CHAPTER 8 Indeterminate Forms and Improper Integrals

8.1 Concepts Review

1. $\lim_{x\to a} f(x); \lim_{x\to a} g(x)$

2. $\dfrac{f'(x)}{g'(x)}$

3. $\sec^2 x; 1; \lim_{x\to 0} \cos x \neq 0$

4. Cauchy's Mean Value

Problem Set 8.1

1. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{2x-\sin x}{x} = \lim_{x\to 0}\frac{2-\cos x}{1} = 1$$

2. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to \pi/2}\frac{\cos x}{\pi/2-x} = \lim_{x\to \pi/2}\frac{-\sin x}{-1} = 1$$

3. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{x-\sin 2x}{\tan x} = \lim_{x\to 0}\frac{1-2\cos 2x}{\sec^2 x} = \frac{1-2}{1} = -1$$

4. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\tan^{-1} 3x}{\sin^{-1} x} = \lim_{x\to 0}\frac{\frac{3}{1+9x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{3}{1} = 3$$

5. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to -2}\frac{x^2+6x+8}{x^2-3x-10} = \lim_{x\to -2}\frac{2x+6}{2x-3}$$
$$= \frac{2}{-7} = -\frac{2}{7}$$

6. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{x^3-3x^2+x}{x^3-2x} = \lim_{x\to 0}\frac{3x^2+6x+1}{3x^2-2} = \frac{1}{-2} = -\frac{1}{2}$$

7. The limit is not of the form $\frac{0}{0}$.

As $x\to 1^-$, $x^2-2x+2\to 1$, and $x^2-1\to 0^-$ so

$$\lim_{x\to 1^-}\frac{x^2-2x+2}{x^2+1} = -\infty$$

8. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1}\frac{\ln x^2}{x^2-1} = \lim_{x\to 1}\frac{\frac{1}{x^2}2x}{2x} = \lim_{x\to 1}\frac{1}{x^2} = 1$$

9. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to \pi/2}\frac{\ln(\sin x)^3}{\pi/2-x} = \lim_{x\to \pi/2}\frac{\frac{1}{\sin^3 x}3\sin^2 x\cos x}{-1}$$
$$= \frac{0}{-1} = 0$$

10. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{2\sin x} = \lim_{x\to 0}\frac{e^x+e^{-x}}{2\cos x} = \frac{2}{2} = 1$$

11. The limit is of the form $\frac{0}{0}$.

$$\lim_{t\to 1}\frac{\sqrt{t}-t^2}{\ln t} = \lim_{t\to 1}\frac{\frac{1}{2\sqrt{t}}-2t}{\frac{1}{t}} = \frac{-\frac{3}{2}}{1} = -\frac{3}{2}$$

12. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^+}\frac{7^{\sqrt{x}}-1}{2^{\sqrt{x}}-1} = \lim_{x\to 0^+}\frac{\frac{7^{\sqrt{x}}\ln 7}{2\sqrt{x}}}{\frac{2^{\sqrt{x}}\ln 2}{2\sqrt{x}}} = \lim_{x\to 0^+}\frac{7^{\sqrt{x}}\ln 7}{2^{\sqrt{x}}\ln 2}$$
$$= \frac{\ln 7}{\ln 2} \approx 2.81$$

13. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\ln\cos 2x}{7x^2} = \lim_{x\to 0}\frac{\frac{-2\sin 2x}{\cos 2x}}{14x} = \lim_{x\to 0}\frac{-2\sin 2x}{14x\cos 2x}$$
$$= \lim_{x\to 0}\frac{-4\cos 2x}{14\cos 2x-28x\sin 2x} = \frac{-4}{14-0} = -\frac{2}{7}$$

14. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^-}\frac{3\sin x}{\sqrt{-x}} = \lim_{x\to 0^-}\frac{3\cos x}{-\frac{1}{2\sqrt{-x}}}$$
$$= \lim_{x\to 0^-} -6\sqrt{-x}\cos x = 0$$

15. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to 0}\frac{\tan x - x}{\sin 2x - 2x} = \lim_{x\to 0}\frac{\sec^2 x - 1}{2\cos 2x - 2}$$
$$= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{-4\sin 2x} = \lim_{x\to 0}\frac{2\sec^4 x + 4\sec^2 x\tan^2 x}{-8\cos 2x}$$
$$= \frac{2+0}{-8} = -\frac{1}{4}$$

16. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to 0}\frac{\sin x - \tan x}{x^2\sin x} = \lim_{x\to 0}\frac{\cos x - \sec^2 x}{2x\sin x + x^2\cos x}$$
$$= \lim_{x\to 0}\frac{-\sin x - 2\sec^2 x\tan x}{2\sin x + 4x\cos x - x^2\sin x}$$
$$= \lim_{x\to 0}\frac{-\cos x - 2\sec^4 x - 4\sec^2 x\tan^2 x}{6\cos x - x^2\cos x - 6x\sin x}$$
$$= \frac{-1-2-0}{6-0-0} = -\frac{1}{2}$$

17. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0^+}\frac{x^2}{\sin x - x} = \lim_{x\to 0^+}\frac{2x}{\cos x - 1} = \lim_{x\to 0^+}\frac{2}{-\sin x}$$

This limit is not of the form $\frac{0}{0}$. As $x \to 0^+, 2 \to 2,$ and $-\sin x \to 0^-,$ so $\lim_{x\to 0^+}\frac{2}{\sin x} = -\infty.$

18. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{e^x - \ln(1+x) - 1}{x^2} = \lim_{x\to 0}\frac{e^x - \frac{1}{1+x}}{2x}$$
$$= \lim_{x\to 0}\frac{e^x + \frac{1}{(1+x)^2}}{2} = \frac{1+1}{2} = 1$$

19. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\tan^{-1} x - x}{8x^3} = \lim_{x\to 0}\frac{\frac{1}{1+x^2} - 1}{24x^2} = \lim_{x\to 0}\frac{\frac{-2x}{(1+x^2)^2}}{48x}$$
$$= \lim_{x\to 0} -\frac{1}{24(1+x^2)^2} = -\frac{1}{24}$$

20. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\cosh x - 1}{x^2} = \lim_{x\to 0}\frac{\sinh x}{2x} = \lim_{x\to 0}\frac{\cosh x}{2} = \frac{1}{2}$$

21. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0^+}\frac{1 - \cos x - x\sin x}{2 - 2\cos x - \sin^2 x}$$
$$= \lim_{x\to 0^+}\frac{-x\cos x}{2\sin x - 2\cos x\sin s}$$
$$= \lim_{x\to 0^+}\frac{x\sin x - \cos x}{2\cos x - 2\cos^2 x + 2\sin^2 x}$$

This limit is not of the form $\frac{0}{0}$.

As $x \to 0^+$, $x\sin x - \cos x \to -1$ and $2\cos x - 2\cos^2 x + 2\sin^2 x \to 0^+$, so

$$\lim_{x\to 0^+}\frac{x\sin x - \cos x}{2\cos x - 2\cos^2 x + 2\sin^2 x} = -\infty$$

22. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^-}\frac{\sin x + \tan x}{e^x + e^{-x} - 2} = \lim_{x\to 0^-}\frac{\cos x + \sec^2 x}{e^x - e^{-x}}$$

This limit is not of the form $\frac{0}{0}$.

As $x \to 0^-, \cos x + \sec^2 x \to 2,$ and $e^x - e^{-x} \to 0^-,$ so $\lim_{x\to 0^-}\frac{\cos x + \sec^2 x}{e^x - e^{-x}} = -\infty.$

23. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\int_0^x \sqrt{1+\sin t}\,dt}{x} = \lim_{x\to 0}\sqrt{1+\sin x} = 1$$

24. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0^+}\frac{\int_0^x \sqrt{t}\cos t\,dt}{x^2}=\lim_{x\to 0^+}\frac{\sqrt{x}\cos x}{2x}$$

$$=\lim_{x\to 0^+}\frac{\cos x}{2\sqrt{x}}=\infty$$

25. It would not have helped us because we proved $\lim_{x\to 0}\frac{\sin x}{x}=1$ in order to find the derivative of $\sin x$.

26. Note that $\sin(1/0)$ is undefined (not zero), so l'Hôpital's Rule cannot be used.

As $x\to 0, \frac{1}{x}\to\infty$ and $\sin\left(\frac{1}{x}\right)$ oscillates rapidly between –1 and 1, so

$$\lim_{x\to 0}\left|\frac{x^2\sin\left(\frac{1}{x}\right)}{\tan x}\right|\le \lim_{x\to 0}\frac{x^2}{\tan x}.$$

$$\frac{x^2}{\tan x}=\frac{x^2\cos x}{\sin x}$$

$$\lim_{x\to 0}\frac{x^2\cos x}{\sin x}=\lim_{x\to 0}\left[\left(\frac{x}{\sin x}\right)x\cos x\right]=0.$$

Thus, $\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\tan x}=0$.

A table of values or graphing utility confirms this.

27. a. $\overline{OB}=\cos t, \overline{BC}=\sin t$ and $\overline{AB}=1-\cos t$, so the area of triangle *ABC* is $\frac{1}{2}\sin t(1-\cos t)$.

The area of the sector *COA* is $\frac{1}{2}t$ while the area of triangle *COB* is $\frac{1}{2}\cos t\sin t$, thus the area of the curved region *ABC* is $\frac{1}{2}(t-\cos t\sin t)$.

$$\lim_{t\to 0^+}\frac{\text{area of triangle } ABC}{\text{area of curved region } ABC}=\lim_{t\to 0^+}\frac{\frac{1}{2}\sin t(1-\cos t)}{\frac{1}{2}(t-\cos t\sin t)}$$

$$=\lim_{t\to 0^+}\frac{\sin t(1-\cos t)}{t-\cos t\sin t}=\lim_{t\to 0^+}\frac{\cos t-\cos^2 t+\sin^2 t}{1-\cos^2 t+\sin^2 t}=\lim_{t\to 0^+}\frac{4\sin t\cos t-\sin t}{4\cos t\sin t}=\lim_{t\to 0^+}\frac{4\cos t-1}{4\cos t}=\frac{3}{4}$$

(L'Hôpital's Rule was applied twice.)

b. The area of the sector *BOD* is $\frac{1}{2}t\cos^2 t$, so the area of the curved region *BCD* is $\frac{1}{2}\cos t\sin t-\frac{1}{2}t\cos^2 t$.

$$\lim_{t\to 0^+}\frac{\text{area of curved region } BCD}{\text{area of curved region } ABC}=\lim_{t\to 0^+}\frac{\frac{1}{2}\cos t(\sin t-t\cos t)}{\frac{1}{2}(t-\cos t\sin t)}$$

$$=\lim_{t\to 0^+}\frac{\cos t(\sin t-t\cos t)}{t-\sin t\cos t}=\lim_{t\to 0^+}\frac{\sin t(2t\cos t-\sin t)}{1-\cos^2 t+\sin^2 t}=\lim_{t\to 0^+}\frac{2t(\cos^2 t-\sin^2 t)}{4\cos t\sin t}=\lim_{t\to 0^+}\frac{t(\cos^2 t-\sin^2 t)}{2\cos t\sin t}$$

$$=\lim_{t\to 0^+}\frac{\cos^2 t-4t\cos t\sin t-\sin^2 t}{2\cos^2 t-2\sin^2 t}=\frac{1-0-0}{2-0}=\frac{1}{2}$$

(L'Hôpital's Rule was applied three times.)

28. a. Note that $\angle DOE$ has measure t radians. Thus the coordinates of E are $(\cos t, \sin t)$. Also, slope $\overline{BC}$ = slope $\overline{CE}$. Thus,

$$\frac{0-y}{(1-t)-0} = \frac{\sin t - 0}{\cos t - (1-t)}$$

$$-y = \frac{(1-t)\sin t}{\cos t + t - 1}$$

$$y = \frac{(t-1)\sin t}{\cos t + t - 1}$$

$$\lim_{t\to 0^+} y = \lim_{t\to 0^+} \frac{(t-1)\sin t}{\cos t + t - 1}$$

This limit is of the form $\frac{0}{0}$.

$$\lim_{t\to 0^+} \frac{(t-1)\sin t}{\cos t + t - 1} = \lim_{t\to 0^+} \frac{\sin t + (t-1)\cos t}{-\sin t + 1} = \frac{0 + (-1)(1)}{-0+1} = -1$$

b. Slope $\overline{AF}$ = slope $\overline{EF}$. Thus,

$$\frac{t}{1-x} = \frac{t - \sin t}{1 - \cos t}$$

$$\frac{t(1-\cos t)}{t - \sin t} = 1 - x$$

$$x = 1 - \frac{t(1+\cos t)}{t - \sin t}$$

$$x = \frac{t\cos t - \sin t}{t - \sin t}$$

$$\lim_{t\to 0^+} x = \lim_{t\to 0^+} \frac{t\cos t - \sin t}{t - \sin t}$$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{t\to 0^+} \frac{t\cos t - \sin t}{t - \sin t} = \lim_{t\to 0^+} \frac{-t\sin t}{1 - \cos t}$$

$$= \lim_{t\to 0^+} \frac{-\sin t - t\cos t}{\sin t} = \lim_{t\to 0^+} \frac{t\sin t - 2\cos t}{\cos t} = \frac{0-2}{1} = -2$$

29. By l'Hôpital's Rule $\left(\frac{0}{0}\right)$, we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^x - 1}{x} = \lim_{x\to 0^+} \frac{e^x}{1} = 1$ and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^x - 1}{x} = \lim_{x\to 0^-} \frac{e^x}{1} = 1$$ so we define $f(0) = 1$.

30. By l'Hôpital's Rule $\left(\frac{0}{0}\right)$, we have $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\ln x}{x-1} = \lim_{x\to 1^+} \frac{1/x}{1} = 1$ and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\ln x}{x-1} = \lim_{x\to 1^-} \frac{1/x}{1} = 1$$ so we define $f(1) = 1$.

31. A should approach $4\pi b^2$, the surface area of a sphere of radius b.

$$\lim_{a\to b^+}\left[2\pi b^2+\frac{2\pi a^2 b\arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]=2\pi b^2+2\pi b\lim_{a\to b^+}\frac{a^2\arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}$$

Focusing on the limit, we have

$$\lim_{a\to b^+}\frac{a^2\arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}=\lim_{a\to b^+}\frac{2a\arcsin\frac{\sqrt{a^2-b^2}}{a}+a^2\left(\frac{b}{a\sqrt{a^2-b^2}}\right)}{\frac{a}{\sqrt{a^2-b^2}}}=\lim_{a\to b^+}\left(2\sqrt{a^2-b^2}\arcsin\frac{\sqrt{a^2-b^2}}{a}+b\right)=b.$$

Thus, $\lim_{a\to b^+} A=2\pi b^2+2\pi b(b)=4\pi b^2$.

32. In order for l'Hôpital's Rule to be of any use, $a(1)^4+b(1)^3+1=0$, so $b=-1-a$.

Using l'Hôpital's Rule,

$$\lim_{x\to 1}\frac{ax^4+bx^3+1}{(x-1)\sin\pi x}=\lim_{x\to 1}\frac{4ax^3+3bx^2}{\sin\pi x+\pi(x-1)\cos\pi x}$$

To use l'Hôpital's Rule here,

$4a(1)^3+3b(1)^2=0$, so $4a+3b=0$, hence $a=3$, $b=-4$.

$$\lim_{x\to 1}\frac{3x^4-4x^3+1}{(x-1)\sin\pi x}=\lim_{x\to 1}\frac{12x^3-12x^2}{\sin\pi x+\pi(x-1)\cos\pi x}=\lim_{x\to 1}\frac{36x^2-24x}{2\pi\cos\pi x-\pi^2(x-1)\sin\pi x}=\frac{12}{-2\pi}=-\frac{6}{\pi}$$

$a=3$, $b=-4$, $c=-\frac{6}{\pi}$

33. If $f'(a)$ and $g'(a)$ both exist, then f and g are both continuous at a. Thus, $\lim_{x\to a} f(x)=0=f(a)$ and $\lim_{x\to a} g(x)=0=g(a)$.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}=\frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)}$$

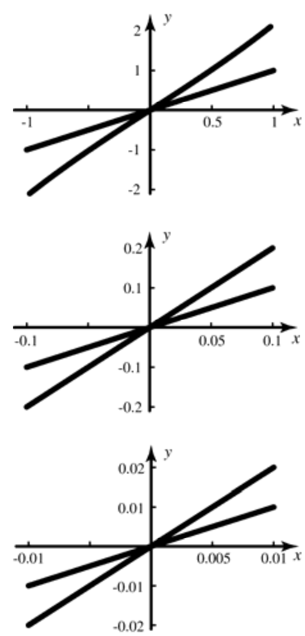
34. $\lim_{x\to 0}\frac{\cos x-1+\frac{x^2}{2}}{x^4}=\frac{1}{24}$

35. $\lim_{x\to 0}\frac{e^x-1-x-\frac{x^2}{2}-\frac{x^3}{6}}{x^4}=\frac{1}{24}$

36. $\lim_{x\to 0}\frac{1-\cos(x^2)}{x^3\sin x}=\frac{1}{2}$

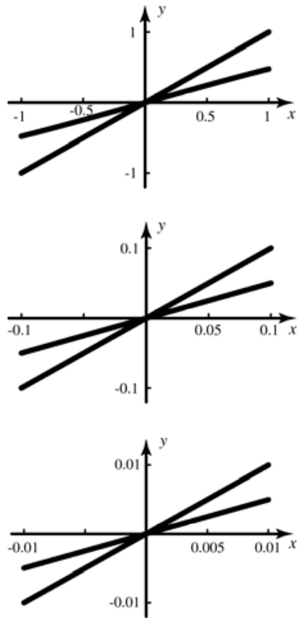
37. $\lim_{x\to 0}\frac{\tan x-x}{\arcsin x-x}=\lim_{x\to 0}\frac{\sec^2 x-1}{\frac{1}{\sqrt{1-x^2}}-1}=2$

38.



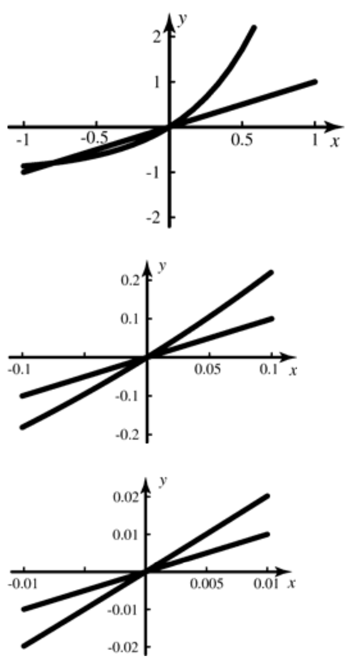
The slopes are approximately $0.02/0.01=2$ and $0.01/0.01=1$. The ratio of the slopes is therefore $2/1=2$, indicating that the limit of the ratio should be about 2. An application of l'Hopital's Rule confirms this.

39.



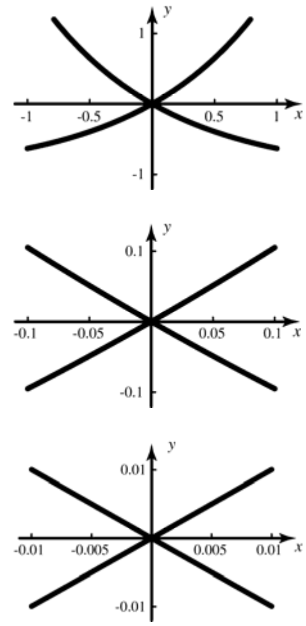
The slopes are approximately $0.005/0.01 = 1/2$ and $0.01/0.01 = 1$. The ratio of the slopes is therefore $1/2$, indicating that the limit of the ratio should be about $1/2$. An application of l'Hopital's Rule confirms this.

40.



The slopes are approximately $0.01/0.01 = 1$ and $0.02/0.01 = 2$. The ratio of the slopes is therefore $1/2$, indicating that the limit of the ratio should be about $1/2$. An application of l'Hopital's Rule confirms this.

41.



The slopes are approximately $0.01/0.01 = 1$ and $-0.01/0.01 = 1$. The ratio of the slopes is therefore $-1/1 = -1$, indicating that the limit of the ratio should be about -1. An application of l'Hopital's Rule confirms this.

42. If f and g are locally linear at zero, then, since $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, $f(x) \approx px$ and $g(x) \approx qx$, where $p = f'(0)$ and $q = g'(0)$. Then $f(x)/g(x) \approx px/px = p/q$ when x is near 0.

8.2 Concepts Review

1. $\dfrac{f'(x)}{g'(x)}$

2. $\lim_{x\to a}\dfrac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a}\dfrac{g(x)}{\frac{1}{f(x)}}$

3. $\infty-\infty, 0^\circ, \infty^\circ, 1^\infty$

4. $\ln x$

Problem Set 8.2

1. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{\ln x^{1000}}{x}=\lim_{x\to\infty}\frac{\frac{1}{x^{1000}}1000x^{999}}{1}$$
$$=\lim_{x\to\infty}\frac{1000}{x}=0$$

2. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to\infty}\frac{(\ln x)^2}{2^x}=\lim_{x\to\infty}\frac{2(\ln x)\frac{1}{x}}{2^x\ln 2}$$
$$=\lim_{x\to\infty}\frac{2\ln x}{x\cdot 2^x\ln 2}=\lim_{x\to\infty}\frac{2\left(\frac{1}{x}\right)}{2^x\ln 2(1+x\ln 2)}$$
$$=\lim_{x\to\infty}\frac{2}{x\cdot 2^x\ln 2(1+x\ln 2)}=0$$

3. $\lim_{x\to\infty}\dfrac{x^{10000}}{e^x}=0$ (See Example 2).

4. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to\infty}=\frac{3x}{\ln(100x+e^x)}=\lim_{x\to\infty}\frac{3}{\frac{1}{100x+e^x}(100+e^x)}$$
$$=\lim_{x\to\infty}\frac{300x+3e^x}{100+e^x}=\lim_{x\to\infty}\frac{300+3e^x}{e^x}$$
$$=\lim_{x\to\infty}\frac{3e^x}{e^x}=3$$

5. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\frac{\pi}{2}}\frac{3\sec x+5}{\tan x}=\lim_{x\to\frac{\pi}{2}}\frac{3\sec x\tan x}{\sec^2 x}$$
$$=\lim_{x\to\frac{\pi}{2}}\frac{3\tan x}{\sec x}=\lim_{x\to\frac{\pi}{2}}3\sin x=3$$

6. The limit is of the form $\frac{-\infty}{-\infty}$.

$$\lim_{x\to 0^+}\frac{\ln\sin^2 x}{3\ln\tan x}=\lim_{x\to 0^+}\frac{\frac{1}{\sin^2 x}2\sin x\cos x}{\frac{3}{\tan x}\sec^2 x}$$
$$=\lim_{x\to 0^+}\frac{2\cos^2 x}{3}=\frac{2}{3}$$

7. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{\ln(\ln x^{1000})}{\ln x}=\lim_{x\to\infty}\frac{\frac{1}{\ln x^{1000}}\left(\frac{1}{x^{1000}}1000x^{999}\right)}{\frac{1}{x}}$$
$$=\lim_{x\to\infty}\frac{1000}{x\ln x^{1000}}=0$$

8. The limit is of the form $\frac{-\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to\left(\frac{1}{2}\right)^-}\frac{\ln(4-8x)^2}{\tan\pi x}=\lim_{x\to\left(\frac{1}{2}\right)^-}\frac{\frac{1}{(4-8x)^2}2(4-8x)(-8)}{\pi\sec^2\pi x}$$
$$=\lim_{x\to\left(\frac{1}{2}\right)^-}\frac{-16\cos^2\pi x}{\pi(4-8x)}=\lim_{x\to\left(\frac{1}{2}\right)^-}\frac{32\pi\cos\pi x\sin\pi x}{-8\pi}$$
$$=\lim_{x\to\left(\frac{1}{2}\right)^-}-4\cos\pi x\sin\pi x=0$$

9. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\cot x}{\sqrt{-\ln x}}=\lim_{x\to 0^+}\frac{-\csc^2 x}{-\frac{1}{2x\sqrt{-\ln x}}}$$
$$=\lim_{x\to 0^+}\frac{2x\sqrt{-\ln x}}{\sin^2 x}$$
$$=\lim_{x\to 0^+}\left[\frac{2x}{\sin x}\csc x\sqrt{-\ln x}\right]=\infty$$

since $\lim_{x\to 0^+}\dfrac{x}{\sin x}=1$ while $\lim_{x\to 0^+}\csc x=\infty$ and $\lim_{x\to 0^+}\sqrt{-\ln x}=\infty$.

10. The limit is of the form $\frac{\infty}{\infty}$, but the fraction can be simplified.

$$\lim_{x\to 0}\frac{2\csc^2 x}{\cot^2 x}=\lim_{x\to 0}\frac{2}{\cos^2 x}=\frac{2}{1^2}=2$$

11. $\lim_{x\to 0}(x\ln x^{1000})=\lim_{x\to 0}\frac{\ln x^{1000}}{\frac{1}{x}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0}\frac{\ln x^{1000}}{\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{1}{x^{1000}}1000x^{999}}{-\frac{1}{x^2}}$$

$$=\lim_{x\to 0}-1000x=0$$

12. $\lim_{x\to 0}3x^2\csc^2 x=\lim_{x\to 0}3\left(\frac{x}{\sin x}\right)^2=3$ since

$$\lim_{x\to 0}\frac{x}{\sin x}=1$$

13. $\lim_{x\to 0}(\csc^2 x-\cot^2 x)=\lim_{x\to 0}\frac{1-\cos^2 x}{\sin^2 x}$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2 x}=1$$

14. $\lim_{x\to\frac{\pi}{2}}(\tan x-\sec x)=\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{-\sin x}=\frac{0}{-1}=0$$

15. The limit is of the form 0^0.

Let $y=(3x)^{x^2}$, then $\ln y=x^2\ln 3x$

$$\lim_{x\to 0^+}x^2\ln 3x=\lim_{x\to 0^+}\frac{\ln 3x}{\frac{1}{x^2}}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln 3x}{\frac{1}{x^2}}=\lim_{x\to 0^+}\frac{\frac{1}{3x}\cdot 3}{-\frac{2}{x^3}}=\lim_{x\to 0^+}-\frac{x^2}{2}=0$$

$$\lim_{x\to 0^+}(3x)^{x^2}=\lim_{x\to 0^+}e^{\ln y}=1$$

16. The limit is of the form 1^∞.

Let $y=(\cos x)^{\csc x}$, then $\ln y=\csc x(\ln(\cos x))$

$$\lim_{x\to 0}\csc x(\ln(\cos x))=\lim_{x\to 0}\frac{\ln(\cos x)}{\sin x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\ln(\cos x)}{\sin x}=\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{\cos x}$$

$$=\lim_{x\to 0}-\frac{\sin x}{\cos^2 x}=-\frac{0}{1}=0$$

$$\lim_{x\to 0}(\cos x)^{\csc x}=\lim_{x\to 0}e^{\ln y}=1$$

17. The limit is of the form 0^∞, which is not an indeterminate form. $\lim_{x\to(\pi/2)^-}(5\cos x)^{\tan x}=0$

18. $\lim_{x\to 0}\left(\csc^2 x-\frac{1}{x^2}\right)^2=\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)^2=\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^2\sin^2 x}\right)^2$

Consider $\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}$. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule four times.)

$$\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim_{x\to 0}\frac{2x-2\sin x\cos x}{2x\sin^2 x+2x^2\sin x\cos x}=\lim_{x\to 0}\frac{x-\sin x\cos x}{x\sin^2 x+x^2\sin x\cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x+\sin^2 x}{\sin^2 x+4x\sin x\cos x+x^2\cos^2 x-x^2\sin^2 x}=\lim_{x\to 0}\frac{4\sin x\cos x}{6x\cos x^2+6\cos x\sin x-4x^2\cos x\sin x-6x\sin^2 x}$$

$$=\lim_{x\to 0}\frac{4\cos^2 x-4\sin^2 x}{12\cos^2 x-4x^2\cos^2 x-32x\cos x\sin x-12\sin^2 x+4x^2\sin^2 x}=\frac{4}{12}=\frac{1}{3}$$

Thus, $\lim_{x\to 0}\left(\frac{x^2-\sin^2 x}{x^2\sin^2 x}\right)^2=\left(\frac{1}{3}\right)^2=\frac{1}{9}$

19. The limit is of the form 1^∞.

Let $y=(x+e^{x/3})^{3/x}$, then $\ln y=\frac{3}{x}\ln(x+e^{x/3})$.

$$\lim_{x\to 0}\frac{3}{x}\ln(x+e^{x/3})=\lim_{x\to 0}\frac{3\ln(x+e^{x/3})}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{3\ln(x+e^{x/3})}{x}=\lim_{x\to 0}\frac{\frac{3}{x+e^{x/3}}\left(1+\frac{1}{3}e^{x/3}\right)}{1}$$

$$=\lim_{x\to 0}\frac{3+e^{x/3}}{x+e^{x/3}}=\frac{4}{1}=4$$

$$\lim_{x\to 0}(x+e^{x/3})^{3/x}=\lim_{x\to 0}e^{\ln y}=e^4$$

20. The limit is of the form $(-1)^0$.
The limit does not exist.

21. The limit is of the form 1^0, which is not an indeterminate form.

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\cos x}=1$$

22. The limit is of the form ∞^∞, which is not an indeterminate form.

$$\lim_{x\to\infty}x^x=\infty$$

23. The limit is of the form ∞^0. Let

$y=x^{1/x}$, then $\ln y=\frac{1}{x}\ln x$.

$$\lim_{x\to\infty}\frac{1}{x}\ln x=\lim_{x\to\infty}\frac{\ln x}{x}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=0$$

$$\lim_{x\to\infty}x^{1/x}=\lim_{x\to\infty}e^{\ln y}=1$$

24. The limit is of the form 1^∞.

Let $y=(\cos x)^{1/x^2}$, then $\ln y=\frac{1}{x^2}\ln(\cos x)$.

$$\lim_{x\to 0}\frac{1}{x^2}\ln(\cos x)=\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}$$

The limit is of the form $\frac{0}{0}$.

(Apply l'Hôpital's rule twice.)

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}=\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{2x}=\lim_{x\to 0}\frac{-\tan x}{2x}$$

$$=\lim_{x\to 0}\frac{-\sec^2 x}{2}=\frac{-1}{2}=-\frac{1}{2}$$

$$\lim_{x\to 0}(\cos x)^{1/x^2}=\lim_{x\to 0}e^{\ln y}=e^{-1/2}=\frac{1}{\sqrt{e}}$$

25. The limit is of the form 0^∞, which is not an indeterminate form.

$$\lim_{x\to 0^+}(\tan x)^{2/x}=0$$

26. The limit is of the form $\infty+\infty$, which is not an indeterminate form.

$$\lim_{x\to-\infty}(e^{-x}-x)=\lim_{x\to\infty}(e^x+x)=\infty$$

27. The limit is of the form 0^0. Let

$y=(\sin x)^x$, then $\ln y=x\ln(\sin x)$.

$$\lim_{x\to 0^+}x\ln(\sin x)=\lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cos x}{-\frac{1}{x^2}}$$

$$=\lim_{x\to 0^+}\left[\frac{x}{\sin x}(-x\cos x)\right]=1\cdot 0=0$$

$$\lim_{x\to 0^+}(\sin x)^x=\lim_{x\to 0^+}e^{\ln y}=1$$

28. The limit is of the form 1^∞. Let

$y=(\cos x-\sin x)^{1/x}$, then $\ln y=\frac{1}{x}\ln(\cos x-\sin x)$.

$$\lim_{x\to 0}\frac{1}{x}\ln(\cos x-\sin x)=\lim_{x\to 0}\frac{\ln(\cos x-\sin x)}{x}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\cos x-\sin x}(-\sin x-\cos x)}{1}$$

$$=\lim_{x\to 0}\frac{-\sin x-\cos x}{\cos x-\sin x}=-1$$

$$\lim_{x\to 0}(\cos x-\sin x)^{1/x}=\lim_{x\to 0}e^{\ln y}=e^{-1}$$

29. The limit is of the form $\infty - \infty$.

$$\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \sin x}{x\sin x}$$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

30. The limit is of the form 1^∞.

Let $y = \left(1+\frac{1}{x}\right)^x$, then $\ln y = x\ln\left(1+\frac{1}{x}\right)$.

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^1 = e$$

31. The limit is of the form 3^∞, which is not an indeterminate form.

$$\lim_{x\to 0^+}(1+2e^x)^{1/x} = \infty$$

32. The limit is of the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{1}{x-1} - \frac{x}{\ln x}\right) = \lim_{x\to 1}\frac{\ln x - x^2 + x}{(x-1)\ln x}$$

The limit is of the form $\frac{0}{0}$.

Apply l'Hôpital's Rule twice.

$$\lim_{x\to 1}\frac{\ln x - x^2 + x}{(x-1)\ln x} = \lim_{x\to 1}\frac{\frac{1}{x} - 2x + 1}{\ln x + \frac{x-1}{x}}$$

$$= \lim_{x\to 1}\frac{1-2x^2+x}{x\ln x + x - 1} = \lim_{x\to 1}\frac{-4x+1}{\ln x + 2} = \frac{-3}{2} = -\frac{3}{2}$$

33. The limit is of the form 1^∞.

Let $y = (\cos x)^{1/x}$, then $\ln y = \frac{1}{x}\ln(\cos x)$.

$$\lim_{x\to 0}\frac{1}{x}\ln(\cos x) = \lim_{x\to 0}\frac{\ln(\cos x)}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x} = \lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{1} = \lim_{x\to 0} -\frac{\sin x}{\cos x} = 0$$

$$\lim_{x\to 0}(\cos x)^{1/x} = \lim_{x\to 0} e^{\ln y} = 1$$

34. The limit is of the form $0\cdot -\infty$.

$$\lim_{x\to 0^+}(x^{1/2}\ln x) = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$

35. Since cos x oscillates between –1 and 1 as $x \to \infty$, this limit is not of an indeterminate form previously seen.

Let $y = e^{\cos x}$, then $\ln y = (\cos x)\ln e = \cos x$

$\lim_{x\to\infty}\cos x$ does not exist, so $\lim_{x\to\infty} e^{\cos x}$ does not exist.

36. The limit is of the form $\infty - \infty$.

$$\lim_{x\to\infty}[\ln(x+1) - \ln(x-1)] = \lim_{x\to\infty}\ln\frac{x+1}{x-1}$$

$$\lim_{x\to\infty}\frac{x+1}{x-1} = \lim_{x\to\infty}\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = 1, \text{ so } \lim_{x\to\infty}\ln\frac{x+1}{x-1} = 0$$

37. The limit is of the form $\frac{0}{-\infty}$, which is not an indeterminate form.

$$\lim_{x\to 0^+}\frac{x}{\ln x} = 0$$

38. The limit is of the form $-\infty\cdot\infty$, which is not an indeterminate form.

$$\lim_{x\to 0^+}(\ln x\cot x) = -\infty$$

39. $\sqrt{1+e^{-t}} > 1$ for all t, so

$\int_1^x \sqrt{1+e^{-t}}\,dt > \int_1^x dt = x-1$.

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\int_1^x \sqrt{1+e^{-t}}\,dt}{x} = \lim_{x\to\infty} \frac{\sqrt{1+e^{-x}}}{1} = 1$$

40. This limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1^+} \frac{\int_1^x \sin t\,dt}{x-1} = \lim_{x\to 1^+} \frac{\sin x}{1} = \sin(1)$$

41. a. Let $y = \sqrt[n]{a}$, then $\ln y = \frac{1}{n}\ln a$.

$$\lim_{n\to\infty} \frac{1}{n}\ln a = 0$$

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} e^{\ln y} = 1$$

b. The limit is of the form ∞^0.

Let $y = \sqrt[n]{n}$, then $\ln y = \frac{1}{n}\ln n$.

$$\lim_{n\to\infty} \frac{1}{n}\ln n = \lim_{n\to\infty} \frac{\ln n}{n}$$

This limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\ln y} = 1$$

c. $$\lim_{n\to\infty} n\left(\sqrt[n]{a}-1\right) = \lim_{n\to\infty} \frac{\sqrt[n]{a}-1}{\frac{1}{n}}$$

This limit is of the form $\frac{0}{0}$,

since $\lim_{n\to\infty} \sqrt[n]{a} = 1$ by part a.

$$\lim_{n\to\infty} \frac{\sqrt[n]{a}-1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}\sqrt[n]{a}\ln a}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \sqrt[n]{a}\ln a = \ln a$$

d. $$\lim_{n\to\infty} n\left(\sqrt[n]{n}-1\right) = \lim_{n\to\infty} \frac{\sqrt[n]{n}-1}{\frac{1}{n}}$$

This limit is of the form $\frac{0}{0}$,

since $\lim_{n\to\infty} \sqrt[n]{n} = 1$ by part b.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n}-1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\left(\frac{1}{n^2}\right)(1-\ln n)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \sqrt[n]{n}(\ln n - 1) = \infty$$

42. a. The limit is of the form 0^0.

Let $y = x^x$, then $\ln y = x\ln x$.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

b. The limit is of the form 1^0, since

$\lim_{x\to 0^+} x^x = 1$ by part a.

Let $y = (x^x)^x$, then $\ln y = x\ln(x^x)$.

$$\lim_{x\to 0^+} x\ln(x^x) = 0$$

$$\lim_{x\to 0^+} (x^x)^x = \lim_{x\to 0^+} e^{\ln y} = 1$$

Note that 1^0 is not an indeterminate form.

c. The limit is of the form 0^1, since

$\lim_{x\to 0^+} x^x = 1$ by part a.

Let $y = x^{(x^x)}$, then $\ln y = x^x \ln x$

$$\lim_{x\to 0^+} x^x \ln x = -\infty$$

$$\lim_{x\to 0^+} x^{(x^x)} = \lim_{x\to 0^+} e^{\ln y} = 0$$

Note that 0^1 is not an indeterminate form.

d. The limit is of the form 1^0, since

$\lim_{x\to 0^+}(x^x)^x = 1$ by part b.

Let $y = ((x^x)^x)^x$, then $\ln y = x\ln((x^x)^x)$.

$\lim_{x\to 0^+} x\ln((x^x)^x) = 0$

$\lim_{x\to 0^+}((x^x)^x)^x = \lim_{x\to 0^+} e^{\ln y} = 1$

Note that 1^0 is not an indeterminate form.

e. The limit is of the form 0^0, since

$\lim_{x\to 0^+}(x^{(x^x)}) = 0$ by part c.

Let $y = x^{(x^{(x^x)})}$, then $\ln y = x^{(x^x)}\ln x$.

$$\lim_{x\to 0^+} x^{(x^x)}\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^{(x^x)}}}$$

The limit is of the form $\frac{-\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^{(x^x)}}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-x(x^x)\left[x^x(\ln x+1)\ln x+\frac{x^x}{x}\right]}{(x^{(x^x)})^2}}$$

$$= \lim_{x\to 0^+}\frac{-x^{(x^x)}}{x^x x(\ln x)^2 + x^x x\ln x + x^x}$$

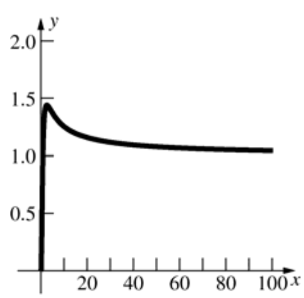
$$= \frac{0}{1\cdot 0 + 1\cdot 0 + 1} = 0$$

Note: $\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+}\frac{(\ln x)^2}{\frac{1}{x}}$

$$= \lim_{x\to 0^+}\frac{\frac{2}{x}\ln x}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -2x\ln x = 0$$

$$\lim_{x\to 0^+} x^{(x^{(x^x)})} = \lim_{x\to 0^+} e^{\ln y} = 1$$

43.



$$\ln y = \frac{\ln x}{x}$$

$$\lim_{x\to 0^+}\frac{\ln x}{x} = -\infty, \text{ so } \lim_{x\to 0^+} x^{1/x} = \lim_{x\to 0^+} e^{\ln y} = 0$$

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0, \text{ so } \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = 1$$

$$y = x^{1/x} = e^{\frac{1}{x}\ln x}$$

$$y' = \left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)e^{\frac{1}{x}\ln x}$$

$y' = 0$ when $x = e$.

y is maximum at $x = e$ since $y' > 0$ on $(0, e)$ and $y' < 0$ on (e, ∞). When $x = e$, $y = e^{1/e}$.

44. a. The limit is of the form $(1+1)^{\infty} = 2^{\infty}$, which is not an indeterminate form.

$\lim_{x\to 0^+}(1^x + 2^x)^{1/x} = \infty$

b. The limit is of the form $(1+1)^{-\infty} = 2^{-\infty}$, which is not an indeterminate form.

$\lim_{x\to 0^-}(1^x + 2^x)^{1/x} = 0$

c. The limit is of the form ∞^0.

Let $y = (1^x + 2^x)^{1/x}$, then

$$\ln y = \frac{1}{x}\ln(1^x + 2^x)$$

$$\lim_{x\to\infty}\frac{1}{x}\ln(1^x + 2^x) = \lim_{x\to\infty}\frac{\ln(1^x + 2^x)}{x}$$

The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to\infty}\frac{\ln(1^x + 2^x)}{x} = \lim_{x\to\infty}\frac{\frac{1}{1^x+2^x}(1^x\ln 1 + 2^x\ln 2)}{1}$$

$$= \lim_{x\to\infty}\frac{2^x\ln 2}{1^x + 2^x} = \lim_{x\to\infty}\frac{2^x(\ln 2)^2}{1^x\ln 1 + 2^x\ln 2} = \ln 2$$

$$\lim_{x\to\infty}(1^x + 2^x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2$$

d. The limit is of the form 1^0, since $1^x = 1$ for all x. This is not an indeterminate form.

$\lim_{x\to -\infty}(1^x + 2^x)^{1/x} = 1$

45. $\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}$

$= \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^k + \left(\frac{2}{n}\right)^k + \cdots \left(\frac{n}{n}\right)^k\right]$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\cdot\left(\frac{i}{n}\right)^k$

The summation has the form of a Reimann sum for $f(x) = x^k$ on the interval $[0,1]$ using a regular partition and evaluating the function at each right endpoint. Thus, $\Delta x_i = \frac{1}{n}$, $\overline{x}_i = \frac{i}{n}$, and $f(\overline{x}_i) = \left(\frac{i}{n}\right)^k$. Therefore,

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\cdot\left(\frac{i}{n}\right)^k$$

$$= \int_0^1 x^k\,dx = \left[\frac{1}{k+1}x^{k+1}\right]_0^1$$

$$= \frac{1}{k+1}$$

46. Let $y = \left(\sum_{i=1}^{n} c_i x_i^{\,t}\right)^{1/t}$, then $\ln y = \frac{1}{t}\ln\left(\sum_{i=1}^{n} c_i x_i^{\,t}\right)$.

$$\lim_{t\to 0^+} \frac{1}{t}\ln\left(\sum_{i=1}^{n} c_i x_i^{\,t}\right) = \lim_{t\to 0^+} \frac{\ln\left(\sum_{i=1}^{n} c_i x_i^{\,t}\right)}{t}$$

The limit is of the form $\frac{0}{0}$, since $\sum_{i=1}^{n} c_i = 1.$

$$\lim_{t\to 0^+} \frac{\ln\left(\sum_{i=1}^{n} c_i x_i^{\,t}\right)}{t} = \lim_{t\to 0^+} \frac{1}{\sum_{i=1}^{n} c_i x_i^{\,t}} \sum_{i=1}^{n} c_i x_i^{\,t} \ln x_i$$

$$= \sum_{i=1}^{n} c_i \ln x_i = \sum_{i=1}^{n} \ln x_i^{\,c_i}$$

$$\lim_{t\to 0^+}\left(\sum_{i=1}^{n} c_i x_i^{\,t}\right)^{1/t} = \lim_{t\to 0^+} e^{\ln y}$$

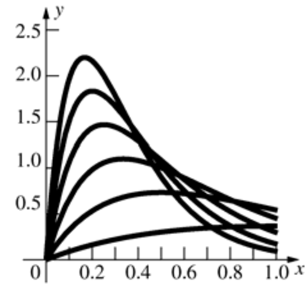
$$= e^{\sum_{i=1}^{n} \ln x_i^{\,c_i}} = x_1^{\,c_1} x_2^{\,c_2} \ldots x_n^{\,c_n} = \prod_{i=1}^{n} x_i^{\,c_i}$$

47. a. $\lim_{t\to 0^+}\left(\frac{1}{2}2^t + \frac{1}{2}5^t\right)^{1/t} = \sqrt{2}\sqrt{5} \approx 3.162$

b. $\lim_{t\to 0^+}\left(\frac{1}{5}2^t + \frac{4}{5}5^t\right)^{1/t} = \sqrt[5]{2}\cdot\sqrt[5]{5^4} \approx 4.163$

c. $\lim_{t\to 0^+}\left(\frac{1}{10}2^t + \frac{9}{10}5^t\right)^{1/t} = \sqrt[10]{2}\cdot\sqrt[10]{5^9} \approx 4.562$

48. a.



b. $n^2 x e^{-nx} = \frac{n^2 x}{e^{nx}}$, so the limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty} \frac{n^2 x}{e^{nx}} = \lim_{n\to\infty} \frac{2nx}{xe^{nx}}$$

This limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty} \frac{2nx}{xe^{nx}} = \lim_{n\to\infty} \frac{2x}{x^2 e^{nx}} = 0$$

c. $\int_0^1 xe^{-x}dx = \left[-xe^{-x} - e^{-x}\right]_0^1 = 1 - \frac{2}{e}$

$\int_0^1 4xe^{-2x}dx = \left[-2xe^{-2x} - e^{-2x}\right]_0^1 = 1 - \frac{3}{e^2}$

$\int_0^1 9xe^{-3x}dx = \left[-3xe^{-3x} - e^{-3x}\right]_0^1 = 1 - \frac{4}{e^3}$

$\int_0^1 16xe^{-4x}dx = \left[-4xe^{-4x} - e^{-4x}\right]_0^1 = 1 - \frac{5}{e^4}$

$\int_0^1 25xe^{-5x} = \left[-5xe^{-5x} - e^{-5x}\right]_0^1 = 1 - \frac{6}{e^5}$

$\int_0^1 36e^{-6x}dx = \left[-6xe^{-6x} - e^{-6x}\right]_0^1 = 1 - \frac{7}{e^6}$

d. Guess: $\lim_{n\to\infty}\int_0^1 n^2 xe^{-nx}dx = 1$

$$\int_0^1 n^2 xe^{-nx}dx = \left[-nxe^{-nx} - e^{-nx}\right]_0^1$$

$$= -(n+1)e^{-n} + 1 = 1 - \frac{n+1}{e^n}$$

$$\lim_{n\to\infty}\int_0^1 n^2 xe^{-nx}dx = \lim_{n\to\infty}\left(1 - \frac{n+1}{e^n}\right)$$

$= 1 - \lim_{n\to\infty}\frac{n+1}{e^n}$ if this last limit exists. The limit is of the form $\frac{\infty}{\infty}$.

$\lim_{n\to\infty}\frac{n+1}{e^n} = \lim_{n\to\infty}\frac{1}{e^n} = 0,$ so

$\lim_{n\to\infty}\int_0^1 n^2 xe^{-nx}dx = 1$.

49. Note $f(x) > 0$ on $[0, \infty)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{x^{25}}{e^x}+\frac{x^3}{e^x}+\left(\frac{2}{e}\right)^x\right)=0$$

Therefore there is no absolute minimum.

$$f'(x) = (25x^{24}+3x^2+2^x \ln 2)e^{-x}$$
$$-(x^{25}+x^3+2^x)e^{-x}$$
$$=(-x^{25}+25x^{24}-x^3+3x^2-2^x+2^x\ln 2)e^{-x}$$

Solve for x when $f'(x)=0$. Using a numerical method, $x \approx 25$.

A graph using a computer algebra system verifies that an absolute maximum occurs at about $x = 25$.

8.3 Concepts Review

1. converge

2. $\lim_{b\to\infty}\int_0^b \cos x\,dx$

3. $\int_{-\infty}^0 f(x)dx; \int_0^\infty f(x)dx$

4. $p > 1$

Problem Set 8.3

In this section and the chapter review, it is understood that $[g(x)]_a^\infty$ means $\lim_{b\to\infty}[g(x)]_a^b$ and likewise for similar expressions.

1. $\int_{100}^\infty e^x dx = \left[e^x\right]_{100}^\infty = \infty - e^{100} = \infty$

The integral diverges.

2. $\int_{-\infty}^{5}\frac{dx}{x^4} = \left[-\frac{1}{3x^3}\right]_{-\infty}^{-5} = -\frac{1}{3(-125)} - 0 = \frac{1}{375}$

3. $\int_1^\infty 2xe^{-x^2}dx = \left[-e^{-x^2}\right]_1^\infty = 0-(-e^{-1}) = \frac{1}{e}$

4. $\int_{-\infty}^1 e^{4x}dx = \left[\frac{1}{4}e^{4x}\right]_{-\infty}^1 = \frac{1}{4}e^4 - 0 = \frac{1}{4}e^4$

5. $\int_9^\infty \frac{x\,dx}{\sqrt{1+x^2}} = \left[\sqrt{1+x^2}\right]_9^\infty = \infty - \sqrt{82} = \infty$

The integral diverges.

6. $\int_1^\infty \frac{dx}{\sqrt{\pi x}} = \left[2\sqrt{\frac{x}{\pi}}\right]_1^\infty = \infty - \frac{2}{\sqrt{\pi}} = \infty$

The integral diverges.

7. $\int_1^\infty \frac{dx}{x^{1.00001}} = \left[-\frac{1}{0.00001x^{0.00001}}\right]_1^\infty$

$$= 0 - \left(-\frac{1}{0.00001}\right) = \frac{1}{0.00001} = 100{,}000$$

8. $\int_{10}^\infty \frac{x}{1+x^2}dx = \frac{1}{2}\left[\ln(1+x^2)\right]_{10}^\infty$

$$= \infty - \frac{1}{2}\ln|101| = \infty$$

The integral diverges.

9. $\int_1^\infty \frac{dx}{x^{0.99999}} = \left[\frac{x^{0.00001}}{0.00001}\right]_1^\infty = \infty - 100{,}000 = \infty$

The integral diverges.

10. $\int_1^\infty \frac{x}{(1+x^2)^2}dx = \left[-\frac{1}{2(1+x^2)}\right]_1^\infty$

$$= 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$$

11. $\int_e^\infty \frac{1}{x\ln x}dx = [\ln(\ln x)]_e^\infty = \infty - 0 = \infty$

The integral diverges.

12. $\int_e^\infty \frac{\ln x}{x}dx = \left[\frac{1}{2}(\ln x)^2\right]_e^\infty = \infty - \frac{1}{2} = \infty$

The integral diverges.

13. Let $u = \ln x,\ du = \frac{1}{x}dx, dv = \frac{1}{x^2}dx, v = -\frac{1}{x}$.

$$\int_2^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_2^b \frac{\ln x}{x^2}dx$$
$$= \lim_{b\to\infty}\left[-\frac{\ln x}{x}\right]_2^b + \lim_{b\to\infty}\int_2^b \frac{1}{x^2}dx$$
$$= \lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^b = \frac{\ln 2 + 1}{2}$$

14. $\int_1^\infty xe^{-x}dx$

$u = x, du = dx$

$dv = e^{-x}dx, v = -e^{-x}$

$$\int_1^\infty xe^{-x}d = \left[-xe^{-x}\right]_1^\infty + \int_1^\infty e^{-x}dx$$
$$= \left[-xe^{-x} - e^{-x}\right]_1^\infty = 0 - 0 - (-e^{-1} - e^{-1}) = \frac{2}{e}$$

15. $\int_{-\infty}^1 \frac{dx}{(2x-3)^3} = \left[-\frac{1}{4(2x-3)^2}\right]_{-\infty}^1$

$$= -\frac{1}{4} - (-0) = -\frac{1}{4}$$

16. $\int_4^\infty \frac{dx}{(\pi - x)^{2/3}} = \left[-3(\pi - x)^{1/3}\right]_4^\infty = \infty + 3\sqrt[3]{\pi - 4} = \infty$

The integral diverges.

17. $\int_{-\infty}^\infty \frac{x}{\sqrt{x^2+9}}dx = \int_{-\infty}^0 \frac{x}{\sqrt{x^2+9}}dx + \int_0^\infty \frac{x}{\sqrt{x^2+9}}dx = \left[\sqrt{x^2+9}\right]_{-\infty}^0 + \left[\sqrt{x^2+9}\right]_0^\infty = (3-\infty)+(\infty-3)$

The integral diverges since both $\int_{-\infty}^0 \frac{x}{\sqrt{x^2+9}}dx$ and $\int_0^\infty \frac{x}{\sqrt{x^2+9}}dx$ diverge.

18. $\int_{-\infty}^\infty \frac{dx}{(x^2+16)^2} = \int_{-\infty}^0 \frac{dx}{(x^2+16)^2} + \int_0^\infty \frac{dx}{(x^2+16)^2}$

$\int \frac{dx}{(x^2+16)^2} = \frac{1}{128}\tan^{-1}\frac{x}{4} + \frac{x}{32(x^2+16)}$ by using the substitution $x = 4\tan\theta$.

$\int_{-\infty}^0 \frac{dx}{(x^2+16)^2} = \left[\frac{1}{128}\tan^{-1}\frac{x}{4} + \frac{x}{32(x^2+16)}\right]_{-\infty}^0 = 0 - \left[\frac{1}{128}\left(-\frac{\pi}{2}\right)+0\right] = \frac{\pi}{256}$

$\int_0^\infty \frac{dx}{(x^2+16)^2} = \left[\frac{1}{128}\tan^{-1}\frac{x}{4} + \frac{x}{32(x^2+16)}\right]_0^\infty = \frac{1}{128}\left(\frac{\pi}{2}\right)+0-(0) = \frac{\pi}{256}$

$\int_{-\infty}^\infty \frac{dx}{(x^2+16)^2} = \frac{\pi}{256}+\frac{\pi}{256} = \frac{\pi}{128}$

19. $\int_{-\infty}^\infty \frac{1}{x^2+2x+10}dx = \int_{-\infty}^\infty \frac{1}{(x+1)^2+9}dx = \int_{-\infty}^0 \frac{1}{(x+1)^2+9}dx + \int_0^\infty \frac{1}{(x+1)^2+9}dx$

$\int \frac{1}{(x+1)^2+9}dx = \frac{1}{3}\tan^{-1}\frac{x+1}{3}$ by using the substitution $x + 1 = 3\tan\theta$.

$\int_{-\infty}^0 \frac{1}{(x+1)^2+9}dx = \left[\frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_{-\infty}^0 = \frac{1}{3}\tan^{-1}\frac{1}{3} - \frac{1}{3}\left(-\frac{\pi}{2}\right) = \frac{1}{6}\left(\pi + 2\tan^{-1}\frac{1}{3}\right)$

$\int_0^\infty \frac{1}{(x+1)^2+9}dx = \left[\frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_0^\infty = \frac{1}{3}\left(\frac{\pi}{2}\right) - \frac{1}{3}\tan^{-1}\frac{1}{3} = \frac{1}{6}\left(\pi - 2\tan^{-1}\frac{1}{3}\right)$

$\int_{-\infty}^\infty \frac{1}{x^2+2x+10}dx = \frac{1}{6}\left(\pi + 2\tan^{-1}\frac{1}{3}\right) + \frac{1}{6}\left(\pi - 2\tan^{-1}\frac{1}{3}\right) = \frac{\pi}{3}$

20. $\int_{-\infty}^\infty \frac{x}{e^{2|x|}}dx = \int_{-\infty}^0 \frac{x}{e^{-2x}}dx + \int_0^\infty \frac{x}{e^{2x}}dx$

For $\int_{-\infty}^0 \frac{x}{e^{-2x}}dx = \int_{-\infty}^0 xe^{2x}dx$, use $u = x, du = dx, dv = e^{2x}dx, v = \frac{1}{2}e^{2x}$.

$\int_{-\infty}^0 xe^{2x}dx = \left[\frac{1}{2}xe^{2x}\right]_{-\infty}^0 - \frac{1}{2}\int_{-\infty}^0 e^{2x}dx = \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_{-\infty}^0 = 0 - \frac{1}{4} - (0) = -\frac{1}{4}$

For $\int_0^\infty \frac{x}{e^{2x}}dx = \int_0^\infty xe^{-2x}dx$, use $u = x, du = dx, dv = e^{-2x}dx, v = -\frac{1}{2}e^{-2x}$.

$\int_0^\infty xe^{-2x}dx = \left[-\frac{1}{2}xe^{-2x}\right]_0^\infty + \frac{1}{2}\int_0^\infty e^{-2x}dx = \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^\infty = 0 - \left(0 - \frac{1}{4}\right) = \frac{1}{4}$

$\int_{-\infty}^\infty \frac{x}{e^{2|x|}}dx = -\frac{1}{4}+\frac{1}{4} = 0$

21. $\int_{-\infty}^{\infty} \text{sech}\, x\, dx = \int_{-\infty}^{0} \text{sech}\, x\, dx = \int_{0}^{\infty} \text{sech}\, x\, dx$

$= [\tan^{-1}(\sinh x)]_{-\infty}^{0} + [\tan^{-1}(\sinh x)]_{0}^{\infty}$

$= \left[0 - \left(-\frac{\pi}{2}\right)\right] + \left[\frac{\pi}{2} - 0\right] = \pi$

22. $\int_{1}^{\infty} \text{csch}\, x\, dx = \int_{1}^{\infty} \frac{1}{\sinh x} dx = \int_{1}^{\infty} \frac{2}{e^x - e^{-x}} dx$

$= \int_{1}^{\infty} \frac{2e^x}{e^{2x} - 1} dx$

Let $u = e^x, du = e^x dx$.

$\int_{1}^{\infty} \frac{2e^x}{e^{2x} - 1} dx = \int_{e}^{\infty} \frac{2}{u^2 - 1} du = \int_{e}^{\infty} \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$

$= [\ln(u-1) - \ln(u+1)]_{e}^{\infty} = \left[\ln \frac{u-1}{u+1}\right]_{e}^{\infty}$

$= 0 - \ln \frac{e-1}{e+1} \approx 0.7719$

$\left(\lim_{b \to \infty} \ln \frac{b-1}{b+1} = 0 \text{ since } \lim_{b \to \infty} \frac{b-1}{b+1} = 1\right)$

23. $\int_{0}^{\infty} e^{-x} \cos x\, dx = \left[\frac{1}{2e^x}(\sin x - \cos x)\right]_{0}^{\infty}$

$= 0 - \frac{1}{2}(0 - 1) = \frac{1}{2}$

(Use Formula 68 with $a = -1$ and $b = 1$.)

24. $\int_{0}^{\infty} e^{-x} \sin x\, dx = \left[-\frac{1}{2e^x}(\cos x + \sin x)\right]_{0}^{\infty}$

$= 0 + \frac{1}{2}(1 + 0) = \frac{1}{2}$

(Use Formula 67 with $a = -1$ and $b = 1$.)

25. The area is given by

$\int_{1}^{\infty} \frac{2}{4x^2 - 1} dx = \int_{1}^{\infty} \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right) dx$

$= \frac{1}{2}\left[\ln|2x-1| - \ln|2x+1|\right]_{1}^{\infty} = \frac{1}{2}\left[\ln\left|\frac{2x-1}{2x+1}\right|\right]_{1}^{\infty}$

$= \frac{1}{2}\left(0 - \ln\left(\frac{1}{3}\right)\right) = \frac{1}{2}\ln 3$

Note:. $\lim_{x \to \infty} \ln = \left|\frac{2x-1}{2x+1}\right| = 0$ since $\lim_{x \to \infty}\left(\frac{2x-1}{2x+1}\right) = 1 \cdot$

26. The area is

$\int_{1}^{\infty} \frac{1}{x^2 + x} dx = \int_{1}^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \left[\ln|x| - \ln|x+1|\right]_{1}^{\infty} = \left[\ln\left|\frac{x}{x+1}\right|\right]_{1}^{\infty} = 0 - \ln\frac{1}{2} = \ln 2$

.

27. The integral would take the form

$k\int_{3960}^{\infty} \frac{1}{x} dx = [k \ln x]_{3960}^{\infty} = \infty$

which would make it impossible to send anything out of the earth's gravitational field.

28. At $x = 1080$ mi, $F = 165$, so $k = 165(1080)^2 \approx 1.925 \times 10^8$. So the work done in mi-lb is

$1.925 \times 10^8 \int_{1080}^{\infty} \frac{1}{x^2} dx = 1.925 \times 10^8 \left[-x^{-1}\right]_{1080}^{\infty}$

$= \frac{1.925 \times 10^8}{1080} \approx 1.782 \times 10^5$ mi-lb.

29. $FP = \int_{0}^{\infty} e^{-rt} f(t)\, dt = \int_{0}^{\infty} 100{,}000 e^{-0.08t}$

$= \left[-\frac{1}{0.08} 100{,}000 e^{-0.08t}\right]_{0}^{\infty} = 1{,}250{,}000$

The present value is $1,250,000.

30. $FP = \int_{0}^{\infty} e^{-0.08t}(100{,}000 + 1000t) dt$

$= \left[-1{,}250{,}000 e^{-0.08t} - 12{,}500 t e^{-0.08t} - 156{,}250 e^{-0.08t}\right]_{0}^{\infty} = 1{,}406{,}250$

The present value is $1,406,250.

31. **a.** $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{a} 0\, dx + \int_{a}^{b} \frac{1}{b-a} dx + \int_{b}^{\infty} 0\, dx$

$= 0 + \frac{1}{b-a}[x]_{a}^{b} + 0 = \frac{1}{b-a}(b-a)$

b.

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$
$$= \int_{-\infty}^{a} x \cdot 0\, dx + \int_{a}^{b} x \frac{1}{b-a}\, dx + \int_{b}^{\infty} x \cdot 0\, dx$$
$$= 0 + \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b + 0$$
$$= \frac{b^2 - a^2}{2(b-a)}$$
$$= \frac{(b+a)(b-a)}{2(b-a)}$$
$$= \frac{a+b}{2}$$
$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, dx$$
$$= \int_{-\infty}^{a} (x-\mu)^2 \cdot 0\, dx + \int_{a}^{b} (x-\mu)^2 \frac{1}{b-a}\, dx + \int_{b}^{\infty} (x-\mu)^2 \cdot 0\, dx$$
$$= 0 + \frac{1}{b-a}\left[\frac{(x-\mu)^3}{3}\right]_a^b + 0$$
$$= \frac{1}{b-a}\frac{(b-\mu)^3 - (a-\mu)^3}{3}$$
$$= \frac{1}{b-a}\frac{b^3 - 3b^2\mu + 3b\mu^2 - a^3 + 3a^2\mu - 3a\mu^2}{3}$$

Next, substitute $\mu = (a+b)/2$ to obtain

$$\sigma^2 = \frac{1}{3(b-a)}\left[\tfrac{1}{4}b^3 - \tfrac{3}{4}b^2 a + \tfrac{3}{4}ba^2 - \tfrac{1}{4}a^3\right]$$
$$= \frac{1}{12(b-a)}(b-a)^3$$
$$= \frac{(b-a)^2}{12}$$

c.

$$P(X < 2) = \int_{-\infty}^{2} f(x)\, dx$$
$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{2} \frac{1}{10-0}\, dx$$
$$= \frac{2}{10} = \frac{1}{5}$$

32. **a.**

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}\, dx$$

In the second integral, let $u = (x/\theta)^\beta$. Then,
$du = (\beta/\theta)(t/\theta)^{\beta-1}\, dt$. When $x = 0, u = 0$ and when $x \to \infty, u \to \infty$. Thus,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{\infty} \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}\, dx$$
$$= \int_{0}^{\infty} e^{-u}\, du = \left[-e^{-u}\right]_0^\infty = -0 + e^0 = 1$$

b. $\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_0^{\infty} \frac{\beta}{\theta} x \left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)}\,dx\partial$

$= \frac{2}{3}\int_0^{\infty} x^2 e^{-(x/3)^2}\,dx = \frac{3}{2}\sqrt{\pi}$

$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{0} (x-\mu)^2 \cdot 0\,dx + \frac{2}{9}\int_0^{\infty} (x-\mu)^2 x e^{-(x^2/9)}\,dx$

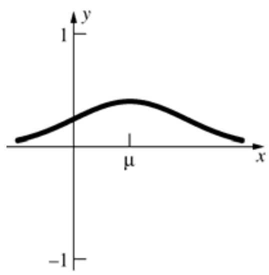
$= \frac{3}{2}\sqrt{\pi} - \mu = \frac{3}{2}\sqrt{\pi} - \frac{3}{2}\sqrt{\pi} = 0$

c. The probability of being less than 2 is

$\int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{2} \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}\,dx = 0 + \left[-e^{-(x/\theta)^\beta}\right]_0^2$

$= 1 - e^{-(2/\theta)^\beta} = 1 - e^{-(2/3)^2} \approx 0.359$

33.



$f'(x) = -\frac{x-\mu}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} + \frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

$= \left(\frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} - \frac{1}{\sigma^3\sqrt{2\pi}}\right) e^{-(x-\mu)^2/2\sigma^2} =$

$\frac{1}{\sigma^5\sqrt{2\pi}}[(x-\mu)^2 - \sigma^2] e^{-(x-\mu)^2/2\sigma^2}$

$f''(x) = 0$ when $(x-\mu)^2 = \sigma^2$ so $x = \mu \pm \sigma$ and the distance from μ to each inflection point is σ.

34. **a.** $\int_{-\infty}^{\infty} f(x)dx = \int_M^{\infty} \frac{CM^k}{x^{k+1}}\,dx = CM^k\left[-\frac{1}{kx^k}\right]_M^{\infty} = CM^k\left(0 + \frac{1}{kM^k}\right) = \frac{C}{k}$. Thus, $\frac{C}{k} = 1$ when $C = k$.

b. $\mu = \int_{-\infty}^{\infty} x f(x)dx = \int_M^{\infty} x \frac{kM^k}{x^{k+1}}\,dx = kM^k \int_M^{\infty} \frac{1}{x^k}\,dx = kM^k\left(\lim_{b\to\infty}\int_M^b \frac{1}{x^k}\,dx\right)$

This integral converges when $k > 1$.

When $k > 1$, $\mu = kM^k\left(\lim_{b\to\infty}\left[-\frac{1}{(k-1)x^{k-1}}\right]_M^b\right) = kM^k\left(-0 + \frac{1}{(k-1)M^{k-1}}\right) = \frac{kM}{k-1}$

The mean is finite only when $k > 1$.

c. Since the mean is finite only when $k > 1$, the variance is only defined when $k > 1$.

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_M^{\infty}\left(x - \frac{kM}{k-1}\right)^2 \frac{kM^k}{x^{k+1}}dx = kM^k \int_M^{\infty}\left(x^2 - \frac{2kM}{k-1}x + \frac{k^2M^2}{(k-1)^2}\right)\frac{1}{x^{k+1}}dx$$

$$= kM^k \int_M^{\infty}\frac{1}{x^{k-1}}dx - \frac{2k^2M^{k+1}}{k-1}\int_M^{\infty}\frac{1}{x^k}dx + \frac{k^3M^{k+2}}{(k-1)^2}\int_M^{\infty}\frac{1}{x^{k+1}}dx$$

The first integral converges only when $k - 1 > 1$ or $k > 2$. The second integral converges only when $k > 1$, which is taken care of by requiring $k > 2$.

$$\sigma^2 = kM^k\left[-\frac{1}{(k-2)x^{k-2}}\right]_M^{\infty} - \frac{2k^2M^{k+1}}{k-1}\left[-\frac{1}{(k-1)x^{k-1}}\right]_M^{\infty} + \frac{k^3M^{k+2}}{(k-1)^2}\left[-\frac{1}{kx^k}\right]_M^{\infty}$$

$$= kM^k\left(-0 + \frac{1}{(k-2)M^{k-2}}\right) - \frac{2k^2M^{k+1}}{k-1}\left(-0 + \frac{1}{(k-1)M^{k-1}}\right) + \frac{k^3M^{k+2}}{(k-1)^2}\left(-0 + \frac{1}{kM^k}\right)$$

$$= \frac{kM^2}{k-2} - \frac{2k^2M^2}{(k-1)^2} + \frac{k^2M^2}{(k-1)^2}$$

$$= kM^2\left(\frac{1}{k-2} - \frac{k}{(k-1)^2}\right) = kM^2\left(\frac{k^2 - 2k + 1 - k^2 + 2k}{(k-2)(k-1)^2}\right) = \frac{kM^2}{(k-2)(k-1)^2}$$

35. We use the results from problem 34:

a. To have a probability density function (34 a.) we need $C = k$; so $C = 3$. Also, $\mu = \frac{kM}{k-1}$ (34 b.) and since, in our problem, $\mu = 20{,}000$ and $k = 3$, we have

$20000 = \frac{3}{2}M$ or $M = \frac{4\times 10^4}{3}$.

b. By 34 c., $\sigma^2 = \frac{kM^2}{(k-2)(k-1)^2}$ so that

$$\sigma^2 = \frac{3}{4}\left(\frac{4\times 10^4}{3}\right)^2 = \frac{4\times 10^8}{3}$$

c. $\int_{10^5}^{\infty} f(x)\,dx = \left(\frac{4\times 10^4}{3}\right)^3 \lim_{t\to\infty}\int_{10^5}^{t}\frac{3}{x^4}dx =$

$$-\left(\frac{4\times 10^4}{3}\right)^3 \lim_{t\to\infty}\left[\frac{1}{x^3}\right]_{10^5}^{t}$$

$$= \left(\frac{4\times 10^4}{3}\right)^3 \lim_{t\to\infty}\left[\frac{1}{10^{15}} - \frac{1}{t^3}\right] = \frac{64}{27\times 10^3}$$

≈ 0.0024

Thus $6/25$ of one percent earn over \$100,000.

36. $u = Ar\int_a^{\infty}\frac{dx}{(r^2+x^2)^{3/2}}$

$$= \frac{A}{r}\left[\frac{x}{\sqrt{r^2+x^2}}\right]_a^{\infty} = \frac{A}{r}\left(1 - \frac{a}{\sqrt{r^2+a^2}}\right)$$

Note that $\int \frac{dx}{(r^2+x^2)^{3/2}} = \frac{x}{r^2\sqrt{r^2+x^2}}$ by using the substitution $x = r\tan\theta$.

37. a. $\int_{-\infty}^{\infty}\sin x\,dx = \int_{-\infty}^{0}\sin x\,dx + \int_0^{\infty}\sin x\,dx$

$= \lim_{a\to\infty}[-\cos x]_0^a + \lim_{a\to -\infty}[-\cos x]_a^0$

Both do not converge since $-\cos x$ is oscillating between –1 and 1, so the integral diverges.

b. $\lim_{a\to\infty}\int_{-a}^{a}\sin x\,dx = \lim_{a\to\infty}[-\cos x]_{-a}^{a}$

$= \lim_{a\to\infty}[-\cos a + \cos(-a)]$

$= \lim_{a\to\infty}[-\cos a + \cos a] = \lim_{a\to\infty} 0 = 0$

38. a. The total mass of the wire is

$\int_0^{\infty}\frac{1}{1+x^2}dx = \frac{\pi}{2}$ from Example 4.

b. $\int_0^{\infty}\frac{x}{1+x^2}dx = \left[\frac{1}{2}\ln\left|1+x^2\right|\right]_0^{\infty}$ which diverges. Thus, the wire does not have a center of mass.

39. For example, the region under the curve $y = \frac{1}{x}$ to the right of $x = 1$.
Rotated about the x-axis the volume is $\pi\int_1^\infty \frac{1}{x^2}dx = \pi$. Rotated about the y-axis, the volume is $2\pi\int_1^\infty x\cdot\frac{1}{x}dx$ which diverges.

40. a. Suppose $\lim_{x\to\infty} f(x) = M \neq 0$, so the limit exists but is non-zero. Since $\lim_{x\to\infty} f(x) = M$, there is some $N > 0$ such that when $x \geq N$, $|f(x) - M| \leq \frac{M}{2}$, or

$$M - \frac{M}{2} \leq f(x) \leq M + \frac{M}{2}$$

Since $f(x)$ is nonnegative, $M > 0$, thus $\frac{M}{2} > 0$ and

$$\int_0^\infty f(x)dx = \int_0^N f(x)dx + \int_N^\infty f(x)dx$$

$$\geq \int_0^N f(x)dx + \int_N^\infty \frac{M}{2}dx = \int_0^N f(x)dx + \left[\frac{Mx}{2}\right]_N^\infty = \infty$$

so the integral diverges. Thus, if the limit exists, it must be 0.

b. For example, let $f(x)$ be given by

$$f(x) = \begin{cases} 2n^2x - 2n^3 + 1 & \text{if } n - \frac{1}{2n^2} \leq x \leq n \\ -2n^2x + 2n^3 + 1 & \text{if } n < x \leq n + \frac{1}{2n^2} \\ 0 & \text{otherwise} \end{cases}$$

for every positive integer n.

$$f\left(n - \frac{1}{2n^2}\right) = 2n^2\left(n - \frac{1}{2n^2}\right) - 2n^3 + 1$$

$$= 2n^3 - 1 - 2n^3 + 1 = 0$$

$$f(n) = 2n^2(n) - 2n^3 + 1 = 1$$

$$\lim_{x\to n^+} f(n) = \lim_{x\to n^+}(-2n^2x + 2n^3 + 1) = 1 = f(n)$$

$$f\left(n + \frac{1}{2n^2}\right) = -2n^2\left(n + \frac{1}{2n^2}\right) + 2n^3 + 1$$

$$= -2n^3 - 1 + 2n^3 + 1 = 0$$

Thus, f is continuous at $n - \frac{1}{2n^2}$, n, and $n + \frac{1}{2n^2}$.

Note that the intervals $\left[n, n + \frac{1}{2n^2}\right]$ and $\left[n + 1 - \frac{1}{2(n+1)^2}, n + 1\right]$ will never overlap since $\frac{1}{2n^2} \leq \frac{1}{2}$ and $\frac{1}{2(n+1)^2} \leq \frac{1}{8}$.

The graph of f consists of a series of isosceles triangles, each of height 1, vertices at $\left(n - \frac{1}{2n^2}, 0\right)$, $(n, 1)$, and $\left(n + \frac{1}{2n^2}, 0\right)$, based on the x-axis, and centered over each integer n.

$\lim_{x\to\infty} f(x)$ does not exist, since $f(x)$ will be 1 at each integer, but 0 between the triangles.

Each triangle has area

$$\frac{1}{2}bh = \frac{1}{2}\left[n + \frac{1}{2n^2} - \left(n - \frac{1}{2n^2}\right)\right](1)$$

$$= \frac{1}{2}\left(\frac{1}{n^2}\right) = \frac{1}{2n^2}$$

$\int_0^\infty f(x)dx$ is the area in all of the triangles, thus

$$\int_0^\infty f(x)dx = \sum_{n=1}^\infty \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2}$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=2}^\infty \frac{1}{n^2} \leq \frac{1}{2} + \frac{1}{2}\int_1^\infty \frac{1}{x^2}dx$$

$$= \frac{1}{2} + \frac{1}{2}\left[-\frac{1}{x}\right]_1^\infty = \frac{1}{2} + \frac{1}{2}(-0 + 1) = 1$$

(By viewing $\sum_{n=2}^\infty \frac{1}{n^2}$ as a lower Riemann sum for $\frac{1}{x^2}$)

Thus, $\int_0^\infty f(x)dx$ converges, although $\lim_{x\to\infty} f(x)$ does not exist.

41. $\int_1^{100} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{100} = 0.99$

$\int_1^{100} \frac{1}{x^{1.1}} dx = \left[-\frac{1}{0.1x^{0.1}} \right]_1^{100} \approx 3.69$

$\int_1^{100} \frac{1}{x^{1.01}} dx = \left[-\frac{1}{0.01x^{0.01}} \right]_1^{100} \approx 4.50$

$\int_1^{100} \frac{1}{x} dx = [\ln x]_1^{100} = \ln 100 \approx 4.61$

$\int_1^{100} \frac{1}{x^{0.99}} dx = \left[\frac{x^{0.01}}{0.01} \right]_1^{100} \approx 4.71$

42. $\int_0^{10} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi}\left[\tan^{-1} x \right]_0^{10}$

$\approx \frac{1.4711}{\pi} \approx 0.468$

$\int_0^{50} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi}\left[\tan^{-1} x \right]_0^{50}$

$\approx \frac{1.5508}{\pi} \approx 0.494$

$\int_0^{100} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi}\left[\tan^{-1} x \right]_0^{100}$

$\approx \frac{1.5608}{\pi} \approx 0.497$

43. $\int_0^{1} \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) dx \approx 0.3413$

$\int_0^{2} \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) dx \approx 0.4772$

$\int_0^{3} \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) dx \approx 0.4987$

$\int_0^{4} \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2) dx \approx 0.5000$

8.4 Concepts Review

1. unbounded

2. 2

3. $\lim_{b \to 4^-} \int_0^b \frac{1}{\sqrt{4-x}} dx$

4. $p < 1$

Problem Set 8.4

1. $\int_1^3 \frac{dx}{(x-1)^{1/3}} = \lim_{b \to 1^+} \left[\frac{3(x-1)^{2/3}}{2} \right]_b^3$

$= \frac{3}{2}\sqrt[3]{2^2} - \lim_{b \to 1^+} \frac{3(b-1)^{2/3}}{2} = \frac{3}{\sqrt[3]{2}} - 0 = \frac{3}{\sqrt[3]{2}}$

2. $\int_1^3 \frac{dx}{(x-1)^{4/3}} = \lim_{b \to 1^+} \left[-\frac{3}{(x-1)^{1/3}} \right]_b^3$

$= -\frac{3}{\sqrt[3]{2}} + \lim_{b \to 1^+} \frac{3}{(x-1)^{1/3}} = -\frac{3}{\sqrt[3]{2}} + \infty$

The integral diverges.

3. $\int_3^{10} \frac{dx}{\sqrt{x-3}} = \lim_{b \to 3^+} \left[2\sqrt{x-3} \right]_b^{10}$

$= 2\sqrt{7} - \lim_{b \to 3^+} 2\sqrt{b-3} = 2\sqrt{7}$

4. $\int_0^9 \frac{dx}{\sqrt{9-x}} = \lim_{b \to 9^-} \left[-2\sqrt{9-x} \right]_0^b$

$= \lim_{b \to 9^-} -2\sqrt{9-b} + 2\sqrt{9} = 6$

5. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b \to 1^-} \left[\sin^{-1} x \right]_0^b$

$= \lim_{b \to 1^-} \sin^{-1} b - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

6. $\int_{100}^{\infty} \frac{x}{\sqrt{1+x^2}} dx = \lim_{b \to \infty} \left[\sqrt{1+x^2} \right]_{100}^b$

$= \lim_{b \to \infty} \sqrt{1+b^2} + \sqrt{10{,}001} = \infty$

The integral diverges.

7. $\int_{-1}^3 \frac{1}{x^3} dx = \lim_{b \to 0^-} \int_{-1}^b \frac{1}{x^3} dx + \lim_{b \to 0^+} \int_b^3 \frac{1}{x^3} dx$

$= \lim_{b \to 0^-} \left[-\frac{1}{2x^2} \right]_{-1}^b + \lim_{b \to 0^+} \left[-\frac{1}{2x^2} \right]_b^3$

$= \left(\lim_{b \to 0^-} -\frac{1}{2b^2} + \frac{1}{2} \right) + \left(-\frac{1}{18} + \lim_{b \to 0^+} \frac{1}{2b^2} \right)$

$= \left(-\infty + \frac{1}{2} \right) + \left(-\frac{1}{8} + \infty \right)$

The integral diverges.

8. $\int_5^{-5} \frac{1}{x^{2/3}}dx = \lim_{b\to 0^+}\int_5^b \frac{1}{x^{2/3}}dx + \lim_{b\to 0^-}\int_b^{-5}\frac{1}{x^{2/3}}dx$

$= \lim_{b\to 0^+}\left[3x^{1/3}\right]_5^b + \lim_{b\to 0^-}\left[3x^{1/3}\right]_b^{-5}$

$= \lim_{b\to 0^+} 3b^{1/3} - 3\sqrt[3]{5} + 3\sqrt[3]{-5} - \lim_{b\to 0^-} 3b^{1/3}$

$= 0 - 3\sqrt[3]{5} + 3\sqrt[3]{5} - 0 = 3\sqrt[3]{-5} - 3\sqrt[3]{5} = -6\sqrt[3]{5}$

9. $\int_{-1}^{128} x^{-5/7}dx$

$= \lim_{b\to 0^-}\int_{-1}^{b} x^{-5/7}dx + \lim_{b\to 0^+}\int_b^{128} x^{-5/7}dx$

$= \lim_{b\to 0^-}\left[\frac{7}{2}x^{2/7}\right]_{-1}^{b} + \lim_{b\to 0^+}\left[\frac{7}{2}x^{2/7}\right]_b^{128}$

$= \lim_{b\to 0^-}\frac{7}{2}b^{2/7} - \frac{7}{2}(-1)^{2/7} + \frac{7}{2}(128)^{2/7} - \lim_{b\to 0^+}\frac{7}{2}b^{2/7}$

$= 0 - \frac{7}{2} + \frac{7}{2}(4) - 0 = \frac{21}{2}$

10. $\int_0^1 \frac{x}{\sqrt[3]{1-x^2}}dx = \lim_{b\to 1^-}\int_0^b \frac{x}{\sqrt[3]{1-x^2}}dx$

$= \lim_{b\to 1^-}\left[-\frac{3}{4}(1-x^2)^{2/3}\right]_0^b$

$= \lim_{b\to 1^-} -\frac{3}{4}(1-b^2)^{2/3} + \frac{3}{4} = -0 + \frac{3}{4} = \frac{3}{4}$

11. $\int_0^4 \frac{dx}{(2-3x)^{1/3}} = \lim_{b\to \frac{2}{3}^-}\int_0^b \frac{dx}{(2-3x)^{1/3}} + \lim_{b\to \frac{2}{3}^+}\int_b^4 \frac{dx}{(2-3x)^{1/3}} = \lim_{b\to \frac{2}{3}^-}\left[-\frac{1}{2}(2-3x)^{2/3}\right]_0^b + \lim_{b\to \frac{2}{3}^+}\left[-\frac{1}{2}(2-3x)^{2/3}\right]_b^4$

$= \lim_{b\to \frac{2}{3}^-} -\frac{1}{2}(2-3b)^{2/3} + \frac{1}{2}(2)^{2/3} - \frac{1}{2}(-10)^{2/3} + \lim_{b\to \frac{2}{3}^+}\frac{1}{2}(2-3b)^{2/3}$

$= 0 + \frac{1}{2}2^{2/3} - \frac{1}{2}10^{2/3} + 0 = \frac{1}{2}(2^{2/3} - 10^{2/3})$

12. $\int_{\sqrt{5}}^{\sqrt{8}} \frac{x}{(16-2x^2)^{2/3}}dx = \lim_{b\to\sqrt{8}^-}\left[-\frac{3}{4}(16-2x^2)^{1/3}\right]_{\sqrt{5}}^{b} = \lim_{b\to\sqrt{8}^-} -\frac{3}{4}(16-2b^2)^{1/3} + \frac{3}{4}\sqrt[3]{6} = \frac{3}{4}\sqrt[3]{6}$

13. $\int_0^{-4}\frac{x}{16-2x^2}dx = \lim_{b\to -\sqrt{8}^+}\int_0^b \frac{x}{16-2x^2}dx + \lim_{b\to -\sqrt{8}^-}\int_b^{-4}\frac{x}{16-2x^2}dx$

$= \lim_{b\to -\sqrt{8}^+}\left[-\frac{1}{4}\ln\left|16-2x^2\right|\right]_0^b + \lim_{b\to -\sqrt{8}^-}\left[-\frac{1}{4}\ln\left|16-2x^2\right|\right]_b^{-4}$

$= \lim_{b\to -\sqrt{8}^+} -\frac{1}{4}\ln\left|16-2b^2\right| + \frac{1}{4}\ln 16 - \frac{1}{4}\ln 16 + \lim_{b\to -\sqrt{8}^-}\frac{1}{4}\ln\left|16-2b^2\right|$

$= \left[-(-\infty) + \frac{1}{4}\ln 16\right] + \left[-\frac{1}{4}\ln 16 + (-\infty)\right]$

The integral diverges.

14. $\int_0^3 \frac{x}{\sqrt{9-x^2}}dx = \lim_{b\to 3^-}\left[-\sqrt{9-x^2}\right]_0^b = \lim_{b\to 3^-} -\sqrt{9-b^2} + \sqrt{9} = 3$

15. $\int_{-2}^{-1}\frac{dx}{(x+1)^{4/3}} = \lim_{b\to -1^-}\left[-\frac{3}{(x+1)^{1/3}}\right]_{-2}^{b} = \lim_{b\to -1^-} -\frac{3}{(b+1)^{1/3}} + \frac{3}{(-1)^{1/3}} = -(-\infty) - 3$

The integral diverges.

16. Note that $\int \frac{dx}{x^2+x-2} = \int \frac{dx}{(x-1)(x+2)} = \int \left[\frac{1}{3(x-1)} - \frac{1}{3(x+2)}\right] dx$ by using a partial fraction decomposition.

$$\int_0^3 \frac{dx}{x^2+x-2} = \lim_{b\to 1^-} \int_0^b \frac{dx}{x^2+x-2} + \lim_{b\to 1^+} \int_b^3 \frac{dx}{x^2+x-2}$$

$$= \lim_{b\to 1^-} \left[\frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2|\right]_0^b + \lim_{b\to 1^+} \left[\frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2|\right]_b^3$$

$$= \lim_{b\to 1^-} \left[\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\right]_0^b + \lim_{b\to 1^+} \left[\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\right]_b^3 = \lim_{b\to 1^-} \frac{1}{3}\ln\left|\frac{b-1}{b+2}\right| - \frac{1}{3}\ln\frac{1}{2} + \frac{1}{3}\ln\frac{2}{5} - \lim_{b\to 1^+} \frac{1}{3}\ln\left|\frac{b-1}{b+2}\right|$$

$$= \left(-\infty - \frac{1}{3}\ln\frac{1}{2}\right) + \left(\frac{1}{3}\ln\frac{2}{5} + \infty\right)$$

The integral diverges.

17. Note that $\frac{1}{x^3-x^2-x+1} = \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)}$

$$\int_0^3 \frac{dx}{x^3-x^2-x+1} = \lim_{b\to 1^-} \int_0^b \frac{dx}{x^3-x^2-x+1} + \lim_{b\to 1^+} \int_b^3 \frac{dx}{x^3-x^2-x+1}$$

$$= \lim_{b\to 1^-} \left[-\frac{1}{2(x-1)} - \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1|\right]_0^b + \lim_{b\to 1^+} \left[-\frac{1}{2(x-1)} - \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1|\right]_b^3$$

$$\lim_{b\to 1^-} \left[\left(-\frac{1}{2(b-1)} + \frac{1}{4}\ln\left|\frac{b+1}{b-1}\right|\right) + \left(-\frac{1}{2} + 0\right)\right] + \lim_{b\to 1^+} \left[-\frac{1}{4} + \frac{1}{4}\ln 2 - \left(-\frac{1}{2(b-1)} + \frac{1}{4}\ln\left|\frac{b+1}{b-1}\right|\right)\right]$$

$$= \left(\infty + \infty - \frac{1}{2}\right) + \left(-\frac{1}{4} + \frac{1}{4}\ln 2 + \infty - \infty\right)$$

The integral diverges.

18. Note that $\frac{x^{1/3}}{x^{2/3}-9} = \frac{1}{x^{1/3}} + \frac{9}{x^{1/3}(x^{2/3}-9)}$.

$$\int_0^{27} \frac{x^{1/3}}{x^{2/3}-9} dx = \lim_{b\to 27^-} \left[\frac{3}{2}x^{2/3} + \frac{27}{2}\ln\left|x^{2/3}-9\right|\right]_0^b = \lim_{b\to 27^-} \left(\frac{3}{2}b^{2/3} + \frac{27}{2}\ln\left|b^{2/3}-9\right|\right) - \left(0 + \frac{27}{2}\ln 9\right)$$

$$= \frac{27}{2} - \infty - \frac{27}{2}\ln 9$$

The integral diverges.

19. $\int_0^{\pi/4} \tan 2x dx = \lim_{b\to \frac{\pi}{4}^-} \left[-\frac{1}{2}\ln|\cos 2x|\right]_0^b$

$$= \lim_{b\to \frac{\pi}{4}^-} -\frac{1}{2}\ln|\cos 2b| + \frac{1}{2}\ln 1 = -(-\infty) + 0$$

The integral diverges.

20. $\int_0^{\pi/2} \csc x dx = \lim_{b\to 0^+} \left[\ln|\csc x - \cot x|\right]_b^{\pi/2}$

$$= \ln|1-0| - \lim_{b\to 0^+} \ln|\csc b - \cot b|$$

$$= 0 - \lim_{b\to 0^+} \ln\left|\frac{1-\cos b}{\sin b}\right|$$

$\lim_{b\to 0^+} \frac{1-\cos b}{\sin b}$ is of the form $\frac{0}{0}$.

$$\lim_{b\to 0^+} \frac{1-\cos b}{\sin b} = \lim_{b\to 0^+} \frac{\sin b}{\cos b} = \frac{0}{1} = 0$$

Thus, $\lim_{b\to 0^+} \ln\left|\frac{1-\cos b}{\sin b}\right| = -\infty$ and the integral diverges.

21. $\int_0^{\pi/2} \frac{\sin x}{1-\cos x}dx = \lim_{b\to 0^+}\left[\ln|1-\cos x|\right]_b^{\pi/2}$

$= \ln 1 - \lim_{b\to 0^+} \ln|1-\cos b| = 0-(-\infty)$

The integral diverges.

22. $\int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin x}}dx = \lim_{b\to 0^+}\left[\frac{3}{2}\sin^{2/3} x\right]_b^{\pi/2}$

$= \frac{3}{2}(1)^{2/3} - \frac{3}{2}(0)^{2/3} = \frac{3}{2}$

23. $\int_0^{\pi/2} \tan^2 x \sec^2 x\, dx = \lim_{b\to \frac{\pi}{2}^-}\left[\frac{1}{3}\tan^3 x\right]_0^b$

$= \lim_{b\to \frac{\pi}{2}^-} \frac{1}{3}\tan^3 b - \frac{1}{3}(0)^3 = \infty$

The integral diverges.

24. $\int_0^{\pi/4} \frac{\sec^2 x}{(\tan x-1)^2}dx = \lim_{b\to \frac{\pi}{4}^-}\left[-\frac{1}{\tan x - 1}\right]_0^b$

$= \lim_{b\to \frac{\pi}{4}^-} -\frac{1}{\tan b-1} + \frac{1}{0-1} = -(-\infty)-1$

The integral diverges.

25. Since $\frac{1-\cos x}{2} = \sin^2 \frac{x}{2}$,

$\frac{1}{\cos x - 1} = -\frac{1}{2}\csc^2 \frac{x}{2}$.

$\int_0^{\pi} \frac{dx}{\cos x - 1} = \lim_{b\to 0^+}\left[\cot \frac{x}{2}\right]_b^{\pi}$

$= \cot\frac{\pi}{2} - \lim_{b\to 0^+} \cot\frac{b}{2} = 0-\infty$

The integral diverges.

26. $\int_{-3}^{-1} \frac{dx}{x\sqrt{\ln(-x)}} = \lim_{b\to -1^-}\left[2\sqrt{\ln(-x)}\right]_{-3}^b$

$= \lim_{b\to -1^-} 2\sqrt{\ln(-b)} - 2\sqrt{\ln 3} = 0 - 2\sqrt{\ln 3}$

$= -2\sqrt{\ln 3}$

27. $\int_0^{\ln 3} \frac{e^x dx}{\sqrt{e^x-1}} = \lim_{b\to 0^+}\left[2\sqrt{e^x-1}\right]_b^{\ln 3}$

$= 2\sqrt{3-1} - \lim_{b\to 0^+} 2\sqrt{e^b-1} = 2\sqrt{2} - 0 = 2\sqrt{2}$

28. Note that $\sqrt{4x-x^2} = \sqrt{4-(x^2-4x+4)} = \sqrt{2^2-(x-2)^2}$. (by completing the square)

$\int_2^4 \frac{dx}{\sqrt{4x-x^2}} = \lim_{b\to 4^-}\int_2^b \frac{dx}{\sqrt{4x-x^2}} = \lim_{b\to 4^-}\left[\sin^{-1}\frac{x-2}{2}\right]_2^b = \lim_{b\to 4^-} \sin^{-1}\frac{b-2}{2} - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

29. $\int_1^e \frac{dx}{x\ln x} = \lim_{b\to 1^+}[\ln(\ln x)]_b^e = \ln(\ln e) - \lim_{b\to 1^+} \ln(\ln b) = \ln 1 - \ln 0 = 0 + \infty$

The integral diverges.

30. $\int_1^{10} \frac{dx}{x\ln^{100} x} = \lim_{b\to 1^+}\left[-\frac{1}{99\ln^{99} x}\right]_b^{10} = -\frac{1}{99\ln^{99} 10} + \lim_{b\to 1^+}\frac{1}{99\ln^{99} b} = -\frac{1}{99\ln^{99} 10} + \infty$

The integral diverges.

31. $\int_{2c}^{4c} \frac{dx}{\sqrt{x^2-4c^2}} = \lim_{b\to 2c^+}\left[\ln\left|x+\sqrt{x^2-4c^2}\right|\right]_b^{4c} = \ln\left[(4+2\sqrt{3})c\right] - \lim_{b\to 2c^+}\ln\left|b+\sqrt{b^2-4c^2}\right|$

$= \ln\left[(4+2\sqrt{3})c\right] - \ln 2c = \ln(2+\sqrt{3})$

32. $\int_c^{2c} \frac{x\,dx}{\sqrt{x^2+xc-2c^2}} = \int_c^{2c} \frac{x\,dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}} = \int_c^{2c} \frac{\left(x+\frac{c}{2}\right)dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}} - \frac{c}{2}\int_0^{2c} \frac{dx}{\sqrt{\left(x+\frac{c}{2}\right)^2-\frac{9}{4}c^2}}$

$= \lim_{b\to c^+}\left[\sqrt{x^2+xc-2c^2} - \frac{c}{2}\ln\left|x+\frac{c}{2}+\sqrt{x^2+xc-2c^2}\right|\right]_b^{2c}$

$= \sqrt{4c^2} - \frac{c}{2}\ln\left|\frac{5c}{2}+\sqrt{4c^2}\right| - \lim_{b\to c^+}\left[\sqrt{b^2+bc-2c^2} - \frac{c}{2}\ln\left|b+\frac{c}{2}+\sqrt{b^2+bc-2c^2}\right|\right]$

$= 2c - \frac{c}{2}\ln\frac{9c}{2} - \left(0 - \frac{c}{2}\ln\left|\frac{3c}{2}+0\right|\right) = 2c - \frac{c}{2}\ln\frac{9c}{2} + \frac{c}{2}\ln\frac{3c}{2} = 2c - \frac{c}{2}\ln 3$

33. For $0 < c < 1$, $\frac{1}{\sqrt{x}(1+x)}$ is continuous. Let $u = \frac{1}{1+x}$, $du = -\frac{1}{(1+x)^2}dx$.

$dv = \frac{1}{\sqrt{x}}dx, v = 2\sqrt{x}$.

$\int_c^1 \frac{1}{\sqrt{x}(1+x)}dx = \left[\frac{2\sqrt{x}}{1+x}\right]_c^1 + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2} = \frac{2}{2} - \frac{2\sqrt{c}}{1+c} + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2} = 1 - \frac{2\sqrt{c}}{1+c} + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2}$

Thus, $\lim_{c\to 0}\int_c^1 \frac{1}{\sqrt{x}(1+x)}dx = \lim_{c\to 0}\left[1 - \frac{2\sqrt{c}}{1+c} + 2\int_c^1 \frac{\sqrt{x}dx}{(1+x)^2}\right] = 1 - 0 + 2\int_0^1 \frac{\sqrt{x}dx}{(1+x)^2}$

This last integral is a proper integral.

34. Let $u = \frac{1}{\sqrt{1+x}}, du = -\frac{1}{2(1+x)^{3/2}}dx$

$dv = \frac{1}{\sqrt{x}}dx, v = 2\sqrt{x}$.

For $0 < c < 1$, $\int_c^1 \frac{dx}{\sqrt{x(1+x)}} = \left[\frac{2\sqrt{x}}{\sqrt{1+x}}\right]_c^1 + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx = \frac{2\sqrt{1}}{\sqrt{2}} - \frac{2\sqrt{c}}{\sqrt{1+c}} + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx$

Thus, $\int_0^1 \frac{dx}{\sqrt{x(1+x)}} = \lim_{c\to 0}\int_c^1 \frac{dx}{\sqrt{x(1+x)}} = \lim_{c\to 0}\left[\sqrt{2} - \frac{2\sqrt{c}}{\sqrt{1+c}} + \int_c^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx\right] = \sqrt{2} - 0 + \int_0^1 \frac{\sqrt{x}}{(1+x)^{3/2}}dx$

This is a proper integral.

35. $\int_{-3}^3 \frac{x}{\sqrt{9-x^2}}dx = \int_{-3}^0 \frac{x}{\sqrt{9-x^2}}dx + \int_0^3 \frac{x}{\sqrt{9-x^2}}dx = \lim_{b\to -3^+}\left[-\sqrt{9-x^2}\right]_b^0 + \lim_{b\to 3^-}\left[-\sqrt{9-x^2}\right]_0^b$

$= -\sqrt{9} + \lim_{b\to -3^+}\sqrt{9-b^2} - \lim_{b\to 3^-}\sqrt{9-b^2} + \sqrt{9} = -3 + 0 - 0 + 3 = 0$

36. $\int_{-3}^3 \frac{x}{9-x^2}dx = \int_{-3}^0 \frac{x}{9-x^2}dx + \int_0^3 \frac{x}{9-x^2}dx = \lim_{b\to 3^+}\left[-\frac{1}{2}\ln\left|9-x^2\right|\right]_b^0 + \lim_{b\to 3^-}\left[-\frac{1}{2}\ln\left|9-x^2\right|\right]_0^b$

$= -\ln 3 + \lim_{b\to -3^+}\frac{1}{2}\ln\left|9-b^2\right| - \lim_{b\to 3^-}\frac{1}{2}\ln\left|9-b^2\right| + \ln 3 = (-\ln 3 - \infty) + (\infty + \ln 3)$

The integral diverges.

37. $\int_{-4}^{4}\frac{1}{16-x^2}dx=\int_{-4}^{0}\frac{1}{16-x^2}dx+\int_{0}^{4}\frac{1}{16-x^2}dx=\lim_{b\to-4^+}\left[\frac{1}{8}\ln\left|\frac{x+4}{x-4}\right|\right]_b^0+\lim_{b\to4^-}\left[\frac{1}{8}\ln\left|\frac{x+4}{x-4}\right|\right]_0^b$

$=\frac{1}{8}\ln 1-\lim_{b\to-4^+}\frac{1}{8}\ln\left|\frac{b+4}{b-4}\right|+\lim_{b\to4^-}\frac{1}{8}\ln\left|\frac{b+4}{b-4}\right|-\frac{1}{8}\ln 1=(0+\infty)+(\infty-0)$

The integral diverges.

38. $\int_{-1}^{1}\frac{1}{x\sqrt{-\ln|x|}}dx=\int_{-1}^{-1/2}\frac{1}{x\sqrt{-\ln|x|}}dx+\int_{-1/2}^{0}\frac{1}{x\sqrt{-\ln|x|}}dx+\int_{0}^{1/2}\frac{1}{x\sqrt{-\ln|x|}}dx+\int_{1/2}^{1}\frac{1}{x\sqrt{-\ln|x|}}dx$

$=\lim_{b\to-1^+}\left[-2\sqrt{-\ln|x|}\right]_b^{-1/2}+\lim_{b\to0^-}\left[-2\sqrt{-\ln|x|}\right]_{-1/2}^{b}+\lim_{b\to0^+}\left[-2\sqrt{-\ln|x|}\right]_b^{1/2}+\lim_{b\to1^-}\left[-2\sqrt{-\ln|x|}\right]_{1/2}^{b}$

$=(-2\sqrt{\ln 2}+0)+(-\infty+2\sqrt{\ln 2})+(-2\sqrt{\ln 2}+\infty)+(0+2\sqrt{\ln 2})$

The integral diverges.

39. $\int_0^\infty\frac{1}{x^p}dx=\int_0^1\frac{1}{x^p}dx+\int_1^\infty\frac{1}{x^p}dx$

If $p>1$, $\int_0^1\frac{1}{x^p}dx=\left[\frac{1}{-p+1}x^{-p+1}\right]_0^1$ diverges

since $\lim_{x\to0^+}x^{-p+1}=\infty$.

If $p<1$ and $p\neq 0$, $\int_1^\infty\frac{1}{x^p}dx=\left[\frac{1}{-p+1}x^{-p+1}\right]_1^\infty$

diverges since $\lim_{x\to\infty}x^{-p+1}=\infty$.

If $p=0$, $\int_0^\infty dx=\infty$.

If $p=1$, both $\int_0^1\frac{1}{x}dx$ and $\int_1^\infty\frac{1}{x}dx$ diverge.

40. $\int_0^\infty f(x)dx$

$=\lim_{b\to1^-}\int_0^b f(x)dx+\lim_{b\to1^+}\int_b^c f(x)dx+\lim_{b\to\infty}\int_c^b f(x)dx$

where $1<c<\infty$.

41. $\int_0^8(x-8)^{-2/3}dx=\lim_{b\to8^-}\left[3(x-8)^{1/3}\right]_0^b$

$=3(0)-3(-2)=6$

42. $\int_0^1\left(\frac{1}{x}-\frac{1}{x^3+x}\right)dx$

$=\lim_{b\to0^-}\int_b^1\frac{x}{x^2+1}dx=\lim_{b\to0^-}\left[\frac{1}{2}\ln\left|x^2+1\right|\right]_b^1$

$=\frac{1}{2}\ln 2-\lim_{b\to0^-}\frac{1}{2}\ln\left|b^2+1\right|=\frac{1}{2}\ln 2$

43. a. $\int_0^1 x^{-2/3}dx=\lim_{b\to0^+}\left[3x^{1/3}\right]_b^1=3$

b. $V=\pi\int_0^1 x^{-4/3}dx=\lim_{b\to0^+}\pi\left[-3x^{-1/3}\right]_b^1$

$=-3\pi+3\pi\lim_{b\to0}b^{-1/3}$

The limit tends to infinity as $b\to0$, so the volume is infinite.

44. Since $\ln x<0$ for $0<x<1$, $b>1$

$\int_0^b\ln x\,dx=\lim_{c\to0^-}\int_c^1\ln x\,dx+\int_1^b\ln x\,dx$

$=\lim_{c\to0^+}\left[x\ln x-x\right]_c^1+\left[x\ln x-x\right]_1^b$

$=-1-\lim_{c\to0^+}(c\ln c-c)+b\ln b-b+1$

$=b\ln b-b$

Thus, $b\ln b-b=0$ when $b=e$.

45. $\int_0^1\frac{\sin x}{x}dx$ is not an improper integral since

$\frac{\sin x}{x}$ is bounded in the interval $0\le x\le 1$.

46. For $x\ge1$, $\frac{1}{1+x^4}<1$ so $\frac{1}{x^4(1+x^4)}<\frac{1}{x^4}$.

$\int_1^\infty\frac{1}{x^4}dx=\lim_{b\to\infty}\left[-\frac{1}{3x^3}\right]_1^b=-\lim_{b\to\infty}\frac{1}{3b^3}+\frac{1}{3}$

$=-0+\frac{1}{3}=\frac{1}{3}$

Thus, by the Comparison Test $\int_1^\infty\frac{1}{x^4(1+x^4)}dx$

converges.

47. For $x \ge 1$, $x^2 \ge x$ so $-x^2 \le -x$, thus

$e^{-x^2} \le e^{-x}$.

$$\int_1^\infty e^{-x}dx = \lim_{b\to\infty}[-e^{-x}]_1^b = -\lim_{b\to\infty}\frac{1}{e^b} + e^{-1}$$

$$= -0 + \frac{1}{e} = \frac{1}{e}$$

Thus, by the Comparison Test, $\int_1^\infty e^{-x^2}dx$ converges.

48. Since $\sqrt{x+2} - 1 \le \sqrt{x+2}$ we know that

$\frac{1}{\sqrt{x+2}-1} \ge \frac{1}{\sqrt{x+2}}$. Consider $\int_0^\infty \frac{1}{\sqrt{x+2}}dx$

$$\int_2^\infty \frac{1}{\sqrt{x+2}}dx = \lim_{b\to\infty}\int_2^b \frac{1}{\sqrt{x+2}}dx$$

$$= \lim_{b\to\infty}\left[2\sqrt{x+2}\right]_2^\infty = \lim_{b\to\infty}2\left(\sqrt{b+2}-2\right) = \infty$$

Thus, by the Comparison Test of Problem 46, we conclude that $\int_0^\infty \frac{1}{\sqrt{x+2}}dx$ diverges.

49. Since $x^2\ln(x+1) \ge x^2$, we know that

$\frac{1}{x^2\ln(x+1)} \le \frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^\infty = 1$

we can apply the Comparison Test of Problem 46 to conclude that $\int_1^\infty \frac{1}{x^2\ln(x+1)}dx$ converges.

50. If $0 \le f(x) \le g(x)$ on $[a, b]$ and either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ or $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = \infty$, then the convergence of $\int_a^b g(x)dx$ implies the convergence of $\int_a^b f(x)dx$ and the divergence of $\int_a^b f(x)dx$ implies the divergence of $\int_a^b g(x)dx$.

51. a. From Example 2 of Section 8.2, $\lim_{x\to\infty}\frac{x^a}{e^x} = 0$ for a any positive real number.

Thus $\lim_{x\to\infty}\frac{x^{n+1}}{e^x} = 0$ for any positive real number n, hence there is a number M such that $0 < \frac{x^{n+1}}{e^x} \le 1$ for $x \ge M$. Divide the inequality by x^2 to get that $0 < \frac{x^{n-1}}{e^x} \le \frac{1}{x^2}$ for $x \ge M$.

b. $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = -\lim_{b\to\infty}\frac{1}{b} + \frac{1}{1}$

$= -0 + 1 = 1$

$\int_1^\infty x^{n-1}e^{-x}dx = \int_1^M x^{n-1}e^{-x}dx + \int_M^\infty x^{n-1}e^{-x}dx$

$\le \int_1^M x^{n-1}e^{-x}dx + \int_1^\infty \frac{1}{x^2}dx$

$= 1 + \int_1^M x^{n-1}e^{-x}dx$

by part a and Problem 46. The remaining integral is finite, so $\int_1^\infty x^{n-1}e^{-x}dx$ converges.

52. $\int_0^1 e^{-x}dx = \left[-e^{-x}\right]_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e}$, so the integral converges when $n = 1$. For $0 \le x \le 1$, $0 \le x^{n-1} \le 1$ for $n > 1$. Thus,

$\frac{x^{n-1}}{e^x} = x^{n-1}e^{-x} \le e^{-x}$. By the comparison test from Problem 50, $\int_0^1 x^{n-1}e^{-x}dx$ converges.

53. a. $\Gamma(1) = \int_0^\infty x^0 e^{-x}dx = \left[-e^{-x}\right]_0^\infty = 1$

b. $\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx$

Let $u = x^n, dv = e^{-x}dx$,

$du = nx^{n-1}dx, v = -e^{-x}$.

$\Gamma(n+1) = [-x^n e^{-x}]_0^\infty + \int_0^\infty nx^{n-1}e^{-x}dx$

$= 0 + n\int_0^\infty x^{n-1}e^{-x}dx = n\Gamma(n)$

c. From parts a and b,

$\Gamma(1) = 1, \Gamma(2) = 1\cdot\Gamma(1) = 1,$

$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!$.

Suppose $\Gamma(n) = (n-1)!$, then by part b,

$\Gamma(n+1) = n\Gamma(n) = n[(n-1)!] = n!$.

54. $n=1,\ \int_0^\infty e^{-x}dx = 1 = 0! = (1-1)!$

$n=2,\ \int_0^\infty xe^{-x}dx = 1 = 1! = (2-1)!$

$n=3,\ \int_0^\infty x^2e^{-x}dx = 2 = 2! = (3-1)!$

$n=4,\ \int_0^\infty x^3e^{-x}dx = 6 = 3! = (4-1)!$

$n=5,\ \int_0^\infty x^4e^{-x}dx = 24 = 4! = (5-1)!$

55. **a.** $\int_{-\infty}^\infty f(x)dx = \int_0^\infty Cx^{\alpha-1}e^{-\beta x}dx$

Let $y=\beta x$, so $x=\frac{y}{\beta}$ and $dx=\frac{1}{\beta}dy$.

$$\int_0^\infty Cx^{\alpha-1}e^{-\beta x}dx = \int_0^\infty C\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y}\frac{1}{\beta}dy = \frac{C}{\beta^\alpha}\int_0^\infty y^{\alpha-1}e^{-y}dy = C\beta^{-\alpha}\Gamma(\alpha)$$

$C\beta^{-\alpha}\Gamma(\alpha)=1$ when $C=\frac{1}{\beta^{-\alpha}\Gamma(\alpha)}=\frac{\beta^\alpha}{\Gamma(\alpha)}$.

b. $\mu = \int_{-\infty}^\infty xf(x)dx = \int_0^\infty x\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\beta x}dx$

Let $y=\beta x$, so $x=\frac{y}{\beta}$ and $dx=\frac{1}{\beta}dy$.

$$\mu = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty\left(\frac{y}{\beta}\right)^\alpha e^{-y}\frac{1}{\beta}dy = \frac{1}{\beta\Gamma(\alpha)}\int_0^\infty y^\alpha e^{-y}dy = \frac{1}{\beta\Gamma(\alpha)}\Gamma(\alpha+1) = \frac{1}{\beta\Gamma(\alpha)}\alpha\Gamma(\alpha) = \frac{\alpha}{\beta}$$

(Recall that $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ for $\alpha>0$.)

c. $\sigma^2 = \int_{-\infty}^\infty (x-\mu)^2 f(x)dx = \int_0^\infty\left(x-\frac{\alpha}{\beta}\right)^2\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty\left(x^2-\frac{2\alpha}{\beta}x+\frac{\alpha^2}{\beta^2}\right)x^{\alpha-1}e^{-\beta x}dx$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha+1}e^{-\beta x}dx - \frac{2\alpha\beta^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\beta x}dx + \frac{\alpha^2\beta^{\alpha-2}}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-\beta x}dx$$

In all three integrals, let $y=\beta x$, so $x=\frac{y}{\beta}$ and $dx=\frac{1}{\beta}dy$.

$$\sigma^2 = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty\left(\frac{y}{\beta}\right)^{\alpha+1}e^{-y}\frac{1}{\beta}dy - \frac{2\alpha\beta^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty\left(\frac{y}{\beta}\right)^\alpha e^{-y}\frac{1}{\beta}dy + \frac{\alpha^2\beta^{\alpha-2}}{\Gamma(\alpha)}\int_0^\infty\left(\frac{y}{\beta}\right)^{\alpha-1}e^{-y}\frac{1}{\beta}dy$$

$$= \frac{1}{\beta^2\Gamma(\alpha)}\int_0^\infty y^{\alpha+1}e^{-y}dy - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\int_0^\infty y^\alpha e^{-y}dy + \frac{\alpha^2}{\beta^2\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y}dy$$

$$= \frac{1}{\beta^2\Gamma(\alpha)}\Gamma(\alpha+2) - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\Gamma(\alpha+1) + \frac{\alpha^2}{\beta^2\Gamma(\alpha)}\Gamma(\alpha) = \frac{1}{\beta^2\Gamma(\alpha)}(\alpha+1)\alpha\Gamma(\alpha) - \frac{2\alpha}{\beta^2\Gamma(\alpha)}\alpha\Gamma(\alpha) + \frac{\alpha^2}{\beta^2}$$

$$= \frac{\alpha^2+\alpha}{\beta^2} - \frac{2\alpha^2}{\beta^2} + \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

56. a. $L\{t^{\alpha}\}(s)=\int_0^{\infty} t^{\alpha}e^{-st}dt$

Let $t=\frac{x}{s}$, so $dt=\frac{1}{s}dx$, then

$$\int_0^{\infty} t^{\alpha}e^{-st}dt=\int_0^{\infty}\left(\frac{x}{s}\right)^{\alpha} e^{-x}\frac{1}{s}dx=\int_0^{\infty}\frac{1}{s^{\alpha+1}}x^{\alpha}e^{-x}dx=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

If $s\le 0$, $t^{\alpha}e^{-st}\to\infty$ as $t\to\infty$, so the integral does not converge. Thus, the transform is defined only when $s>0$.

b. $L\{e^{\alpha t}\}(s)=\int_0^{\infty} e^{\alpha t}e^{-st}dt=\int_0^{\infty} e^{(\alpha-s)t}dt=\left[\frac{1}{\alpha-s}e^{(\alpha-s)t}\right]_0^{\infty}=\frac{1}{\alpha-s}\left[\lim_{b\to\infty} e^{(\alpha-s)b}-1\right]$

$$\lim_{b\to\infty} e^{(\alpha-s)b}=\begin{cases}\infty & \text{if } \alpha>s\\ 0 & \text{if } s>\alpha\end{cases}$$

Thus, $L\{e^{\alpha t}\}(s)=\frac{-1}{\alpha-s}=\frac{1}{s-\alpha}$ when $s>\alpha$. (When $s\le\alpha$, the integral does not converge.)

c. $L\{\sin(\alpha t)\}(s)=\int_0^{\infty}\sin(\alpha t)e^{-st}dt$

Let $I=\int_0^{\infty}\sin(\alpha t)e^{-st}dt$ and use integration by parts with $u=\sin(\alpha t)$, $du=\alpha\cos(\alpha t)dt$,

$dv=e^{-st}dt$, and $v=-\frac{1}{s}e^{-st}$.

Then $I=\left[-\frac{1}{s}\sin(\alpha t)e^{-st}\right]_0^{\infty}+\frac{\alpha}{s}\int_0^{\infty}\cos(\alpha t)e^{-st}dt$

Use integration by parts on this integral with

$u=\cos(\alpha t)$, $du=-\alpha\sin(\alpha t)dt$, $dv=e^{-st}dt$, and $v=-\frac{1}{s}e^{-st}$.

$$I=\left[-\frac{1}{s}\sin(\alpha t)e^{-st}\right]_0^{\infty}+\frac{\alpha}{s}\left(\left[-\frac{1}{s}\cos(\alpha t)e^{-st}\right]_0^{\infty}-\frac{\alpha}{s}\int_0^{\infty}\sin(\alpha t)e^{-st}dt\right)$$

$$=-\frac{1}{s}\left[e^{-st}\left(\sin(\alpha t)+\frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^{\infty}-\frac{\alpha^2}{s^2}I$$

Thus,

$$I\left(1+\frac{\alpha^2}{s^2}\right)=-\frac{1}{s}\left[e^{-st}\left(\sin(\alpha t)+\frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^{\infty}$$

$$I=-\frac{1}{s\left(1+\frac{\alpha^2}{s^2}\right)}\left[e^{-st}\left(\sin(\alpha t)+\frac{\alpha}{s}\cos(\alpha t)\right)\right]_0^{\infty}=-\frac{s}{s^2+\alpha^2}\left[\lim_{b\to\infty} e^{-sb}\left(\sin(\alpha b)+\frac{\alpha}{s}\cos(\alpha b)\right)-\frac{\alpha}{s}\right]$$

$$\lim_{b\to\infty} e^{-sb}\left(\sin(\alpha b)+\frac{\alpha}{s}\cos(\alpha b)\right)=\begin{cases}0 & \text{if } s>0\\ \infty & \text{if } s\le 0\end{cases}$$

Thus, $I=\frac{\alpha}{s^2+\alpha^2}$ when $s>0$.

57. a. The integral is the area between the curve $y^2 = \frac{1-x}{x}$ and the *x*-axis from $x = 0$ to $x = 1$.

$y^2 = \frac{1-x}{x}; xy^2 = 1-x; x(y^2+1) = 1$

$x = \frac{1}{y^2+1}$

As $x \to 0, y = \sqrt{\frac{1-x}{x}} \to \infty,$ while when $x = 1,\ y = \sqrt{\frac{1-1}{1}} = 0,$ thus the area is

$\int_0^\infty \frac{1}{y^2+1} dy = \lim_{b\to\infty} [\tan^{-1} y]_0^b$

$= \lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2}$

b. The integral is the area between the curve $y^2 = \frac{1+x}{1-x}$ and the *x*-axis from $x = -1$ to $x = 1$.

$y^2 = \frac{1+x}{1-x}; y^2 - xy^2 = 1+x; y^2 - 1 = x(y^2+1);$

$x = \frac{y^2-1}{y^2+1}$

When $x = -1,\ y = \sqrt{\frac{1+(-1)}{1-(-1)}} = \sqrt{\frac{0}{2}} = 0,$ while as $x \to 1, y = \sqrt{\frac{1+x}{1-x}} \to \infty.$

The area in question is the area to the right of the curve $y = \sqrt{\frac{1+x}{1-x}}$ and to the left of the line $x = 1$. Thus, the area is

$\int_0^\infty \left(1 - \frac{y^2-1}{y^2+1}\right) dy = \int_0^\infty \frac{2}{y^2+1} dy$

$= \lim_{b\to\infty} \left[2\tan^{-1} y\right]_0^b$

$\lim_{b\to\infty} 2\tan^{-1} b - 2\tan^{-1} 0 = 2\left(\frac{\pi}{2}\right) = \pi$

58. For $0 < x < 1,\ x^p > x^q$ so $2x^p > x^p + x^q$ and $\frac{1}{x^p + x^q} > \frac{1}{2x^p}$. For $1 < x,\ x^q > x^p$ so $2x^q > x^p + x^q$ and $\frac{1}{x^p + x^q} > \frac{1}{2x^q}$.

$\int_0^\infty \frac{1}{x^p + x^q} dx = \int_0^1 \frac{1}{x^p + x^q} dx + \int_1^\infty \frac{1}{x^p + x^q} dx$

Both of these integrals must converge.

$\int_0^1 \frac{1}{x^p + x^q} dx > \int_0^1 \frac{1}{2x^p} dx = \frac{1}{2}\int_0^1 \frac{1}{x^p} dx$ which converges if and only if $p < 1$.

$\int_1^\infty \frac{1}{x^p + x^q} dx > \int_1^\infty \frac{1}{2x^q} dx = \frac{1}{2}\int_1^\infty \frac{1}{x^q} dx$ which converges if and only if $q > 1$. Thus, $0 < p < 1$ and $1 < q$.

8.5 Chapter Review

Concepts Test

1. True: See Example 2 of Section 8.2.

2. True: Use l'Hôpital's Rule.

3. False: $\lim_{x\to\infty} \frac{1000x^4 + 1000}{0.001x^4 + 1} = \frac{1000}{0.001} = 10^6$

4. False: $\lim_{x\to\infty} xe^{-1/x} = \infty$ since $e^{-1/x} \to 1$ and $x \to \infty$ as $x \to \infty$.

5. False: For example, if $f(x) = x$ and $g(x) = e^x$, $\lim_{x\to\infty} \frac{x}{e^x} = 0.$

6. False: See Example 7 of Section 8.2.

7. True: Take the inner limit first.

8. True: Raising a small number to a large exponent results in an even smaller number.

9. True: Since $\lim_{x\to a} f(x) = -1 \neq 0,$ it serves only to affect the sign of the limit of the product.

10. False: Consider $f(x)=(x-a)^2$ and $g(x)=\frac{1}{(x-a)^2}$, then $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$, while $\lim_{x\to a}[f(x)g(x)]=1$.

11. False: Consider $f(x)=3x^2$ and $g(x)=x^2+1$, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{3x^2}{x^2+1}$$

$$=\lim_{x\to\infty}\frac{3}{1+\frac{1}{x^2}}=3, \text{ but}$$

$$\lim_{x\to\infty}[f(x)-3g(x)]$$

$$=\lim_{x\to\infty}[3x^2-3(x^2+1)]$$

$$=\lim_{x\to\infty}[-3]=-3$$

12. True: As $x\to a$, $f(x)\to 2$ while $\frac{1}{|g(x)|}\to\infty.$

13. True: See Example 7 of Section 8.2.

14. True: Let $y=[1+f(x)]^{1/f(x)}$, then

$$\ln y=\frac{1}{f(x)}\ln[1+f(x)].$$

$$\lim_{x\to a}\frac{1}{f(x)}\ln[1+f(x)]=\lim_{x\to a}\frac{\ln[1+f(x)]}{f(x)}$$

This limit is of the form $\frac{0}{0}$.

$$\lim_{x\to a}\frac{\ln[1+f(x)]}{f(x)}=\lim_{x\to a}\frac{\frac{1}{1+f(x)}f'(x)}{f'(x)}$$

$$=\lim_{x\to a}\frac{1}{1+f(x)}=1$$

$$\lim_{x\to a}[1+f(x)]^{1/f(x)}=\lim_{x\to a}e^{\ln y}=e^1=e$$

15. True: Use repeated applications of l'Hôpital's Rule.

16. True: $e^0=1$ and $p(0)$ is the constant term.

17. False: Consider $f(x)=3x^2+x+1$ and $g(x)=4x^3+2x+3$; $f'(x)=6x+1$ $g'(x)=12x^2+2$, and so

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{6x+1}{12x^2+2}=\frac{1}{2} \text{ while}$$

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{3x^2+x+1}{4x^3+2x+3}=\frac{1}{3}$$

18. False: $p>1$. See Example 4 of Section 8.4.

19. True: $\int_0^\infty \frac{1}{x^p}dx=\int_0^1\frac{1}{x^p}dx+\int_1^\infty\frac{1}{x^p}dx$;

$\int_0^1\frac{1}{x^p}dx$ diverges for $p\ge 1$ and

$\int_1^\infty\frac{1}{x^p}dx$ diverges for $p\le 1$.

20. False: Consider $\int_0^\infty\frac{1}{x+1}dx$.

21. True: $\int_{-\infty}^\infty f(x)dx=\int_{-\infty}^0 f(x)dx+\int_0^\infty f(x)dx$

If f is an even function, then $f(-x)=f(x)$ so

$\int_{-\infty}^0 f(x)dx=\int_0^\infty f(x)dx.$

Thus, both integrals making up $\int_{-\infty}^\infty f(x)dx$ converge so their sum converges.

22. False: See Problem 37 of Section 8.3.

23. True: $\int_0^\infty f'(x)dx=\lim_{b\to\infty}\int_0^b f'(x)dx$

$=\lim_{b\to\infty}[f(x)]_0^b=\lim_{b\to\infty}f(b)-f(0)$

$=0-f(0)=-f(0)$.

$f(0)$ must exist and be finite since $f'(x)$ is continuous on $[0,\infty)$.

24. True: $\int_0^\infty f(x)dx\le\int_0^\infty e^{-x}dx=\lim_{b\to\infty}[-e^{-x}]_0^b$

$=\lim_{b\to\infty}-e^{-b}+1=1$, so $\int_0^\infty f(x)dx$ must converge.

25. False: The integrand is bounded on the interval $\left[0,\frac{\pi}{4}\right]$.

Sample Test Problems

1. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{4x}{\tan x}=\lim_{x\to 0}\frac{4}{\sec^2 x}=4$$

2. The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\tan 2x}{\sin 3x}=\lim_{x\to 0}\frac{2\sec^2 2x}{3\cos 3x}=\frac{2}{3}$$

3. The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0}\frac{\sin x-\tan x}{\frac{1}{3}x^2}=\lim_{x\to 0}\frac{\cos x-\sec^2 x}{\frac{2}{3}x}$$

$$=\lim_{x\to 0}\frac{-\sin x-2\sec x(\sec x\tan x)}{\frac{2}{3}}=0$$

4. $\lim_{x\to 0}\frac{\cos x}{x^2}=\infty$ (L'Hôpital's Rule does not apply since cos(0) = 1.)

5. $\lim_{x\to 0}2x\cot x=\lim_{x\to 0}\frac{2x\cos x}{\sin x}$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{2x\cos x}{\sin x}=\lim_{x\to 0}\frac{2\cos x-2x\sin x}{\cos x}$$

$$=\frac{2-0}{1}=2$$

6. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 1^-}\frac{\ln(1-x)}{\cot \pi x}=\lim_{x\to 1^-}\frac{-\frac{1}{1-x}}{-\pi\csc^2 \pi x}$$

$$=\lim_{x\to 1^-}\frac{\sin^2 \pi x}{\pi(1-x)}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 1^-}\frac{\sin^2 \pi x}{\pi(1-x)}=\lim_{x\to 1^-}\frac{2\pi\sin \pi x\cos \pi x}{-\pi}=0$$

7. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{t\to\infty}\frac{\ln t}{t^2}=\lim_{t\to\infty}\frac{\frac{1}{t}}{2t}=\lim_{t\to\infty}\frac{1}{2t^2}=0$$

8. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{2x^3}{\ln x}=\lim_{x\to\infty}\frac{6x^2}{\frac{1}{x}}=\lim_{x\to\infty}6x^3=\infty$$

9. As $x\to 0, \sin x\to 0$, and $\frac{1}{x}\to\infty$. A number less than 1, raised to a large power, is a very small number $\left(\left(\frac{1}{2}\right)^{32}=2.328\times 10^{-10}\right)$ so

$$\lim_{x\to 0^+}(\sin x)^{1/x}=0.$$

10. $\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=0$$

11. The limit is of the form 0^0.

Let $y=x^x$, then $\ln y=x\ln x$.

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=0$$

$$\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{\ln y}=1$$

12. The limit is of the form 1^∞.

Let $y=(1+\sin x)^{2/x}$, then $\ln y=\frac{2}{x}\ln(1+\sin x)$.

$$\lim_{x\to 0}\frac{2}{x}\ln(1+\sin x)=\lim_{x\to 0}\frac{2\ln(1+\sin x)}{x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{2\ln(1+\sin x)}{x}=\lim_{x\to 0}\frac{\frac{2}{1+\sin x}\cos x}{1}$$

$$=\lim_{x\to 0}\frac{2\cos x}{1+\sin x}=\frac{2}{1}=2$$

$$\lim_{x\to 0}(1+\sin x)^{2/x}=\lim_{x\to 0}e^{\ln y}=e^2$$

13. $\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}}$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$

14. The limit is of the form ∞^0.

Let $y = t^{1/t}$, then $\ln y = \frac{1}{t}\ln t$.

$$\lim_{t\to\infty}\frac{1}{t}\ln t = \lim_{t\to\infty}\frac{\ln t}{t}$$

The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{t\to\infty}\frac{\ln t}{t} = \lim_{t\to\infty}\frac{\frac{1}{t}}{1} = \lim_{t\to\infty}\frac{1}{t} = 0$$

$$\lim_{t\to\infty} t^{1/t} = \lim_{t\to\infty} e^{\ln y} = 1$$

15. $\lim_{x\to 0^+}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x-\sin x}{x\sin x}$

The limit is of the form $\frac{0}{0}$. (Apply l'Hôpital's Rule twice.)

$$\lim_{x\to 0^+}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0^+}\frac{1-\cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

16. The limit is of the form $\frac{\infty}{\infty}$. (Apply l'Hôpital's Rule three times.)

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 3x}{\tan x} = \lim_{x\to\frac{\pi}{2}}\frac{3\sec^2 3x}{\sec^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{3\cos^2 x}{\cos^2 3x} = \lim_{x\to\frac{\pi}{2}}\frac{\cos x\sin x}{\cos 3x\sin 3x}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x - \sin^2 x}{3(\cos^2 3x - \sin^2 3x)} = -\frac{1}{3(0-1)} = \frac{1}{3}$$

17. The limit is of the form 1^∞.

Let $y = (\sin x)^{\tan x}$, then $\ln y = \tan x \ln(\sin x)$.

$$\lim_{x\to\frac{\pi}{2}}\tan x\ln(\sin x) = \lim_{x\to\frac{\pi}{2}}\frac{\sin x\ln(\sin x)}{\cos x}$$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\frac{\pi}{2}}\frac{\sin x\ln(\sin x)}{\cos x} = \lim_{x\to\frac{\pi}{2}}\frac{\cos x\ln(\sin x) + \frac{\sin x}{\sin x}\cos x}{\sin x}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\cos x(1+\ln(\sin x))}{\sin x} = \frac{0}{1} = 0$$

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x} = \lim_{x\to\frac{\pi}{2}} e^{\ln y} = 1$$

18. $\lim_{x\to\frac{\pi}{2}}\left(x\tan x - \frac{\pi}{2}\sec x\right) = \lim_{x\to\frac{\pi}{2}}\frac{x\sin x - \frac{\pi}{2}}{\cos x}$

The limit is of the form $\frac{0}{0}$.

$$\lim_{x\to\frac{\pi}{2}}\frac{x\sin x - \frac{\pi}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}}\frac{\sin x + x\cos x}{\sin x} = \frac{1}{1} = 1$$

19. $\int_0^\infty \frac{dx}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_0^\infty = 0 + 1 = 1$

20. $\int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

21. $\int_{-\infty}^1 e^{2x}dx = \left[\frac{1}{2}e^{2x}\right]_{-\infty}^1 = \frac{1}{2}e^2 - 0 = \frac{1}{2}e^2$

22. $\int_{-1}^1 \frac{dx}{1-x} = \lim_{b\to 1}[-\ln(1-x)]_{-1}^b$

$= -\lim_{b\to 1}\ln(1-b) + \ln 2 = \infty$

The integral diverges.

23. $\int_0^\infty \frac{dx}{x+1} = [\ln(x+1)]_0^\infty = \infty - 0 = \infty$

The integral diverges.

24. $\int_{\frac{1}{2}}^{2}\frac{dx}{x(\ln x)^{1/5}}=\lim_{b\to 1^-}\int_{\frac{1}{2}}^{b}\frac{dx}{x(\ln x)^{1/5}}+\lim_{b\to 1^+}\int_{b}^{2}\frac{dx}{x(\ln x)^{1/5}}=\lim_{b\to 1^-}\left[\frac{5}{4}(\ln x)^{4/5}\right]_{\frac{1}{2}}^{b}+\lim_{b\to 1^+}\left[\frac{5}{4}(\ln x)^{4/5}\right]_{b}^{2}$

$=\left(\frac{5}{4}(0)-\frac{5}{4}\left(\ln\frac{1}{2}\right)^{4/5}\right)+\left(\frac{5}{4}(\ln 2)^{4/5}-\frac{5}{4}(0)\right)=\frac{5}{4}(\ln 2)^{4/5}-\frac{5}{4}\left(\ln\frac{1}{2}\right)^{4/5}=\frac{5}{4}[(\ln 2)^{4/5}-(-\ln 2)^{4/5}]$

$=\frac{5}{4}[(\ln 2)^{4/5}-(\ln 2)^{4/5}]=0$

25. $\int_1^{\infty}\frac{dx}{x^2+x^4}=\int_1^{\infty}\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)dx=\left[-\frac{1}{x}-\tan^{-1}x\right]_1^{\infty}=0-\frac{\pi}{2}+1+\tan^{-1}1=1+\frac{\pi}{4}-\frac{\pi}{2}=1-\frac{\pi}{4}$

26. $\int_{-\infty}^{1}\frac{dx}{(2-x)^2}=\left[\frac{1}{2-x}\right]_{-\infty}^{1}=\frac{1}{1}-0=1$

27. $\int_{-2}^{0}\frac{dx}{2x+3}=\lim_{b\to -\frac{3}{2}^-}\int_{-2}^{b}\frac{dx}{2x+3}+\lim_{b\to -\frac{3}{2}^+}\int_{b}^{0}\frac{dx}{2x+3}=\lim_{b\to -\frac{3}{2}^-}\left[\frac{1}{2}\ln|2x+3|\right]_{-2}^{b}+\lim_{b\to -\frac{3}{2}^+}\left[\frac{1}{2}\ln|2x+3|\right]_{b}^{0}$

$=\left(\lim_{b\to -\frac{3}{2}^-}\frac{1}{2}\ln|2b+3|-\frac{1}{2}(0)\right)+\left(\frac{1}{2}\ln 3-\lim_{b\to -\frac{3}{2}^+}\frac{1}{2}\ln|2b+3|\right)=(-\infty)+\left(\frac{1}{2}\ln 3+\infty\right)$

The integral diverges.

28. $\int_1^4\frac{dx}{\sqrt{x-1}}=\lim_{b\to 1^+}[2\sqrt{x-1}]_b^4=2\sqrt{3}-\lim_{b\to 1^+}2\sqrt{x-1}=2\sqrt{3}-0=2\sqrt{3}$

29. $\int_2^{\infty}\frac{dx}{x(\ln x)^2}=\left[-\frac{1}{\ln x}\right]_2^{\infty}=-0+\frac{1}{\ln 2}=\frac{1}{\ln 2}$

30. $\int_0^{\infty}\frac{dx}{e^{x/2}}=\left[-\frac{2}{e^{x/2}}\right]_0^{\infty}=-0+\frac{2}{1}=2$

31. $\int_3^5\frac{dx}{(4-x)^{2/3}}=\lim_{b\to 4^-}\int_3^b\frac{dx}{(4-x)^{2/3}}+\lim_{b\to 4^+}\int_b^5\frac{dx}{(4-x)^{2/3}}=\lim_{b\to 4^-}\left[-3(4-x)^{1/3}\right]_3^b+\lim_{b\to 4^+}\left[-3(4-x)^{1/3}\right]_b^5$

$=\lim_{b\to 4^-}-3(4-b)^{1/3}+3(1)^{1/3}-3(-1)^{1/3}+\lim_{b\to 4^+}3(4-b)^{1/3}=0+3+3+0=6$

32. $\int_2^{\infty}xe^{-x^2}dx=\left[-\frac{1}{2}e^{-x^2}\right]_2^{\infty}=0+\frac{1}{2}e^{-4}=\frac{1}{2}e^{-4}$

33. $\int_{-\infty}^{\infty}\frac{x}{x^2+1}dx=\int_{-\infty}^{0}\frac{x}{x^2+1}dx+\int_0^{\infty}\frac{x}{x^2+1}dx$

$=\frac{1}{2}\left[\ln(x^2+1)\right]_{-\infty}^{0}+\frac{1}{2}\left[\ln(x^2+1)\right]_0^{\infty}=$ The integral diverges.

$(0+\infty)+(\infty-0)$

34. $\int_{-\infty}^{\infty}\frac{x}{1+x^4}dx=\int_{-\infty}^{0}\frac{x}{1+x^4}dx+\int_0^{\infty}\frac{x}{1+x^4}dx=\left[\frac{1}{2}\tan^{-1}x^2\right]_{-\infty}^{0}+\left[\frac{1}{2}\tan^{-1}x^2\right]_0^{\infty}$

$=\frac{1}{2}\tan^{-1}0-\frac{1}{2}\left(\frac{\pi}{2}\right)+\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}\tan^{-1}0=0-\frac{\pi}{4}+\frac{\pi}{4}-0=0$

35. $\frac{e^x}{e^{2x}+1} = \frac{e^x}{(e^x)^2+1}$

Let $u = e^x, du = e^x dx$

$$\int_0^\infty \frac{e^x}{e^{2x}+1}dx = \int_1^\infty \frac{1}{u^2+1}du = \left[\tan^{-1} u\right]_1^\infty = \frac{\pi}{2} - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

36. Let $u = x^3, du = 3x^2 dx$

$$\int_{-\infty}^\infty x^2 e^{-x^3} dx = \int_{-\infty}^\infty \frac{1}{3}e^{-u}du = \frac{1}{3}\int_{-\infty}^0 e^{-u}du + \frac{1}{3}\int_0^\infty e^{-u}du = \frac{1}{3}\left[-e^{-u}\right]_{-\infty}^0 + \frac{1}{3}\left[-e^{-u}\right]_0^\infty = \frac{1}{3}(-1+\infty) + \frac{1}{3}(-0+1)$$

The integral diverges.

37. $\int_{-3}^3 \frac{x}{\sqrt{9-x^2}}dx = 0$

See Problem 35 in Section 8.4.

38. let $u = \ln(\cos x)$, then $du = \frac{1}{\cos x}\cdot -\sin x\, dx = -\tan x\, dx$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\tan x}{(\ln \cos x)^2}dx = \int_{\ln\frac{1}{2}}^{-\infty} -\frac{1}{u^2}du = \int_{-\infty}^{\ln\frac{1}{2}} \frac{1}{u^2}du = \left[-\frac{1}{u}\right]_{-\infty}^{\ln\frac{1}{2}} = -\frac{1}{\ln\frac{1}{2}} + 0 = \frac{1}{\ln 2}$$

39. For $p \neq 1, p \neq 0$, $\int_1^\infty \frac{1}{x^p}dx = \left[-\frac{1}{(p-1)x^{p-1}}\right]_1^\infty = \lim_{b\to\infty} \frac{1}{(1-p)b^{p-1}} + \frac{1}{p-1}$

$\lim_{b\to\infty} \frac{1}{b^{p-1}} = 0$ when $p - 1 > 0$ or $p > 1$, and $\lim_{b\to\infty} \frac{1}{b^{p-1}} = \infty$ when $p < 1, p \neq 0$.

When $p = 1$, $\int_1^\infty \frac{1}{x}dx = [\ln x]_1^\infty = \infty - 0$. The integral diverges.

When $p = 0$, $\int_1^\infty 1dx = [x]_1^\infty = \infty - 1$. The integral diverges.

$\int_1^\infty \frac{1}{x^p}dx$ converges when $p > 1$ and diverges when $p \leq 1$.

40. For $p \neq 1, p \neq 0$, $\int_0^1 \frac{1}{x^p}dx = \left[-\frac{1}{(p-1)x^{p-1}}\right]_0^1 = \frac{1}{1-p} + \lim_{b\to 0} \frac{1}{(p-1)b^{p-1}}$

$\lim_{b\to 0} \frac{1}{b^{p-1}}$ converges when $p - 1 < 0$ or $p < 1$.

When $p = 1$, $\int_0^1 \frac{1}{x}dx = [\ln x]_0^1 = 0 - \lim_{b\to 0^+} \ln b = \infty$. The integral diverges.

When $p = 0$, $\int_0^1 1dx = [x]_0^1 = 1 - 0 = 1$

$\int_0^1 \frac{1}{x^p}dx$ converges when $p < 1$ and diverges when $1 \leq p$.

41. For $x \geq 1$, $x^6 + x > x^6$, so $\sqrt{x^6+x} > \sqrt{x^6} = x^3$ and $\frac{1}{\sqrt{x^6+x}} < \frac{1}{x^3}$. Hence, $\int_1^\infty \frac{1}{\sqrt{x^6+x}}dx < \int_1^\infty \frac{1}{x^3}dx$ which converges since $3 > 1$ (see Problem 39). Thus $\int_1^\infty \frac{1}{\sqrt{x^6+x}}dx$ converges.

42. For $x > 1$, $\ln x < e^x$, so $\frac{\ln x}{e^x} < 1$ and

$\frac{\ln x}{e^{2x}} = \frac{\ln x}{(e^x)^2} < \frac{1}{e^x}$.

Hence,

$\int_1^\infty \frac{\ln x}{e^{2x}} dx < \int_1^\infty e^{-x} dx = [-e^{-x}]_1^\infty = -0 + e^{-1} = \frac{1}{e}$.

Thus, $\int_1^\infty \frac{\ln x}{e^{2x}} dx$ converges.

43. For $x > 3$, $\ln x > 1$, so $\frac{\ln x}{x} > \frac{1}{x}$. Hence,

$\int_3^\infty \frac{\ln x}{x} dx > \int_3^\infty \frac{1}{x} dx = [\ln x]_3^\infty = \infty - \ln 3$.

The integral diverges, thus $\int_3^\infty \frac{\ln x}{x} dx$ also diverges.

44. For $x \geq 1$, $\ln x < x$, so $\frac{\ln x}{x} < 1$ and $\frac{\ln x}{x^3} < \frac{1}{x^2}$.

Hence,

$\int_1^\infty \frac{\ln x}{x^3} dx < \int_1^\infty \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^\infty = -0 + 1 = 1$.

Thus, $\int_1^\infty \frac{\ln x}{x^3} dx$ converges.

Review and Preview Problems

1. Original: If $x > 0$, then $x^2 > 0$ (AT)

Converse: If $x^2 > 0$, then $x > 0$

Contrapositive: If $x^2 \leq 0$, then $x \leq 0$ (AT)

2. Original: If $x^2 > 0$, then $x > 0$

Converse: If $x > 0$, then $x^2 > 0$ (AT)

Contrapositive: If $x \leq 0$, then $x^2 \leq 0$

3. Original:
f differentiable at $c \Rightarrow f$ continuous at c (AT)
Converse:
f continuous at $c \Rightarrow f$ differentiable at c
Contrapositive:
f discontinuous at $c \Rightarrow f$ non-differentiable at c (AT)

4. Original:
f continuous at $c \Rightarrow f$ differentiable at c
Converse:
f differentiable at $c \Rightarrow f$ continuous at c (AT)
Contrapositive:
f non-differentiable at $c \Rightarrow f$ discontinuous at c

5. Original:
f right continuous at $c \Rightarrow f$ continuous at c
Converse:
f continuous at $c \Rightarrow f$ right continuous at c (AT)
Contrapositive:
f discontinuous at $c \Rightarrow f$ not right continuous at c

6. Original: $f'(x) \equiv 0 \Rightarrow f(x) = c$ (AT)
Converse: $f(x) = c \Rightarrow f'(x) \equiv 0$ (AT)
Contrapositive: $f(x) \neq c \Rightarrow f'(x) \not\equiv 0$ (AT)

7. Original: $f(x) = x^2 \Rightarrow f'(x) = 2x$ (AT)
Converse: $f'(x) = 2x \Rightarrow f(x) = x^2$
(Could have $f(x) = x^2 + 3$)
Contrapositive: $f'(x) \neq 2x \Rightarrow f(x) \neq x^2$ (AT)

8. Original: $a < b \Rightarrow a^2 < b^2$
Converse: $a^2 < b^2 \Rightarrow a < b$
Contrapositive: $a^2 \geq b^2 \Rightarrow a \geq b$

9. $1 + \frac{1}{2} + \frac{1}{4} = \frac{4}{4} + \frac{2}{4} + \frac{1}{4} = \frac{7}{4}$

10. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} =$
$\frac{32}{32} + \frac{16}{32} + \frac{8}{32} + \frac{4}{32} + \frac{2}{32} + \frac{1}{32} = \frac{63}{32}$

11. $\sum_{i=1}^{4} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{12+6+4+3}{12} = \frac{25}{12}$

12. $\sum_{k=1}^{4} \frac{(-1)^k}{2^k} = \frac{-1}{2} + \frac{1}{4} + \frac{-1}{8} + \frac{1}{16} =$
$\frac{-8+4-2+1}{16} = \frac{-5}{16}$

13. By L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$:

$\lim_{x\to\infty} \frac{x}{2x+1} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$

14. By L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$ twice:

$$\lim_{n\to\infty}\frac{n^2}{2n^2+1}=\lim_{n\to\infty}\frac{2n}{4n}=\frac{2}{4}=\frac{1}{2}$$

15. By L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$ twice:

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$$

16. By L'Hopital's Rule $\left(\frac{\infty}{\infty}\right)$ twice:

$$\lim_{n\to\infty}\frac{n^2}{e^n}=\lim_{n\to\infty}\frac{2n}{e^n}=\lim_{n\to\infty}\frac{2}{e^n}=0$$

17. $\int_1^\infty \frac{1}{x}dx=\lim_{t\to\infty}\int_1^t\frac{1}{x}dx=$

$$\lim_{t\to\infty}\left[\ln x\right]_1^t=\lim_{t\to\infty}\left[\ln t\right]=\infty$$

Integral does not converge.

18. $\int_1^\infty \frac{1}{x^2}dx=\lim_{t\to\infty}\int_1^t\frac{1}{x^2}dx=$

$$\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t=\lim_{t\to\infty}\left[1-\frac{1}{t}\right]=1$$

Integral converges.

19. $\int_1^\infty \frac{1}{x^{1.001}}dx=\lim_{t\to\infty}\int_1^t\frac{1}{x^{1.001}}dx=$

$$\lim_{t\to\infty}\left[-\frac{1000}{x^{0.001}}\right]_1^t=\lim_{t\to\infty}\left[1000-\frac{1000}{t^{0.001}}\right]=1000$$

Integral converges.

20. $\int_1^\infty \frac{x}{x^2+1}dx=\lim_{t\to\infty}\int_1^t\frac{x}{x^2+1}\,dx=$

$u=x^2+1$
$du=2x\,dx$

$$\frac{1}{2}\lim_{t\to\infty}\int_2^{t^2+1}\frac{1}{u}du=\infty$$

Integral does not converge (see problem 17).

21. $\int_1^\infty \frac{x}{x^2+1}dx=\lim_{t\to\infty}\int_1^t\frac{x}{x^2+1}dx=\frac{1}{2}\ln\left(x^2+1\right)\Big|_1^\infty=\infty$

Integral does not converge.

22. $\int_2^\infty \frac{1}{x(\ln x)^2}dx=\lim_{t\to\infty}\int_2^t\frac{1}{x(\ln x)^2}dx=$

$u=\ln x$
$du=\frac{1}{x}dx$

$$\lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u^2}du=\lim_{t\to\infty}\left[-\frac{1}{u}\right]_{\ln 2}^{\ln t}=$$

$$\lim_{t\to\infty}\left[\frac{1}{\ln 2}-\frac{1}{\ln t}\right]=\frac{1}{\ln 2}\approx 1.443$$

Integral converges.