrt{2a^2-x^2} - a\right)dx$$

$$= \int_{-a}^{a} \sqrt{a^2-x^2}dx - \int_{-a}^{a} \sqrt{2a^2-x^2}dx + 2a^2$$

Note that $\int_{-a}^{a} \sqrt{a^2-x^2}dx$ is the area of a semicircle with radius a, so

$$\int_{-a}^{a} \sqrt{a^2-x^2}dx = \frac{\pi a^2}{2}.$$

For $\int_{-a}^{a} \sqrt{2a^2-x^2}dx,$ let

$x = \sqrt{2}a\sin t, dx = \sqrt{2}a\cos t\,dt$

$$\int_{-a}^{a} \sqrt{2a^2-x^2}dx = \int_{-\pi/4}^{\pi/4} 2a^2\cos^2 t\,dt$$

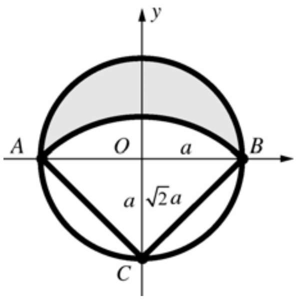
$$= a^2\int_{-\pi/4}^{\pi/4} (1+\cos 2t)dt = a^2\left[t + \frac{1}{2}\sin 2t\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi a^2}{2} + a^2$$

$$A = \frac{\pi a^2}{2} - \left(\frac{\pi a^2}{2} + a^2\right) + 2a^2 = a^2$$

Thus, the area of the lune is equal to the area of the square.

b. Without using calculus, consider the following labels on the figure.



Area of the lune = Area of the semicircle of radius a at O + Area (ΔABC) – Area of the sector ABC.

$$A = \frac{1}{2}\pi a^2 + a^2 - \frac{1}{2}\left(\frac{\pi}{2}\right)(\sqrt{2}a)^2$$

$$= \frac{1}{2}\pi a^2 + a^2 - \frac{1}{2}\pi a^2 = a^2$$

Note that since BC has length $\sqrt{2}a$, the measure of angle OCB is $\frac{\pi}{4}$, so the measure of angle ACB is $\frac{\pi}{2}$.

34. Using reasoning similar to Problem 33 b, the area is

$$\frac{1}{2}\pi a^2 + \frac{1}{2}(2a)\sqrt{b^2 - a^2} - \frac{1}{2}\left(2\sin^{-1}\frac{a}{b}\right)b^2$$

$$= \frac{1}{2}\pi a^2 + a\sqrt{b^2 - a^2} - b^2 \sin^{-1}\frac{a}{b}.$$

35. $\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$; $\quad y = \int -\frac{\sqrt{a^2 - x^2}}{x}dx$

$x = a \sin t$, $dx = a \cos t\, dt$

$$y = \int -\frac{a\cos t}{a\sin t}a\cos t\, dt = -a\int \frac{\cos^2 t}{\sin t}dt$$

$$= -a\int \frac{1-\sin^2 t}{\sin t}dt = a\int(\sin t - \csc t)dt$$

$$= a\left(-\cos t - \ln|\csc t - \cot t|\right) + C$$

$$\cos t = \frac{\sqrt{a^2 - x^2}}{a}, \csc t = \frac{a}{x}, \cot t = \frac{\sqrt{a^2 - x^2}}{x}$$

$$y = a\left(-\frac{\sqrt{a^2 - x^2}}{a} - \ln\left|\frac{a}{x} - \frac{\sqrt{a^2 - x^2}}{x}\right|\right) + C$$

$$= -\sqrt{a^2 - x^2} - a\ln\left|\frac{a - \sqrt{a^2 - x^2}}{x}\right| + C$$

Since $y = 0$ when $x = a$,
$0 = 0 - a \ln 1 + C$, so $C = 0$.

$$y = -\sqrt{a^2 - x^2} - a\ln\left|\frac{a - \sqrt{a^2 - x^2}}{x}\right|$$

7.5 Concepts Review

1. proper

2. $x - 1 + \frac{5}{x+1}$

3. $a = 2$; $b = 3$; $c = -1$

4. $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 1}$

Problem Set 7.5

1. $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$1 = A(x + 1) + Bx$
$A = 1$, $B = -1$

$$\int \frac{1}{x(x+1)}dx = \int \frac{1}{x}dx - \int \frac{1}{x+1}dx$$

$$= \ln|x| - \ln|x+1| + C$$

2. $\frac{2}{x^2 + 3x} = \frac{2}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$

$2 = A(x + 3) + Bx$

$A = \frac{2}{3}, B = -\frac{2}{3}$

$$\int \frac{2}{x^2 + 3x}dx = \frac{2}{3}\int \frac{1}{x}dx - \frac{2}{3}\int \frac{B}{x+3}dx$$

$$= \frac{2}{3}\ln|x| - \frac{2}{3}\ln|x+3| + C$$

3. $\frac{3}{x^2 - 1} = \frac{3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$3 = A(x - 1) + B(x + 1)$

$A = -\frac{3}{2}, B = \frac{3}{2}$

$$\int \frac{3}{x^2 - 1}dx = -\frac{3}{2}\int \frac{1}{x+1}dx + \frac{3}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{3}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C$$

4. $\frac{5x}{2x^3+6x^2}=\frac{5x}{2x^2(x+3)}=\frac{5}{2x(x+3)}$

$=\frac{A}{x}+\frac{B}{x+3}$

$\frac{5}{2}=A(x+3)+Bx$

$A=\frac{5}{6}, B=-\frac{5}{6}$

$\int\frac{5x}{2x^3+6x^2}=\frac{5}{6}\int\frac{1}{x}dx-\frac{5}{6}\int\frac{1}{x+3}dx$

$=\frac{5}{6}\ln|x|-\frac{5}{6}\ln|x+3|+C$

5. $\frac{x-11}{x^2+3x-4}=\frac{x-11}{(x+4)(x-1)}=\frac{A}{x+4}+\frac{B}{x-1}$

$x-11=A(x-1)+B(x+4)$

$A=3, B=-2$

$\int\frac{x-11}{x^2+3x-4}dx=3\int\frac{1}{x+4}dx-2\int\frac{1}{x-1}dx$

$=3\ln|x+4|-2\ln|x-1|+C$

6. $\frac{x-7}{x^2-x-12}=\frac{x-7}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$

$x-7=A(x+3)+B(x-4)$

$A=-\frac{3}{7}, B=\frac{10}{7}$

$\int\frac{x-7}{x^2-x-12}dx=-\frac{3}{7}\int\frac{1}{x-4}dx+\frac{10}{7}\int\frac{1}{x+3}dx$

$=-\frac{3}{7}\ln|x-4|+\frac{10}{7}\ln|x+3|+C$

7. $\frac{3x-13}{x^2+3x-10}=\frac{3x-13}{(x+5)(x-2)}=\frac{A}{x+5}+\frac{B}{x-2}$

$3x-13=A(x-2)+B(x+5)$

$A=4, B=-1$

$\int\frac{3x-13}{x^2+3x-10}dx=4\int\frac{1}{x+5}dx-\int\frac{1}{x-2}dx$

$=4\ln|x+5|-\ln|x-2|+C$

8. $\frac{x+\pi}{x^2-3\pi x+2\pi^2}=\frac{x+\pi}{(x-2\pi)(x-\pi)}=\frac{A}{x-2\pi}+\frac{B}{x-\pi}$

$x+\pi=A(x-\pi)+B(x-2\pi)$

$A=3, B=-2$

$\int\frac{x+\pi}{x^2-3\pi x+2\pi^2}dx=\int\frac{3}{x-2\pi}dx-\int\frac{2}{x-\pi}dx$

$=3\ln|x-2\pi|-2\ln|x-\pi|+C$

9. $\frac{2x+21}{2x^2+9x-5}=\frac{2x+21}{(2x-1)(x+5)}=\frac{A}{2x-1}+\frac{B}{x+5}$

$2x+21=A(x+5)+B(2x-1)$

$A=4, B=-1$

$\int\frac{2x+21}{2x^2+9x-5}dx=\int\frac{4}{2x-1}dx-\int\frac{1}{x+5}dx$

$=2\ln|2x-1|-\ln|x+5|+C$

10. $\frac{2x^2-x-20}{x^2+x-6}=\frac{2(x^2+x-6)-3x-8}{x^2+x-6}$

$=2-\frac{3x+8}{x^2+x-6}$

$\frac{3x+8}{x^2+x-6}=\frac{3x+8}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$

$3x+8=A(x-2)+B(x+3)$

$A=\frac{1}{5}, B=\frac{14}{5}$

$\int\frac{2x^2-x-20}{x^2+x-6}dx$

$=\int 2\,dx-\frac{1}{5}\int\frac{1}{x+3}dx-\frac{14}{5}\int\frac{1}{x-2}dx$

$=2x-\frac{1}{5}\ln|x+3|-\frac{14}{5}\ln|x-2|+C$

11. $\frac{17x-3}{3x^2+x-2}=\frac{17x-3}{(3x-2)(x+1)}=\frac{A}{3x-2}+\frac{B}{x+1}$

$17x-3=A(x+1)+B(3x-2)$

$A=5, B=4$

$\int\frac{17x-3}{3x^2+x-2}dx=\int\frac{5}{3x-2}dx+\int\frac{4}{x+1}dx=\frac{5}{3}\ln|3x-2|+4\ln|x+1|+C$

12. $\dfrac{5-x}{x^2-x(\pi+4)+4\pi}=\dfrac{5-x}{(x-\pi)(x-4)}=\dfrac{A}{x-\pi}+\dfrac{B}{x-4}$

$5 - x = A(x - 4) + B(x - \pi)$

$A=\dfrac{5-\pi}{\pi-4}, B=\dfrac{1}{4-\pi}$

$\displaystyle\int\frac{5-x}{x^2-x(\pi+4)+4\pi}dx=\frac{5-\pi}{\pi-4}\int\frac{1}{x-\pi}dx+\frac{1}{4-\pi}\int\frac{1}{x-4}dx=\frac{5-\pi}{\pi-4}\ln|x-\pi|+\frac{1}{4-\pi}\ln|x-4|+C$

13. $\dfrac{2x^2+x-4}{x^3-x^2-2x}=\dfrac{2x^2+x-4}{x(x+1)(x-2)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-2}$

$2x^2+x-4=A(x+1)(x-2)+Bx(x-2)+Cx(x+1)$

$A = 2, B = –1, C = 1$

$\displaystyle\int\frac{2x^2+x-4}{x^3-x^2-2x}dx=\int\frac{2}{x}dx-\int\frac{1}{x+1}dx+\int\frac{1}{x-2}dx=2\ln|x|-\ln|x+1|+\ln|x-2|+C$

14. $\dfrac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)}=\dfrac{A}{2x-1}+\dfrac{B}{3x+2}+\dfrac{C}{x-3}$

$7x^2+2x-3=A(3x+2)(x-3)+B(2x-1)(x-3)+C(2x-1)(3x+2)$

$A=\dfrac{1}{35}, B=-\dfrac{1}{7}, C=\dfrac{6}{5}$

$\displaystyle\int\frac{7x^2+2x-3}{(2x-1)(3x+2)(x-3)}dx=\frac{1}{35}\int\frac{1}{2x-1}dx-\frac{1}{7}\int\frac{1}{3x+2}dx+\frac{6}{5}\int\frac{1}{x-3}dx$

$=\dfrac{1}{70}\ln|2x-1|-\dfrac{1}{21}\ln|3x+2|+\dfrac{6}{5}\ln|x-3|+C$

15. $\dfrac{6x^2+22x-23}{(2x-1)(x^2+x-6)}=\dfrac{6x^2+22x-23}{(2x-1)(x+3)(x-2)}=\dfrac{A}{2x-1}+\dfrac{B}{x+3}+\dfrac{C}{x-2}$

$6x^2+22x-23=A(x+3)(x-2)+B(2x-1)(x-2)+C(2x-1)(x+3)$

$A = 2, B = –1, C = 3$

$\displaystyle\int\frac{6x^2+22x-23}{(2x-1)(x^2+x-6)}dx=\int\frac{2}{2x-1}dx-\int\frac{1}{x+3}dx+\int\frac{3}{x-2}dx=\ln|2x-1|-\ln|x+3|+3\ln|x-2|+C$

16. $\dfrac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32}=\dfrac{1}{4}\left(\dfrac{x^3-6x^2+11x-6}{x^3-7x^2+14x-8}\right)=\dfrac{1}{4}\left(1+\dfrac{x^2-3x+2}{x^3-7x^2+14x-8}\right)$

$=\dfrac{1}{4}\left(1+\dfrac{(x-1)(x-2)}{(x-1)(x-2)(x-4)}\right)=\dfrac{1}{4}\left(1+\dfrac{1}{x-4}\right)$

$\displaystyle\int\frac{x^3-6x^2+11x-6}{4x^3-28x^2+56x-32}dx=\int\frac{1}{4}dx+\frac{1}{4}\int\frac{1}{x-4}dx=\frac{1}{4}x+\frac{1}{4}\ln|x-4|+C$

17. $\dfrac{x^3}{x^2+x-2} = x-1+\dfrac{3x-2}{x^2+x-2}$

$\dfrac{3x-2}{x^2+x-2} = \dfrac{3x-2}{(x+2)(x-1)} = \dfrac{A}{x+2}+\dfrac{B}{x-1}$

$3x - 2 = A(x - 1) + B(x + 2)$

$A = \dfrac{8}{3}, B = \dfrac{1}{3}$

$$\int \frac{x^3}{x^2+x-2}dx = \int (x-1)\,dx + \frac{8}{3}\int \frac{1}{x+2}dx + \frac{1}{3}\int \frac{1}{x-1}dx = \frac{1}{2}x^2 - x + \frac{8}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

18. $\dfrac{x^3+x^2}{x^2+5x+6} = x-4+\dfrac{14x+24}{(x+3)(x+2)}$

$\dfrac{14x+24}{(x+3)(x+2)} = \dfrac{A}{x+3}+\dfrac{B}{x+2}$

$14x + 24 = A(x + 2) + B(x + 3)$

$A = 18, B = -4$

$$\int \frac{x^3+x^2}{x^2+5x+6}dx = \int (x-4)\,dx + \int \frac{18}{x+3}dx - \int \frac{4}{x+2}dx = \frac{1}{2}x^2 - 4x + 18\ln|x+3| - 4\ln|x+2| + C$$

19. $\dfrac{x^4+8x^2+8}{x^3-4x} = x+\dfrac{12x^2+8}{x(x+2)(x-2)}$

$\dfrac{12x^2+8}{x(x+2)(x-2)} = \dfrac{A}{x}+\dfrac{B}{x+2}+\dfrac{C}{x-2}$

$12x^2+8 = A(x+2)(x-2)+Bx(x-2)+Cx(x+2)$

$A = -2, B = 7, C = 7$

$$\int \frac{x^4+8x^2+8}{x^3-4x}dx = \int x\,dx - 2\int \frac{1}{x}dx + 7\int \frac{1}{x+2}dx + 7\int \frac{1}{x-2}dx = \frac{1}{2}x^2 - 2\ln|x| + 7\ln|x+2| + 7\ln|x-2| + C$$

20. $\dfrac{x^6+4x^3+4}{x^3-4x^2} = x^3+4x^2+16x+68+\dfrac{272x^2+4}{x^3-4x^2}$

$\dfrac{272x^2+4}{x^2(x-4)} = \dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-4}$

$272x^2+4 = Ax(x-4)+B(x-4)+Cx^2$

$A = -\dfrac{1}{4}, B = -1, C = \dfrac{1089}{4}$

$$\int \frac{x^6+4x^3+4}{x^3-4x^2}dx = \int (x^3+4x^2+16x+68)\,dx - \frac{1}{4}\int \frac{1}{x}dx - \int \frac{1}{x^2}dx + \frac{1089}{4}\int \frac{1}{x-4}dx$$

$$= \frac{1}{4}x^4 + \frac{4}{3}x^3 + 8x^2 + 68x - \frac{1}{4}\ln|x| + \frac{1}{x} + \frac{1089}{4}\ln|x-4| + C$$

21. $\dfrac{x+1}{(x-3)^2} = \dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}$

$x + 1 = A(x - 3) + B$

$A = 1, B = 4$

$$\int \frac{x+1}{(x-3)^2}dx = \int \frac{1}{x-3}dx + \int \frac{4}{(x-3)^2}dx = \ln|x-3| - \frac{4}{x-3} + C$$

22. $\frac{5x+7}{x^2+4x+4} = \frac{5x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$

$5x + 7 = A(x + 2) + B$

$A = 5, B = –3$

$$\int \frac{5x+7}{x^2+4x+4}dx = \int \frac{5}{x+2}dx - \int \frac{3}{(x+2)^2}dx = 5\ln|x+2| + \frac{3}{x+2} + C$$

23. $\frac{3x+2}{x^3+3x^2+3x+1} = \frac{3x+2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$

$3x+2 = A(x+1)^2 + B(x+1) + C$

$A = 0, B = 3, C = –1$

$$\int \frac{3x+2}{x^3+3x^2+3x+1}dx = \int \frac{3}{(x+1)^2}dx - \int \frac{1}{(x+1)^3}dx = -\frac{3}{x+1} + \frac{1}{2(x+1)^2} + C$$

24. $\frac{x^6}{(x-2)^2(1-x)^5} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{1-x} + \frac{D}{(1-x)^2} + \frac{E}{(1-x)^3} + \frac{F}{(1-x)^4} + \frac{G}{(1-x)^5}$

$A = 128, B = –64, C = 129, D = –72, E = 30, F = –8, G = 1$

$$\int \frac{x^6}{(x-2)^2(1-x)^5}dx = \int \left[\frac{128}{x-2} - \frac{64}{(x-2)^2} + \frac{129}{1-x} - \frac{72}{(1-x)^2} + \frac{30}{(1-x)^3} - \frac{8}{(1-x)^4} + \frac{1}{(1-x)^5}\right]dx$$

$$= 128\ln|x-2| + \frac{64}{x-2} - 129\ln|1-x| + \frac{72}{1-x} - \frac{15}{(1-x)^2} + \frac{8}{3(1-x)^3} - \frac{1}{4(1-x)^4} + C$$

25. $\frac{3x^2-21x+32}{x^3-8x^2+16x} = \frac{3x^2-21x+32}{x(x-4)^2} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$

$3x^2 - 21x + 32 = A(x-4)^2 + Bx(x-4) + Cx$

$A = 2, B = 1, C = –1$

$$\int \frac{3x^2-21x+32}{x^3-8x^2+16}dx = \int \frac{2}{x}dx + \int \frac{1}{x-4}dx - \int \frac{1}{(x-4)^2}dx = 2\ln|x| + \ln|x-4| + \frac{1}{x-4} + C$$

26. $\frac{x^2+19x+10}{2x^4+5x^3} = \frac{x^2+19x+10}{x^3(2x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{2x+5}$

$A = –1, B = 3, C = 2, D = 2$

$$\int \frac{x^2+19x+10}{2x^4+5x^3}dx = \int \left(-\frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3} + \frac{2}{2x+5}\right)dx = -\ln|x| - \frac{3}{x} - \frac{1}{x^2} + \ln|2x+5| + C$$

27. $\frac{2x^2+x-8}{x^3+4x} = \frac{2x^2+x-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

$A = –2, B = 4, C = 1$

$$\int \frac{2x^2+x-8}{x^3+4x}dx = -2\int \frac{1}{x}dx + \int \frac{4x+1}{x^2+4}dx = -2\int \frac{1}{x}dx + 2\int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}dx$$

$$= -2\ln|x| + 2\ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

28. $\frac{3x+2}{x(x+2)^2+16x}=\frac{3x+2}{x(x^2+4x+20)}=\frac{A}{x}+\frac{Bx+C}{x^2+4x+20}$

$A=\frac{1}{10}, B=-\frac{1}{10}, C=\frac{13}{5}$

$\int\frac{3x+2}{x(x+2)^2+16x}dx=\frac{1}{10}\int\frac{1}{x}dx+\int\frac{-\frac{1}{10}x+\frac{13}{5}}{x^2+4x+20}dx=\frac{1}{10}\int\frac{1}{x}dx+\frac{14}{5}\int\frac{1}{(x+2)^2+16}dx-\frac{1}{20}\int\frac{2x+4}{x^2+4x+20}dx$

$=\frac{1}{10}\ln|x|+\frac{7}{10}\tan^{-1}\left(\frac{x+2}{4}\right)-\frac{1}{20}\ln\left|x^2+4x+20\right|+C$

29. $\frac{2x^2-3x-36}{(2x-1)(x^2+9)}=\frac{A}{2x-1}+\frac{Bx+C}{x^2+9}$

$A = –4, B = 3, C = 0$

$\int\frac{2x^2-3x-36}{(2x-1)(x^2+9)}dx=-4\int\frac{1}{2x-1}dx+\int\frac{3x}{x^2+9}dx=-2\ln|2x-1|+\frac{3}{2}\ln\left|x^2+9\right|+C$

30. $\frac{1}{x^4-16}=\frac{1}{(x-2)(x+2)(x^2+4)}$

$=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4}$

$A=\frac{1}{32}, B=-\frac{1}{32}, C=0, D=-\frac{1}{8}$

$\int\frac{1}{x^4-16}dx=\frac{1}{32}\int\frac{1}{x-2}dx-\frac{1}{32}\int\frac{1}{x+2}dx-\frac{1}{8}\int\frac{1}{x^2+4}dx=\frac{1}{32}\ln|x-2|-\frac{1}{32}\ln|x+2|-\frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right)+C$

31. $\frac{1}{(x-1)^2(x+4)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+4}+\frac{D}{(x+4)^2}$

$A=-\frac{2}{125}, B=\frac{1}{25}, C=\frac{2}{125}, D=\frac{1}{25}$

$\int\frac{1}{(x-1)^2(x+4)^2}dx=-\frac{2}{125}\int\frac{1}{x-1}dx+\frac{1}{25}\int\frac{1}{(x-1)^2}dx+\frac{2}{125}\int\frac{1}{x+4}dx+\frac{1}{25}\int\frac{1}{(x+4)^2}dx$

$=-\frac{2}{125}\ln|x-1|-\frac{1}{25(x-1)}+\frac{2}{125}\ln|x+4|-\frac{1}{25(x+4)}+C$

32. $\frac{x^3-8x^2-1}{(x+3)(x^2-4x+5)}=1+\frac{-7x^2+7x-16}{(x+3)(x^2-4x+5)}$

$\frac{-7x^2+7x-16}{(x+3)(x^2-4x+5)}=\frac{A}{x+3}+\frac{Bx+C}{x^2-4x+5}$

$A=-\frac{50}{13}, B=-\frac{41}{13}, C=\frac{14}{13}$

$\int\frac{x^3-8x^2-1}{(x+3)(x^2-4x+5)}dx=\int\left[1-\frac{50}{13}\left(\frac{1}{x+3}\right)+\frac{-\frac{41}{13}x+\frac{14}{13}}{x^2-4x+5}\right]dx$

$=\int dx-\frac{50}{13}\int\frac{1}{x+3}dx-\frac{68}{13}\int\frac{1}{(x-2)^2+1}dx-\frac{41}{26}\int\frac{2x-4}{x^2-4x+5}dx$

$=x-\frac{50}{13}\ln|x+3|-\frac{68}{13}\tan^{-1}(x-2)-\frac{41}{26}\ln\left|x^2-4x+5\right|+C$

33. $x = \sin t$, $dx = \cos t\, dt$

$$\int \frac{(\sin^3 t - 8\sin^2 t - 1)\cos t}{(\sin t + 3)(\sin^2 t - 4\sin t + 5)} dt = \int \frac{x^3 - 8x^2 - 1}{(x+3)(x^2 - 4x + 5)} dx$$

$$= x - \frac{50}{13}\ln|x+3| - \frac{68}{13}\tan^{-1}(x-2) - \frac{41}{26}\ln\left|x^2 - 4x + 5\right| + C$$

which is the result of Problem 32.

$$\int \frac{(\sin^3 t - 8\sin^2 t - 1)\cos t}{(\sin t + 3)(\sin^2 t - 4\sin t + 5)} dt = \sin t - \frac{50}{13}\ln|\sin t + 3| - \frac{68}{13}\tan^{-1}(\sin t - 2) - \frac{41}{26}\ln\left|\sin^2 t - 4\sin t + 5\right| + C$$

34. $x = \sin t$, $dx = \cos t\, dt$

$$\int \frac{\cos t}{\sin^4 t - 16} dt = \int \frac{1}{x^4 - 16} dx = \frac{1}{32}\ln|x-2| - \frac{1}{32}\ln|x+2| - \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + C$$

which is the result of Problem 30.

$$\int \frac{\cos t}{\sin^4 t - 16} dt = \frac{1}{32}\ln|\sin t - 2| - \frac{1}{32}\ln|\sin t + 2| - \frac{1}{16}\tan^{-1}\left(\frac{\sin t}{2}\right) + C$$

35. $\dfrac{x^3 - 4x}{(x^2+1)^2} = \dfrac{Ax + B}{x^2 + 1} + \dfrac{Cx + D}{(x^2+1)^2}$

$A = 1, B = 0, C = -5, D = 0$

$$\int \frac{x^3 - 4x}{(x^2+1)^2} dx = \int \frac{x}{x^2+1} dx - 5\int \frac{x}{(x^2+1)^2} dx = \frac{1}{2}\ln\left|x^2 + 1\right| + \frac{5}{2(x^2+1)} + C$$

36. $x = \cos t$, $dx = -\sin t\, dt$

$$\int \frac{(\sin t)(4\cos^2 t - 1)}{(\cos t)(1 + 2\cos^2 t + \cos^4 t)} dt = -\int \frac{4x^2 - 1}{x(1 + 2x^2 + x^4)} dx$$

$$\frac{4x^2 - 1}{x(1 + 2x^2 + x^4)} = \frac{4x^2 - 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2+1)^2}$$

$A = -1$, $B = 1$, $C = 0$, $D = 5$, $E = 0$

$$-\int \left[-\frac{1}{x} + \frac{x}{x^2+1} + \frac{5x}{(x^2+1)^2}\right] dx = \ln|x| - \frac{1}{2}\ln\left|x^2 + 1\right| + \frac{5}{2(x^2+1)} + C = \ln|\cos t| - \frac{1}{2}\ln\left|\cos^2 t + 1\right| + \frac{5}{2(\cos^2 t + 1)} + C$$

37. $\dfrac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x} = \dfrac{x(2x^2 + 5x + 16)}{x(x^4 + 8x^2 + 16)} = \dfrac{2x^2 + 5x + 16}{(x^2+4)^2} = \dfrac{Ax + B}{x^2 + 4} + \dfrac{Cx + D}{(x^2+4)^2}$

$A = 0$, $B = 2$, $C = 5$, $D = 8$

$$\int \frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x} dx = \int \frac{2}{x^2+4} dx + \int \frac{5x + 8}{(x^2+4)^2} dx = \int \frac{2}{x^2+4} dx + \int \frac{5x}{(x^2+4)^2} dx + \int \frac{8}{(x^2+4)^2} dx$$

To integrate $\displaystyle\int \frac{8}{(x^2+4)^2} dx$, let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$.

$$\int \frac{8}{(x^2+4)^2} dx = \int \frac{16\sec^2\theta}{16\sec^4\theta} d\theta = \int \cos^2\theta\, d\theta = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C = \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{x}{x^2+4} + C$$

$$\int \frac{2x^3 + 5x^2 + 16x}{x^5 + 8x^3 + 16x} dx = \tan^{-1}\frac{x}{2} - \frac{5}{2(x^2+4)} + \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{x}{x^2+4} + C = \frac{3}{2}\tan^{-1}\frac{x}{2} + \frac{2x - 5}{2(x^2+4)} + C$$

38. $\dfrac{x-17}{x^2+x-12}=\dfrac{x-17}{(x+4)(x-3)}=\dfrac{A}{x+4}+\dfrac{B}{x-3}$

$A = 3, B = -2$

$$\int_4^6 \frac{x-17}{x^2+x-12}dx=\int_4^6\left(\frac{3}{x+4}-\frac{2}{x-3}\right)dx=\Big[3\ln|x+4|-2\ln|x-3|\Big]_4^6=(3\ln 10-2\ln 3)-(3\ln 8-2\ln 1)$$

$$=3\ln 10-2\ln 3-3\ln 8 \approx -1.53$$

39. $u = \sin\theta$, $du = \cos\theta\, d\theta$

$$\int_0^{\pi/4}\frac{\cos\theta}{(1-\sin^2\theta)(\sin^2\theta+1)^2}d\theta=\int_0^{1/\sqrt{2}}\frac{1}{(1-u^2)(u^2+1)^2}du=\int_0^{1/\sqrt{2}}\frac{1}{(1-u)(1+u)(u^2+1)^2}du$$

$$\frac{1}{(1-u^2)(u^2+1)^2}=\frac{A}{1-u}+\frac{B}{1+u}+\frac{Cu+D}{u^2+1}+\frac{Eu+F}{(u^2+1)^2}$$

$$A=\frac{1}{8}, B=\frac{1}{8}, C=0, D=\frac{1}{4}, E=0, F=\frac{1}{2}$$

$$\int_0^{1/\sqrt{2}}\frac{1}{(1-u^2)(u^2+1)^2}du=\frac{1}{8}\int_0^{1/\sqrt{2}}\frac{1}{1-u}du+\frac{1}{8}\int_0^{1/\sqrt{2}}\frac{1}{1+u}du+\frac{1}{4}\int_0^{1/\sqrt{2}}\frac{1}{u^2+1}du+\frac{1}{2}\int_0^{1/\sqrt{2}}\frac{1}{(u^2+1)^2}du$$

$$=\left[-\frac{1}{8}\ln|1-u|+\frac{1}{8}\ln|1+u|+\frac{1}{4}\tan^{-1}u+\frac{1}{4}\left(\tan^{-1}u+\frac{u}{u^2+1}\right)\right]_0^{1/\sqrt{2}}=\left[\frac{1}{8}\ln\left|\frac{1+u}{1-u}\right|+\frac{1}{2}\tan^{-1}u+\frac{u}{4(u^2+1)}\right]_0^{1/\sqrt{2}}$$

$$=\frac{1}{8}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|+\frac{1}{2}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{1}{6\sqrt{2}}\approx 0.65$$

(To integrate $\int\frac{1}{(u^2+1)^2}du$, let $u = \tan t$.)

40. $\dfrac{3x+13}{x^2+4x+3}=\dfrac{3x+13}{(x+3)(x+1)}=\dfrac{A}{x+3}+\dfrac{B}{x+1}$

$A = -2, B = 5$

$$\int_1^5\frac{3x+13}{x^2+4x+3}dx=\Big[-2\ln|x+3|+5\ln|x+1|\Big]_1^5=-2\ln 8+5\ln 6+2\ln 4-5\ln 2=5\ln 3-2\ln 2\approx 4.11$$

41. $\dfrac{dy}{dt}=y(1-y)$ so that

$$\int\frac{1}{y(1-y)}dy=\int 1\,dt=t+C_1$$

a. Using partial fractions:

$$\frac{1}{y(1-y)}=\frac{A}{y}+\frac{B}{1-y}=\frac{A(1-y)+By}{y(1-y)}\Rightarrow$$

$$A+(B-A)y=1+0y\Rightarrow A=1,\ B-A=0\Rightarrow A=1,\ B=1\Rightarrow\frac{1}{y(1-y)}=\frac{1}{y}+\frac{1}{1-y}$$

Thus: $t+C_1=\int\left(\frac{1}{y}+\frac{1}{1-y}\right)dy=\ln y-\ln(1-y)=\ln\left(\frac{y}{1-y}\right)$ so that

$$\frac{y}{1-y}=e^{t+C_1}=\underset{(C=e^{C_1})}{Ce^t}\text{ or } y(t)=\frac{e^t}{\frac{1}{C}+e^t}$$

Since $y(0)=0.5,\ 0.5=\dfrac{1}{\frac{1}{C}+1}$ or $C=1$; thus $y(t)=\dfrac{e^t}{1+e^t}$

b. $y(3)=\dfrac{e^3}{1+e^3}\approx 0.953$

42. $\frac{dy}{dt} = \frac{1}{10} y(12-y)$ so that

$$\int \frac{1}{y(12-y)} dy = \int \frac{1}{10} dt = \frac{1}{10} t + C_1$$

a. Using partial fractions:

$$\frac{1}{y(12-y)} = \frac{A}{y} + \frac{B}{12-y} = \frac{A(12-y)+By}{y(12-y)} \Rightarrow$$

$$12A + (B-A)y = 1 + 0y \Rightarrow 12A = 1,\ B - A = 0$$

$$\Rightarrow A = \frac{1}{12},\ B = \frac{1}{12}$$

$$\Rightarrow \frac{1}{y(12-y)} = \frac{1}{12y} + \frac{1}{12(12-y)}$$

Thus: $\frac{1}{10} t + C_1 = \int \left(\frac{1}{12y} + \frac{1}{12(12-y)} \right) dy =$

$\frac{1}{12}\left[\ln y - \ln(12-y)\right] = \frac{1}{12} \ln\left(\frac{y}{12-y} \right)$ so that

$$\frac{y}{12-y} = e^{1.2t+12C_1} \underset{(C=e^{12C_1})}{=} Ce^{1.2t} \text{ or}$$

$$y(t) = \frac{12e^{1.2t}}{\frac{1}{C}+e^{1.2t}}$$

Since $y(0) = 2.0,\ 2.0 = \frac{12}{\frac{1}{C}+1}$ or $C = 0.2$;

thus $y(t) = \frac{12e^{1.2t}}{5+e^{1.2t}}$

b. $y(3) = \frac{12e^{3.6}}{5+e^{3.6}} \approx 10.56$

43. $\frac{dy}{dt} = 0.0003y(8000-y)$ so that

$$\int \frac{1}{y(8000-y)} dy = \int 0.0003\,dt = 0.0003t + C_1$$

a. Using partial fractions:

$$\frac{1}{y(8000-y)} = \frac{A}{y} + \frac{B}{8000-y} = \frac{A(8000-y)+By}{y(8000-y)}$$

$$\Rightarrow 8000A + (B-A)y = 1 + 0y$$

$$\Rightarrow 8000A = 1,\ B - A = 0$$

$$\Rightarrow A = \frac{1}{8000},\ B = \frac{1}{8000}$$

$$\Rightarrow \frac{1}{y(8000-y)} = \frac{1}{8000}\left[\frac{1}{y} + \frac{1}{(8000-y)} \right]$$

Thus:

$$0.0003t + C_1 = \frac{1}{8000} \int \left(\frac{1}{y} + \frac{1}{(8000-y)} \right) dy =$$

$$\frac{1}{8000}\left[\ln y - \ln(8000-y)\right] = \frac{1}{8000} \ln\left(\frac{y}{8000-y} \right)$$

so that

$$\frac{y}{8000-y} = e^{2.4t+8000C_1} \underset{(C=e^{8000C_1})}{=} Ce^{2.4t} \text{ or}$$

$$y(t) = \frac{8000e^{2.4t}}{\frac{1}{C}+e^{2.4t}}$$

Since $y(0) = 1000,\ 1000 = \frac{8000}{\frac{1}{C}+1}$ or $C = \frac{1}{7}$;

thus $y(t) = \frac{8000e^{2.4t}}{7+e^{2.4t}}$

b. $y(3) = \frac{8000e^{7.2}}{7+e^{7.2}} \approx 7958.4$

44. $\frac{dy}{dt} = 0.001y(4000-y)$ so that

$$\int \frac{1}{y(4000-y)} dy = \int 0.001\,dt = 0.001t + C_1$$

a. Using partial fractions:

$$\frac{1}{y(4000-y)} = \frac{A}{y} + \frac{B}{4000-y}$$

$$= \frac{A(4000-y)+By}{y(4000-y)}$$

$$\Rightarrow 4000A + (B-A)y = 1 + 0y$$

$$\Rightarrow 4000A = 1,\ B - A = 0$$

$$\Rightarrow A = \frac{1}{4000},\ B = \frac{1}{4000}$$

$$\Rightarrow \frac{1}{y(4000-y)} = \frac{1}{4000}\left[\frac{1}{y} + \frac{1}{(4000-y)} \right]$$

Thus:

$$0.001t + C_1 = \frac{1}{4000} \int \left(\frac{1}{y} + \frac{1}{(4000-y)} \right) dy =$$

$$\frac{1}{4000}\left[\ln y - \ln(4000-y)\right] = \frac{1}{4000} \ln\left(\frac{y}{4000-y} \right)$$

so that

$$\frac{y}{4000-y} = e^{4t+4000C_1} \underset{(C=e^{4000C_1})}{=} Ce^{4t} \text{ or}$$

$$y(t) = \frac{4000e^{4t}}{\frac{1}{C}+e^{4t}}$$

Since $y(0) = 100,\ 100 = \frac{4000}{\frac{1}{C}+1}$ or $C = \frac{1}{39}$;

thus $y(t) = \frac{4000e^{4t}}{39+e^{4t}}$

b. $y(3) = \frac{4000e^{12}}{39+e^{12}} \approx 3999.04$

45. $\frac{dy}{dt} = ky(L - y)$ so that

$\int \frac{1}{y(L-y)} dy = \int k\, dt = kt + C_1$

Using partial fractions:

$\frac{1}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y} = \frac{A(L-y)+By}{y(L-y)} \Rightarrow$

$LA + (B - A)y = 1 + 0y \Rightarrow LA = 1,\ B - A = 0 \Rightarrow$

$A = \frac{1}{L},\ B = \frac{1}{L} \Rightarrow \frac{1}{y(L-y)} = \frac{1}{L}\left[\frac{1}{y} + \frac{1}{L-y}\right]$

Thus: $kt + C_1 = \frac{1}{L}\int\left(\frac{1}{y} + \frac{1}{L-y}\right)dy =$

$\frac{1}{L}\left[\ln y - \ln(L-y)\right] = \frac{1}{L}\ln\left(\frac{y}{L-y}\right)$ so that

$\frac{y}{L-y} = e^{kLt+LC_1} = \underset{(C=e^{LC_1})}{Ce^{kLt}}$ or $y(t) = \frac{Le^{kLt}}{\frac{1}{C}+e^{kLt}}$

If $y_0 = y(0) = \frac{L}{\frac{1}{C}+1}$ then $\frac{1}{\text{C}} = \frac{L-y_0}{y_0}$; so our final formula is $y(t) = \frac{Le^{kLt}}{\left(\frac{L-y_0}{y_0}\right)+e^{kLt}}$.

(Note: if $y_0 < L$, then $u = \frac{L-y_0}{y_0} > 0$ and $\frac{e^{kLt}}{u+e^{kLt}} < 1$; thus $y(t) < L$ for all t)

46. Since $y'(0) = ky_0(L - y_0)$ is negative if $y_0 > L$, the population would be decreasing at time $t = 0$. Further, since

$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{L}{\left(\frac{L-y_0}{y_0e^{kLt}}\right)+1} = \frac{L}{0+1} = L$

(no matter how y_0 and L compare), and since $\frac{L-y_0}{y_0e^{kLt}}$ is monotonic as $t \to \infty$, we conclude that the population would *decrease* toward a limiting value of L.

47. If $y_0 < L$, then $y'(0) = ky_0(L - y_0) > 0$ and the population is increasing initially.

48. The graph will be concave up for values of t that make $y''(t) > 0$. Now

$y''(t) = \frac{dy'}{dt} = \frac{d}{dt}\left[ky(L-y)\right] =$

$k\left[-yy' + (L-y)y'\right] = k\left[ky(L-y)\right]\left[L-2y\right]$

Thus if $y_0 < L$, then $y(t) < L$ for all positive t (see note at the end of problem 45 solution) and so the graph will be concave up as long as $L - 2y > 0$; that is, as long as the population is less than half the capacity.

49. a. $\frac{dy}{dt} = ky(16 - y)$

$\frac{dy}{y(16-y)} = kdt$

$\int \frac{dy}{y(16-y)} = \int kdt$

$\frac{1}{16}\int\left(\frac{1}{y} + \frac{1}{16-y}\right)dy = kt + C$

$\frac{1}{16}\left(\ln|y| - \ln|16-y|\right) = kt + C$

$\ln\left|\frac{y}{16-y}\right| = 16kt + C$

$\frac{y}{16-y} = Ce^{16kt}$

$y(0) = 2$: $\frac{1}{7} = C;\ \frac{y}{16-y} = \frac{1}{7}e^{16kt}$

$y(50) = 4$: $\frac{1}{3} = \frac{1}{7}e^{800k}$, so $k = \frac{1}{800}\ln\frac{7}{3}$

$\frac{y}{16-y} = \frac{1}{7}e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}$

$7y = 16e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t} - ye^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}$

$y = \frac{16e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}{7+e^{\left(\frac{1}{50}\ln\frac{7}{3}\right)t}} = \frac{16}{1+7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}$

b. $y(90) = \frac{16}{1+7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)90}} \approx 6.34$ billion

c. $9 = \frac{16}{1+7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t}}$

$7e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t} = \frac{16}{9} - 1$

$e^{-\left(\frac{1}{50}\ln\frac{7}{3}\right)t} = \frac{1}{9}$

$-\left(\frac{1}{50}\ln\frac{7}{3}\right)t = \ln\frac{1}{9}$

$t = -50\left(\frac{\ln\frac{1}{9}}{\ln\frac{7}{3}}\right) \approx 129.66$

The population will be 9 billion in 2055.

50. **a.** $\frac{dy}{dt} = ky(10-y)$

$\frac{dy}{y(10-y)} = kdt$

$\frac{1}{10}\int\left(\frac{1}{y}+\frac{1}{10-y}\right)dy = \int kdt$

$\ln\left|\frac{y}{10-y}\right| = 10kt + C$

$\frac{y}{10-y} = Ce^{10kt}$

$y(0) = 2$: $\frac{1}{4} = C$; $\frac{y}{10-y} = \frac{1}{4}e^{10kt}$

$y(50) = 4$: $\frac{2}{3} = \frac{1}{4}e^{500k}$, $k = \frac{1}{500}\ln\frac{8}{3}$

$\frac{y}{10-y} = \frac{1}{4}e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}$

$4y = 10e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t} - ye^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}$

$y = \frac{10e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}{4+e^{\left(\frac{1}{50}\ln\frac{8}{3}\right)t}} = \frac{10}{1+4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}$

b. $y(90) = \frac{10}{1+4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)90}} \approx 5.94$ billion

c. $9 = \frac{10}{1+4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t}}$

$4e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t} = \frac{10}{9} - 1$

$e^{-\left(\frac{1}{50}\ln\frac{8}{3}\right)t} = \frac{1}{36}$

$-\left(\frac{1}{50}\ln\frac{8}{3}\right)t = \ln\frac{1}{36}$

$t = -50\left(\frac{\ln\frac{1}{36}}{\ln\frac{8}{3}}\right) \approx 182.68$

The population will be 9 billion in 2108.

51. **a.** Separating variables, we obtain

$\frac{dx}{(a-x)(b-x)} = k\,dt$

$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$

$A = -\frac{1}{a-b}$, $B = \frac{1}{a-b}$

$\int\frac{dx}{(a-x)(b-x)}$

$= \frac{1}{a-b}\int\left(-\frac{1}{a-x}+\frac{1}{b-x}\right)dx = \int k\,dt$

$\frac{\ln|a-x| - \ln|b-x|}{a-b} = kt + C$

$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| = kt + C$

$\frac{a-x}{b-x} = Ce^{(a-b)kt}$

Since $x = 0$ when $t = 0$, $C = \frac{a}{b}$, so

$a - x = (b-x)\frac{a}{b}e^{(a-b)kt}$

$a\left(1-e^{(a-b)kt}\right) = x\left(1-\frac{a}{b}e^{(a-b)kt}\right)$

$x(t) = \frac{a(1-e^{(a-b)kt})}{1-\frac{a}{b}e^{(a-b)kt}} = \frac{ab(1-e^{(a-b)kt})}{b-ae^{(a-b)kt}}$

b. Since $b > a$ and $k > 0$, $e^{(a-b)kt} \to 0$ as $t \to \infty$. Thus,

$x \to \frac{ab(1)}{b-0} = a$.

c. $x(t) = \frac{8(1-e^{-2kt})}{4-2e^{-2kt}}$

$x(20) = 1$, so $4 - 2e^{-40k} = 8 - 8e^{-40k}$

$6e^{-40k} = 4$

$k = -\frac{1}{40}\ln\frac{2}{3}$

$e^{-2kt} = e^{t/20\ln 2/3} = e^{\ln(2/3)^{t/20}} = \left(\frac{2}{3}\right)^{t/20}$

$x(t) = \frac{4\left(1-\left(\frac{2}{3}\right)^{t/20}\right)}{2-\left(\frac{2}{3}\right)^{t/20}}$

$x(60) = \frac{4\left(1-\left(\frac{2}{3}\right)^3\right)}{2-\left(\frac{2}{3}\right)^3} = \frac{38}{23} \approx 1.65$ grams

d. If $a = b$, the differential equation is, after separating variables

$$\frac{dx}{(a-x)^2} = k\,dt$$

$$\int \frac{dx}{(a-x)^2} = \int k\,dt$$

$$\frac{1}{a-x} = kt + C$$

$$\frac{1}{kt+C} = a - x$$

$$x(t) = a - \frac{1}{kt+C}$$

Since $x = 0$ when $t = 0$, $C = \frac{1}{a}$, so

$$x(t) = a - \frac{1}{kt+\frac{1}{a}} = a - \frac{a}{akt+1}$$

$$= a\left(1 - \frac{1}{akt+1}\right) = a\left(\frac{akt}{akt+1}\right).$$

52. Separating variables, we obtain

$$\frac{dy}{(y-m)(M-y)} = k\,dt.$$

$$\frac{1}{(y-m)(M-y)} = \frac{A}{y-m} + \frac{B}{M-y}$$

$$A = \frac{1}{M-m}, B = \frac{1}{M-m}$$

$$\int \frac{dy}{(y-m)(M-y)} = \frac{1}{M-m}\int\left(\frac{1}{y-m} + \frac{1}{M-y}\right)dy$$

$$= \int k\,dt$$

$$\frac{\ln|y-m| - \ln|M-y|}{M-m} = kt + C$$

$$\frac{1}{M-m}\ln\left|\frac{y-m}{M-y}\right| = kt + C$$

$$\frac{y-m}{M-y} = Ce^{(M-m)kt}$$

$$y - m = (M-y)Ce^{(M-m)kt}$$

$$y(1 + Ce^{(M-m)kt}) = m + MCe^{(M-m)kt}$$

$$y = \frac{m + MCe^{(M-m)kt}}{1 + Ce^{(M-m)kt}} = \frac{me^{-(M-m)kt} + MC}{e^{-(M-m)kt} + C}$$

As $t \to \infty, e^{-(M-m)kt} \to 0$ since $M > m$.

Thus $y \to \frac{MC}{C} = M$ as $t \to \infty$.

53. Separating variables, we obtain

$$\frac{dy}{(A-y)(B+y)} = k\,dt$$

$$\frac{1}{(A-y)(B+y)} = \frac{C}{A-y} + \frac{D}{B+y}$$

$$C = \frac{1}{A+B}, D = \frac{1}{A+B}$$

$$\int \frac{dy}{(A-y)(B+y)} = \frac{1}{A+B}\int\left(\frac{1}{A-y} + \frac{1}{B+y}\right)dy$$

$$= \int k\,dt$$

$$\frac{-\ln(A-y) + \ln(B+y)}{A+B} = kt + C$$

$$\frac{1}{A+B}\ln\left|\frac{B+y}{A-y}\right| = kt + C$$

$$\frac{B+y}{A-y} = Ce^{(A+B)kt}$$

$$B + y = (A-y)Ce^{(A+B)kt}$$

$$y(1 + Ce^{(A+B)kt}) = ACe^{(A+B)kt} - B$$

$$y(t) = \frac{ACe^{(A+B)kt} - B}{1 + Ce^{(A+B)kt}}$$

54. $u = \sin x$, $du = \cos x\,dx$

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(\sin^2 x + 1)^2}dx = \int_{\frac{1}{2}}^{1} \frac{1}{u(u^2+1)^2}du$$

$$\frac{1}{u(u^2+1)^2} = \frac{A}{u} + \frac{Bu+C}{u^2+1} + \frac{Du+E}{(u^2+1)^2}$$

$A = 1$, $B = -1$, $C = 0$, $D = -1$, $E = 0$

$$\int_{\frac{1}{2}}^{1} \frac{1}{u(u^2+1)^2}du$$

$$= \int_{\frac{1}{2}}^{1} \frac{1}{u}du - \int_{\frac{1}{2}}^{1} \frac{u}{u^2+1}du - \int_{\frac{1}{2}}^{1} \frac{u}{(u^2+1)^2}du$$

$$= \left[\ln u - \frac{1}{2}\ln(u^2+1) + \frac{1}{2(u^2+1)}\right]_{\frac{1}{2}}^{1}$$

$$= 0 - \frac{1}{2}\ln 2 + \frac{1}{4} - \left(\ln\frac{1}{2} - \frac{1}{2}\ln\frac{5}{4} + \frac{2}{5}\right) \approx 0.308$$

7.6 Concepts Review

1. substitution

2. 53

3. approximation

4. 0

Problem Set 7.6

Note: Throughout this section, the notation Fxxx refers to integration formula number xxx in the back of the book.

1. Integration by parts.

$u = x \qquad dv = e^{-5x}$

$du = 1dx \qquad v = -\frac{1}{5}e^{-5x}$

$$\int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \int -\frac{1}{5}e^{-5x}\,dx$$

$$= -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$$

$$= -\frac{1}{5}e^{-5x}\left(x + \frac{1}{5}\right) + C$$

2. Substitution

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \ln|u| + C = \ln\left(x^2+9\right) + C$$

$u = x^2+9$, $du = 2x\,dx$

3. Substitution

$$\int_1^2 \frac{\ln x}{x}\,dx = \int_0^{\ln 2} u\,du = \left[\frac{u^2}{2}\right]_0^{\ln 2} = \frac{(\ln 2)^2}{2} \approx 0.2402$$

$u = \ln x$, $du = \frac{1}{x}dx$

4. Partial fractions

$$\int \frac{x}{x^2-5x+6}\,dx = \int \frac{x}{(x-3)(x-2)}\,dx$$

$$\frac{x}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)} =$$

$$\frac{A(x-2)+B(x-3)}{(x-3)(x-2)} = \frac{(A+B)x+(-2A-3B)}{(x-3)(x-2)} \Rightarrow$$

$A + B = 1, -2A - 3B = 0 \Rightarrow A = 3, B = -2$

$$\int \frac{x}{x^2-5x+6}\,dx = \int \frac{3}{(x-3)} - \frac{2}{(x-2)}\,dx =$$

$$3\ln|x-3| - 2\ln|x-2| = \ln\left|\frac{(x-3)^3}{(x-2)^2}\right| + C$$

5. Trig identity $\cos^2 u = \frac{1+\cos 2u}{2}$ and substitution.

$$\int \cos^4 2x\,dx = \int \left(\frac{1+\cos 4x}{2}\right)^2 dx =$$

$$\frac{1}{4}\int \left[1 + 2\cos 4x + \cos^2 4x\right] dx =$$

$u = 4x$, $du = 4\,dx$

$$\frac{1}{4}\left[x + \frac{1}{2}\sin 4x + \int \left(\frac{1+\cos 8x}{2}\right) dx\right] =$$

$v = 8x$, $dv = 8dx$

$$\frac{1}{4}\left[x + \frac{1}{2}\sin 4x + \frac{1}{2}x + \frac{1}{16}\sin 8x\right] + C =$$

$$\frac{1}{64}\left[24x + 8\sin 4x + \sin 8x\right] + C$$

6. Substitution

$$\int \sin^3 x\cos x\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

$u = \sin x$, $du = \cos x\,dx$

7. Partial fractions

$$\int \frac{1}{x^2+6x+8}\,dx = \int \frac{1}{(x+4)(x+2)}\,dx$$

$$\frac{1}{(x+4)(x+2)} = \frac{A}{(x+4)} + \frac{B}{(x+2)} =$$

$$\frac{A(x+2)+B(x+4)}{(x+4)(x+2)} = \frac{(A+B)x+(2A+4B)}{(x+4)(x+2)} \Rightarrow$$

$A + B = 0,\ 2A + 4B = 1 \Rightarrow A = -\frac{1}{2}, B = \frac{1}{2}$

$$\int_1^2 \frac{1}{x^2+6x+8} = \frac{1}{2}\int_1^2 \left(\frac{1}{x+2} - \frac{1}{x+4}\right) dx$$

$$= \frac{1}{2}\left[\ln|x+2| - \ln|x+4|\right]_1^2 = \frac{1}{2}\left[\ln\left|\frac{(x+2)}{(x+4)}\right|\right]_1^2$$

$$= \frac{1}{2}\left(\ln\frac{4}{6} - \ln\frac{3}{5}\right) = \frac{1}{2}\ln\frac{10}{9} \approx 0.0527$$

8. Partial fractions

$$\int \frac{1}{1-t^2}\,dt = \int \frac{1}{(1-t)(1+t)}\,dt$$

$$\frac{1}{(1-t)(1+t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} =$$

$$\frac{A(1+t)+B(1-t)}{(1-t)(1+t)} = \frac{(A-B)t+(A+B)}{(1-t)(1+t)} \Rightarrow$$

$$A+B=1,\ \ A-B=0 \Rightarrow A=\frac{1}{2}, B=\frac{1}{2}$$

$$\int_0^{1/2} \frac{1}{1-t^2}\,dt = \frac{1}{2}\int_0^{1/2}\left(\frac{1}{1-t}+\frac{1}{1+t}\right)dx$$

$$\frac{1}{2}\left[-\ln|1-t| + \ln|1+t|\right]_0^{1/2} =$$

$$\frac{1}{2}\left[\ln\left|\frac{(1+t)}{(1-t)}\right|\right]_0^{1/2} \approx 0.5493$$

9. Substitution

$$\int_0^5 x\sqrt{x+2}\,dx \underset{\substack{u=\sqrt{x+2}\\ u^2=x+2\\ 2u\,du=dx}}{=} \int_{\sqrt{2}}^{\sqrt{7}} (u^2-2)(u)2u\,du =$$

$$\int_{\sqrt{2}}^{\sqrt{7}} 2u^4 - 4u^2\,du = 2\left[\frac{u^5}{5} - \frac{2u^3}{3}\right]_{\sqrt{2}}^{\sqrt{7}} =$$

$$\frac{2}{15}\left[3u^5 - 10u^3\right]_{\sqrt{2}}^{\sqrt{7}} = \frac{2}{15}\left[77\sqrt{7}+8\sqrt{2}\right] \approx 28.67$$

10. Substitution

$$\int_3^4 \frac{1}{t-\sqrt{2t}}\,dt \underset{\substack{u=\sqrt{2t},\ u^2=2t\\ u\,du\,=\,dt}}{=} \int_{\sqrt{6}}^{\sqrt{8}} \frac{u}{\frac{u^2}{2}-u}\,du = 2\int_{\sqrt{6}}^{\sqrt{8}} \frac{1}{u-2}\,du =$$

$$2\left[\ln|u-2|\right]_{\sqrt{6}}^{\sqrt{8}} = 2\ln\left|\frac{\sqrt{8}-2}{\sqrt{6}-2}\right| \approx 1.223$$

11. Use of symmetry; this is an odd function, so

$$\int_{-\pi/2}^{\pi/2} \cos^2 x \sin x\,dx = 0$$

12. Use of symmetry; substitution

$$\int_0^{2\pi} |\sin 2x|\,dx = 8\int_0^{\pi/4} \sin 2x\ dx \underset{\substack{u=2x\\ du=2\,dx}}{=}$$

$$4\int_0^{\pi/2} \sin u\,du = 4\left[-\cos u\right]_0^{\pi/2} = 4$$

13. a. Formula 96

$$\int x\sqrt{3x+1}\,dx \underset{\substack{F96\\ a=3,b=1}}{=} \frac{2}{135}(9x-2)(3x+1)^{3/2} + C$$

b. Substitution; Formula 96

$$\int\limits_{u=e^x,\ du=e^x dx} e^x\sqrt{3e^x+1}\ dx = \int u\sqrt{3u+1}\,du \underset{\substack{F96\\ a=3,b=1}}{=}$$

$$\frac{2}{135}\left(9e^x-2\right)\left(3e^x+1\right)^{3/2} + C$$

14. a. Formula 96

$$\int 2t(3-4t)\,dt = 2\int t(3-4t)\,dt \underset{\substack{F96\\ a=-4,\\ b=3}}{=}$$

$$2\left[\frac{2}{240}(-12t-6)(3-4t)^{3/2}\right] + C =$$

$$-\frac{1}{10}(2t+1)(3-4t)^{3/2} + C$$

b. Substitution; Formula 96

$$\int\limits_{u=\cos t,\ du=-\sin t\,dt} \cos t\sqrt{3-4\cos t}\,\sin t\,dt = -\int u\sqrt{3-4u}\,du \underset{\text{part a.}}{=}$$

$$\frac{1}{20}(2\cos t+1)(3-4\cos t)^{3/2} + C$$

15. a. Substitution, Formula 18

$$\int\limits_{u=4x,\,du=4dx} \frac{dx}{9-16x^2} = \frac{1}{4}\int \frac{du}{9-u^2} \underset{\substack{F18\\ a=3}}{=}$$

$$\frac{1}{4}\left[\frac{1}{6}\ln\left|\frac{u+3}{u-3}\right|\right] + C = \frac{1}{24}\ln\left|\frac{4x+3}{4x-3}\right| + C$$

b. Substitution, Formula 18

$$\int\limits_{u=4e^x,\,du=4e^x dx} \frac{e^x}{9-16e^{2x}}\ dx = \frac{1}{4}\int \frac{du}{9-u^2} \underset{\text{part a.}}{=}$$

$$\frac{1}{24}\ln\left|\frac{4e^x+3}{4e^x-3}\right| + C$$

16. a. Substitution, Formula 18

$$\int \frac{dx}{5x^2-11} = -\int\limits_{\substack{u=\sqrt{5}x,\\ du=\sqrt{5}\,dx}} \frac{dx}{11-5x^2} = -\frac{\sqrt{5}}{5}\int \frac{du}{11-u^2} \underset{\substack{F18\\ a=\sqrt{11}}}{=}$$

$$\frac{-\sqrt{5}}{5}\frac{\sqrt{11}}{22}\ln\left|\frac{\sqrt{5}x+\sqrt{11}}{\sqrt{5}x-\sqrt{11}}\right| + C$$

$$= \frac{\sqrt{55}}{110}\ln\left|\frac{\sqrt{5}x-\sqrt{11}}{\sqrt{5}x+\sqrt{11}}\right| + C$$

b. Substitution, Formula 18

$$\int \underset{\substack{u=\sqrt{5}x^2,\\ du=2\sqrt{5}\,x\,dx}}{\frac{x\,dx}{5x^4-11}} = -\frac{\sqrt{5}}{10}\int \frac{du}{11-u^2} \underset{\substack{F18\\ a=\sqrt{11}}}{=}$$

$$\frac{\sqrt{55}}{220}\ln\left|\frac{\sqrt{5}x^2-\sqrt{11}}{\sqrt{5}x^2+\sqrt{11}}\right|+C$$

17. a. Substitution, Formula 57

$$\int \underset{\substack{u=\sqrt{2}\,x\\ du=\sqrt{2}\,dx}}{x^2\sqrt{9-2x^2}\,dx} = \frac{\sqrt{2}}{4}\int u^2\sqrt{9-u^2}\,du \underset{\substack{F57\\ a=3}}{=}$$

$$\frac{1}{16}\left(x(4x^2-9)\sqrt{9-2x^2}\right)+\frac{81\sqrt{2}}{32}\sin^{-1}\left(\frac{\sqrt{2}x}{3}\right)+C$$

b. Substitution, Formula 57

$$\int \underset{\substack{u=\sqrt{2}\sin x\\ du=\sqrt{2}\cos x\,dx}}{\sin^2 x\cos x\sqrt{9-2\sin^2 x}\,dx} = \frac{\sqrt{2}}{4}\int u^2\sqrt{9-u^2}\,du$$

$$\underset{\substack{F57\\ a=3}}{=} \frac{1}{16}\left(\sin x(4\sin^2 x-9)\sqrt{9-2\sin^2 x}\right)$$

$$+\frac{81\sqrt{2}}{32}\sin^{-1}\left(\frac{\sqrt{2}\sin x}{3}\right)+C$$

18. a. Substitution, Formula 55

$$\int \underset{\substack{u=\sqrt{3}t\\ du=\sqrt{3}\,dt}}{\frac{\sqrt{16-3t^2}}{t}\,dt} = \int \frac{\sqrt{16-u^2}}{u}\,du \underset{\substack{F55\\ a=4}}{=}$$

$$\sqrt{16-3t^2}-4\ln\left|\frac{4+\sqrt{16-3t^2}}{\sqrt{3}\,t}\right|+C$$

b. Substitution, Formula 55

$$\int \frac{\sqrt{16-3t^6}}{t}\,dt = \int \underset{\substack{u=\sqrt{3}t^3\\ du=3\sqrt{3}\,t^2dt}}{\frac{t^2\sqrt{16-3t^6}}{t^3}\,dt} =$$

$$\frac{1}{3}\int \frac{\sqrt{16-u^2}}{u}\,du \underset{\substack{F55\\ a=4}}{=}$$

$$\frac{1}{3}\left\{\sqrt{16-3t^6}-4\ln\left|\frac{4+\sqrt{16-3t^6}}{\sqrt{3}\,t^3}\right|\right\}+C$$

19. a. Substitution, Formula 45

$$\int \underset{\substack{u=\sqrt{3}x\\ du=\sqrt{3}\,dx}}{\frac{dx}{\sqrt{5+3x^2}}} = \frac{\sqrt{3}}{3}\int \frac{du}{\sqrt{5+u^2}} \underset{\substack{F45\\ a=\sqrt{5}}}{=}$$

$$\frac{\sqrt{3}}{3}\ln\left|\sqrt{3}x+\sqrt{5+3x^2}\right|+C$$

b. Substitution, Formula 45

$$\int \underset{\substack{u=\sqrt{3}x^2\\ du=2\sqrt{3}\,x\,dx}}{\frac{x}{\sqrt{5+3x^4}}\,dx} = \frac{\sqrt{3}}{6}\int \frac{du}{\sqrt{5+u^2}} \underset{\substack{F45\\ a=\sqrt{5}}}{=}$$

$$\frac{\sqrt{3}}{6}\ln\left|\sqrt{3}x^2+\sqrt{5+3x^4}\right|+C$$

20. a. Substitution; Formula 48

$$\int \underset{\substack{u=\sqrt{5}\,t\\ du=\sqrt{5}\,dt}}{t^2\sqrt{3+5t^2}\,dt} = \frac{\sqrt{5}}{25}\int u^2\sqrt{3+u^2}\,du \underset{\substack{F48\\ a=\sqrt{3}}}{=}$$

$$\frac{\sqrt{5}}{25}\left\{\begin{array}{l}\left(\frac{\sqrt{5}}{8}t\right)(10t^2+3)\left(\sqrt{3+5t^2}\right)-\\ \frac{9}{8}\ln\left|\sqrt{5}\,t+\sqrt{3+5t^2}\right|\end{array}\right\}+C=$$

$$\frac{1}{200}\left\{5t(10t^2+3)\sqrt{3+5t^2}-9\sqrt{5}\ln\left|\sqrt{5}t+\sqrt{3+5t^2}\right|\right\}+C$$

b. Substitution; Formula 48

$$\int t^8\sqrt{3+5t^6}\,dt = \int \underset{\substack{u=\sqrt{5}\,t^3\\ du=3\sqrt{5}\,t^2dt}}{t^6\sqrt{3+5t^6}\,t^2dt} =$$

$$\frac{\sqrt{5}}{75}\int u^2\sqrt{3+u^2}\,du \underset{\substack{F48\\ a=\sqrt{3}}}{=}$$

$$\frac{\sqrt{5}}{75}\left\{\begin{array}{l}\left(\frac{\sqrt{5}}{8}t^3\right)(10t^6+3)\left(\sqrt{3+5t^6}\right)-\\ \frac{9}{8}\ln\left|\sqrt{5}\,t^3+\sqrt{3+5t^6}\right|\end{array}\right\}+C=$$

$$\frac{1}{600}\left\{\begin{array}{l}5t^3(10t^6+3)\sqrt{3+5t^6}-\\ 9\sqrt{5}\ln\left|\sqrt{5}t^3+\sqrt{3+5t^6}\right|\end{array}\right\}+C$$

21. a. Complete the square; substitution; Formula 45.

$$\int \frac{dt}{\sqrt{t^2+2t-3}} = \int \frac{dt}{\sqrt{(t+1)^2-4}} \underset{\substack{u=t+1\\du=dt}}{=} \int \frac{du}{\sqrt{u^2-4}} \underset{\substack{F45\\a=2}}{=}$$

$$\ln\left|(t+1)+\sqrt{t^2+2t-3}\right|+C$$

b. Complete the square; substitution; Formula 45.

$$\int \frac{dt}{\sqrt{t^2+3t-5}} = \int \frac{dt}{\sqrt{(t+\frac{3}{2})^2-\frac{29}{4}}} \underset{\substack{u=t+3/2\\du=dt}}{=}$$

$$\int \frac{du}{\sqrt{u^2-\frac{29}{4}}} \underset{\substack{F45\\a=\sqrt{29}/2}}{=} \ln\left|(t+\tfrac{3}{2})+\sqrt{t^2+3t-5}\right|+C$$

22. a. Complete the square; substitution; Formula 47.

$$\int \frac{\sqrt{x^2+2x-3}}{x+1}dx = \int \frac{\sqrt{(x+1)^2-4}}{x+1}dx \underset{\substack{u=x+1\\du=dx}}{=}$$

$$\int \frac{\sqrt{u^2-4}}{u}du \underset{\substack{F47\\a=2}}{=}$$

$$\sqrt{x^2+2x-3}-2\sec^{-1}\left(\frac{x+1}{2}\right)+C$$

b. Complete the square; substitution; Formula 47.

$$\int \frac{\sqrt{x^2-4x}}{x-2}dx = \int \frac{\sqrt{(x-2)^2-4}}{x-2}dx \underset{\substack{u=x-2\\du=dx}}{=}$$

$$\int \frac{\sqrt{u^2-4}}{u}du \underset{\substack{F47\\a=2}}{=}$$

$$\sqrt{x^2-4x}-2\sec^{-1}\left(\frac{x-2}{2}\right)+C$$

23. a. Formula 98

$$\int \frac{y}{\sqrt{3y+5}}dy \underset{\substack{F98\\a=3,b=5}}{=} \frac{2}{27}(3y-10)\sqrt{3y+5}+C$$

b. Substitution, Formula 98

$$\int \frac{\sin t\cos t}{\sqrt{3\sin t+5}} \underset{\substack{u=\sin t\\du=\cos t\,dt}}{=} \int \frac{u}{\sqrt{3u+5}}du \underset{\substack{F98\\a=3,b=5}}{=}$$

$$\frac{2}{27}(3\sin t-10)\sqrt{3\sin t+5}+C$$

24. a. Formula 100a

$$\int \frac{dz}{z\sqrt{5-4z}} \underset{\substack{F100a\\a=-4\\b=5}}{=} \frac{\sqrt{5}}{5}\ln\left|\frac{\sqrt{5-4z}-\sqrt{5}}{\sqrt{5-4z}+\sqrt{5}}\right|+C$$

b. Substitution, Formula 100a

$$\int \frac{\sin x}{\cos x\sqrt{5-4\cos x}}dx \underset{\substack{u=\cos x\\du=-\sin x\,dx}}{=} -\int \frac{du}{u\sqrt{5-4u}} \underset{\substack{F100a\\a=-4\\b=5}}{=}$$

$$-\frac{\sqrt{5}}{5}\ln\left|\frac{\sqrt{5-4\cos x}-\sqrt{5}}{\sqrt{5-4\cos x}+\sqrt{5}}\right|+C=$$

$$\frac{\sqrt{5}}{5}\ln\left|\frac{\sqrt{5-4\cos x}+\sqrt{5}}{\sqrt{5-4\cos x}-\sqrt{5}}\right|+C$$

25. Substitution; Formula 84

$$\int \sinh^2 3t\,dt \underset{\substack{u=3t\\du=3dt}}{=} \frac{1}{3}\int \sinh^2 u\,du \underset{F84}{=}$$

$$\frac{1}{3}\left(\frac{1}{4}\sinh 6t-\frac{3}{2}t\right)+C=\frac{1}{12}\left(\sinh 6t-6t\right)+C$$

26. Substitution; Formula 82

$$\int \frac{\operatorname{sech}\sqrt{x}}{\sqrt{x}}\,dx \underset{\substack{u=\sqrt{x}\\du=\frac{1}{2\sqrt{x}}dx}}{=} 2\int \operatorname{sech} u\,du \underset{F82}{=}$$

$$2\tan^{-1}\left|\sinh\sqrt{x}\right|+C$$

27. Substitution; Formula 98

$$\int \frac{\cos t\sin t}{\sqrt{2\cos t+1}}dt \underset{\substack{u=\cos t\\du=-\sin t\,dt}}{=} -\int \frac{u}{\sqrt{2u+1}}du \underset{\substack{F98\\a=2\\b=1}}{=}$$

$$-\frac{1}{6}(2\cos t-2)\sqrt{2\cos t+1}+C=$$

$$\frac{1}{3}(1-\cos t)\sqrt{2\cos t+1}+C$$

28. Substitution; Formula 96

$$\int \cos t\sin t\sqrt{4\cos t-1}\,dt \underset{\substack{u=\cos t\\du=-\sin t\,dt}}{=} -\int u\sqrt{4u-1}\,du \underset{\substack{F96\\a=4\\b=-1}}{=}$$

$$-\frac{1}{60}(6\cos t+1)(4\cos t-1)^{3/2}+C$$

29. Substitution; Formula 99, Formula 98

$$\int \frac{\cos^2 t \sin t}{\sqrt{\cos t+1}}\,dt \underset{\substack{u=\cos t\\ du=-\sin t\,dt}}{=} -\int \frac{u^2}{\sqrt{u+1}}\,du \underset{\substack{F99\\ n=2\\ a=1\\ b=1}}{=}$$

$$-\frac{2}{5}\left[u^2\sqrt{u+1}-2\int \frac{u}{\sqrt{u+1}}\,du\right] \underset{F98}{=}$$

$$-\frac{2}{5}\left[u^2\sqrt{u+1}-2\left(\frac{2}{3}(u-2)\sqrt{u+1}\right)\right]+C=$$

$$-\frac{2}{5}\sqrt{\cos t+1}\left[\cos^2 t-\frac{4}{3}(\cos t-2)\right]+C$$

30. Formula 95, Formula 17

$$\int \frac{1}{(9+x^2)^3}\,dx \underset{\substack{F95\\ n=3\\ a=3}}{=}$$

$$\frac{1}{36}\left[\frac{x}{(9+x^2)^2}+3\int \frac{dx}{(9+x^2)^2}\right] \underset{\substack{F95\\ n=2\\ a=3}}{=}$$

$$\frac{1}{36}\left[\frac{x}{(9+x^2)^2}+3\left[\frac{1}{18}\left(\frac{x}{(9+x^2)}+\int \frac{dx}{9+x^2}\right)\right]\right]$$

$$\underset{\substack{F17\\ a=3}}{=} \frac{1}{36}\left\{\frac{x}{(9+x^2)^2}+\frac{x}{6\cdot(9+x^2)}+\tan^{-1}\left(\frac{x}{3}\right)\right\}+C$$

31. Using a CAS, we obtain:

$$\int_0^{\pi} \frac{\cos^2 x}{1+\sin x}\,dx = \pi-2 \approx 1.14159$$

32. Using a CAS, we obtain:

$$\int_0^1 \operatorname{sech}\sqrt[3]{x}\,dx \approx 0.76803$$

33. Using a CAS, we obtain:

$$\int_0^{\pi/2} \sin^{12} x\,dx = \frac{231\pi}{2048} \approx 0.35435$$

34. Using a CAS, we obtain:

$$\int_0^{\pi} \cos^4 \frac{x}{2}\,dx = \frac{3\pi}{8} \approx 1.17810$$

35. Using a CAS, we obtain:

$$\int_1^4 \frac{\sqrt{t}}{1+t^8}\,dt \approx 0.11083$$

36. Using a CAS, we obtain:

$$\int_0^3 x^4 e^{-x/2}\,dx = 768-3378e^{-3/2} \approx 14.26632$$

37. Using a CAS, we obtain:

$$\int_0^{\pi/2} \frac{1}{1+2\cos^5 x}\,dx \approx 1.10577$$

38. Using a CAS, we obtain:

$$\int_{-\pi/4}^{\pi/4} \frac{x^3}{4+\tan x}\,dx \approx -0.00921$$

39. Using a CAS, we obtain:

$$\int_2^3 \frac{x^2+2x-1}{x^2-2x+1}\,dx = 4\ln(2)+2 \approx 4.77259$$

40. Using a CAS, we obtain:

$$\int_1^3 \frac{du}{u\sqrt{2u-1}} = 2\tan^{-1}\left(\sqrt{5}\right)-\frac{\pi}{2} \approx 0.72973$$

41. $\int_0^c \frac{1}{x+1}\,dx \underset{F3}{=} \left[\ln|x+1|\right]_0^c = \ln(c+1)$

$\ln(c+1)=1 \Rightarrow c+1=e \Rightarrow$

$c=e-1 \approx 1.71828$

42. Formula 17

$$\int_0^c \frac{2}{x^2+1}\,dx \underset{F17}{=} \left[2\tan^{-1} x\right]_0^c = 2\tan^{-1} c$$

$$2\tan^{-1} c = 1 \Rightarrow \tan^{-1} c = \frac{1}{2} \Rightarrow$$

$$c = \tan\frac{1}{2} \approx 0.5463$$

43. Substitution; Formula 65

$$\int \ln(x+1)\,dx \underset{\substack{u=x+1\\ du=dx}}{=} \int \ln u\,du \underset{F65}{=}$$

$(x+1)\left[\ln(x+1)-1\right]$. Thus

$$\int_0^c \ln(x+1)\,dx = (x+1)\left[\ln(x+1)-1\right]_0^c =$$

$(c+1)\ln(c+1)-c$ and

$(c+1)\ln(c+1)-c=1 \Rightarrow \ln(c+1)=1 \Rightarrow$

$c+1=e \Rightarrow c=e-1 \approx 1.71828$

44. Substitution ; Formula 3

$$\int_0^c \frac{x}{x^2+1}\,dx \underset{\substack{u=x^2+1\\ du=2x\,dx}}{=} \frac{1}{2}\int_1^{c^2+1} \frac{1}{u}\,du =$$

$$\frac{1}{2}\left[\ln u\right]_1^{c^2+1} = \frac{1}{2}\ln(c^2+1)$$

$$\frac{1}{2}\ln(c^2+1)=1 \Rightarrow c^2+1=e^2 \Rightarrow$$

$$c=\sqrt{e^2-1} \approx 2.528$$

45. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 0.59601$

46. Integration by parts; partial fractions; Formula 17

a. $\int \ln(x^3+1)\,dx = x\ln(x^3+1) - 3\int \frac{x^3}{x^3+1}dx =$

$u = \ln(x^3+1)$, $du = \frac{3x^2}{x^3+1}$, $dv = dx, v = x$

$x\ln(x^3+1) - 3\int\left(1 - \frac{1}{x^3+1}\right)dx =$

$x\ln(x^3+1) - 3x + 3\int\left(\frac{1}{(x+1)(x^2-x+1)}\right)dx$

b. $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} =$

$\frac{(A+B)x^2 + (B+C-A)x + (A+C)}{(x+1)(x^2-x+1)} \Rightarrow$

$A + C = 1 \quad B + C = A \quad A = -B \quad \Rightarrow$

$A = \frac{1}{3} \quad B = -\frac{1}{3} \quad C = \frac{2}{3}.$

Therefore

$3\int\frac{1}{(x+1)(x^2-x+1)}dx =$

$\int\frac{1}{x+1}dx - \int\frac{x-2}{x^2-x+1}dx =$

$\ln|x+1| - \int\frac{x-2}{(x-\frac{1}{2})^2 + \frac{3}{4}}dx =$

$u = x - \frac{1}{2}$, $du = dx$

$\ln|x+1| - \int\frac{u-\frac{3}{2}}{u^2+\frac{3}{4}}du \underset{F17}{=}$

$\ln|x+1| - \frac{1}{2}\ln\left|x^2-x+1\right| + \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right)$

c. Summarizing

$\int_0^c \ln(x^3+1)\,dx =$

$$\left[\begin{array}{l} x\ln(x^3+1) - 3x + \ln(x+1) - \\ \frac{1}{2}\ln(x^2-x+1) + \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right)\end{array}\right]_0^c =$$

$$\left\{\begin{array}{l} c(\ln(c^3+1)-3) + \ln\left(\frac{c+1}{\sqrt{c^2-c+1}}\right) + \\ \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}(c-\frac{1}{2})\right) + \frac{\sqrt{3}\pi}{6}\end{array}\right\}$$

Using Newton's Method , with

$$G(c) = \left\{\begin{array}{l} c(\ln(c^3+1)-3) + \ln\left(\frac{c+1}{\sqrt{c^2-c+1}}\right) + \\ \sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}(c-\frac{1}{2})\right) + \frac{\sqrt{3}\pi}{6} - 1\end{array}\right\}$$

and $G'(c) = \ln(c^3+1)$ we get

n	1	2	3	4	5
a_n	2.0000	1.6976	1.6621	1.6615	1.6615

Therefore

$\int_0^c \ln(x^3+1)\,dx = 1 \Rightarrow c \approx 1.6615$

47. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 0.16668$

48. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 0.2509$

49. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 9.2365$

50. There is no antiderivative that can be expressed in terms of elementary functions; an approximation for the integral, as well as a process such as Newton's Method, must be used. Several approaches are possible. $c \approx 1.96$

51. $f(x)=8-x \quad g(x)=cx \quad a=0 \quad b=\frac{8}{c+1}$

a. $\int_a^b x(f(x)-g(x))\,dx=\int_0^{8/(c+1)} 8x-(c+1)x^2\,dx=$

$\left[4x^2-\left(\frac{c+1}{3}\right)x^3\right]_0^{8/(c+1)}=\frac{256}{(c+1)^2}-\frac{512}{3(c+1)^2}=$

$\frac{256}{3(c+1)^2}$

b. $\int_0^{c+1}(f(x)-g(x))\,dx=\int_0^{8/(c+1)} 8-(c+1)x\,dx=$

$\left[8x-\left(\frac{c+1}{2}\right)x^2\right]_0^{8/(c+1)}=\frac{64}{(c+1)}-\frac{32}{(c+1)}=$

$\frac{32}{(c+1)}$

c. $\bar{x}=\left(\frac{256}{3(c+1)^2}\right)\left(\frac{c+1}{32}\right)=\frac{8}{3(c+1)}$

$\bar{x}=2\Rightarrow\frac{8}{3(c+1)}=2\Rightarrow c=\frac{1}{3}$

52. $f(x)=c \quad g(x)=x \quad a=0 \quad b=c$

a. $\int_a^b x(f(x)-g(x))\,dx=\int_0^c cx-x^2\,dx=$

$\left[\frac{cx^2}{2}-\frac{x^3}{3}\right]_0^c=\frac{c^3}{6}$

b. $\int_a^b(f(x)-g(x))\,dx=\int_0^c(c-x)\,dx=$

$\left[cx-\frac{x^2}{2}\right]_0^c=\frac{c^2}{2}$

c. $\bar{x}=\left(\frac{c^3}{6}\right)\left(\frac{2}{c^2}\right)=\frac{c}{3}$

$\bar{x}=2\Rightarrow c=6$

53. $f(x)=6e^{-x/3} \quad g(x)=0 \quad a=0 \quad b=c$

a. $\int_a^b x(f(x)-g(x))\,dx=6\int_0^c xe^{-x/3}\,dx=$

$u=x$
$dv=e^{-x/3}$
$du=dx$
$v=-3e^{-x/3}$

$6\left[-3xe^{-x/3}\right]_0^c+18\int_0^c e^{-x/3}\,dx=$

$\left[-18e^{-x/3}(x+3)\right]_0^c=-18e^{-c/3}(c+3)+54$

b. $\int_a^b(f(x)-g(x))\,dx=\int_0^c 6e^{-x/3}\,dx=$

$-18\left(e^{-c/3}-1\right)$

c. For notational convenience, let

$u=-18e^{-c/3}$; then

$\bar{x}=\frac{u(c+3)+54}{u+18}=\frac{cu}{u+18}+\frac{3(u+18)}{u+18}=$

$\frac{cu}{u+18}+3$

$\bar{x}=2\Rightarrow\frac{cu}{u+18}=-1\Rightarrow\frac{c}{1+\frac{18}{u}}=-1\Rightarrow$

$c=\frac{1}{e^{-c/3}}-1\Rightarrow\frac{1}{c+1}=e^{-c/3}$

Let

$h(c)=\frac{1}{c+1}-e^{-c/3}\ , \quad h'(c)=\frac{1}{3}e^{-c/3}-\frac{1}{(c+1)^2}$

and apply Newton's Method

n	1	2	3	4	5	6
a_n	2.0000	5.0000	5.6313	5.7103	5.7114	5.7114

$c\approx 5.7114$

54. $f(x)=c\sin\left(\frac{\pi x}{2c}\right) \quad g(x)=x \quad a=0 \quad b=c$

(Note: the value for b is obtained by setting $c\sin\left(\frac{\pi x}{2c}\right)=x$ This requires that $\frac{x}{c}$ be a zero for the function $h(u)=u-\sin\left(\frac{\pi}{2}u\right)$. Applying Newton's Method to h we discover that the zeros of h are -1, 0, and 1. Since we are dealing with positive values, we conclude that $\frac{x}{c}$ =1 or x = c.)

a. $\int_a^b x(f(x)-g(x))\,dx = \int_0^c \left[cx\sin\left(\frac{\pi x}{2c}\right)-x^2\right]dx$

$= \int_0^c \underset{u=\frac{\pi}{2c}x,\ du=\frac{\pi}{2c}dx}{cx\sin\left(\frac{\pi x}{2c}\right)}\,dx - \left[\frac{x^3}{3}\right]_0^c$

$= \int_0^{\pi/2} c\left(\frac{2c}{\pi}u\right)\sin u\left(\frac{2c}{\pi}\right)du - \left[\frac{c^3}{3}\right]$

$\underset{F40}{=} \frac{4c^3}{\pi^2}\left[\sin u - u\cos u\right]_0^{\pi/2} - \left[\frac{c^3}{3}\right] = \frac{4c^3}{\pi^2}-\frac{c^3}{3}$

$= c^3\left(\frac{4}{\pi^2}-\frac{1}{3}\right)$

b. $\int_a^b (f(x)-g(x))\,dx = \int_0^c \left[c\sin\left(\frac{\pi x}{2c}\right)-x\right]dx =$

$\left[-\frac{2c^2}{\pi}\cos\left(\frac{\pi x}{2c}\right)-\frac{x^2}{2}\right]_0^c = \frac{2c^2}{\pi}-\frac{c^2}{2} =$

$c^2\left(\frac{2}{\pi}-\frac{1}{2}\right)$

c. $\bar{x} = \frac{c^3\left(\frac{12-\pi^2}{3\pi^2}\right)}{c^2\left(\frac{4-\pi}{2\pi}\right)} = c\left[\frac{2(12-\pi^2)}{3\pi(4-\pi)}\right]$

$\bar{x} = 2 \Rightarrow c = \frac{3\pi(4-\pi)}{12-\pi^2} \approx 3.798$

55. a. $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

$\therefore \frac{d}{dx}erf(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$

b. $Si(x) = \int_0^x \frac{\sin t}{t}dt$

$\therefore \frac{d}{dx}Si(x) = \frac{\sin x}{x}$

56. a. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$

$\therefore \frac{d}{dx}S(x) = \sin\left(\frac{\pi x^2}{2}\right)$

b. $C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right)dt$

$\therefore \frac{d}{dx}C(x) = \cos\left(\frac{\pi x^2}{2}\right)$

57. a. (See problem 55 a.) . Since $erf'(x) > 0$ for all x , $erf(x)$ is increasing on $(0,\infty)$.

b. $erf''(x) = \frac{-4x}{\sqrt{\pi}}e^{-x^2}$ which is negative on $(0,\infty)$, so $erf(x)$ is not concave up anywhere on the interval.

58. a. (See problem 56 a.) Since $S'(x) = \sin\left(\frac{\pi}{2}x^2\right)$, $S'(x) > 0$ when $0 < \frac{\pi}{2}x^2 < \pi$ or $0 < x^2 < 2$; thus $S(x)$ is increasing on $\left(0,\sqrt{2}\right)$.

b. Since $S''(x) = \pi x\cos\left(\frac{\pi}{2}x^2\right)$, $S''(x) > 0$ when $0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ and $\frac{3\pi}{2} < \frac{\pi}{2}x^2 < 2\pi$, or $0 < x^2 < 1$ and $3 < x^2 < 4$. Thus $S(x)$ is concave up on $(0,1)\cup(\sqrt{3},2)$.

59. a. (See problem 56 b.) Since $C'(x) = \cos\left(\frac{\pi}{2}x^2\right)$, $C'(x) > 0$ when $0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \frac{\pi}{2}x^2 < 2\pi$; thus $C(x)$ is increasing on $(0,1)\cup(\sqrt{3},2)$.

b. Since $C''(x) = -\pi x\sin\left(\frac{\pi}{2}x^2\right)$, $C''(x) > 0$ when $\pi < \frac{\pi}{2}x^2 < 2\pi$. Thus $C(x)$ is concave up on $(\sqrt{2},2)$.

60. From problem 58 we know that $S(x)$ is concave up on $(0,1)$ and concave down on $(1,\sqrt{3})$ so the first point of inflection occurs at $x = 1$. Now $S(1) = \int_0^1 \sin\left(\frac{\pi}{2}t^2\right)dt$. Since the integral cannot be integrated directly, we must use some approximation method. Methods may vary but the result will be $S(1) \approx 0.43826$. Thus the first point of inflection is $(1, 0.43826)$

7.7 Chapter Review

Concepts Test

1. True: The resulting integrand will be of the form sin u.

2. True: The resulting integrand will be of the form $\frac{1}{a^2+u^2}$.

3. False: Try the substitution $u = x^4, du = 4x^3\, dx$

4. False: Use the substitution $u = x^2 - 3x + 5$, $du = (2x - 3)dx$.

5. True: The resulting integrand will be of the form $\frac{1}{a^2+u^2}$.

6. True: The resulting integrand will be of the form $\frac{1}{\sqrt{a^2-x^2}}$.

7. True: This integral is most easily solved with a partial fraction decomposition.

8. False: This improper fraction should be reduced first, then a partial fraction decomposition can be used.

9. True: Because both exponents are even positive integers, half-angle formulas are used.

10. False: Use the substitution $u = 1 + e^x,\ du = e^x dx$

11. False: Use the substitution $u = -x^2 - 4x, du = (-2x - 4)dx$

12. True: This substitution eliminates the radical.

13. True: Then expand and use the substitution $u = \sin x$, $du = \cos x\, dx$

14. True: The trigonometric substitution $x = 3\sin t$ will eliminate the radical.

15. True: Let $u = \ln x$ $\quad dv = x^2 dx$
$du = \frac{1}{x}dx$ $\quad v = \frac{1}{3}x^3$

16. False: Use a product identity.

17. False: $\frac{x^2}{x^2-1} = 1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$

18. True: $\frac{x^2+2}{x(x^2-1)} = -\frac{2}{x} + \frac{3}{2(x+1)} + \frac{3}{2(x-1)}$

19. True: $\frac{x^2+2}{x(x^2+1)} = \frac{2}{x} + \frac{-x}{x^2+1}$

20. False: $\frac{x+2}{x^2(x^2-1)}$
$= -\frac{1}{x} - \frac{2}{x^2} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$

21. False: To complete the square, add $\frac{b^2}{4a}$.

22. False: Polynomials can be factored into products of linear and quadratic polynomials with real coefficients.

23. True: Polynomials with the same values for all x will have identical coefficients for like degree terms.

24. True: Let $u = 2x$; then $du = 2dx$ and
$\int x^2\sqrt{25-4x^2}\,dx = \frac{1}{8}\int u^2\sqrt{25-u^2}\,du$
which can be evaluated using Formula 57.

25. False: It can, however, be solved by the substitution $u = 25 - 4x^2$; then $du = -8x\,dx$ and
$\int x\sqrt{25-4x^2}\,dx = -\frac{1}{8}\int \sqrt{u}\,du =$
$-\frac{1}{12}(25-4x^2)^{3/2} + C$

26. True: Since (see Section 7.6, prob 55 a.)
$erf'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2} > 0$ for all x,
$erf(x)$ is an increasing function.

27. True: by the First Fundamental Theorem of Calculus.

28. False: Since (see Section 7.6, prob 55 b.)
$Si'(x) = \frac{\sin x}{x}$, which is negative on, say, $(\pi, 2\pi)$, $Si(x)$ will be decreasing on that same interval.

Sample Test Problems

1. $\int_0^4 \frac{t}{\sqrt{9+t^2}}dt = \left[\sqrt{9+t^2}\right]_0^4 = 5-3 = 2$

2. $\int \cot^2(2\theta)d\theta = \int \frac{\cos^2 2\theta}{\sin^2 2\theta}d\theta$

$= \int \frac{1-\sin^2 2\theta}{\sin^2 2\theta}d\theta = \int (\csc^2 2\theta - 1)d\theta$

$= -\frac{1}{2}\cot 2\theta - \theta + C$

3. $\int_0^{\pi/2} e^{\cos x}\sin x\,dx = \left[-e^{\cos x}\right]_0^{\pi/2} = e-1 \approx 1.718$

4. $\int_0^{\pi/4} x\sin 2x\,dx = \left[\frac{\sin 2x}{4} - \frac{x}{2}\cos 2x\right]_0^{\pi/4} = \frac{1}{4}$

(Use integration by parts with $u = x$, $dv = \sin 2x\,dx$.)

5. $\int \frac{y^3+y}{y+1}dy = \int\left(y^2 - y + 2 - \frac{2}{1+y}\right)dy$

$= \frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y - 2\ln|1+y| + C$

6. $\int \sin^3(2t)dt = \int [1-\cos^2(2t)]\sin(2t)dt$

$= -\frac{1}{2}\cos(2t) + \frac{1}{6}\cos^3(2t) + C$

7. $\int \frac{y-2}{y^2-4y+2}dy = \frac{1}{2}\int \frac{2y-4}{y^2-4y+2}dy$

$= \frac{1}{2}\ln\left|y^2-4y+2\right| + C$

8. $\int_0^{3/2} \frac{dy}{\sqrt{2y+1}} = \left[\sqrt{2y+1}\right]_0^{3/2} = 2-1 = 1$

9. $\int \frac{e^{2t}}{e^t-2}dt = e^t + 2\ln\left|e^t-2\right| + C$

(Use the substitution $u = e^t - 2$, $du = e^t dt$

which gives the integral $\int \frac{u+2}{u}du$.)

10. $\int \frac{\sin x + \cos x}{\tan x}dx = \int\left(\cos x + \frac{\cos^2 x}{\sin x}\right)dx$

$= \int\left(\cos x + \frac{1-\sin^2 x}{\sin x}\right)dx$

$= \int(\cos x + \csc x - \sin x)dx$

$= \sin x + \ln|\csc x - \cot x| + \cos x + C$

(Use Formula 15 for $\int \csc x\,dx$.)

11. $\int \frac{dx}{\sqrt{16+4x-2x^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x-1}{3}\right) + C$

(Complete the square.)

12. $\int x^2e^x dx = e^x(2-2x+x^2) + C$

Use integration by parts twice.

13. $y = \sqrt{\frac{2}{3}}\tan t,\ dy = \sqrt{\frac{2}{3}}\sec^2 t\,dt$

$\int \frac{dy}{\sqrt{2+3y^2}} = \int \frac{\sqrt{\frac{2}{3}}\sec^2 t}{\sqrt{2}\sec t}dt$

$= \frac{1}{\sqrt{3}}\int \sec t\,dt = \frac{1}{\sqrt{3}}\ln|\sec t + \tan t| + C_1$

$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{y^2+\frac{2}{3}}}{\sqrt{\frac{2}{3}}} + \frac{y}{\sqrt{\frac{2}{3}}}\right| + C_1$

$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{y^2+\frac{2}{3}}+y}{\sqrt{\frac{2}{3}}}\right| + C_1$

$= \frac{1}{\sqrt{3}}\ln\left|\sqrt{y^2+\frac{2}{3}}+y\right| + C$

Note that $\tan t = \frac{y}{\sqrt{\frac{2}{3}}}$, so $\sec t = \frac{\sqrt{y^2+\frac{2}{3}}}{\sqrt{\frac{2}{3}}}$.

14. $\int \frac{w^3}{1-w^2}dw = -\frac{1}{2}w^2 - \frac{1}{2}\ln\left|1-w^2\right| + C$

Divide the numerator by the denominator.

15. $\int \frac{\tan x}{\ln|\cos x|}dx = -\ln\left|\ln|\cos x|\right| + C$

Use the substitution $u = \ln|\cos x|$.

16. $\int \frac{3dt}{t^3-1} = \int \frac{1}{t-1}dt - \int \frac{t+2}{t^2+t+1}dt$

$= \int \frac{1}{t-1}dt - \frac{1}{2}\int \frac{2t+4}{t^2+t+1}dt$

$= \int \frac{1}{t-1}dt - \frac{1}{2}\int \frac{2t+1+3}{t^2+t+1}dt$

$= \int \frac{1}{t-1}dt - \frac{1}{2}\int \frac{2t+1}{t^2+t+1}dt - \frac{3}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}dt$

$= \ln|t-1| - \frac{1}{2}\ln\left|t^2+t+1\right| - \sqrt{3}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C$

17. $\int \sinh x\, dx = \cosh x + C$

18. $u = \ln y,\ du = \frac{1}{y}dy$

$\int \frac{(\ln y)^5}{y}dy = \int u^5 du = \frac{1}{6}(\ln y)^6 + C$

19. $u = x \qquad dv = \cot^2 x\, dx$

$du = dx \qquad v = -\cot x - x$

$\int x\cot^2 x\, dx = -x\cot x - x^2 - \int(-\cot x - x)dx$

$= -x\cot x - \frac{1}{2}x^2 + \ln|\sin x| + C$

Use $\cot^2 x = \csc^2 x - 1$ for $\int \cot^2 x\, dx$.

20. $u = \sqrt{x},\ du = \frac{1}{2}x^{-1/2}dx$

$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin u\, du$

$= -2\cos\sqrt{x} + C$

21. $u = \ln t^2, du = \frac{2}{t}dt$

$\int \frac{\ln t^2}{t}dt = \frac{[\ln(t^2)]^2}{4} + C$

22. $u = \ln(y^2+9) \qquad dv = dy$

$du = \frac{2y}{y^2+9}dy \qquad v = y$

$\int \ln(y^2+9)dy = y\ln(y^2+9) - \int \frac{2y^2}{y^2+9}dy$

$= y\ln(y^2+9) - \int\left(2 - \frac{18}{y^2+9}\right)dy$

$= y\ln(y^2+9) - 2y + 6\tan^{-1}\left(\frac{y}{3}\right) + C$

23. $\int e^{t/3}\sin 3t\, dt = \frac{-3e^{t/3}(9\cos 3t - \sin 3t)}{82} + C$

Use integration by parts twice.

24. $\int \frac{t+9}{t^3+9t}dt = \int \frac{1}{t}dt + \int \frac{-t+1}{t^2+9}dt$

$= \int \frac{1}{t}dt - \int \frac{t}{t^2+9}dt + \int \frac{1}{t^2+9}dt$

$= \ln|t| - \frac{1}{2}\ln\left|t^2+9\right| + \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$

25. $\int \sin\frac{3x}{2}\cos\frac{x}{2}dx = -\frac{\cos x}{2} - \frac{\cos 2x}{4} + C$

Use a product identity.

26. $\int \cos^4\left(\frac{x}{2}\right)dx = \int\left(\frac{1+\cos x}{2}\right)^2 dx$

$= \frac{1}{4}\int dx + \frac{1}{4}\int 2\cos x\, dx + \frac{1}{4}\int \cos^2 x\, dx$

$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos x\, dx + \frac{1}{8}\int(1+\cos 2x)dx$

$= \frac{3}{8}x + \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C$

27. $\int \tan^3 2x \sec 2x\, dx = \frac{1}{2}\int(\sec^2 2x - 1)d(\sec 2x)$

$= \frac{1}{6}\sec^3(2x) - \frac{1}{2}\sec(2x) + C$

28. $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx$

$\int \frac{\sqrt{x}}{1+\sqrt{x}}dx = \int \frac{2x}{1+\sqrt{x}}\left(\frac{1}{2\sqrt{x}}dx\right) = 2\int \frac{u^2}{1+u}du$

$= 2\int \frac{(u+1)(u-1)+1}{u+1}du = 2\int\left(u - 1 + \frac{1}{u+1}\right)du$

$= 2\left(\frac{u^2}{2} - u + \ln|u+1|\right) + C$

$= x - 2\sqrt{x} + 2\ln\left(1+\sqrt{x}\right) + C$

29. $\int \tan^{3/2} x \sec^4 x\,dx = \int \tan^{3/2} x(1+\tan^2 x)\sec^2 x\,dx = \int \tan^{3/2} x \sec^2 x\,dx + \int \tan^{7/2} x \sec^2 x\,dx$

$= \frac{2}{5}\tan^{5/2} x + \frac{2}{9}\tan^{9/2} x + C$

30. $u = t^{1/6}+1, (u-1)^6 = t, 6(u-1)^5 du = dt$

$\int \frac{dt}{t(t^{1/6}+1)} = \int \frac{6(u-1)^5 du}{(u-1)^6 u} = \int \frac{6du}{u(u-1)} = -6\int \frac{1}{u}du + 6\int \frac{1}{u-1}du = -6\ln\left|t^{1/6}+1\right| + 6\ln\left|t^{1/6}\right| + C$

31. $u = 9-e^{2y},\ du = -2e^{2y}dy$

$\int \frac{e^{2y}}{\sqrt{9-e^{2y}}}dy = -\frac{1}{2}\int u^{-1/2}du = -\sqrt{u}+C = -\sqrt{9-e^{2y}}+C$

32. $\int \cos^5 x\sqrt{\sin x}dx = \int (1-\sin^2 x)^2(\sin^{1/2} x)\cos x\,dx = \int \sin^{1/2} x\cos x\,dx - 2\int \sin^{5/2} x\cos x\,dx + \int \sin^{9/2} x\cos x\,dx$

$= \frac{2}{3}\sin^{3/2} x - \frac{4}{7}\sin^{7/2} x + \frac{2}{11}\sin^{11/2} x + C$

33. $\int e^{\ln(3\cos x)}dx = \int 3\cos x\,dx = 3\sin x + C$

34. $y = 3\sin t,\ dy = 3\cos t\,dt$

$\int \frac{\sqrt{9-y^2}}{y}dy = \int \frac{3\cos t}{3\sin t}\cdot 3\cos t\,dt$

$= 3\int \frac{1-\sin^2 t}{\sin t} = 3\int (\csc t - \sin t)\,dt$

$= 3\left[\ln\left|\csc t - \cot t\right| + \cos t\right] + C$

$= 3\ln\left|\frac{3}{y} - \frac{\sqrt{9-y^2}}{y}\right| + \sqrt{9-y^2} + C$

Note that $\sin t = \frac{y}{3}$, so $\csc t = \frac{3}{y}$ and

$\cot t = \frac{\sqrt{9-y^2}}{y}$.

35. $u = e^{4x},\ du = 4e^{4x}dx$

$\int \frac{e^{4x}}{1+e^{8x}}dx = \frac{1}{4}\int \frac{du}{1+u^2} = \frac{1}{4}\tan^{-1}(e^{4x}) + C$

36. $x = a\tan t,\ dx = a\sec^2 t\,dt$

$\int \frac{\sqrt{x^2+a^2}}{x^4}dx = \int \frac{a\sec t}{a^4\tan^4 t}a\sec^2 t\,dt$

$= \frac{1}{a^2}\int \frac{\sec^3 t}{\tan^4 t}dt = \frac{1}{a^2}\int \frac{\cos t}{\sin^4 t}dt$

$= \frac{1}{a^2}\left(-\frac{1}{3}\frac{1}{\sin^3 t}\right) + C = -\frac{1}{3a^2}\csc^3 t + C$

$= -\frac{1}{3a^2}\frac{(x^2+a^2)^{3/2}}{x^3} + C$

Note that $\tan t = \frac{x}{a}$, so $\csc t = \frac{\sqrt{x^2+a^2}}{x}$.

37. $u = \sqrt{w+5}, u^2 = w+5,\ 2u\,du = dw$

$\int \frac{w}{\sqrt{w+5}}dw = 2\int (u^2-5)du = \frac{2}{3}u^3 - 10u + C$

$= \frac{2}{3}(w+5)^{3/2} - 10(w+5)^{1/2} + C$

38. $u = 1+\cos t,\ du = -\sin t\,dt$

$\int \frac{\sin t\,dt}{\sqrt{1+\cos t}} = -\int \frac{du}{\sqrt{u}} = -2\sqrt{1+\cos t} + C$

39. $u = \cos^2 y, du = -2\cos y\sin y\,dy$

$\int \frac{\sin y\cos y}{9+\cos^4 y}dy = -\frac{1}{2}\int \frac{du}{9+u^2}$

$= -\frac{1}{6}\tan^{-1}\left(\frac{\cos^2 y}{3}\right) + C$

40. $\int \frac{dx}{\sqrt{1-6x-x^2}} = \int \frac{dx}{\sqrt{10-(x+3)^2}}$

$= \sin^{-1}\left(\frac{x+3}{\sqrt{10}}\right) + C$

41. $\frac{4x^2+3x+6}{x^2(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}$

$A = 1, B = 2, C = -1, D = 2$

$\int \frac{4x^2+3x+6}{x^2(x^2+3)}dx = \int \frac{1}{x}dx + 2\int \frac{1}{x^2}dx + \int \frac{-x+2}{x^2+3}dx$

$= \int \frac{1}{x}dx + 2\int \frac{1}{x^2}dx - \frac{1}{2}\int \frac{2x}{x^2+3}dx + 2\int \frac{1}{x^2+3}dx$

$= \ln|x| - \frac{2}{x} - \frac{1}{2}\ln\left|x^2+3\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

42. $x = 4\tan t,\ dx = 4\sec^2 t\,dt$

$\int \frac{dx}{(16+x^2)^{3/2}} = \frac{1}{16}\int \cos t\,dt = \frac{1}{16}\sin t + C = \frac{1}{16}\left(\frac{x}{\sqrt{x^2+16}}\right) + C = \frac{x}{16\sqrt{x^2+16}} + C$

43. **a.** $\frac{3-4x^2}{(2x+1)^3} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3}$

b. $\frac{7x-41}{(x-1)^2(2-x)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2-x} + \frac{D}{(2-x)^2} + \frac{E}{(2-x)^3}$

c. $\frac{3x+1}{(x^2+x+10)^2} = \frac{Ax+B}{x^2+x+10} + \frac{Cx+D}{(x^2+x+10)^2}$

d. $\frac{(x+1)^2}{(x^2-x+10)^2(1-x^2)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2} + \frac{Ex+F}{x^2-x+10} + \frac{Gx+H}{(x^2-x+10)^2}$

e. $\frac{x^5}{(x+3)^4(x^2+2x+10)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} + \frac{Ex+F}{x^2+2x+10} + \frac{Gx+H}{(x^2+2x+10)^2}$

f. $\frac{(3x^2+2x-1)^2}{(2x^2+x+10)^3} = \frac{Ax+B}{2x^2+x+10} + \frac{Cx+D}{(2x^2+x+10)^2} + \frac{Ex+F}{(2x^2+x+10)^3}$

44. **a.** $V = \pi\int_1^2 \left[\frac{1}{\sqrt{3x-x^2}}\right]^2 dx = \pi\int_1^2 \frac{1}{3x-x^2}dx$

$\frac{1}{3x-x^2} = \frac{A}{x} + \frac{B}{3-x}$

$A = \frac{1}{3}, B = \frac{1}{3}$

$V = \pi\int_1^2 \frac{1}{3}\left(\frac{1}{x} + \frac{1}{3-x}\right)dx = \frac{\pi}{3}\Big[\ln|x| - \ln|3-x|\Big]_1^2 = \frac{\pi}{3}(\ln 2 + \ln 2) = \frac{2\pi}{3}\ln 2 \approx 1.4517$

b. $V = 2\pi\int_1^2 \frac{x}{\sqrt{3x-x^2}}\,dx = -\pi\int_1^2 \frac{-2x+3-3}{\sqrt{3x-x^2}}\,dx = -\pi\int_1^2 \frac{3-2x}{\sqrt{3x-x^2}}\,dx + 3\pi\int_1^2 \frac{1}{\sqrt{3x-x^2}}\,dx$

$= -\pi\left[2\sqrt{3x-x^2}\right]_1^2 + 3\pi\int_1^2 \frac{1}{\sqrt{\frac{9}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx = \left[-2\pi\sqrt{3x-x^2} + 3\pi\sin^{-1}\left(\frac{2x-3}{3}\right)\right]_1^2$

$= -2\pi\sqrt{2} + 3\pi\sin^{-1}\frac{1}{3} + 2\pi\sqrt{2} - 3\pi\sin^{-1}\left(-\frac{1}{3}\right) = 6\pi\sin^{-1}\frac{1}{3} \approx 6.4058$

45. $y = \frac{x^2}{16},\ y' = \frac{x}{8}$

$L = \int_0^4 \sqrt{1+\left(\frac{x}{8}\right)^2}\,dx = \int_0^4 \sqrt{1+\frac{x^2}{64}}\,dx$

$x = 8\tan t,\ dx = 8\sec^2 t$

$L = \int_0^{\tan^{-1}\frac{1}{2}} \sec t \cdot 8\sec^2 t\,dt = 8\int_0^{\tan^{-1}\frac{1}{2}} \sec^3 t\,dt = 4\left[\sec t\tan t + \ln|\sec t + \tan t|\right]_0^{\tan^{-1}\frac{1}{2}}$

$= 4\left[\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right) + \ln\left|\frac{1}{2}+\frac{\sqrt{5}}{2}\right|\right] = \sqrt{5} + 4\ln\left(\frac{1+\sqrt{5}}{2}\right) \approx 4.1609$ Note: Use Formula 28 for $\int \sec^3 t\,dt$.

46. $V = \pi\int_0^3 \frac{1}{(x^2+5x+6)^2}\,dx = \pi\int_0^3 \frac{1}{(x+3)^2(x+2)^2}\,dx$

$\frac{1}{(x+3)^2(x+2)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$

$A = 2,\ B = 1,\ C = -2,\ D = 1$

$V = \pi\int_0^3\left[\frac{2}{x+3} + \frac{1}{(x+3)^2} - \frac{2}{x+2} + \frac{1}{(x+2)^2}\right]dx = \pi\left[2\ln|x+3| - \frac{1}{x+3} - 2\ln|x+2| - \frac{1}{x+2}\right]_0^3$

$= \pi\left[\left(2\ln 6 - \frac{1}{6} - 2\ln 5 - \frac{1}{5}\right) - \left(2\ln 3 - \frac{1}{3} - 2\ln 2 - \frac{1}{2}\right)\right] = \pi\left(\frac{7}{15} + 2\ln\frac{4}{5}\right) \approx 0.06402$

47. $V = 2\pi\int_0^3 \frac{x}{x^2+5x+6}\,dx$

$\frac{x}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$

$A = -2,\ B = 3$

$V = 2\pi\int_0^3\left[-\frac{2}{x+2} + \frac{3}{x+3}\right]dx = 2\pi\left[-2\ln(x+2) + 3\ln(x+3)\right]_0^3$

$= 2\pi[(-2\ln 5 + 3\ln 6) - (-2\ln 2 + 3\ln 3)] = 2\pi\left(3\ln 2 + 2\ln\frac{2}{5}\right) = 2\pi\ln\frac{32}{25} \approx 1.5511$

48. $V = 2\pi\int_0^2 4x^2\sqrt{2-x}\,dx$

$u = 2 - x \qquad du = -dx$

$x = 2 - u \qquad dx = -du$

$V = 2\pi\int_2^0 4(2-u)^2\sqrt{u}(-du) = 8\pi\int_0^2 (4u^{1/2} - 4u^{3/2} + u^{5/2})du = 8\pi\left[\frac{8}{3}u^{3/2} - \frac{8}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_0^2$

$= 8\pi\left(\frac{16\sqrt{2}}{3} - \frac{32\sqrt{2}}{5} + \frac{16\sqrt{2}}{7}\right) = 8\pi\left(\frac{128\sqrt{2}}{105}\right) = \frac{1024\sqrt{2}\pi}{105} \approx 43.3287$

49. $V = 2\pi\int_0^{\ln 3} 2(e^x - 1)(\ln 3 - x)dx = 4\pi\int_0^{\ln 3}[(\ln 3)e^x - xe^x - \ln 3 + x]dx$

Note that $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$ by using integration by parts.

$$V = 4\pi\left[(\ln 3)e^x - xe^x + e^x - (\ln 3)x + \frac{1}{2}x^2\right]_0^{\ln 3} = 4\pi\left[\left(3\ln 3 - 3\ln 3 + 3 - (\ln 3)^2 + \frac{1}{2}(\ln 3)^2\right) - (\ln 3 + 1)\right]$$

$$= 4\pi\left[2 - \ln 3 - \frac{1}{2}(\ln 3)^2\right] \approx 3.7437$$

50. $A = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{18}{x^2\sqrt{x^2+9}}dx$

$x = 3\tan t,\ dx = 3\sec^2 t\,dt$

$$A = \int_{\pi/6}^{\pi/3} \frac{18}{27\tan^2 t\sec t}3\sec^2 t\,dt = 2\int_{\pi/6}^{\pi/3}\frac{\cos t}{\sin^2 t}dt = 2\left[-\frac{1}{\sin t}\right]_{\pi/6}^{\pi/3} = 2\left(-\frac{2}{\sqrt{3}} + 2\right) = 4\left(1 - \frac{1}{\sqrt{3}}\right) \approx 1.6906$$

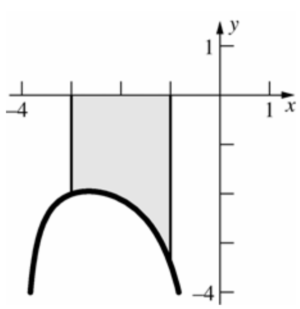
51. $A = -\int_{-6}^{0}\frac{t}{(t-1)^2}dt$

$$\frac{t}{(t-1)^2} = \frac{A}{(t-1)} + \frac{B}{(t-1)^2}$$

$A = 1, B = 1$

$$A = -\int_{-6}^{0}\left[\frac{1}{t-1} + \frac{1}{(t-1)^2}\right]dt = -\left[\ln|t-1| - \frac{1}{t-1}\right]_{-6}^{0} = -\left[(0+1) - \left(\ln 7 + \frac{1}{7}\right)\right] = \ln 7 - \frac{6}{7} \approx 1.0888$$

52.



$$V = \pi\int_{-3}^{-1}\left(\frac{6}{x\sqrt{x+4}}\right)^2 dx = \pi\int_{-3}^{-1}\frac{36}{x^2(x+4)}dx$$

$$\frac{36}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$A = -\frac{9}{4}, B = 9, C = \frac{9}{4}$

$$V = \pi\int_{-3}^{-1}\left[-\frac{9}{4x} + \frac{9}{x^2} + \frac{9}{4(x+4)}\right]dx = \frac{9\pi}{4}\int_{-3}^{-1}\left(-\frac{1}{x} + \frac{4}{x^2} + \frac{1}{x+4}\right)dx = \frac{9\pi}{4}\left[-\ln|x| - \frac{4}{x} + \ln|x+4|\right]_{-3}^{-1}$$

$$= \frac{9\pi}{4}\left[(4 + \ln 3) - \left(-\ln 3 + \frac{4}{3}\right)\right] = \frac{9\pi}{4}\left(\frac{8}{3} + 2\ln 3\right) = \frac{3\pi}{2}(4 + 3\ln 3) \approx 34.3808$$

53. The length is given by

$$\int_{\pi/6}^{\pi/3}\sqrt{1 + [f'(x)]^2}\,dx = \int_{\pi/6}^{\pi/3}\sqrt{1 + \frac{\cos^2 x}{\sin^2 x}}\,dx = \int_{\pi/6}^{\pi/3}\sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,dx = \int_{\pi/6}^{\pi/3}\frac{1}{\sin x}dx = \int_{\pi/6}^{\pi/3}\csc x\,dx$$

$$= \left[\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3} = \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \ln\left|2 - \sqrt{3}\right| = \ln\left(\frac{1}{\sqrt{3}}\right) - \ln(2 - \sqrt{3}) = \ln\left(\frac{2\sqrt{3} + 3}{3}\right) \approx 0.768$$

54. a. First substitute $u = 2x$, $du = 2\,dx$ to obtain $\int \frac{\sqrt{81-4x^2}}{x}\,dx = \int \frac{\sqrt{81-u^2}}{u}\,du$, then use Formula 55:

$$\int \frac{\sqrt{81-4x^2}}{x}\,dx = \sqrt{81-4x^2} - 9\ln\left|\frac{9+\sqrt{81-4x^2}}{2x}\right| + C$$

b. First substitute $u = e^x$, $du = e^x\,dx$ to obtain $\int e^x\left(9-e^{2x}\right)^{3/2}dx = \int\left(9-u^2\right)^{3/2}du$, then use Formula 62:

$$\int e^x\left(9-e^{2x}\right)^{3/2}dx = \frac{e^x}{8}\left(45-2e^{2x}\right)\sqrt{9-e^{2x}} + \frac{243}{8}\sin^{-1}\left(\frac{e^x}{3}\right) + C$$

55. a. First substitute $u = \sin x$, $du = \cos x\,dx$ to obtain $\int \cos x\sqrt{\sin^2 x+4}\;dx = \int\sqrt{u^2+4}\,du$, then use Formula 44:

$$\int \cos x\sqrt{\sin^2 x+4}\;dx = \frac{\sin x}{2}\sqrt{\sin^2 x+4} + 2\ln\left|\sin x+\sqrt{\sin^2 x+4}\right| + C$$

b. First substitute $u = 2x$, $du = 2dx$ to obtain $\int \frac{1}{1-4x^2}dx = \frac{1}{2}\int\frac{du}{1-u^2}$.

Then use Formula 18: $\int \frac{1}{1-4x^2}dx = \frac{1}{4}\ln\left|\frac{2x+1}{2x-1}\right| + C$.

56. By the First Fundamental Theorem of Calculus,

$$Si'(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad Si''(x) = \begin{cases} \dfrac{x\cos x - \sin x}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

57. Using partial fractions (see Section 7.6, prob 46 b.):

$$\frac{1}{1+x^3} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{(A+B)x^2+(B+C-A)x+(A+C)}{(x+1)(x^2-x+1)} \Rightarrow$$

$$A+C=1 \quad B+C=A \quad A=-B \quad \Rightarrow A=\frac{1}{3} \quad B=-\frac{1}{3} \quad C=\frac{2}{3}.$$

Therefore:

$$\int\frac{1}{1+x^3}dx = \frac{1}{3}\left[\int\frac{1}{x+1}dx - \int\frac{x-2}{x^2-x+1}dx\right] = \frac{1}{3}\left[\ln|x+1| - \underbrace{\int\frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}dx}_{u=x-\frac{1}{2},\ du=dx}\right]$$

$$= \left[\ln|x+1| - \int\frac{u-\frac{3}{2}}{u^2+\frac{3}{4}}du\right] \underset{F17}{=} \frac{1}{3}\left[\ln\left|\frac{x+1}{\sqrt{x^2-x+1}}\right| + \sqrt{3}\tan^{-1}\left(\tfrac{2}{\sqrt{3}}(x-\tfrac{1}{2})\right)\right]$$

so $\int_0^c \frac{1}{1+x^3}dx = \frac{1}{3}\left[\ln\left|\frac{c+1}{\sqrt{c^2-c+1}}\right| + \sqrt{3}\left[\tan^{-1}\left(\tfrac{2}{\sqrt{3}}(c-\tfrac{1}{2})\right)+\frac{\pi}{6}\right]\right]$.

Letting $G(c) = \frac{1}{3}\left[\ln\left|\frac{c+1}{\sqrt{c^2-c+1}}\right| + \sqrt{3}\left[\tan^{-1}\left(\tfrac{2}{\sqrt{3}}(c-\tfrac{1}{2})\right)+\frac{\pi}{6}\right]\right] - 0.5$ and $G'(c) = \frac{1}{1+c^3}$ we apply Newton's

Method to find the value of c such that $\int_0^c \frac{1}{1+x^3}dx = 0.5$:

n	1	2	3	4	5	6
a_n	1.0000	0.3287	0.5090	0.5165	0.5165	0.5165

Thus $c \approx 0.5165$.

Review and Preview Problems

1. $\lim_{x\to 2}\frac{x^2+1}{x^2-1}=\frac{2^2+1}{2^2-1}=\frac{5}{3}$

2. $\lim_{x\to 3}\frac{2x+1}{x+5}=\frac{2(3)+1}{3+5}=\frac{7}{8}$

3. $\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x+3)(x-3)}{x-3}=$
$\lim_{x\to 3}(x+3)=3+3=6$

4. $\lim_{x\to 2}\frac{x^2-5x+6}{x-2}=\lim_{x\to 2}\frac{(x-2)(x-3)}{x-2}=$
$\lim_{x\to 2}(x-3)=2-3=-1$

5. $\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{2\sin x\cos x}{x}=$
$\lim_{x\to 0}2\left(\frac{\sin x}{x}\right)\cos x=2(1)(1)=2$

6. $\lim_{x\to 0}\frac{\tan 3x}{x}=\lim_{x\to 0}\left(\frac{\sin 3x}{\cos 3x}\right)\left(\frac{3}{3x}\right)=$
$\lim_{x\to 0}3\left(\frac{\sin 3x}{3x}\right)\left(\frac{1}{\cos 3x}\right)=3(1)(1)=3$

7. $\lim_{x\to\infty}\frac{x^2+1}{x^2-1}=\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}=\frac{1+0}{1-0}=1$ or:

$\lim_{x\to\infty}\frac{x^2+1}{x^2-1}=\lim_{x\to\infty}1+\frac{2}{x^2-1}=1+0=1$

8. $\lim_{x\to\infty}\frac{2x+1}{x+5}=\lim_{x\to\infty}\frac{2+\frac{1}{x}}{1+\frac{5}{x}}=\frac{2+0}{1+0}=2$

9. $\lim_{x\to\infty}e^{-x}=\lim_{x\to\infty}\frac{1}{e^x}=0$

10. $\lim_{x\to\infty}e^{-x^2}=\lim_{x\to\infty}\frac{1}{e^{x^2}}=0$

11. $\lim_{x\to\infty}e^{2x}=\infty$ (has no finite value)

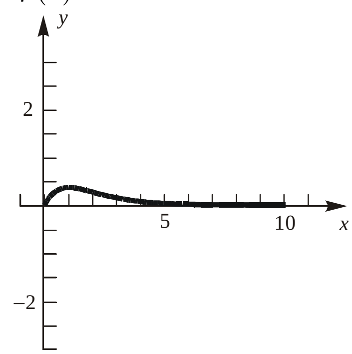
12. $\lim_{x\to-\infty}e^{-2x}\underset{(u=-x)}{=}\lim_{u\to\infty}e^{2u}=\infty$
(has no finite value)

13. $\lim_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}$

14. Note that, if $\theta=\sec^{-1}x$, then
$\sec\theta=x\Rightarrow\cos\theta=\frac{1}{x}\Rightarrow\theta=\cos^{-1}\frac{1}{x}$. Hence

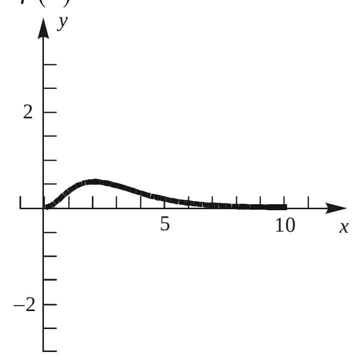
$$\lim_{x\to\infty}\sec^{-1}x=\lim_{x\to\infty}\cos^{-1}\frac{1}{x}=1$$

15. $f(x)=xe^{-x}$



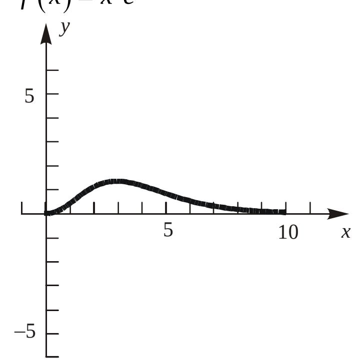
We would conjecture $\lim_{x\to\infty}xe^{-x}=0$.

16. $f(x)=x^2e^{-x}$



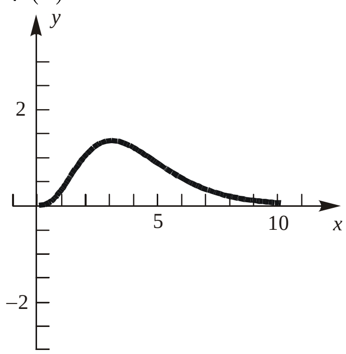
We would conjecture $\lim_{x\to\infty}x^2e^{-x}=0$.

17. $f(x)=x^3e^{-x}$



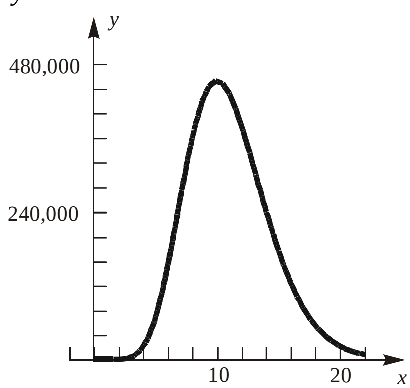
We would conjecture $\lim_{x\to\infty}x^3e^{-x}=0$.

18. $f(x) = x^4 e^{-x}$



We would conjecture $\lim_{x\to\infty} x^{10}e^{-x} = 0$.

19. $y = x^{10}e^{-x}$



We would conjecture $\lim_{x\to\infty} x^2 e^{-x} = 0$.

20. Based on the results from problems 15-19, we would conjecture

$\lim_{x\to\infty} x^n e^{-x} = 0$

21. $\int_0^a e^{-x}\,dx = \left[-e^{-x}\right]_0^a = 1 - e^{-a}$

a	1	2	4	8	16
$1-e^{-a}$	0.632	0.865	0.982	0.9997	0.9999+

22. $\int_0^a xe^{-x^2}\,dx \underset{\substack{u=-x^2 \\ du=-2x\,dx}}{=} -\frac{1}{2}\left[e^{-x^2}\right] = 1 - \frac{e^{-a^2}}{2}$

a	$1-\frac{1}{2e^{a^2}}$
1	0.81606028
2	0.93233236
4	0.999999944
8	$1-(8.02\times10^{-29})$
16	1

23. $\int_0^a \frac{x}{1+x^2}\,dx = \frac{1}{2}\left[\ln(1+x^2)\right]_0^a = \ln\left(\sqrt{1+a^2}\right)$

$u=x^2$
$du=2x\,dx$

a	1	2	4	8	16
$\ln\left(\sqrt{1+a^2}\right)$	0.3466	0.8047	1.4166	2.0872	2.7745

24. $\int_0^a \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^a = \ln(1+a)$

a	1	2	4	8	16
$\ln(1+a)$	0.6931	1.0986	1.6094	2.1972	2.8332

25. $\int_1^a \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^a = 1 - \frac{1}{a}$

a	2	4	8	16
$1-\frac{1}{a}$	0.5	0.75	0.875	0.9375

26. $\int_1^a \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^a = \frac{1}{2}\left[1 - \frac{1}{a^2}\right]$

a	2	4	8	16
$\frac{1}{2}\left[1-\frac{1}{a^2}\right]$	0.375	0.46875	0.4921875	0.498046875

27. $\int_a^4 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_a^4 = 4 - 2\sqrt{a}$

a	1	$1/2$	$1/4$	$1/8$	$1/16$
$4-2\sqrt{a}$	2	2.58579	3	3.29289	3.5

28. $\int_a^4 \frac{1}{x}\,dx = \left[\ln x\right]_a^4 = \ln\frac{4}{a}$

a	1	$1/2$	$1/4$	$1/8$	$1/16$
$\ln\frac{4}{a}$	1.38629	2.07944	2.77259	3.46574	4.15888