CHAPTER 9 Infinite Series

9.1 Concepts Review

1. a sequence

2. $\lim_{n\to\infty} a_n$ exists (finite sense)

3. bounded above

4. –1; 1

Problem Set 9.1

1. $a_1 = \frac{1}{2}, a_2 = \frac{2}{5}, a_3 = \frac{3}{8}, a_4 = \frac{4}{11}, a_5 = \frac{5}{14}$

$\lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3};$

converges

2. $a_1 = \frac{5}{2}, a_2 = \frac{8}{3}, a_3 = \frac{11}{4}, a_4 = \frac{14}{5}, a_5 = \frac{17}{6}$

$\lim_{n\to\infty} \frac{3n+2}{n+1} = \lim_{n\to\infty} \frac{3+\frac{2}{n}}{1+\frac{1}{n}} = 3;$

converges

3. $a_1 = \frac{6}{3} = 2, a_2 = \frac{18}{9} = 2, a_3 = \frac{38}{17},$

$a_4 = \frac{66}{27} = \frac{22}{9}, a_5 = \frac{102}{39} = \frac{34}{13}$

$\lim_{n\to\infty} \frac{4n^2+2}{n^2+3n-1} = \lim_{n\to\infty} \frac{4+\frac{2}{n^2}}{1+\frac{3}{n}-\frac{1}{n^2}} = 4;$

converges

4. $a_1 = 5, a_2 = \frac{14}{3}, a_3 = \frac{29}{5}, a_4 = \frac{50}{7}, a_5 = \frac{77}{9}$

$\lim_{n\to\infty} \frac{3n^2+2}{2n-1} = \lim_{n\to\infty} \frac{3n+\frac{2}{n}}{2-\frac{1}{n}} = \infty;$

diverges

5. $a_1 = \frac{7}{8}, a_2 = \frac{26}{27}, a_3 = \frac{63}{64}, \ a_4 = \frac{124}{125}, a_5 = \frac{215}{216}$

$\lim_{n\to\infty} \frac{n^3+3n^2+3n}{(n+1)^3} = \lim_{n\to\infty} \frac{n^3+3n^2+3n}{n^3+3n^2+3n+1}$

$= \lim_{n\to\infty} \frac{1+\frac{3}{n}+\frac{3}{n^2}}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}} = 1$

6. $a_1 = \frac{\sqrt{5}}{3}, a_2 = \frac{\sqrt{14}}{5}, a_3 = \frac{\sqrt{29}}{7},$

$a_4 = \frac{\sqrt{50}}{9} = \frac{5\sqrt{2}}{9}, a_5 = \frac{\sqrt{77}}{11}$

$\lim_{n\to\infty} \frac{\sqrt{3n^2+2}}{2n+1} = \lim_{n\to\infty} \frac{\sqrt{3+\frac{2}{n^2}}}{2+\frac{1}{n}} = \frac{\sqrt{3}}{2};$

converges

7. $a_1 = -\frac{1}{3}, a_2 = \frac{2}{4} = \frac{1}{2}, a_3 = -\frac{3}{5}, a_4 = \frac{4}{6} = \frac{2}{3},$

$a_5 = -\frac{5}{7}$

$\lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1,$ but since it alternates between positive and negative, the sequence diverges.

8. $a_1 = -1, a_2 = \frac{2}{3}, a_3 = -\frac{3}{5}, a_4 = \frac{4}{7}, a_5 = -\frac{5}{9}$

$\cos(n\pi) = \begin{cases} -1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}$

$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2},$ but since $\cos(n\pi)$ alternates between 1 and –1, the sequence diverges.

9. $a_1 = -1, a_2 = \frac{1}{2}, a_3 = -\frac{1}{3}, a_4 = \frac{1}{4}, a_5 = -\frac{1}{5}$

$\cos(n\pi) = (-1)^n$, so $-\frac{1}{n} \le \frac{\cos(n\pi)}{n} \le \frac{1}{n}$.

$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, so by the Squeeze Theorem, the sequence converges to 0.

10. $a_1 = e^{-1}\sin 1 \approx 0.3096, a_2 = e^{-2}\sin 2 \approx 0.1231,$

$a_3 = e^{-3}\sin 3 \approx 0.0070, a_4 = e^{-4}\sin 4 \approx -0.0139,$

$a_5 = e^{-5}\sin 5 \approx -0.0065$

$-1 \le \sin n \le 1$ for all n, so

$-e^{-n} \le e^{-n}\sin n \le e^{-n}$.

$\lim_{n\to\infty} -e^{-n} = \lim_{n\to\infty} e^{-n} = 0$, so by the Squeeze Theorem, the sequence converges to 0.

11. $a_1 = \frac{e^2}{3} \approx 2.4630, a_2 = \frac{e^4}{9} \approx 6.0665,$

$a_3 = \frac{e^6}{17} \approx 23.7311, a_4 = \frac{e^8}{27} \approx 110.4059,$

$a_5 = \frac{e^{10}}{39} \approx 564.7812$

Consider

$$\lim_{x\to\infty} \frac{e^{2x}}{x^2+3x-1} = \lim_{x\to\infty} \frac{2e^{2x}}{2x+3} = \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty$$

by using l'Hôpital's Rule twice. The sequence diverges.

12. $a_1 = \frac{e^2}{4} \approx 1.8473, a_2 = \frac{e^4}{16} \approx 3.4124,$

$a_3 = \frac{e^6}{64} \approx 6.3036, a_4 = \frac{e^8}{256} \approx 11.6444,$

$a_5 = \frac{e^{10}}{1024} \approx 21.510$

$\frac{e^{2n}}{4^n} = \left(\frac{e^2}{4}\right)^n, \frac{e^2}{4} > 1$ so the sequence diverges.

13. $a_1 = -\frac{\pi}{5} \approx -0.6283, a_2 = \frac{\pi^2}{25} \approx 0.3948,$

$a_3 = -\frac{\pi^3}{125} \approx -0.2481, a_4 = \frac{\pi^4}{625} \approx 0.1559,$

$a_5 = -\frac{\pi^5}{3125} \approx -0.0979$

$\frac{(-\pi)^n}{5^n} = \left(-\frac{\pi}{5}\right)^n, -1 < -\frac{\pi}{5} < 1,$ thus the sequence converges to 0.

14. $a_1 = \frac{1}{4} + \sqrt{3} \approx 1.9821, a_2 = \frac{1}{16} + 3 = 3.0625,$

$a_3 = \frac{1}{64} + 3\sqrt{3} \approx 5.2118, a_4 = \frac{1}{256} + 9 \approx 9.0039,$

$a_5 = \frac{1}{1024} + 9\sqrt{3} \approx 15.589$

$\left(\frac{1}{4}\right)^n$ converges to 0 since $-1 < \frac{1}{4} < 1$.

$3^{n/2} = \left(\sqrt{3}\right)^n$ diverges since $\sqrt{3} \approx 1.732 > 1$.

Thus, the sum diverges.

15. $a_1 = 2.99, a_2 = 2.9801, a_3 \approx 2.9703,$

$a_4 \approx 2.9606, a_5 \approx 2.9510$

$(0.99)^n$ converges to 0 since $-1 < 0.99 < 1$, thus $2 + (0.99)^n$ converges to 2.

16. $a_1 = \frac{1}{e} \approx 0.3679, a_2 = \frac{2^{100}}{e^2} \approx 1.72 \times 10^{29},$

$a_3 = \frac{3^{100}}{e^3} \approx 2.57 \times 10^{46}, a_4 = \frac{4^{100}}{e^4} \approx 2.94 \times 10^{58},$

$a_5 = \frac{5^{100}}{e^5} \approx 5.32 \times 10^{67}$

Consider $\lim_{x\to\infty} \frac{x^{100}}{e^x}$. By Example 2 of Section 8.2, $\lim_{x\to\infty} \frac{x^{100}}{e^x} = 0$. Thus, $\lim_{n\to\infty} \frac{n^{100}}{e^n} = 0$; converges

17. $a_1 = \frac{\ln 1}{\sqrt{1}} = 0, a_2 = \frac{\ln 2}{\sqrt{2}} \approx 0.4901,$

$a_3 = \frac{\ln 3}{\sqrt{3}} \approx 0.6343, a_4 = \frac{\ln 4}{2} \approx 0.6931,$

$a_5 = \frac{\ln 5}{\sqrt{5}} \approx 0.7198$

Consider $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$ by using l'Hôpital's Rule. Thus, $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$; converges.

18. $a_1 = \frac{\ln 1}{\sqrt{2}} = 0, a_2 = \frac{\ln \frac{1}{2}}{2} \approx -0.3466,$

$a_3 = \frac{\ln \frac{1}{3}}{\sqrt{6}} \approx -0.4485, a_4 = \frac{\ln \frac{1}{4}}{2\sqrt{2}} \approx -0.4901,$

$a_5 = \frac{\ln \frac{1}{5}}{\sqrt{10}} \approx -0.5089$

Consider $\lim_{x\to\infty} \frac{\ln \frac{1}{x}}{\sqrt{2x}} = \lim_{x\to\infty} \frac{-\ln x}{\sqrt{2x}} = \lim_{x\to\infty} \frac{-\frac{1}{x}}{\frac{1}{\sqrt{2x}}}$

$= \lim_{x\to\infty} -\frac{\sqrt{2}}{\sqrt{x}} = 0$ by using l'Hôpital's Rule. Thus,

$\lim_{n\to\infty} \frac{\ln \frac{1}{n}}{\sqrt{2n}} = 0;$ converges

19. $a_1 = \left(1+\frac{2}{1}\right)^{1/2} = \sqrt{3} \approx 1.7321,$

$a_2 = \left(1+\frac{2}{2}\right)^{2/2} = 2,$

$a_3 = \left(1+\frac{2}{3}\right)^{3/2} = \left(\frac{5}{3}\right)^{3/2} \approx 2.1517,$

$a_4 = \left(1+\frac{2}{4}\right)^{4/2} = \left(\frac{3}{2}\right)^{2} = \frac{9}{4},$

$a_5 = \left(1+\frac{2}{5}\right)^{5/2} = \left(\frac{7}{5}\right)^{5/2} \approx 2.3191$

Let $\frac{2}{n} = h,$ then as $n \to \infty, h \to 0$ and

$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{n/2} = \lim_{h\to 0}(1+h)^{1/h} = e$ by Theorem 6.5A; converges

20. $a_1 = 2^{1/2} \approx 1.4142, a_2 = 4^{1/4} = 2^{1/2} \approx 1.4142,$

$a_3 = 6^{1/6} \approx 1.3480, a_4 = 8^{1/8} = 2^{3/8} \approx 1.2968,$

$a_5 = 10^{1/10} \approx 1.2589$

Consider $\lim_{x\to\infty} (2x)^{1/2x}$. This limit is of the form ∞^0. Let $y = (2x)^{1/2x}$, then $\ln y = \frac{1}{2x} \ln 2x.$

$\lim_{x\to\infty} \frac{1}{2x} \ln 2x = \lim_{x\to\infty} \frac{\ln 2x}{2x}$

This limit is of the form $\frac{\infty}{\infty}$.

$\lim_{x\to\infty} \frac{\ln 2x}{2x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2} = \lim_{x\to\infty} \frac{1}{2x} = 0$

$\lim_{x\to\infty} (2x)^{1/2x} = \lim_{x\to\infty} e^{\ln y} = 1$

Thus $\lim_{n\to\infty} (2n)^{1/2n} = 1;$ converges

21. $a_n = \frac{n}{n+1}$ or $a_n = 1 - \frac{1}{n+1};$

$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1;$ converges

22. $a_n = \frac{n}{2^{n+1}}$

Consider $\frac{x}{2^x}$. Now, $\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$

by l'Hôpital's Rule. Thus, $\lim_{n\to\infty} \frac{n}{2^{n+1}} = 0;$

converges

23. $a_n = (-1)^n \frac{n}{2n-1}; \lim_{n\to\infty} \frac{n}{2n-1}$

$= \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$, but due to $(-1)^n$, the terms of the sequence alternate between positive and negative, so the sequence diverges.

24. $a_n = \frac{1}{1-\frac{n-1}{n}} = n;$

$\lim_{n\to\infty} n = \infty$; diverges

25. $a_n = \frac{n}{n^2 - (n-1)^2} = \frac{n}{n^2 - (n^2 - 2n + 1)} = \frac{n}{2n-1};$

$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2};$ converges

26. $a_n = \dfrac{n}{(n+1)-\frac{1}{n+1}} = \dfrac{n(n+1)}{(n+1)^2-1} = \dfrac{n^2+n}{n^2+2n};$

$\lim\limits_{n\to\infty}\dfrac{n^2+n}{n^2+2n} = \lim\limits_{n\to\infty}\dfrac{1+\frac{1}{n}}{1+\frac{2}{n}} = 1;$ converges

27. $a_n = n\sin\dfrac{1}{n};\ \lim\limits_{n\to\infty} n\sin\dfrac{1}{n} = \lim\limits_{n\to\infty}\dfrac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$ since

$\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1;$ converges

28. $a_n = (-1)^n\dfrac{n^2}{3^n};$

$\lim\limits_{n\to\infty}\dfrac{n^2}{3^n} = \lim\limits_{n\to\infty}\dfrac{2n}{3^n\ln 3} = \lim\limits_{n\to\infty}\dfrac{2}{3^n(\ln 3)^2} = 0$

by using l'Hôpital's Rule twice; converges

29. $a_n = \dfrac{2^n}{n^2};$

$\lim\limits_{n\to\infty}\dfrac{2^n}{n^2} = \lim\limits_{n\to\infty}\dfrac{2^n\ln 2}{2n} = \lim\limits_{n\to\infty}\dfrac{2^n(\ln 2)^2}{2} = \infty;$

diverges

30. $a_n = \dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{n+1-n}{n(n+1)} = \dfrac{1}{n(n+1)};$

$\lim\limits_{n\to\infty}\dfrac{1}{n(n+1)} = 0;$ converges

31. $a_1 = \dfrac{1}{2}, a_2 = \dfrac{5}{4}, a_3 = \dfrac{9}{8}, a_4 = \dfrac{13}{16}$

a_n is positive for all n, and $a_{n+1} < a_n$ for all $n \ge 2$ since $a_{n+1} - a_n = -\dfrac{4n-7}{2^{n+1}}$, so $\{a_n\}$ converges to a limit $L \ge 0$.

32. $a_1 = \dfrac{1}{2};\ a_2 = \dfrac{7}{6};\ a_3 = \dfrac{17}{12};\ a_4 = \dfrac{31}{20}$

$a_n = \dfrac{2n^2-1}{n^2+n} < 2$ for all n, and $a_n < a_{n+1}$ for all n since $a_{n+1} - a_n = \dfrac{2}{n^2+2n}$, so $\{a_n\}$ converges to a limit $L \le 2$.

33. $a_2 = \dfrac{3}{4};\ a_3 = \left(\dfrac{3}{4}\right)\left(\dfrac{8}{9}\right) = \dfrac{2}{3};$

$a_4 = \left(\dfrac{3}{4}\right)\left(\dfrac{8}{9}\right)\left(\dfrac{15}{16}\right) = \dfrac{5}{8};$

$a_5 = \left(\dfrac{3}{4}\right)\left(\dfrac{8}{9}\right)\left(\dfrac{15}{16}\right)\left(\dfrac{24}{25}\right) = \dfrac{3}{5}$

$a_n > 0$ for all n and $a_{n+1} < a_n$ since

$a_{n+1} = a_n\left(1 - \dfrac{1}{(n+1)^2}\right)$ and $1 - \dfrac{1}{(n+1)^2} < 1$, so

$\{a_n\}$ converges to a limit $L \ge 0$.

34. $a_1 = 1;\ a_2 = \dfrac{3}{2};\ a_3 = \dfrac{5}{3};\ a_4 = \dfrac{41}{24}$

$a_n < 2$ for all n since

$$1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \le \frac{1}{2^0} + \frac{1}{2^1} + \cdots + \frac{1}{2^{n+1}}$$
$$< \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2$$

the sum never reaches 2. $a_n < a_{n+1}$ since each term is the previous term plus a positive quantity, so $\{a_n\}$ converges to a limit $L \le 2$.

35. $a_1 = 1, a_2 = 1 + \dfrac{1}{2}(1) = \dfrac{3}{2}, a_3 = 1 + \dfrac{1}{2}\left(\dfrac{3}{2}\right) = \dfrac{7}{4},$

$a_4 = 1 + \dfrac{1}{2}\left(\dfrac{7}{4}\right) = \dfrac{15}{8}$

Suppose that $1 < a_n < 2$, then $\dfrac{1}{2} < \dfrac{1}{2}a_n < 1$, so $\dfrac{3}{2} < 1 + \dfrac{1}{2}a_n < 2$, or $\dfrac{3}{2} < a_{n+1} < 2$. Thus, since $1 < a_2 < 2$, every subsequent term is between $\dfrac{3}{2}$ and 2.

$a_n < 2$ thus $\dfrac{1}{2}a_n < 1$, so $a_n < 1 + \dfrac{1}{2}a_n = a_{n+1}$ and the sequence is nondecreasing, so $\{a_n\}$ converges to a limit $L \le 2$.

36. $a_1 = 2, a_2 = \frac{1}{2}\left(2+\frac{2}{2}\right) = \frac{3}{2},$

$a_3 = \frac{1}{2}\left(\frac{3}{2}+\frac{4}{3}\right) = \frac{17}{12}, a_4 = \frac{1}{2}\left(\frac{17}{12}+\frac{24}{17}\right) = \frac{577}{408}$

Suppose $a_n > \sqrt{2}$, and consider

$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) > \sqrt{2} \Leftrightarrow a_n + \frac{2}{a_n} > 2\sqrt{2} \Leftrightarrow$

$a_n^2 + 2 > 2\sqrt{2}a_n \Leftrightarrow a_n^2 - 2\sqrt{2}a_n + 2 > 0 \Leftrightarrow$

$\left(a_n - \sqrt{2}\right)^2 > 0$, which is always true. Hence, $a_n > \sqrt{2}$ for all *n*. Also,

$a_{n+1} \le a_n \Leftrightarrow \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \le a_n$

$\Leftrightarrow \frac{1}{a_n} \le \frac{1}{2}a_n \Leftrightarrow \sqrt{2} \le a_n$

which is true. Hence, $\sqrt{2} < a_{n+1} \le a_n$ and the series converges to a limit $L \ge \sqrt{2}$.

37.

n	u_n
1	1.73205
2	2.17533
3	2.27493
4	2.29672
5	2.30146
6	2.30249
7	2.30271
8	2.30276
9	2.30277
10	2.30278
11	2.30278

$\lim_{n\to\infty} u_n \approx 2.3028$

38. Suppose that $0 < u_n < \frac{1}{2}\left(1+\sqrt{13}\right)$, then

$3 < 3 + u_n < \frac{1}{2}\left(7+\sqrt{13}\right)$ and

$\sqrt{3} < \sqrt{3+u_n} = u_{n+1} < \sqrt{\frac{1}{2}\left(7+\sqrt{13}\right)} = \frac{1}{2}\left(1+\sqrt{13}\right)$

$\left(\sqrt{\frac{1}{2}\left(7+\sqrt{13}\right)} = \frac{1}{2}\left(1+\sqrt{13}\right)\right.$ can be seen by squaring both sides of the equality and noting that both sides are positive.) Hence, since

$0 < u_1 = \sqrt{3} \approx 1.73 < \frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.3028,$

$\sqrt{3} < u_n < \frac{1}{2}\left(1+\sqrt{13}\right)$ for all *n*; $\{u_n\}$ is bounded above.

$u_{n+1} = \sqrt{3+u_n} > u_n$ if $3 + u_n > u_n^2$ or $u_n^2 - u_n - 3 < 0.$ $u_n^2 - u_n - 3 = 0$ when $u_n = \frac{1}{2}\left(1 \pm \sqrt{13}\right)$, thus $u_{n+1} > u_n$ if $\frac{1}{2}\left(1-\sqrt{13}\right) < u_n < \frac{1}{2}\left(1+\sqrt{13}\right)$, $\frac{1}{2}\left(1-\sqrt{13}\right) < 0$ and $0 < u_n < \frac{1}{2}\left(1+\sqrt{13}\right)$ for all *n*, as shown above, so $\{u_n\}$ is increasing. Hence, by Theorem D, $\{u_n\}$ converges.

39. If $u = \lim_{n\to\infty} u_n$, then $u = \sqrt{3+u}$ or $u^2 = 3+u$;

$u^2 - u - 3 = 0$ when $u = \frac{1}{2}\left(1 \pm \sqrt{13}\right)$ so

$u = \frac{1}{2}\left(1+\sqrt{13}\right) \approx 2.3028$ since *u* > 0 and

$\frac{1}{2}\left(1-\sqrt{13}\right) < 0.$

40. If $a = \lim_{n\to\infty} a_n$ where $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, then

$a = \frac{1}{2}\left(a + \frac{2}{a}\right)$ or $2a^2 = a^2 + 2$; $a^2 = 2$ when

$a = \pm\sqrt{2}$, so $a = \sqrt{2}$, since *a* > 0.

41.

n	u_n
1	0
2	1
3	1.1
4	1.11053
5	1.11165
6	1.11177
7	1.11178
8	1.11178

$\lim_{n\to\infty} u_n \approx 1.1118$

42. Since 1.1 > 1, $1.1^a > 1.1^b$ if *a* > *b*. Thus, since $u_3 = 1.1 > 1 = u_2$, $u_4 = 1.1^{1.1} > 1.1^1 = u_3$. Suppose that $u_n < u_{n+1}$ for all $n \le N$. Then $u_{N+1} = 1.1^{u_N} > 1.1^{u_{N-1}} = u_N$, since $u_N > u_{N-1}$ by the induction hypothesis. Thus, u_n is increasing.

$1.1^{u_n} < 2$ if and only if $u_n \ln 1.1 < \ln 2$;

$u_n < \frac{\ln 2}{\ln 1.1} \approx 7.3.$ Thus, unless $u_n > 7.3$, $u_{n+1} = 1.1^{u_n} < 2.$ This means that $\{u_n\}$ is bounded above by 2, since $u_1 = 0$.

43. As $n \to \infty, \frac{k}{n} \to 0;$ using $\Delta x = \frac{1}{n},$ an equivalent definite integral is

$\int_0^1 \sin x\, dx = [-\cos x]_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1$

≈ 0.4597

44. As $n \to \infty, \frac{k}{n} \to 0;$ using $\Delta x = \frac{1}{n},$ an equivalent definite integral is

$\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

45. $\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1};$

$\frac{1}{n+1} < \varepsilon$ is the same as $\frac{1}{\varepsilon} < n+1.$ For any given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon} - 1$ then

$n \ge N \Rightarrow \left|\frac{n}{n+1} - 1\right| < \varepsilon.$

46. For $n > 0$, $\left|\frac{n}{n^2+1}\right| = \frac{n}{n^2+1}.$ $\frac{n}{n^2+1} < \varepsilon$ is the same as $\frac{n^2+1}{n} = n + \frac{1}{n} > \frac{1}{\varepsilon}.$

Since $n + \frac{1}{n} > n$, it suffices to take $n > \frac{1}{\varepsilon}$. So for any given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon},$ then

$n \ge N \Rightarrow \left|\frac{n}{n^2+1}\right| < \varepsilon.$

47. Recall that every rational number can be written as either a terminating or a repeating decimal. Thus if the sequence 1, 1.4, 1.41, 1.414, … has a limit within the rational numbers, the terms of the sequence would eventually either repeat or terminate, which they do not since they are the decimal approximations to $\sqrt{2},$ which is irrational. Within the real numbers, the least upper bound is $\sqrt{2}.$

48. Suppose that $\{a_n\}$ is a nondecreasing sequence, and U is an upper bound for $\{a_n\},$ so $S = \{a_n : n \in \mathbb{N}\}$ is bounded above. By the completeness property, S has a least upper bound, which we call A. Then $A \le U$ by definition and $a_n \le A$ for all n. Suppose that $\lim_{n\to\infty} a_n \ne A,$ i.e., that $\{a_n\}$ either does not converge, or does not converge to A. Then there is some $\varepsilon > 0$ such that $A - a_n > \varepsilon$ for all n, since if $A - a_N \le \varepsilon,$ $A - a_n \le \varepsilon$ for $n \ge N$ since $\{a_n\}$ is nondecreasing and $a_n \le A$ for all n. However, if $A - a_n > \varepsilon$ for all n, $a_n < A - \frac{\varepsilon}{2} < A$ for all n, which contradicts A being the least upper bound for the set S. For the second part of Theorem D, suppose that $\{a_n\}$ is a nonincreasing sequence, and L is a lower bound for $\{a_n\}.$ Then $\{-a_n\}$ is a nondecreasing sequence and $-L$ is an upper bound for $\{-a_n\}.$ By what was just proven, $\{-a_n\}$ converges to a limit $A \le -L$, so $\{a_n\}$ converges to a limit $B = -A \ge L$.

49. If $\{b_n\}$ is bounded, there are numbers N and M with $N \le |b_n| \le M$ for all n. Then

$|a_n N| \le |a_n b_n| \le |a_n M|.$

$\lim_{n\to\infty} |a_n N| = |N| \lim_{n\to\infty} |a_n| = 0$ and

$\lim_{n\to\infty} |a_n M| = |M| \lim_{n\to\infty} |a_n| = 0,$ so $\lim_{n\to\infty} |a_n b_n| = 0$

by the Squeeze Theorem, and by Theorem C, $\lim_{n\to\infty} a_n b_n = 0.$

50. Suppose $\{a_n + b_n\}$ converges. Then, by Theorem A

$\lim_{n\to\infty} [(a_n + b_n) - a_n] = \lim_{n\to\infty} (a_n + b_n) - \lim_{n\to\infty} a_n.$

But since $(a_n + b_n) - a_n = b_n,$ this would mean that $\{b_n\}$ converges. Thus $\{a_n + b_n\}$ diverges.

51. No. Consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Both $\{a_n\}$ and $\{b_n\}$ diverge, but

$a_n + b_n = (-1)^n + (-1)^{n+1} = (-1)^n(1 + (-1)) = 0$ so $\{a_n + b_n\}$ converges.

52. a. $f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8,$
$f_7 = 13, f_8 = 21, f_9 = 34, f_{10} = 55$

b. Using the formula,

$$f_1 = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] = 1$$

$$f_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1 + 2\sqrt{5} + 5 - (1 - 2\sqrt{5} + 5)}{4}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{4\sqrt{5}}{4}\right] = 1.$$

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n}=\lim_{n\to\infty}\frac{\phi^{n+1}-(-1)^{n+1}\phi^{-n-1}}{\phi^n-(-1)^n\phi^{-n}}$$

$$=\lim_{n\to\infty}\frac{\phi^{n+1}-\frac{(-1)^{n+1}}{\phi^{n+1}}}{\phi^n-\frac{(-1)^n}{\phi^n}}=\lim_{n\to\infty}\frac{\phi-\frac{(-1)^{n+1}}{\phi^{2n+1}}}{1-\frac{(-1)^n}{\phi^{2n}}}=\phi$$

c. $\phi^2-\phi-1=\left[\frac{1}{2}\left(1+\sqrt{5}\right)\right]^2-\frac{1}{2}\left(1+\sqrt{5}\right)-1$

$$=\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)-1=0$$

Therefore ϕ satisfies $x^2-x-1=0$.

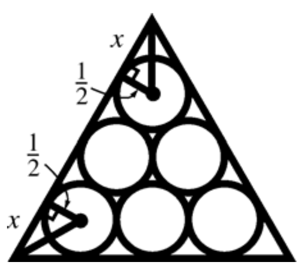
Using the Quadratic Formula on $x^2-x-1=0$ yields

$$x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}.$$

$$\phi=\frac{1+\sqrt{5}}{2};$$

$$-\frac{1}{\phi}=-\frac{2}{1+\sqrt{5}}=-\frac{2(1-\sqrt{5})}{1-5}=\frac{1-\sqrt{5}}{2}$$

53.



From the figure shown, the sides of the triangle have length $n-1+2x$. The small right triangles marked are 30-60-90 right triangles, so $x=\frac{\sqrt{3}}{2}$; thus the sides of the large triangle have lengths $n-1+\sqrt{3}$ and $B_n=\frac{\sqrt{3}}{4}\left(n-1+\sqrt{3}\right)^2$

$=\frac{\sqrt{3}}{4}\left(n^2+2\sqrt{3}n-2n-2\sqrt{3}+4\right)$ while

$$A_n=\frac{n(n+1)}{2}\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{8}(n^2+n)$$

$$\lim_{n\to\infty}\frac{A_n}{B_n}=\lim_{n\to\infty}\frac{\frac{\pi}{8}(n^2+n)}{\frac{\sqrt{3}}{4}(n^2+2\sqrt{3}n-2n-2\sqrt{3}+4)}$$

$$=\lim_{n\to\infty}\frac{\pi\left(1+\frac{1}{n}\right)}{2\sqrt{3}\left(1+\frac{2\sqrt{3}}{n}-\frac{2}{n}-\frac{2\sqrt{3}}{n^2}+\frac{4}{n^2}\right)}=\frac{\pi}{2\sqrt{3}}$$

54. Let $f(x)=\left(1+\frac{1}{x}\right)^x$.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0^+}(1+x)^{1/x}=e,\text{ so}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

55. Let $f(x)=\left(1+\frac{1}{2x}\right)^x$.

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^x=\lim_{x\to 0^+}\left(1+\frac{x}{2}\right)^{1/x}$$

$$=\lim_{x\to 0^+}\left[\left(1+\frac{x}{2}\right)^{2/x}\right]^{1/2}=e^{1/2},\text{ so}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n=e^{1/2}.$$

56. Let $f(x)=\left(1+\frac{1}{x^2}\right)^x$.

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x=\lim_{x\to\infty}\left(1+\left(\frac{1}{x}\right)^2\right)^{\frac{1}{1/x}}$$

Using the fact that $\lim_{x\to\infty}f\left(x\right)=\lim_{x\to 0^+}f\left(\frac{1}{x}\right)$, we can write

$$\lim_{x\to\infty}\left(1+\left(\frac{1}{x}\right)^2\right)^{\frac{1}{1/x}}=\lim_{x\to 0^+}\left(1+x^2\right)^{1/x}$$ which leads to the indeterminate form 1^∞.

Let $y=\left(1+x^2\right)^{1/x}$. Then,

$$\ln y=\ln\left(1+x^2\right)^{1/x}$$

$$\ln y=\frac{1}{x}\ln\left(1+x^2\right)$$

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln\left(1+x^2\right)}{x}=\lim_{x\to 0^+}\frac{\frac{2x}{1+x^2}}{1}$$

$$=\lim_{x\to 0^+}\frac{2x}{1+x^2}=0$$

This gives us

$$\lim_{x\to 0^+}\ln y=0$$

$$\ln\left(\lim_{x\to 0^+}y\right)=0$$

$$\lim_{x\to 0^+}y=e^0=1\quad\text{or}\quad\lim_{x\to 0^+}\left(1+x^2\right)^{1/x}=1$$

Thus, $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n=1$.

57. Let $f(x) = \left(\frac{x-1}{x+1}\right)^x$.

Using the fact that $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right)$, we can write

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x = \lim_{x\to 0^+}\left(\frac{\frac{1}{x}-1}{\frac{1}{x}+1}\right)^{1/x} = \lim_{x\to 0^+}\left(\frac{\frac{1-x}{x}}{\frac{1+x}{x}}\right)^{1/x}$$

$= \lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x}$ which leads to the indeterminate form 1^∞.

Let $y = \left(\frac{1-x}{1+x}\right)^{1/x}$. Then,

$$\ln y = \ln\left(\frac{1-x}{1+x}\right)^{1/x}$$

$$\ln y = \frac{1}{x}\ln\left(\frac{1-x}{1+x}\right)^{1/x}$$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{1}{x}\ln\left(\frac{1-x}{1+x}\right)$$

$$\ln\left[\lim_{x\to 0^+} y\right] = \lim_{x\to 0^+}\frac{\ln\left(\frac{1-x}{1+x}\right)}{x}$$

$$= \lim_{x\to 0^+}\frac{-2}{1-x^2} \quad \text{(l'Hopital's Rule)}$$

-2

This gives us,

$$\ln\left[\lim_{x\to 0^+} y\right] = -2$$

$$\lim_{x\to 0^+} y = e^{-2} \quad \text{or} \quad \lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x} = e^{-2}$$

Thus, $\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n = e^{-2}$.

58. Let $f(x) = \left(\frac{2+x^2}{3+x^2}\right)^x$.

Using the fact that $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right)$, we can write

$$\lim_{x\to\infty}\left(\frac{2+x^2}{3+x^2}\right)^x = \lim_{x\to 0^+}\left(\frac{2+\frac{1}{x^2}}{3+\frac{1}{x^2}}\right)^{1/x}$$

$= \lim_{x\to 0^+}\left(\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+1}{x^2}}\right)^{1/x} = \lim_{x\to 0^+}\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x}$ which leads to the indeterminate form 1^∞.

Let $y = \left(\frac{2x^2+1}{3x^2+1}\right)^{1/x}$. Then,

$$\ln y = \ln\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x}$$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{1}{x}\ln\left(\frac{2x^2+1}{3x^2+1}\right)$$

$$\ln\left[\lim_{x\to 0^+} y\right] = \lim_{x\to 0^+}\frac{\ln\left(\frac{2x^2+1}{3x^2+1}\right)}{x}$$

$$= \lim_{x\to 0^+}\left[\frac{4x}{2x^2+1} - \frac{6x}{3x^2+1}\right] \quad \text{(l'Hopital's Rule)}$$

$= 0$

This gives us,

$$\ln\left[\lim_{x\to 0^+} y\right] = 0$$

$$\lim_{x\to 0^+} y = e^0 = 1 \quad \text{or} \quad \lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x} = 1$$

Thus,

$$\lim_{n\to\infty}\left(\frac{2+n^2}{3+n^2}\right)^n = 1.$$

59. Let $f(x)=\left(\frac{2+x^2}{3+x^2}\right)^{x^2}$

Using the fact that $\lim_{x\to\infty} f(x)=\lim_{x\to 0^+} f\left(\frac{1}{x}\right)$, we can write

$$\lim_{x\to\infty}\left(\frac{2+x^2}{3+x^2}\right)^{x^2} = \lim_{x\to 0^+}\left(\frac{2+\frac{1}{x^2}}{3+\frac{1}{x^2}}\right)^{1/x^2}$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+1}{x^2}}\right)^{1/x^2} = \lim_{x\to 0^+}\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2}$$ which

leads to the indeterminate form 1^∞.

Let $y=\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2}$. Then,

$$\ln y=\ln\left(\frac{2x^2+1}{3x^2+1}\right)^{1/x^2}$$

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{1}{x^2}\ln\left(\frac{2x^2+1}{3x^2+1}\right)$$

$$\ln\left[\lim_{x\to 0^+} y\right]=\lim_{x\to 0^+}\frac{\ln\left(\frac{2x^2+1}{3x^2+1}\right)}{x^2}$$

$$=\lim_{x\to 0^+}\left[\frac{-1}{\left(2x^2+1\right)\left(3x^2+1\right)}\right] \quad \text{(l'Hopital's Rule)}$$

$$=-1$$

This gives us,

$$\ln\left[\lim_{x\to 0^+} y\right]=-1$$

$$\lim_{x\to 0^+} y=e^{-1} \quad \text{or} \quad \lim_{x\to 0^+}\left(\frac{1-x}{1+x}\right)^{1/x^2}=e^{-1}$$

Thus,

$$\lim_{n\to\infty}\left(\frac{2+n^2}{3+n^2}\right)^{n^2}=e^{-1}.$$

9.2 Concepts Review

1. an infinite series

2. $a_1+a_2+\ldots+a_n$

3. $|r|<1; \frac{a}{1-r}$

4. diverges

Problem Set 9.2

1. $\sum_{k=1}^{\infty}\left(\frac{1}{7}\right)^k=\frac{1}{7}+\frac{1}{7}\cdot\frac{1}{7}+\frac{1}{7}\left(\frac{1}{7}\right)^2+\ldots$; a geometric series with $a=\frac{1}{7}, r=\frac{1}{7}$; $S=\frac{\frac{1}{7}}{1-\frac{1}{7}}=\frac{\frac{1}{7}}{\frac{6}{7}}=\frac{1}{6}$

2. $\sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)^{-k-2}=\left(-\frac{1}{4}\right)^{-3}+\left(-\frac{1}{4}\right)^{-4}+\left(-\frac{1}{4}\right)^{-5}+\ldots$
$=(-4)^3+(-4)^4+(-4)^5+\ldots$; a geometric series with
$a=(-4)^3, r=-4$; $|r|=4>1$ so the series diverges.

3. $\sum_{k=0}^{\infty}2\left(\frac{1}{4}\right)^k=2+2\cdot\frac{1}{4}+2\left(\frac{1}{4}\right)^2+\ldots$; a geometric series with a = 2, $r=\frac{1}{4}$; $S=\frac{2}{1-\frac{1}{4}}=\frac{2}{\frac{3}{4}}=\frac{8}{3}$.

$\sum_{k=0}^{\infty}3\left(-\frac{1}{5}\right)^k=3-3\cdot\frac{1}{5}+3\left(\frac{1}{5}\right)^2-\ldots$; a geometric series with a = 3, $r=-\frac{1}{5}$;

$$S=\frac{3}{1-\left(-\frac{1}{5}\right)}=\frac{3}{\frac{6}{5}}=\frac{5}{2}$$

Thus, by Theorem B,

$$\sum_{k=0}^{\infty}\left[2\left(\frac{1}{4}\right)^k+3\left(-\frac{1}{5}\right)^k\right]=\frac{8}{3}+\frac{5}{2}=\frac{31}{6}$$

4. $\sum_{k=1}^{\infty}5\left(\frac{1}{2}\right)^k=\frac{5}{2}+\frac{5}{2}\cdot\frac{1}{2}+\frac{5}{2}\left(\frac{1}{2}\right)^2+\ldots$; a geometric series with $a=\frac{5}{2}, r=\frac{1}{2}$; $S=\frac{\frac{5}{2}}{1-\frac{1}{2}}=\frac{\frac{5}{2}}{\frac{1}{2}}=5$.

$\sum_{k=1}^{\infty}3\left(\frac{1}{7}\right)^{k+1}=\frac{3}{49}+\frac{3}{49}\cdot\frac{1}{7}+\frac{3}{49}\left(\frac{1}{7}\right)^2+\ldots$; a geometric series with $a=\frac{3}{49}, r=\frac{1}{7}$;

$$S=\frac{\frac{3}{49}}{1-\frac{1}{7}}=\frac{\frac{3}{49}}{\frac{6}{7}}=\frac{1}{14}$$

Thus, by Theorem B,

$$\sum_{k=1}^{\infty}\left[2\left(\frac{1}{4}\right)^k-3\left(\frac{1}{7}\right)^{k+1}\right]=5-\frac{1}{14}=\frac{69}{14}.$$

5. $\sum_{k=1}^{\infty}\frac{k-5}{k+2}=-\frac{4}{3}-\frac{3}{4}-\frac{2}{5}-\frac{1}{6}+0+\frac{1}{8}+\frac{2}{9}+\ldots;$

$\lim_{k\to\infty}\frac{k-5}{k+2}=\lim_{k\to\infty}\frac{1-\frac{5}{k}}{1+\frac{2}{k}}=1\neq 0;$ the series diverges.

6. $\sum_{k=1}^{\infty}\left(\frac{9}{8}\right)^k=\frac{9}{8}+\frac{9}{8}\cdot\frac{9}{8}+\frac{9}{8}\left(\frac{9}{8}\right)^2+\ldots;$ a geometric series with $a=\frac{9}{8}, r=\frac{9}{8}; \left|\frac{9}{8}\right|>1,$ so the series diverges.

7. $\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right)=\left(\frac{1}{2}-\frac{1}{1}\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{3}\right)+\ldots;$

$S_n=\left(\frac{1}{2}-1\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n-2}\right)+\left(\frac{1}{n}-\frac{1}{n-1}\right)=-1+\frac{1}{n};$

$\lim_{n\to\infty}S_n=\lim_{n\to\infty}-1+\frac{1}{n}=-1,$ so $\sum_{k=2}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1}\right)=-1$

8. $\sum_{k=1}^{\infty}\frac{3}{k}=3\sum_{k=1}^{\infty}\frac{1}{k}$ which diverges since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

9. $\sum_{k=1}^{\infty}\frac{k!}{100^k}=\frac{1}{100}+\frac{2}{10{,}000}+\frac{6}{1{,}000{,}000}+\ldots$

Consider $\{a_n\}$, where $a_{n+1}=\frac{n+1}{100}a_n, a_1=\frac{1}{100}$. $a_n>0$ for all n, and for $n>99$, $a_{n+1}>a_n$, so the sequence is eventually an increasing sequence, hence $\lim_{n\to\infty}a_n\neq 0$. The sequence can also be described by $a_n=\frac{n!}{100^n}$, hence $\sum_{k=1}^{\infty}\frac{k!}{100^k}$ diverges.

10. $\sum_{k=1}^{\infty}\frac{2}{(k+2)k}=\frac{2}{3}+\frac{2}{8}+\frac{2}{15}+\ldots=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+2}\right)=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots$

$S_n=\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)$

$=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}=\frac{3}{2}-\frac{2n+3}{n^2+3n+2}$

$\lim_{n\to\infty}S_n=\frac{3}{2}-\lim_{n\to\infty}\frac{2n+3}{n^2+3n+2}=\frac{3}{2}-\lim_{n\to\infty}\frac{\frac{2}{n}+\frac{3}{n^2}}{1+\frac{3}{n}+\frac{2}{n^2}}=\frac{3}{2},$ so $\sum_{k=1}^{\infty}\frac{2}{(k+2)k}=\frac{3}{2}.$

11. $\sum_{k=1}^{\infty}\left(\frac{e}{\pi}\right)^{k+1}=\left(\frac{e}{\pi}\right)^2+\left(\frac{e}{\pi}\right)^2\cdot\frac{e}{\pi}+\left(\frac{e}{\pi}\right)^2\left(\frac{e}{\pi}\right)^2+\ldots;$ a geometric series with $a=\left(\frac{e}{\pi}\right)^2, r=\frac{e}{\pi}<1;$

$S=\frac{\left(\frac{e}{\pi}\right)^2}{1-\frac{e}{\pi}}=\frac{\left(\frac{e}{\pi}\right)^2}{\frac{\pi-e}{\pi}}=\frac{e^2}{\pi(\pi-e)}\approx 5.5562$

12. $\sum_{k=1}^{\infty}\frac{4^{k+1}}{7^{k-1}}=\frac{16}{1}+16\cdot\frac{4}{7}+16\left(\frac{4}{7}\right)^2+\ldots;$ a geometric series with $a=16$, $r=\frac{4}{7}<1;$ $S=\frac{16}{1-\frac{4}{7}}=\frac{16}{\frac{3}{7}}=\frac{112}{3}$

13. $\sum_{k=2}^{\infty}\left(\frac{3}{(k-1)^2}-\frac{3}{k^2}\right)=\left(\frac{3}{1}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{3}{9}\right)+\left(\frac{3}{9}-\frac{3}{16}\right)+\dots;$

$S_n=\left(3-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{3}{16}\right)+\dots+\left(\frac{3}{(n-2)^2}-\frac{3}{(n-1)^2}\right)+\left(\frac{3}{(n-1)^2}-\frac{3}{n^2}\right)$

$=3-\frac{3}{n^2};\ \lim_{n\to\infty}S_n=3-\lim_{n\to\infty}\frac{3}{n^2}=3,$ so

$\sum_{k=2}^{\infty}\left(\frac{3}{(k-1)^2}-\frac{3}{k^2}\right)=3.$

14. $\sum_{k=6}^{\infty}\frac{2}{k-5}=\frac{2}{1}+\frac{2}{2}+\frac{2}{3}+\frac{2}{4}+\dots$

$=2\sum_{k=1}^{\infty}\frac{1}{k}$ which diverges since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

15. $0.22222\ldots=\sum_{k=1}^{\infty}\frac{2}{10}\left(\frac{1}{10}\right)^{k-1}$

$=\frac{\frac{2}{10}}{1-\frac{1}{10}}=\frac{2}{9}$

16. $0.21212121\ldots=\sum_{k=1}^{\infty}\frac{21}{100}\left(\frac{1}{100}\right)^{k-1}$

$=\frac{\frac{21}{100}}{1-\frac{1}{100}}=\frac{21}{99}=\frac{7}{33}$

17. $0.013013013...=\sum_{k=1}^{\infty}\frac{13}{1000}\left(\frac{1}{1000}\right)^{k-1}$

$=\frac{\frac{13}{1000}}{1-\frac{1}{1000}}=\frac{13}{999}$

18. $0.125125125...=\sum_{k=1}^{\infty}\frac{125}{1000}\left(\frac{1}{1000}\right)^{k-1}$

$=\frac{\frac{125}{1000}}{1-\frac{1}{1000}}=\frac{125}{999}$

19. $0.4999...=\frac{4}{10}+\sum_{k=1}^{\infty}\frac{9}{100}\left(\frac{1}{10}\right)^{k-1}$

$=\frac{4}{10}+\frac{\frac{9}{100}}{1-\frac{1}{10}}=\frac{1}{2}$

20. $0.36717171...=\frac{36}{100}+\sum_{k=1}^{\infty}\frac{71}{10,000}\left(\frac{1}{100}\right)^{k-1}$

$=\frac{36}{100}+\frac{\frac{71}{10,000}}{1-\frac{1}{100}}=\frac{727}{1980}$

21. Let $s = 1 - r$, so $r = 1 - s$. Since $0 < r < 2$, $-1 < 1 - r < 1$, so

$|s|<1,$ and $\sum_{k=0}^{\infty}r(1-r)^k=\sum_{k=0}^{\infty}(1-s)s^k$

$=\sum_{k=1}^{\infty}(1-s)s^{k-1}=\frac{1-s}{1-s}=1$

22. $\sum_{k=0}^{\infty}(-1)^k x^k=\sum_{k=0}^{\infty}(-x)^k=\sum_{k=1}^{\infty}(-x)^{k-1};$
if $-1 < x < 1$ then
$-1 < -x < 1$ so $|x|<1$;

$\sum_{k=1}^{\infty}(-x)^{k-1}=\frac{1}{1-(-x)}=\frac{1}{1+x}$

23. $\ln \frac{k}{k+1} = \ln k - \ln(k+1)$

$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \ldots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) \;\; = \ln 1 - \ln(n+1) = -\ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1) = -\infty$, thus $\sum_{k=1}^{\infty} \ln \frac{k}{k+1}$ diverges.

24. $\ln\left(1 - \frac{1}{k^2}\right) = \ln \frac{k^2-1}{k^2} = \ln(k^2-1) - \ln k^2 \;\; = \ln[(k+1)(k-1)] - \ln k^2 = \ln(k+1) + \ln(k-1) - 2\ln k$

$S_n = (\ln 3 + \ln 1 - 2\ln 2) + (\ln 4 + \ln 2 - 2\ln 3) + (\ln 5 + \ln 3 - 2\ln 4) + \ldots$

$+(\ln n + \ln(n-2) - 2\ln(n-1)) + (\ln(n+1) + \ln(n-1) - 2\ln n)$

$= -\ln 2 + \ln(n+1) - \ln n \; = -\ln 2 + \ln \frac{n+1}{n}$

$\lim_{n\to\infty} S_n = -\ln 2 + \lim_{n\to\infty} \ln \frac{n+1}{n} \;\; = -\ln 2 + \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right) = -\ln 2 + \ln 1 = -\ln 2$

25. The ball drops 100 feet, rebounds up $100\left(\frac{2}{3}\right)$ feet, drops $100\left(\frac{2}{3}\right)$ feet, rebounds up $100\left(\frac{2}{3}\right)^2$ feet, drops $100\left(\frac{2}{3}\right)^2$, etc. The total distance it travels is

$100 + 200\left(\frac{2}{3}\right) + 200\left(\frac{2}{3}\right)^2 + 200\left(\frac{2}{3}\right)^3 + \ldots \;\; = -100 + 200 + 200\left(\frac{2}{3}\right) + 200\left(\frac{2}{3}\right)^2 + 200\left(\frac{2}{3}\right)^3 + \ldots$

$= -100 + \sum_{k=1}^{\infty} 200\left(\frac{2}{3}\right)^{k-1} \;\; = -100 + \frac{200}{1-\frac{2}{3}} = 500$ feet

26. Each gets $\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\cdot\frac{1}{4}\right) + \ldots = \sum_{k=1}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^{k-1} \;\; = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$

(This can be seen intuitively, since the size of the leftover piece is approaching 0, and each person gets the same amount.)

27. \$1 billion + 75% of \$1 billion + 75% of 75% of \$1 billion + ... = $\sum_{k=1}^{\infty} (\$1 \text{ billion})0.75^{k-1} = \frac{\$1 \text{ billion}}{1-0.75} = \$4 \text{ billion}$

28. $\sum_{k=1}^{\infty} \$1 \text{ billion } (0.90)^{k-1} \; = \frac{\$1 \text{ billion}}{1-0.90} \; = \$10 \text{ billion}$

29. As the midpoints of the sides of a square are connected, a new square is formed. The new square has sides $\frac{1}{\sqrt{2}}$ times the sides of the old square. Thus, the new square has area $\frac{1}{2}$ the area of the old square. Then in the next step, $\frac{1}{8}$ of each new square is shaded.

Area = $\frac{1}{8}\cdot 1 + \frac{1}{8}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{1}{4} + \ldots = \sum_{k=1}^{\infty} \frac{1}{8}\left(\frac{1}{2}\right)^{k-1} \;\; = \frac{\frac{1}{8}}{1-\frac{1}{2}} = \frac{1}{4}$

The area will be $\frac{1}{4}$.

30. $\frac{1}{9}+\frac{1}{9}\left(\frac{8}{9}\right)+\frac{1}{9}\left(\frac{8}{9}\cdot\frac{8}{9}\right)+\cdots=\sum_{k=1}^{\infty}\frac{1}{9}\left(\frac{8}{9}\right)^{k-1} = \frac{\frac{1}{9}}{1-\frac{8}{9}}=1$; the whole square will be painted.

31. $\frac{3}{4}+\frac{3}{4}\left(\frac{1}{4}\cdot\frac{1}{4}\right)+\frac{3}{4}\left(\frac{1}{4}\cdot\frac{1}{4}\right)\left(\frac{1}{4}\cdot\frac{1}{4}\right)+\ldots=\sum_{k=1}^{\infty}\frac{3}{4}\left(\frac{1}{16}\right)^{k-1} = \frac{\frac{3}{4}}{1-\frac{1}{16}}=\frac{4}{5}$

The original does not need to be equilateral since each smaller triangle will have $\frac{1}{4}$ area of the previous larger triangle.

32. Ratio of inscribed circle to triangle is $\frac{\pi}{3\sqrt{3}}$, so $\sum_{k=1}^{\infty}\frac{\pi}{3\sqrt{3}}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^{k-1} = \frac{\left(\frac{\pi}{4\sqrt{3}}\right)}{1-\frac{1}{4}}=\frac{\pi}{3\sqrt{3}}$

(This can be seen intuitively, since *every* small triangle has a circle inscribed in it.)

33. **a.** We first note that, at each stage, the number of sides is four times the number in the previous stage and the length of each side is one-third the length in the previous stage. Summarizing:

Stage	# of sides	length/ side (in.)	perimeter p_n
0	3	9	27
1	3(4)	$9\left(\frac{1}{3}\right)$	36
⋮	⋮	⋮	⋮
n	$3(4^n)$	$9\left(\frac{1}{3^n}\right)$	$27\left(\frac{4}{3}\right)^n$

The perimeter of the Koch snowflake is $\lim_{n\to\infty} p_n = \lim_{n\to\infty} 27\left(\frac{4}{3}\right)^n$ which is infinite since $\frac{4}{3}>1$.

b. We note the following:

1. The area of an equilateral triangle of side s is $\frac{\sqrt{3}}{4}s^2$

2. The number of new triangles added at each stage is equal to the number of sides the figure had at the previous stage and

3. the area of each new triangle at a given stage is $\frac{\sqrt{3}}{4}(\text{side length at that stage})^2$. Using results from part a. we can summarize:

Stage	Additional triangles (col 2, part *a.*)	Area of each new Δ (see col 3, part *a.*)	Additional area, A_n
0	original	$\frac{\sqrt{3}}{4}\left(9^2\right)$	$\frac{\sqrt{3}}{4}\left(9^2\right)$
1	3	$\frac{\sqrt{3}}{4}\left(3^2\right)$	36
⋮	⋮	⋮	⋮
n	$3\left(4^{n-1}\right)$	$\frac{\sqrt{3}}{4}\left(\frac{9}{3^n}\right)^2$	$3\sqrt{3}\left(\frac{4}{9}\right)^{n-2}$

Thus the area of the Koch snowflake is

$$\sum_{n=0}^{\infty} A_n = \frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + \sum_{n=1}^{\infty} 3\sqrt{3}\left(\frac{4}{9}\right)^{n-1}$$

$$= \frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + \left(\frac{3\sqrt{3}}{\left(1-\frac{4}{9}\right)}\right)$$

$$= \frac{81\sqrt{3}}{4} + \frac{1}{3}\left(\frac{81\sqrt{3}}{4}\right) + \frac{4}{15}\left(\frac{81\sqrt{3}}{4}\right) = \frac{8}{5}\left(\frac{81\sqrt{3}}{4}\right)$$

Note: By generalizing the above argument it can be shown that, no matter what the size of the original equilateral triangle, the area of the Koch snowflake constructed from it will be $\frac{8}{5}$ times the area of the original triangle.

34. We note the following:

1. Each triangle contains the angles $90, \theta, 90-\theta$ 2. The height of each triangle will be the hypotenuse of the succeeding triangle. Summarizing:

# triangle	base	height	area A_n
1	$h\cos\theta$	$h\sin\theta$	$\frac{1}{2}h^2\sin\theta\cos\theta$
2	$h\sin\theta\cos\theta$	$h\sin^2\theta$	$\frac{1}{2}h^2\sin^3\theta\cos\theta$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$h(\sin^{n-1}\theta)\cos\theta$	$h\sin^n\theta$	$\frac{1}{2}h^2\sin^{2n-1}\theta\cos\theta$

Thus the total area of the small triangles is $A = \sum_{n=1}^{\infty} A_n = \frac{h^2}{2}\left(\frac{\cos\theta}{\sin\theta}\right)\sum_{n=2}^{\infty}(\sin^2\theta)^{n-1}$

Now consider the infinite geometric series $S = \sum_{n=1}^{\infty}(\sin^2\theta)^{n-1} = \frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}$

then: $\sum_{n=2}^{\infty}(\sin^2\theta)^{n-1} = S-1 = \frac{1}{\cos^2\theta} - 1 = \frac{\sin^2\theta}{\cos^2\theta}$ Therefore: $A = \frac{h^2}{2}\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{\sin^2\theta}{\cos^2\theta}\right) = \frac{h^2}{2}\tan\theta$

In $\triangle ABC$, height = h and base = $h\tan\theta$; thus the area of $\triangle ABC = \frac{1}{2}(h\tan\theta)h = \frac{h^2}{2}\tan\theta$, the same as A.

35. Both Achilles and the tortoise will have moved.

$$100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \ldots = \sum_{k=1}^{\infty} 100\left(\frac{1}{10}\right)^{k-1}$$

$$= \frac{100}{1-\frac{1}{10}} = 111\frac{1}{9} \text{ yards}$$

Also, one can see this by the following reasoning. In the time it takes the tortoise to run $\frac{d}{10}$ yards, Achilles will run d yards. Solve $d = 100 + \frac{d}{10}. d = \frac{1000}{9} = 111\frac{1}{9}$ yards

36. a. Say Trot and Tom start from the left, Joel from the right. Trot and Joel run towards each other at 30 mph. Since they are 60 miles apart they will meet in 2 hours. Trot will have run 40 miles and Tom will have run 20 miles, so they will be 20 miles apart. Trot and Tom will now be approaching each other at 30 mph, so they will meet after 2/3 hour. Trot will have run another 40/3 miles and will be 80/3 miles from the left. Joel will have run another 20/3 miles and will be at 100/3 miles from the left, so they will be 20/3 miles apart. They will meet after 2/9 hour, during which Trot will have run 40/9 miles, etc. So Trot runs

$$40+\frac{40}{3}+\frac{40}{9}+\cdots=\sum_{k=1}^{\infty}40\left(\frac{1}{3}\right)^{k-1}=\frac{40}{1-\frac{1}{3}}$$

$= 60$ miles.

b. Tom and Joel are approaching each other at 20 mph. They are 60 miles apart, so they will meet in 3 hours. Trot is running at 20 mph during that entire time, so he runs 60 miles.

37. Note that:

1. If we let t_n be the probability that Peter wins on his nth flip, then the total probability that Peter wins is $T=\sum_{n=1}^{\infty}t_n$

2. The probability that neither man wins in their first k flips is $\left(\frac{2}{3}\cdot\frac{2}{3}\right)^k=\left(\frac{4}{9}\right)^k$.

3. The probability that Peter wins on his nth flip requires that (i) he gets a head on the nth flip, and (ii) neither he nor Paul gets a head on their previous n-1 flips. Thus:

$$t_n=\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^{n-1} \text{ and } T=\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)^{n-1}=$$

$$\left(\frac{1/3}{(1-4/9)}\right)=\frac{1}{3}\cdot\frac{9}{5}=\frac{3}{5}$$

38. In this case (see problem 37),

$$t_n=p\left[(1-p)^2\right]^{n-1} \text{ so}$$

$$T=\sum_{n=1}^{\infty}p\left[(1-p)^2\right]^{n-1}=\frac{p}{\left(1-(1-p)^2\right)}$$

$$=\frac{p}{\left(2p-p^2\right)}=\frac{1}{2-p}$$

39. Let X = *number of rolls needed to get first 6* For X to equal n, two things must occur:

1. Mary must get a non-6 (probability = $\frac{5}{6}$) on each of her first n-1 rolls, and

2. Mary rolls a 6 (probability = $\frac{1}{6}$) on the nth roll. Thus,

$$\Pr(X=n)=\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) \text{ and}$$

$$\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)=\frac{1/6}{(1-5/6)}=1$$

40.

$$EV(X)=\sum_{n=1}^{\infty}n\cdot\Pr(X=n)=\sum_{n=1}^{\infty}n\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^{n-1}$$

$$=\frac{1}{6}\left(\frac{1}{p}\right)\sum_{n=1}^{\infty}n\cdot p^n=\frac{1}{6}\left(\frac{6}{5}\right)\left(\frac{5/6}{(1-5/6)^2}\right)$$

$$=\frac{1}{6}\left(\frac{6}{5}\right)\left(\frac{5}{6}\right)\left(\frac{36}{1}\right)=6$$

41. (Proof by contradiction) Assume $\sum_{k=1}^{\infty}ca_k$ converges, and $c\neq 0$. Then $\frac{1}{c}$ is defined, so $\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{\infty}\frac{1}{c}ca_k=\frac{1}{c}\sum_{k=1}^{\infty}ca_k$ would also converge, by Theorem B(i).

42. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots=\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{1}{k}\right)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

43. a. The top block is supported *exactly* at its center of mass. The location of the center of mass of the top n blocks is the average of the locations of their individual centers of mass, so the nth block moves the center of mass left by $\frac{1}{n}$ of the location of its center of mass, that is, $\frac{1}{n}\cdot\frac{1}{2}$ or $\frac{1}{2n}$ to the left. But this is exactly how far the $(n + 1)$st block underneath it is offset.

b. Since $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$, which diverges, there is no limit to how far the top block can protrude.

44. $N = 31$; $S_{31} \approx 4.0272$ and $S_{30} \approx 3.9950$.

45. (Proof by contradiction) Assume $\sum_{k=1}^{\infty}(a_k + b_k)$ converges. Since $\sum_{k=1}^{\infty} b_k$ converges, so would $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(a_k + b_k) + (-1)\sum_{k=1}^{\infty} b_k$, by Theorem B(ii).

46. (Answers may vary). $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(-1)\frac{1}{n}$ both diverge, but $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n}\right)$ converges to 0.

47. Taking vertical strips, the area is

$$1\cdot 1 + 1\cdot\frac{1}{2} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{8} + \cdots = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}.$$

Taking horizontal strips, the area is

$$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \cdots = \sum_{k=1}^{\infty}\frac{k}{2^k}.$$

a. $\sum_{k=1}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1} = \frac{1}{1-\frac{1}{2}} = 2$

b. The moment about $x = 0$ is

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}\cdot(1)k = \sum_{k=0}^{\infty}\frac{k}{2^k} = \sum_{k=1}^{\infty}\frac{k}{2^k} = 2.$$

$$\overline{x} = \frac{\text{moment}}{\text{area}} = \frac{2}{2} = 1$$

48. If $\sum_{k=1}^{\infty} kr^k$ converges, so will $r\sum_{k=1}^{\infty} kr^k$, by Theorem B.

$$rS = r\sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty} kr^{k+1} = \sum_{k=2}^{\infty}(k-1)r^k \text{ while}$$

$$S = \sum_{k=1}^{\infty} kr^k = r + \sum_{k=2}^{\infty} kr^k \text{ so}$$

$$S - rS = r + \sum_{k=2}^{\infty} kr^k - \sum_{k=2}^{\infty}(k-1)r^k$$

$$= r + \sum_{k=2}^{\infty}[k-(k-1)]r^k = r + \sum_{k=2}^{\infty} r^k = \sum_{k=1}^{\infty} r^k$$

Since $|r| < 1$, $\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$, thus

$$S = \frac{1}{1-r}\sum_{k=1}^{\infty} r^k = \frac{r}{(1-r)^2}.$$

49. a. $A = \sum_{n=0}^{\infty} Ce^{-nkt} = \sum_{n=1}^{\infty} C\left(\frac{1}{e^{kt}}\right)^{n-1}$

$$= \frac{C}{1-\frac{1}{e^{kt}}} = \frac{Ce^{kt}}{e^{kt}-1}$$

b. $\frac{1}{2} = e^{-kt} = e^{-6k} \Rightarrow k = \frac{\ln 2}{6} \Rightarrow A = \frac{4}{3}C$;

if $C = 2$ mg, then $A = \frac{8}{3}$ mg.

50. Using partial fractions, $\frac{2^k}{(2^{k+1}-1)(2^k-1)} = \frac{1}{2^k-1} - \frac{1}{2^{k+1}-1}$

$$S_n = \left(\frac{1}{2^1-1} - \frac{1}{2^2-1}\right) + \left(\frac{1}{2^2-1} - \frac{1}{2^3-1}\right) + \ldots + \left(\frac{1}{2^{n-1}-1} - \frac{1}{2^n-1}\right) + \left(\frac{1}{2^n-1} - \frac{1}{2^{n+1}-1}\right)$$

$$= \frac{1}{2-1} - \frac{1}{2^{n+1}-1} = 1 - \frac{1}{2^{n+1}-1}$$

$$\lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty}\frac{1}{2^{n+1}-1} = 1 - 0 = 1$$

51. $\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}} = \frac{f_{k+2} - f_k}{f_k f_{k+1} f_{k+2}} = \frac{1}{f_k f_{k+2}}$

since $f_{k+2} = f_{k+1} + f_k$. Thus,

$\sum_{k=1}^{\infty} \frac{1}{f_k f_{k+2}} = \sum_{k=1}^{\infty} \left(\frac{1}{f_k f_{k+1}} - \frac{1}{f_{k+1} f_{k+2}} \right)$ and

$S_n = \left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} \right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4} \right) + \cdots + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) + \left(\frac{1}{f_n f_{n+1}} - \frac{1}{f_{n+1} f_{n+2}} \right)$

$= \frac{1}{f_1 f_2} - \frac{1}{f_{n+1} f_{n+2}} = \frac{1}{1 \cdot 1} - \frac{1}{f_{n+1} f_{n+2}} = 1 - \frac{1}{f_{n+1} f_{n+2}}$

The terms of the Fibonacci sequence increase without bound, so

$\lim_{n \to \infty} S_n = 1 - \lim_{n \to \infty} \frac{1}{f_{n+1} f_{n+2}} = 1 - 0 = 1$

9.3 Concepts Review

1. bounded above

2. $f(k)$; continuous; positive; nonincreasing

3. convergence or divergence

4. $p > 1$

Problem Set 9.3

1. $\frac{1}{x+3}$ is continuous, positive, and nonincreasing on $[0, \infty)$.

$\int_0^{\infty} \frac{1}{x+3} dx = \left[\ln|x+3| \right]_0^{\infty} = \infty - \ln 3 = \infty$

The series diverges.

2. $\frac{3}{2x-3}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

$\int_2^{\infty} \frac{3}{2x-3} dx = \left[\frac{3}{2} \ln|2x-3| \right]_2^{\infty} = \infty - \frac{3}{2} \ln 1 = \infty$

The series diverges.

3. $\frac{x}{x^2+3}$ is continuous, positive, and nonincreasing on $[2, \infty)$.

$\int_2^{\infty} \frac{x}{x^2+3} dx = \left[\frac{1}{2} \ln\left|x^2+3\right| \right]_2^{\infty} = \infty - \frac{1}{2} \ln 7 = \infty$

The series diverges.

4. $\frac{3}{2x^2+1}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{3}{2x^2+1} dx = \left[\frac{3}{\sqrt{2}} \tan^{-1} \sqrt{2}x \right]_1^{\infty}$

$= \frac{3}{\sqrt{2}} \left(\frac{\pi}{2} - \tan^{-1} \sqrt{2} \right) < \infty$

The series converges.

5. $\frac{2}{\sqrt{x+2}}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{2}{\sqrt{x+2}} dx = \left[4\sqrt{x+2} \right]_1^{\infty} = \infty - 4\sqrt{3} = \infty$

Thus $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k+2}}$ diverges, hence

$\sum_{k=1}^{\infty} \frac{-2}{\sqrt{k+2}} = -\sum_{k=1}^{\infty} \frac{2}{\sqrt{k+2}}$ also diverges.

6. $\frac{3}{(x+2)^2}$ is continuous, positive, and nonincreasing on $[100, \infty)$.

$\int_{100}^{\infty} \frac{3}{(x+2)^2} dx = \left[-\frac{3}{x+2} \right]_{100}^{\infty} = 0 + \frac{3}{102} = \frac{3}{102} < \infty$

The series converges.

7. $\frac{7}{4x+2}$ is continuous, positive, and nonincreasing on. $[2,\infty)$

$$\int_2^\infty \frac{7}{4x+2}dx = \left[\frac{7}{4}\ln|4x+2|\right]_2^\infty = \infty - \frac{7}{4}\ln 10 = \infty$$

The series diverges.

8. $\frac{x^2}{e^x}$ is continuous, positive, and nonincreasing $[2,\infty)$. Using integration by parts twice, with $u = x^i$, i = 1, 2 and $dv = e^{-x}dx$,

$$\int_2^\infty x^2e^{-x}dx = [-x^2e^{-x}]_2^\infty + 2\int_2^\infty xe^{-x}dx$$

$$= [-x^2e^{-x}]_2^\infty + 2\left([-xe^{-x}]_2^\infty + \int_2^\infty e^{-x}dx\right)$$

$$= [-x^2e^{-x} - 2xe^{-x} - 2e^{-x}]_2^\infty$$

$$= 0 + 4e^{-2} + 4e^{-2} + 2e^{-2} = 10e^{-2} < \infty$$

The series converges.

9. $\frac{3}{(4+3x)^{7/6}}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$$\int_1^\infty \frac{3}{(4+3x)^{7/6}}dx = \left[-\frac{6}{(4+3x)^{1/6}}\right]_1^\infty$$

$$= 0 + \frac{6}{7^{1/6}} = 6\cdot 7^{-1/6} < \infty$$

The series converges.

10. $\frac{1000x^2}{1+x^3}$ is continuous, positive, and nonincreasing on $[2,\infty)$.

$$\int_2^\infty \frac{1000x^2}{1+x^3}dx = \left[\frac{1000}{3}\ln\left|1+x^3\right|\right]_2^\infty$$

$$= \infty - \frac{1000}{3}\ln 9 = \infty$$

The series diverges.

11. xe^{-3x^2} is continuous, positive, and nonincreasing on $[1,\infty)$.

$$\int_1^\infty xe^{-3x^2}dx = \left[-\frac{1}{6}e^{-3x^2}\right]_1^\infty = 0 + \frac{1}{6}e^{-3}$$

$$= \frac{1}{6e^3} < \infty$$

The series converges.

12. $\frac{1000}{x(\ln x)^2}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$$\int_5^\infty \frac{1000}{x(\ln x)^2}dx = \left[-\frac{1000}{\ln x}\right]_5^\infty = 0 + \frac{1000}{\ln 5}$$

$$= \frac{1000}{\ln 5} < \infty$$

The series converges.

13. $\lim_{k\to\infty}\frac{k^2+1}{k^2+5} = \lim_{k\to\infty}\frac{1+\frac{1}{k^2}}{1+\frac{5}{k^2}} = 1 \neq 0$, so the series diverges.

14. $\sum_{k=1}^\infty \left(\frac{3}{\pi}\right)^k = \sum_{k=1}^\infty \frac{3}{\pi}\left(\frac{3}{\pi}\right)^{k-1}$; a geometric series with $a = \frac{3}{\pi}, r = \frac{3}{\pi}$; $\left|\frac{3}{\pi}\right| < 1$ so the series converges.

15. $\sum_{k=1}^\infty \left(\frac{1}{2}\right)^k$ is a geometric series with $r = \frac{1}{2}; \left|\frac{1}{2}\right| < 1$ so the series converges.

In $\sum_{k=1}^\infty \frac{k-1}{2k+1}$, $\lim_{k\to\infty}\frac{k-1}{2k+1} = \lim_{k\to\infty}\frac{1-\frac{1}{k}}{2+\frac{1}{k}} = \frac{1}{2} \neq 0$, so the series diverges. Thus, the sum of the series diverges.

16. $\frac{1}{x^2}$ is continuous, positive, and nonincreasing on $[1,\infty)$. $\int_1^\infty \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^\infty = 0 + 1 = 1 < \infty$, so $\sum_{k=1}^\infty \frac{1}{k^2}$ converges.

$\sum_{k=1}^\infty \frac{1}{2^k} = \sum_{k=1}^\infty \left(\frac{1}{2}\right)^k$; a geometric series with $r = \frac{1}{2}; \left|\frac{1}{2}\right| < 1$, so the series converges. Thus, the sum of the series converges.

17. $\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 1 & k = 4j+1 \\ -1 & k = 4j+3, \\ 0 & k \text{ is even} \end{cases}$

where j is any nonnegative integer.

Thus $\lim_{k\to\infty}\left|\sin\left(\frac{k\pi}{2}\right)\right|$ does not exist, hence

$\lim_{k\to\infty}\left|\sin\left(\frac{k\pi}{2}\right)\right| \neq 0$ and the series diverges.

18. As $k \to \infty, \frac{1}{k} \to 0.$ Let $y = \frac{1}{k},$ then

$\lim_{k\to\infty} k\sin\frac{1}{k} = \lim_{y\to 0}\frac{1}{y}\sin y = \lim_{y\to 0}\frac{\sin y}{y} = 1 \neq 0,$ so the series diverges.

19. $x^2e^{-x^3}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$\int_1^\infty x^2e^{-x^3}dx = \left[-\frac{1}{3}e^{-x^3}\right]_1^\infty = 0 + \frac{1}{3}e^{-1} < \infty,$ so the series converges.

20. $S_n = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n-1}\right) = 1-\frac{1}{n-1}$

$\lim_{n\to\infty} S_n = 1 - \lim_{n\to\infty}\frac{1}{n-1} = 1-0 = 1$

The series converges to 1.

21. $\frac{\tan^{-1} x}{1+x^2}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$\int_1^\infty \frac{\tan^{-1} x}{1+x^2}dx = \left[\frac{1}{2}(\tan^{-1} x)^2\right]_1^\infty$

$= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{3\pi^2}{32} < \infty,$ so the series converges.

22. $\frac{1}{1+4x^2}$ is continuous, positive, and nonincreasing on $[1,\infty)$.

$\int_1^\infty \frac{1}{1+4x^2}dx = \left[\frac{1}{2}\tan^{-1}(2x)\right]_1^\infty$

$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\tan^{-1} 2 < \infty,$

so the series converges.

23. $\frac{x}{e^x}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$E = \sum_{k=6}^\infty \frac{k}{e^k} \le \int_5^\infty \frac{x}{e^x}dx = [-xe^{-x}]_5^\infty + \int_5^\infty e^{-x}dx$

$= [-xe^{-x} - e^{-x}]_5^\infty = 0 + 5e^{-5} + e^{-5} = 6e^{-5}$

≈ 0.0404

24. $\frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$E = \sum_{k=6}^\infty \frac{1}{k\sqrt{k}} \le \int_5^\infty \frac{1}{x^{3/2}}dx = \left[-\frac{2}{\sqrt{x}}\right]_5^\infty = 0 + \frac{2}{\sqrt{5}}$

≈ 0.8944

25. $\frac{1}{1+x^2}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$E = \sum_{k=6}^\infty \frac{1}{1+k^2} \le \int_5^\infty \frac{1}{1+x^2}dx = [\tan^{-1} x]_5^\infty$

$= \frac{\pi}{2} - \tan^{-1} 5 \approx 0.1974$

26. $\frac{1}{x(x+1)}$ is continuous, positive, and nonincreasing on $[5,\infty)$.

$E = \sum_{k=6}^\infty \frac{1}{k(k+1)} \le \int_5^\infty \frac{1}{x(x+1)}dx = \int_5^\infty\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$

$= \big[\ln|x| - \ln|x+1|\big]_5^\infty = \left[\ln\left|\frac{x}{x+1}\right|\right]_5^\infty = 0 - \ln\frac{5}{6}$

$= \ln\frac{6}{5} \approx 0.1823$

27. $E_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \int_n^{\infty} \frac{1}{x^2} dx = \lim_{A\to\infty} \int_n^A \frac{1}{x^2} dx =$

$\lim_{A\to\infty} \left[\frac{1}{n} - \frac{1}{A}\right] = \frac{1}{n}$

$\frac{1}{n} < 0.0002 \Rightarrow n > 5000$

28. $E_n = \sum_{k=n+1}^{\infty} \frac{1}{k^3} < \int_n^{\infty} \frac{1}{x^3} dx = \lim_{A\to\infty} \int_n^A \frac{1}{x^3} dx =$

$\lim_{A\to\infty} \left[\frac{1}{2n^2} - \frac{1}{2A^2}\right] = \frac{1}{2n^2}$

$\frac{1}{2n^2} < 0.0002 \Rightarrow n > \frac{1}{\sqrt{0.0004}} = 50$

29. $E_n = \sum_{k=n+1}^{\infty} \frac{1}{1+k^2} < \int_n^{\infty} \frac{1}{1+x^2} dx$

$= \lim_{A\to\infty} \int_n^A \frac{1}{1+x^2} dx = \lim_{A\to\infty} \left[\tan^{-1} A - \tan^{-1} n\right]$

$= \frac{\pi}{2} - \tan^{-1} n$

$\frac{\pi}{2} - \tan^{-1} n < 0.0002 \Rightarrow \tan^{-1} n > \frac{\pi}{2} - 0.0002$

$\Rightarrow n > \tan\left(\frac{\pi}{2} - 0.0002\right) \approx 5000$

30. $E_n = \sum_{k=n+1}^{\infty} \frac{k}{e^{k^2}} < \int_n^{\infty} \frac{x}{e^{x^2}} dx =$

$u = x^2$
$du = 2x\,dx$

$\lim_{A\to\infty} \frac{1}{2} \int_{n^2}^{A} \frac{1}{e^u} du =$

$\left(-\frac{1}{2}\right) \lim_{A\to\infty} \left[\frac{1}{e^A} - \frac{1}{e^{n^2}}\right] = \frac{1}{2e^{n^2}}$

$\frac{1}{2e^{n^2}} < 0.0002 \Rightarrow n > \sqrt{\ln \frac{1}{0.0004}}$

$\approx 2.797 \qquad \therefore n > 2$

31. $E_n = \sum_{k=n+1}^{\infty} \frac{k}{1+k^4} < \int_n^{\infty} \frac{x}{1+x^4} dx = \lim_{A\to\infty} \frac{1}{2} \int_{n^2}^{A} \frac{du}{1+u^2}$

$u=x^2$
$du=2xdx$

$= \frac{1}{2} \lim_{A\to\infty} \left[\tan^{-1} A - \tan^{-1} n^2\right] = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-2}\left(n^2\right)\right]$

$\frac{1}{2}\left[\frac{\pi}{2} - \tan^{-2}\left(n^2\right)\right] < 0.0002$

$\Rightarrow \frac{\pi}{2} - \tan^{-2}\left(n^2\right) < 0.0004 \Rightarrow \tan^{-1}\left(n^2\right) > 1.5703963$

$\Rightarrow n > \sqrt{\tan\left(1.5703963\right)} \approx 50$

32. $E_n = \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} < \int_n^{\infty} \frac{1}{x(x+1)} dx =$

$\lim_{A\to\infty} \int_n^A \left(\frac{1}{x} - \frac{1}{x+1}\right) dx =$

$\lim_{A\to\infty} \left[\ln\left(\frac{A}{A+1}\right) - \ln\left(\frac{n}{n+1}\right)\right] = 0 - \ln\left(\frac{n}{n+1}\right) =$

$\ln\left(\frac{n+1}{n}\right)$

$\ln\left(\frac{n+1}{n}\right) < 0.0002 \Rightarrow 1 + \frac{1}{n} < e^{0.0002} \approx 1.0002$

$\Rightarrow n > \frac{1}{0.0002} = 5000$

33. Consider $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx.$ Let $u = \ln x$,

$du = \frac{1}{x} dx.$

$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\infty} \frac{1}{u^p} du$ which converges for $p > 1$.

34. $\frac{1}{x \ln x \ln(\ln x)}$ is continuous, positive, and nonincreasing on $[3, \infty)$.

$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx$

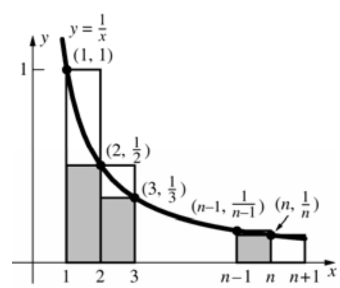
Let $u = \ln(\ln x)$, $du = \frac{1}{x \ln x} dx.$

$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx = \int_{\ln(\ln 3)}^{\infty} \frac{1}{u} du = [\ln u]_{\ln(\ln 3)}^{\infty}$

$= \infty - \ln(\ln(\ln 3)) = \infty$

The series diverges.

35.



The upper rectangles, which extend to $n + 1$ on the right, have area $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. These rectangles are above the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n + 1$. Thus,

$\int_1^{n+1} \frac{1}{x} dx = [\ln x]_1^{n+1} = \ln(n+1) - \ln 1 = \ln(n+1)$

$< 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$

The lower (shaded) rectangles have area $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. These rectangles lie below the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n$. Thus

$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \int_1^n \frac{1}{x} dx = \ln n$, so

$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < 1 + \ln n.$

36. From Problem 35, B_n is the area of the region within the upper rectangles but above the curve $y = \frac{1}{x}$. Each time n is incremented by 1, the added area is a positive amount, thus B_n is increasing.

From the inequalities in Problem 35,

$0 < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n+1) < 1 + \ln n - \ln(n+1)$

$= 1 + \ln \frac{n}{n+1}$

Since $\frac{n}{n+1} < 1, \ln \frac{n}{n+1} < 0$, thus $B_n < 1$ for all n, and B_n is bounded by 1.

37. $\{B_n\}$ is a nondecreasing sequence that is bounded above, thus by the Monotonic Sequence Theorem (Theorem D of Section 9.1), $\lim_{n\to\infty} B_n$ exists. The rationality of γ is a famous unsolved problem.

38. From Problem 35, $\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k} < 1 + \ln n$, thus

$\ln(10{,}000{,}001) \approx 16.1181 < \sum_{k=1}^{10{,}000{,}000} \frac{1}{k}$

$< 1 + \ln(10{,}000{,}000) \approx 17.1181$

39. $\gamma + \ln(n+1) > 20 \Rightarrow \ln(n+1) > 20 - \gamma \approx 19.4228$

$\Rightarrow n + 1 > e^{19.4228} \approx 272{,}404{,}867$

$\Rightarrow n > 272{,}404{,}866$

40. **a.** Each time n is incremented by 1, a positive amount of area is added.

b. The leftmost rectangle has area $1 \cdot f(1) = f(1)$. If each shaded region to the right of $x = 2$ is shifted until it is in the leftmost rectangle, there will be no overlap of the shaded area, since the top of each rectangle is at the bottom of the shaded region to the left. Thus, the total shaded area is less than or equal to the area of the leftmost rectangle, or $B_n \le f(1)$.

c. By parts a and b, $\{B_n\}$ is a nondecreasing sequence that is bounded above, so $\lim_{n\to\infty} B_n$ exists.

d. Let $f(x) = \frac{1}{x}$, then

$\int_1^{n+1} f(x) dx = \int_1^{n+1} \frac{1}{x} dx = \ln(n+1)$ and

$\lim_{n\to\infty} B_n = \gamma$ as defined in Problem 37.

41. Every time n is incremented by 1, a positive amount of area is added, thus $\{A_n\}$ is an increasing sequence. Each curved region has horizontal width 1, and can be moved into the heavily outlined triangle without any overlap. This can be done by shifting the nth shaded region, which goes from $(n, f(n))$ to $(n + 1, f(n + 1))$, as follows:

shift $(n + 1, f(n + 1))$ to $(2, f(2))$ and $(n, f(n))$ to $(1, f(2)–[f(n + 1) – f(n)])$.

The slope of the line forming the bottom of the shaded region between $x = n$ and $x = n + 1$ is

$\frac{f(n+1) - f(n)}{(n+1) - n} = f(n+1) - f(n) > 0$

since f is increasing.

By the Mean Value Theorem,

$f(n+1) - f(n) = f'(c)$ for some c in $(n, n + 1)$.

Since f is concave down, $n < c < n + 1$ means that $f'(c) < f'(b)$ for all b in $[1, n]$. Thus, the nth shaded region will not overlap any other shaded region when shifted into the heavily outlined triangle. Thus, the area of all of the shaded regions is less than or equal to the area of the heavily outlined triangle, so $\lim_{n\to\infty} A_n$ exists.

42. $\ln x$ is continuous, increasing, and concave down on $[1,\infty)$, so the conditions of Problem 41 are met.

a. See the figure in the text for Problem 41. The area under the curve from $x = 1$ to $x = n$ is $\int_1^n \ln x\,dx$ and the area of the nth trapezoid is $\frac{\ln n+\ln(n+1)}{2}$, thus $A_n = \int_1^n \ln x\,dx - \left[\frac{\ln 1+\ln 2}{2}+\ldots+\frac{\ln(n-1)+\ln(n)}{2}\right]$.

Using integration by parts with $u = \ln x$, $du = \frac{1}{x}dx$, $dv = dx$, $v = x$

$$\int_1^n \ln x\,dx = [x\ln x]_1^n - \int_1^n dx = [x\ln x - x]_1^n = n\ln n - n - (\ln 1 - 1) = n\ln n - n + 1$$

The sum of the areas of the n trapezoids is

$$\frac{\ln 1+\ln 2}{2}+\frac{\ln 2+\ln 3}{2}+\ldots+\frac{\ln(n-2)+\ln(n-1)}{2}+\frac{\ln(n-1)+\ln(n)}{2} = \frac{2\ln 2+2\ln 3+\ldots+2\ln(n-1)+\ln n}{2}$$

$$= \ln 2+\ln 3+\ldots+\ln n - \frac{\ln n}{2} = \ln(2\cdot 3\cdot\ldots\cdot n) - \frac{\ln n}{2} = \ln n! - \ln\sqrt{n}$$

Thus, $A_n = n\ln n - n + 1 - \left(\ln n! - \ln\sqrt{n}\right) = n\ln n - n + 1 - \ln n! + \ln\sqrt{n} = \ln n^n - \ln e^n + 1 - \ln n! + \ln\sqrt{n}$

$$= \ln\left(\frac{n}{e}\right)^n + 1 + \ln\frac{\sqrt{n}}{n!} = 1 + \ln\left[\left(\frac{n}{e}\right)^n \frac{\sqrt{n}}{n!}\right]$$

b. By Problem 41, $\lim_{n\to\infty} A_n$ exists, hence part a says that $\lim_{n\to\infty}\left[1+\ln\left[\left(\frac{n}{e}\right)^n\frac{\sqrt{n}}{n!}\right]\right]$ exists.

$$\lim_{n\to\infty}\left[1+\ln\left[\left(\frac{n}{e}\right)^n\frac{\sqrt{n}}{n!}\right]\right] = 1+\lim_{n\to\infty}\ln\left[\left(\frac{n}{e}\right)^n\frac{\sqrt{n}}{n!}\right] = 1+\ln\left[\lim_{n\to\infty}\left(\frac{n}{e}\right)^n\frac{\sqrt{n}}{n!}\right]$$

Since the limit exists, $\lim_{n\to\infty}\left(\frac{n}{e}\right)^n\frac{\sqrt{n}}{n!} = m$. m cannot be 0 since $\lim_{x\to 0^+}\ln x = -\infty$.

Thus, $\lim_{n\to\infty}\frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\frac{\left(\frac{n}{e}\right)^n\sqrt{n}}{n!}} = \frac{1}{\lim_{n\to\infty}\left(\frac{n}{e}\right)^n\frac{\sqrt{n}}{n!}} = \frac{1}{m}$, i.e., the limit exists.

c. From part b, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, thus, $15! \approx \sqrt{30\pi}\left(\frac{15}{e}\right)^{15} \approx 1.3004\times 10^{12}$

The exact value is $15! = 1{,}307{,}674{,}368{,}000$.

43. (Refer to *fig 2* in the text). Let $b_k = \int_k^{k+1} f(x)\,dx$; then from *fig 2*, it is clear that $a_k \ge b_k$ for $k = 1, 2, \ldots, n, \ldots$

Therefore $\sum_{k=n+1}^{t} a_k \ge \sum_{k=n+1}^{t} b_k = \int_{n+1}^{t} f(x)\,dx$ so that

$$E_n = \sum_{k=n+1}^{\infty} a_k = \lim_{t\to\infty}\sum_{k=n+1}^{t} a_k \ge \lim_{t\to\infty}\int_{n+1}^{t} f(x)\,dx = \int_{n+1}^{\infty} f(x)\,dx.$$

9.4 Concepts Review

1. $0 \le a_k \le b_k$

2. $\lim_{k\to\infty} \frac{a_k}{b_k}$

3. $\rho < 1$; $\rho > 1$; $\rho = 1$

4. Ratio; Limit Comparison

Problem Set 9.4

1. $a_n = \frac{n}{n^2+2n+3}; b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+3} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{3}{n^2}} = 1;$$

$0 < 1 < \infty$

$\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

2. $a_n = \frac{3n+1}{n^3-4}; b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^3+n^2}{n^3-4} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1-\frac{4}{n^3}} = 3;$$

$0 < 3 < \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

3. $a_n = \frac{1}{n\sqrt{n+1}} = \frac{1}{\sqrt{n^3+n^2}}; \quad b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+n^2}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+n^2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1; \; 0 < 1 < \infty$$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

4. $a_n = \frac{\sqrt{2n+1}}{n^2}; b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{2n+1}}{n^2} = \lim_{n\to\infty} \sqrt{\frac{2n^4+n^3}{n^4}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{2+\frac{1}{n}}{1}} = \sqrt{2}; 0 < \sqrt{2} < \infty$$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

5. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{8^{n+1}n!}{(n+1)!8^n} = \lim_{n\to\infty} \frac{8}{n+1} = 0 < 1$

The series converges.

6. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1}n^5}{(n+1)^5 5^n} = \lim_{n\to\infty} \frac{5n^5}{(n+1)^5}$

$$= \lim_{n\to\infty} \frac{5n^5}{n^5+5n^4+10n^3+10n^2+5n+1}$$

$$= \lim_{n\to\infty} \frac{5}{1+\frac{5}{n}+\frac{10}{n^2}+\frac{10}{n^3}+\frac{5}{n^4}+\frac{1}{n^5}} = 5 > 1$$

The series diverges.

7. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!n^{100}}{(n+1)^{100}n!} = \lim_{n\to\infty} \frac{n^{100}}{(n+1)^{99}}$

$$= \lim_{n\to\infty} \frac{n}{\left(\frac{n+1}{n}\right)^{99}} = \infty \text{ since } \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{99} = 1$$

The series diverges.

8. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\left(\frac{1}{3}\right)^{n+1}}{n\left(\frac{1}{3}\right)^n} = \lim_{n\to\infty} \frac{n+1}{3n}$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3} = \frac{1}{3} < 1$$

The series converges.

9. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3(2n)!}{(2n+2)!n^3}$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{(2n+2)(2n+1)n^3} = \lim_{n\to\infty} \frac{n^3+3n^2+3n+1}{4n^5+6n^4+2n^3}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{3}{n^3}+\frac{3}{n^4}+\frac{1}{n^5}}{4+\frac{6}{n}+\frac{2}{n^2}} = 0 < 1$$

The series converges.

10. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(3^{n+1}+n+1)n!}{(n+1)!(3^n+n)}$

$$= \lim_{n\to\infty} \frac{3^{n+1}+n+1}{(3^n+n)(n+1)} = \lim_{n\to\infty} \frac{3^{n+1}+n+1}{n3^n+3^n+n^2+n}$$

$$= \lim_{n\to\infty} \frac{3+\frac{n}{3^n}+\frac{1}{3^n}}{n+1+\frac{n^2}{3^n}+\frac{n}{3^n}} = 0 < \infty \text{ since } \lim_{n\to\infty} \frac{n}{3^n} = 0$$

and $\lim_{n\to\infty} \frac{n^2}{3^n} = 0$ which can be seen by using l'Hôpital's Rule. The series converges.

11. $\lim_{n\to\infty} \frac{n}{n+200} = \lim_{n\to\infty} \frac{1}{1+\frac{200}{n}} = 1 \neq 0$

The series diverges; nth-Term Test

12. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!(5+n)}{(6+n)n!}$

$= \lim_{n\to\infty} \frac{(n+1)(5+n)}{6+n} = \lim_{n\to\infty} \frac{n^2+6n+5}{6+n}$

$= \lim_{n\to\infty} \frac{n+6+\frac{5}{n}}{\frac{6}{n}+1} = \infty > 1$

The series diverges; Ratio Test

13. $a_n = \frac{n+3}{n^2\sqrt{n}}; b_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{5/2}+3n^{3/2}}{n^{5/2}} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{1} = 1;$

$0 < 1 < \infty$. $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

14. $a_n = \frac{\sqrt{n+1}}{n^2+1}; b_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n+1}}{n^2+1} = \lim_{n\to\infty} \frac{\sqrt{n^4+n^3}}{n^2+1}$

$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{1+\frac{1}{n^2}} = 1; 0 < 1 < \infty.$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

15. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 n!}{(n+1)!n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{(n+1)n^2}$

$= \lim_{n\to\infty} \frac{n^2+2n+1}{n^3+n^2} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{2}{n^2}+\frac{1}{n^3}}{1+\frac{1}{n}} = 0 < 1$

The series converges; Ratio Test

16. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\ln(n+1)2^n}{2^{n+1}\ln n} = \lim_{n\to\infty} \frac{\ln(n+1)}{2\ln n}$

Using l'Hôpital's Rule,

$\lim_{n\to\infty} \frac{\ln(n+1)}{2\ln n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{2}{n}} = \lim_{n\to\infty} \frac{n}{2(n+1)}$

$= \lim_{n\to\infty} \frac{1}{2+\frac{2}{n}} = \frac{1}{2} < 1.$

The series converges; Ratio Test

17. $a_n = \frac{4n^3+3n}{n^5-4n^2+1}; b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n^5+3n^3}{n^5-4n^2+1} = \lim_{n\to\infty} \frac{4+\frac{3}{n^2}}{1-\frac{4}{n^3}+\frac{1}{n^5}} = 4;$

$0 < 4 < \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

18. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)^2+1]3^n}{3^{n+1}(n^2+1)}$

$= \lim_{n\to\infty} \frac{n^2+2n+2}{3n^2+3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{2}{n^2}}{3+\frac{3}{n^2}} = \frac{1}{3} < 1$

The series converges; Ratio Test

19. $a_n = \frac{1}{n(n+1)} = \frac{1}{n^2+n}; b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1;$

$0 < 1 < \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

20. $a_n = \frac{n}{(n+1)^2} = \frac{n}{n^2+2n+1}; b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1;$

$0 < 1 < \infty$

$\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges; Limit Comparison Test

21. $a_n = \frac{n+1}{n(n+2)(n+3)} = \frac{n+1}{n^3+5n^2+6n}; b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+5n^2+6n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{5}{n}+\frac{6}{n^2}} = 1;$

$0 < 1 < \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges; Limit Comparison Test

22. $a_n = \frac{n}{n^2+1}; b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1; 0 < 1 < \infty$$

$\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges;
Limit Comparison Test

23. $a_n = \frac{n}{3^n}; \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)3^n}{3^{n+1}n}$

$$= \lim_{n\to\infty} \frac{n+1}{3n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3} = \frac{1}{3} < 1$$

The series converges; Ratio Test

24. $a_n = \frac{3^n}{n!};$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}n!}{(n+1)!3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

The series converges; Ratio Test

25. $a_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}; \frac{1}{x^{3/2}}$ is continuous, positive, and nonincreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^{3/2}}\,dx = \left[-\frac{2}{\sqrt{x}}\right]_1^{\infty} = 0 + 2 = 2 < \infty$$

The series converges; Integral Test

26. $a_n = \frac{\ln n}{n^2}; \frac{\ln x}{x^2}$ is continuous, positive, and nonincreasing on $[2, \infty)$. Use integration by parts with $u = \ln x$ and $dv = \frac{1}{x^2}dx$ for

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_2^{\infty} + \int_2^{\infty} \frac{1}{x^2}\,dx$$

$$= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^{\infty} = 0 + \frac{\ln 2}{2} + \frac{1}{2} < \infty$$

$\left(\lim_{x\to\infty} \frac{\ln x}{x} = 0 \text{ by l'Hôpital's Rule.}\right)$

The series converges; Integral Test

27. $0 \le \sin^2 n \le 1$ for all n, so

$2 \le 2 + \sin^2 n \le 3 \Rightarrow \frac{1}{2} \ge \frac{1}{2+\sin^2 n} \ge \frac{1}{3}$ for all n.

Thus, $\lim_{n\to\infty} \frac{1}{2+\sin^2 n} \ne 0$ and the series diverges; nth-Term Test

28. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5(3^n+1)}{(3^{n+1}+1)5} = \lim_{n\to\infty} \frac{1+\frac{1}{3^n}}{3+\frac{1}{3^n}}$

$$= \frac{1}{3} < 1$$

The series converges; Ratio Test

29. $-1 \le \cos n \le 1$ for all n, so

$3 \le 4 + \cos n \le 5 \Rightarrow \frac{3}{n^3} \le \frac{4+\cos n}{n^3} \le \frac{5}{n^3}$ for all n.

$\sum_{n=1}^{\infty} \frac{5}{n^3}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{4+\cos n}{n^3}$ converges;
Comparison Test

30. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{2n+2}n!}{(n+1)!5^{2n}} = \lim_{n\to\infty} \frac{25}{n+1} = 0 < 1$

The series converges; Ratio Test

31. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}(2n)!}{(2n+2)!n^n}$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(2n+2)(2n+1)n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{2(n+1)(2n+1)n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{2(2n+1)n^n} = \lim_{n\to\infty} \left[\frac{1}{4n+2}\left(\frac{n+1}{n}\right)^n\right]$$

$$= \left[\lim_{n\to\infty} \frac{1}{4n+2}\right]\left[\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n\right] = 0 \cdot e = 0 < 1$$

(The limits can be separated since both limits exist.) The series converges; Ratio Test

32. Let $y = \left(1 - \frac{1}{x}\right)^x; \ln y = x \ln\left(1 - \frac{1}{x}\right)$

$$\lim_{x\to\infty} x \ln\left(1 - \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1/x^2}{\left(1-\frac{1}{x}\right)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{1}{\left(1-\frac{1}{x}\right)} = -1$$

Thus $\lim_{x\to\infty} y = e^{-1}$, so $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$.

The series diverges; nth-Term Test

33. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(4^{n+1}+n+1)n!}{(n+1)!(4^n+n)}$

$= \lim_{n\to\infty} \frac{4^{n+1}+n+1}{(n+1)(4^n+n)} = \lim_{n\to\infty} \frac{4+\frac{n}{4^n}+\frac{1}{4^n}}{(n+1)\left(1+\frac{n}{4^n}\right)}$

$= \lim_{n\to\infty} \frac{4+\frac{n}{4^n}+\frac{1}{4^n}}{1+n+\frac{n}{4^n}+\frac{n^2}{4^n}} = 0$

since $\lim_{n\to\infty} \frac{n^2}{4^n} = 0$, $\lim_{n\to\infty} \frac{n}{4^n} = 0$, and

$\lim_{n\to\infty} \frac{1}{4^n} = 0$. The series converges; Ratio Test

34. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)(2+n5^n)}{[2+(n+1)5^{n+1}]n}$

$= \lim_{n\to\infty} \frac{2n+n^2 5^n+2+n5^n}{2n+n^2 5^{n+1}+n5^{n+1}}$

$= \lim_{n\to\infty} \frac{\frac{2}{n5^n}+1+\frac{2}{n^2 5^n}+\frac{1}{n}}{\frac{2}{n5^n}+5+\frac{5}{n}} = \frac{1}{5} < 1$

The series converge; Ratio Test

35. Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Thus, there is some positive integer N such that $0 < a_n < 1$ for all $n \geq N$. $a_n < 1 \Rightarrow a_n{}^2 < a_n$, thus $\sum_{n=N}^{\infty} a_n{}^2 < \sum_{n=N}^{\infty} a_n$. Hence $\sum_{n=N}^{\infty} a_n{}^2$ converges, and $\sum a_n{}^2$ also converges, since adding a finite number of terms does not affect the convergence or divergence of a series.

36. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by Example 7, thus

$\lim_{n\to\infty} \frac{n!}{n^n} = 0$ by the nth-Term Test.

37. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then there is some positive integer N such that $0 \leq \frac{a_n}{b_n} < 1$ for all $n \geq N$. Thus, for $n \geq N$, $a_n < b_n$. By the Comparison Test, since $\sum_{n=N}^{\infty} b_n$ converges, $\sum_{n=N}^{\infty} a_n$ also converges. Thus, $\sum a_n$ converges since adding a finite number of terms will not affect the convergence or divergence of a series.

38. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then there is some positive integer N such that $\frac{a_n}{b_n} > 1$ for all $n \geq N$. Thus, for $n \geq N$, $a_n > b_n$ and by the Comparison Test, since $\sum_{n=N}^{\infty} b_n$ diverges, $\sum_{n=N}^{\infty} a_n$ also diverges. Thus, $\sum a_n$ diverges since adding a finite number of terms will not affect the convergence or divergence of a series.

39. If $\lim_{n\to\infty} na_n = 1$ then there is some positive integer N such that $a_n \geq 0$ for all $n \geq N$, Let $b_n = \frac{1}{n}$, so $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} na_n = 1 < \infty$.

Since $\sum_{n=N}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=N}^{\infty} a_n$ diverges by the Limit Comparison Test.
Thus $\sum a_n$ diverges since adding a finite number of terms will not affect the convergence or divergence of a series.

40. Consider $f(x) = x - \ln(1+x)$, then

$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ on $(0, \infty)$.

$f(0) = 0 - \ln 1 = 0$, so since $f(x)$ is increasing, $f(x) > 0$ on $(0, \infty)$, i.e., $x > \ln(1+x)$ for $x > 0$. Thus, since a_n is a series of positive terms, $\sum \ln(1+a_n) < \sum a_n$, hence if $\sum a_n$ converges, $\sum \ln(1+a_n)$ also converges.

41. Suppose that $\lim_{n\to\infty} (a_n)^{1/n} = R$ where $a_n > 0$.
If $R < 1$, there is some number r with $R < r < 1$ and some positive integer N such that $\left|(a_n)^{1/n} - R\right| < r - R$ for all $n \geq N$. Thus,

$R - r < (a_n)^{1/n} - R < r - R$ or

$-r < (a_n)^{1/n} < r < 1$. Since $a_n > 0$,

$0 < (a_n)^{1/n} < r$ and $0 < a_n < r^n$ for all $n \geq N$.

Thus, $\sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} r^n$, which converges since $|r| < 1$. Thus, $\sum_{n=N}^{\infty} a_n$ converges so $\sum a_n$ also converges.
If $R > 1$, there is some number r with $1 < r < R$ and some positive integer N such that

$\left|(a_n)^{1/n} - R\right| < R - r$ for all $n \geq N$. Thus,

$r - R < (a_n)^{1/n} - R < R - r$ or

$r < (a_n)^{1/n} < 2R - r$ for all $n \geq N$. Hence

$r^n < a_n$ for all $n \geq N$, so $\sum_{n=N}^{\infty} r^n < \sum_{n=N}^{\infty} a_n$, and

since $\sum_{n=N}^{\infty} r^n$ diverges ($r > 1$), $\sum_{n=N}^{\infty} a_n$ also diverges, so $\sum a_n$ diverges.

42. **a.** $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{1}{\ln n}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{1}{\ln n}$

$= 0 < 1$

The series converges.

b. $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n}{3n+2}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty}\frac{n}{3n+2} = \lim_{n\to\infty}\frac{1}{3+\frac{2}{n}} = \frac{1}{3} < 1$

The series converges.

c. $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{1}{2}+\frac{1}{n}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{n}\right) = \frac{1}{2} < 1$

The series converges.

43. **a.** $\ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$

$S_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \ldots$

$+ (\ln n - \ln(n-1)) + (\ln(n+1) - \ln n)$

$= -\ln 1 + \ln(n+1) = \ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\ln(n+1) = \infty$

Since the partial sums are unbounded, the series diverges.

b. $\ln\frac{(n+1)^2}{n(n+2)} = 2\ln(n+1) - \ln n - \ln(n+2)$

$S_n = (2\ln 2 - \ln 1 - \ln 3) + (2\ln 3 - \ln 2 - \ln 4)$

$+ (2\ln 4 - \ln 3 - \ln 5) + \ldots$

$+ (2\ln n - \ln(n-1) - \ln(n+1))$

$+ (2\ln(n+1) - \ln n - \ln(n+2))$

$= \ln 2 - \ln 1 + \ln (n + 1) - \ln (n + 2)$

$= \ln 2 + \ln\frac{n+1}{n+2}$

$\lim_{n\to\infty} S_n = \ln 2 + \lim_{n\to\infty}\ln\frac{n+1}{n+2} = \ln 2$

Since the partial sums converge, the series converges.

c. $\left(\frac{1}{\ln x}\right)^{\ln x}$ is continuous, positive, and nonincreasing on $[2, \infty)$, thus $\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}\right)^{\ln n}$ converges if and only if $\int_2^{\infty}\left(\frac{1}{\ln x}\right)^{\ln x} dx$ converges.

Let $u = \ln x$, so $x = e^u$ and $dx = e^u du$.

$\int_2^{\infty}\left(\frac{1}{\ln x}\right)^{\ln x} dx = \int_{\ln 2}^{\infty}\left(\frac{1}{u}\right)^u e^u du = \int_{\ln 2}^{\infty}\left(\frac{e}{u}\right)^u du$

This integral converges if and only if the associated series, $\sum_{n=1}^{\infty}\left(\frac{e}{n}\right)^n$ converges. With $a_n = \left(\frac{e}{n}\right)^n$, the Root Test (Problem 41)

gives $\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{e}{n}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty}\frac{e}{n} = 0 < 1$

Thus, $\sum_{n=1}^{\infty}\left(\frac{e}{n}\right)^n$ converges, so $\int_{\ln 2}^{\infty}\left(\frac{e}{u}\right)^u du$ converges, whereby $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$ converges.

d. $\left(\frac{1}{\ln(\ln x)}\right)^{\ln x}$ is continuous, positive, and nonincreasing on $[3, \infty)$, thus $\sum_{n=3}^{\infty}\left(\frac{1}{\ln(\ln n)}\right)^{\ln n}$ converges if and only if $\int_3^{\infty}\left(\frac{1}{\ln(\ln x)}\right)^{\ln x} dx$ converges.

Let $u = \ln x$, so $x = e^u$ and $dx = e^u du$.

$\int_3^{\infty}\left(\frac{1}{\ln(\ln x)}\right)^{\ln x} dx$

$= \int_{\ln 3}^{\infty}\left(\frac{1}{\ln u}\right)^u e^u du = \int_{\ln 3}^{\infty}\left(\frac{e}{\ln u}\right)^u du.$

This integral converges if and only if the associated series, $\sum_{n=2}^{\infty}\left(\frac{e}{\ln n}\right)^n$ converges.

With $a_n=\left(\frac{e}{\ln n}\right)^n$, the Root Test (Problem 41) gives

$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left[\left(\frac{e}{\ln n}\right)^n\right]^{1/n}.$$

$$=\lim_{n\to\infty}\frac{e}{\ln n}=0<1$$

Thus, $\sum_{n=2}^{\infty}\left(\frac{e}{\ln n}\right)^n$ converges, so

$\int_{\ln 3}^{\infty}\left(\frac{e}{\ln u}\right)^u du$ converges, whereby

$\sum_{n=3}^{\infty}\frac{1}{(\ln(\ln n))^{\ln n}}$ converges.

e. $a_n=1/n;\ b_n=1/(\ln n)^4$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1/n}{1/(\ln n)^4}=\lim_{n\to\infty}\frac{(\ln n)^4}{n}$$

$$=\lim_{n\to\infty}\frac{4(\ln n)^3(1/n)}{1}=\lim_{n\to\infty}\frac{4(\ln n)^3}{n}$$

$$=\lim_{n\to\infty}\frac{12(\ln n)^2(1/n)}{1}=\lim_{n\to\infty}\frac{12(\ln n)^2}{n}$$

$$=\lim_{n\to\infty}\frac{24(\ln n)}{n}=\lim_{n\to\infty}\frac{24(1/n)}{1}$$

$$=\lim_{n\to\infty}\frac{24}{n}=0$$

$\sum_{n=2}^{\infty}\frac{1}{n}$ diverges $\Rightarrow\sum_{n=2}^{\infty}\frac{1}{(\ln n)^4}$ diverges

f. $\left(\frac{\ln x}{x}\right)^2$ is continuous, positive, and nonincreasing on $[3,\infty)$. Using integration by parts twice,

$$\int_3^{\infty}\left(\frac{\ln x}{x}\right)^2 dx=\left[-\frac{(\ln x)^2}{x}\right]_3^{\infty}+\int_3^{\infty}\frac{2\ln x}{x^2}dx$$

$$=\left[-\frac{(\ln x)^2}{x}\right]_3^{\infty}+\left[-\frac{2\ln x}{x}\right]_3^{\infty}+\int_3^{\infty}\frac{2}{x^2}dx$$

$$=\left[-\frac{(\ln x)^2}{x}-\frac{2\ln x}{x}-\frac{2}{x}\right]_3^{\infty}\approx 1.8<\infty$$

Thus, $\sum_{n=3}^{\infty}\left(\frac{\ln x}{x}\right)^2$ converges.

44. The degree of $p(n)$ must be at least 2 less than the degree of $q(n)$. If $p(n)$ and $q(n)$ have the same degree, r, then

$p(n)=c_rn^r+c_{r-1}n^{r-1}+\ldots+c_1n+c_0$ and

$q(n)=d_rn^r+d_{r-1}n^{r-1}+\ldots+d_1n+d_0$ where $c_r, d_r\neq 0$ and

$$\lim_{n\to\infty}\frac{p(n)}{q(n)}=\lim_{n\to\infty}\frac{c_rn^r+c_{r-1}n^{r-1}+\ldots+c_1n+c_0}{d_rn^r+d_{r-1}n^{r-1}+\ldots+d_1n+d_0}=\lim_{n\to\infty}\frac{c_r+\frac{c_{r-1}}{n}+\ldots+\frac{c_1}{n^{r-1}}+\frac{c_0}{n^r}}{d_r+\frac{d_{r-1}}{n}+\ldots+\frac{d_1}{n^{r-1}}+\frac{d_0}{n^r}}=\frac{c_r}{d_r}\neq 0.$$

Thus, the series diverges by the nth-Term Test. If the degree of $p(n)$ is r and the degree of $q(n)$ is s, then the Limit Comparison Test with $a_n=\frac{p(n)}{q(n)}, b_n=\frac{1}{n^{s-r}}$ will give $\lim_{n\to\infty}\frac{a_n}{b_n}=L$ with $0<L<\infty$, since $\frac{a_n}{b_n}=\frac{n^{s-r}p(n)}{q(n)}$ and the degrees of $n^{s-r}p(n)$ and $q(n)$ are the same, similar to the previous case. Since $0<L<\infty$, a_n and b_n either both converge or both diverge.

If $s\geq r+2$, then $s-r\geq 2$ so $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{s-r}}\leq\sum_{n=1}^{\infty}\frac{1}{n^2}$. Thus $\sum_{n=1}^{\infty}b_n$, and hence $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{p(n)}{q(n)}$ converges.

If $s<r+2$, then $s-r\leq 1$ so $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{s-r}}\geq\sum_{n=1}^{\infty}\frac{1}{n}$. Thus $\sum_{n=1}^{\infty}b_n$, and hence $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{p(n)}{q(n)}$ diverges.

45. Let $a_n = \frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}\right)$ and $b_n = \frac{1}{n^p}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}\right) = \sum_{n=1}^{\infty}\frac{1}{n^p}$ which converges if $p > 1$. Thus, by the Limit Comparison Test, if $\sum_{n=1}^{\infty} b_n$ converges for $p > 1$, so does $\sum_{n=1}^{\infty} a_n$. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\right)$ also converges. For $p \le 1$, since $1+\frac{1}{2^p}+\ldots+\frac{1}{n^p} > 1$, $\frac{1}{n^p}\left(1+\frac{1}{2^p}+\ldots+\frac{1}{n^p}\right) > \frac{1}{n^p}$. Hence, since $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges for $p \le 1$, $\sum_{n=1}^{\infty}\frac{1}{n^p}\left(1+\frac{1}{2^p}+\ldots+\frac{1}{n^p}\right)$ also diverges. The series converges for $p > 1$ and diverges for $p \le 1$.

46. a. Let $a_n = \sin^2\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n^2}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n^2\sin^2\left(\frac{1}{n}\right) = \lim_{u\to 0^+}\left(\frac{1}{u}\right)^2\sin^2 u = \lim_{u\to 0^+}\left(\frac{\sin u}{u}\right)^2 = 1$ using the substitution $u = \frac{1}{n}$. Since $0 < 1 < \infty$, both $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ converge.

b. Let $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{n\sin\left(\frac{1}{n}\right)}{\cos\left(\frac{1}{n}\right)}$

$= \lim_{u\to 0}\frac{\left(\frac{1}{u}\right)\sin u}{\cos u} = \lim_{u\to 0}\left(\frac{\sin u}{u}\cos u\right) = 1$ using the substitution $u = \frac{1}{n}$. Since $0 < 1 < \infty$, both $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$ diverge.

c. $$\sum_{n=1}^{\infty}\sqrt{n}\left(1-\cos\frac{1}{n}\right) = \sum_{n=1}^{\infty}\sqrt{n}\left(1-\cos\frac{1}{n}\right)\left(\frac{1+\cos\frac{1}{n}}{1+\cos\frac{1}{n}}\right) = \sum_{n=1}^{\infty}\frac{\sqrt{n}\left(1-\cos^2\frac{1}{n}\right)}{1+\cos\frac{1}{n}} = \sum_{n=1}^{\infty}\frac{\sqrt{n}\sin^2\frac{1}{n}}{1+\cos\frac{1}{n}} < \sum_{n=1}^{\infty}\sqrt{n}\sin^2\frac{1}{n}$$

Let $a_n = \sqrt{n}\sin^2\frac{1}{n}$ and $b_n = \frac{1}{n^{3/2}}$.

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} n^2\sin^2\frac{1}{n} = \lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)^2 = 1$, since $\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{u\to 0^+}\frac{\sin u}{u} = 1$ with $u = \frac{1}{n}$.

Thus, by the Limit Comparison Test, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty}\sqrt{n}\sin^2\frac{1}{n}$ converges, and hence, $\sum_{n=1}^{\infty}\sqrt{n}\left(1-\cos\frac{1}{n}\right)$ converges by the Comparison Test.

9.5 Concepts Review

1. $\lim_{n\to\infty} a_n = 0$

2. absolutely; conditionally

3. the alternating harmonic series

4. rearranged

Problem Set 9.5

1. $a_n = \frac{2}{3n+1}; \frac{2}{3n+1} > \frac{2}{3n+4}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{2}{3n+1} = 0.$ $S_9 \approx 0.363$. The error made by using S_9 is not more than $a_{10} \approx 0.065$.

2. $a_n = \frac{1}{\sqrt{n}}; \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} = \frac{1}{\sqrt{n}} = 0.$ $S_9 \approx 0.76695$. The error made by using S_9 is not more than $a_{10} \approx 0.31623$.

3. $a_n = \frac{1}{\ln(n+1)}; \frac{1}{\ln(n+1)} > \frac{1}{\ln(n+2)}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0.$ $S_9 \approx 1.137$. The error made by using S_9 is not more than $a_{10} \approx 0.417$.

4. $a_n = \frac{n}{n^2+1}; \frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$, so $a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{n}{n^2+1} = 0.$ $S_9 \approx 0.32153$. The error made by using S_9 is not more than $a_{10} \approx 0.09901$.

5. $a_n = \frac{\ln n}{n}; \frac{\ln n}{n} > \frac{\ln(n+1)}{n+1}$ is equivalent to $\ln \frac{n^{n+1}}{(n+1)^n} > 0$ or $\frac{n^{n+1}}{(n+1)^n} > 1$ which is true for $n > 2$. $S_9 \approx -0.041$. The error made by using S_9 is not more than $a_{10} \approx 0.230$.

6. $a_n = \frac{\ln n}{\sqrt{n}}; \frac{\ln n}{\sqrt{n}} > \frac{\ln(n+1)}{\sqrt{n+1}}$ for $n \ge 7$, so $a_n > a_{n+1}$ for $n \ge 7$. $S_9 \approx 0.17199$. The error made by using S_9 is not more than $a_{10} \approx 0.72814$.

7. $\frac{|u_{n+1}|}{|u_n|} = \frac{\left|\left(-\frac{3}{4}\right)^{n+1}\right|}{\left|\left(-\frac{3}{4}\right)^{n}\right|} = \frac{3}{4} < 1$, so the series converges absolutely.

8. $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which converges since $\frac{3}{2} > 1$, thus $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\sqrt{n}}$ converges absolutely.

9. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n}; \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$, so the series converges absolutely.

10. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \frac{(n+1)^2}{en^2}$; $\lim_{n\to\infty} \frac{(n+1)^2}{en^2} = \frac{1}{e} \approx 0.36788 < 1$, so the series converges absolutely.

11. $n(n+1) = n^2 + n > n^2$ for all $n > 0$, thus $\frac{1}{n(n+1)} < \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges since 2 > 1, thus $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$ converges absolutely.

12. $\frac{|u_{n+1}|}{|u_n|} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1}$; $\lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$, so the series converges absolutely.

13. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ which converges since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. The series is conditionally convergent since $\frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

14. $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{5n^{1.1}} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges since 1.1 > 1. The series is absolutely convergent.

15. $\lim_{n\to\infty}\frac{n}{10n+1}=\frac{1}{10}\neq 0.$ Thus the sequence of partial sums does not converge; the series diverges.

16. $\frac{n}{10n^{1.1}+1}>\frac{n+1}{10(n+1)^{1.1}+1}$, so $a_n > a_{n+1}$;

$\lim_{n\to\infty}\frac{n}{10n^{1.1}+1}=\lim_{n\to\infty}\frac{1}{10n^{0.1}+\frac{1}{n}}=0$. The alternating series converges.

Let $a_n=\frac{n}{10n^{1.1}+1}$ and $b_n=\frac{1}{n^{0.1}}$. Then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^{1.1}}{10n^{1.1}+1}=\frac{1}{10};\ \ 0<\frac{1}{10}<\infty;$ so

both $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ diverge, since $\sum_{n=1}^{\infty}\frac{1}{n^{0.1}}$ diverges. The series is conditionally convergent.

17. $\lim_{n\to\infty}\frac{1}{n\ln n}=0;\ \frac{1}{n\ln n}>\frac{1}{(n+1)\ln(n+1)}$ is equivalent to $(n+1)^{n+1}>n^n$ which is true for all $n > 0$ so $a_n > a_{n+1}$. The alternating series converges.

$\sum_{n=2}^{\infty}|u_n|=\sum_{n=2}^{\infty}\frac{1}{n\ln n};\ \frac{1}{x\ln x}$ is continuous, positive, and nonincreasing on $[2,\ \infty)$.

Using $u = \ln x,\ du=\frac{1}{x}dx,$

$\int_2^{\infty}\frac{1}{x\ln x}dx=\int_{\ln 2}^{\infty}\frac{1}{u}du=\left[\ln|u|\right]_{\ln 2}^{\infty}=\infty.$ Thus,

$\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges and $\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\ln n}$ is conditionally convergent.

18. $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{1}{n(1+\sqrt{n})}\le\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which converges since $\frac{3}{2}>1.$ The series is absolutely convergent.

19. $\frac{|u_{n+1}|}{|u_n|}=\frac{\frac{(n+1)^4}{2^{n+1}}}{\frac{n^4}{2^n}}=\frac{(n+1)^4}{2n^4};$

$\lim_{n\to\infty}\frac{(n+1)^4}{2n^4}=\frac{1}{2}<1.$

The series is absolutely convergent.

20. $a_n=\frac{1}{\sqrt{n^2-1}};\ \frac{1}{\sqrt{n^2-1}}>\frac{1}{\sqrt{n^2+2n}}$, so $a_n > a_{n+1};\ \lim_{n\to\infty}\frac{1}{\sqrt{n^2-1}}=0,$ hence the alternating series converges.

Let $b_n=\frac{1}{n},$ then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2-1}}=\lim_{n\to\infty}\frac{1}{\sqrt{1-\frac{1}{n^2}}}=1;$

$0 < 1 < \infty$.

Thus, since $\sum_{n=2}^{\infty}b_n=\sum_{n=2}^{\infty}\frac{1}{n}$ diverges,

$\sum_{n=2}^{\infty}a_n=\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-1}}$ also diverges. The series converges conditionally.

21. $a_n=\frac{n}{n^2+1};\ \frac{n}{n^2+1}>\frac{n+1}{(n+1)^2+1}$ is equivalent to $n^2+n-1>0,$ which is true for $n > 1$, so $a_n > a_{n+1};\ \lim_{n\to\infty}\frac{n}{n^2+1}=0,$ hence the alternating series converges. Let $b_n=\frac{1}{n},$ then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=1; 0<1<\infty.$ Thus, since

$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ also diverges. The series is conditionally convergent.

22. $a_n=\frac{n-1}{n};\ \lim_{n\to\infty}\frac{n-1}{n}=1\neq 0$

The series is divergent.

23. $\cos n\pi=(-1)^n=\frac{1}{(-1)}(-1)^{n+1}$ so the series is

$-1\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, -1 times the alternating harmonic series. The series is conditionally convergent.

24. $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}=1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\cdots,$ since

$\sin\frac{n\pi}{2}=\begin{cases}0 & n\text{ even}\\(-1)^{\frac{n-1}{2}} & n\text{ odd}\end{cases}.$

Thus, $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)^2}$.

$\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

$(2n-1)^2>n^2$ for $n > 1$, thus

$\sum_{n=2}^{\infty}\frac{1}{(2n-1)^2}<\sum_{n=2}^{\infty}\frac{1}{n^2}$, which converges since $2 > 1$. The series is absolutely convergent.

25. $|\sin n|\le 1$ for all n, so

$\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\frac{|\sin n|}{n\sqrt{n}}\le\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which converges since $\frac{3}{2}>1.$ Thus the series is absolutely convergent.

26. $n\sin\left(\frac{1}{n}\right)=\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$. As $n\to\infty, \frac{1}{n}\to 0$ and

$\lim_{k\to 0}\frac{\sin k}{k}=1,$ so $\lim_{n\to\infty}n\sin\left(\frac{1}{n}\right)=1.$ The series diverges.

27. $a_n=\frac{1}{\sqrt{n(n+1)}};$ $\frac{1}{\sqrt{n(n+1)}}>\frac{1}{\sqrt{(n+1)(n+2)}}$ and

$\lim_{n\to\infty}\frac{1}{\sqrt{n(n+1)}}=0$ so the alternating series converges.

Let $b_n=\frac{1}{n},$ then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1;$

$0 < 1 < \infty$.

Thus, since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$ also diverges.

The series is conditionally convergent.

28. $a_n=\frac{1}{\sqrt{n+1}+\sqrt{n}};$

$\frac{1}{\sqrt{n+1}+\sqrt{n}}>\frac{1}{\sqrt{n+2}+\sqrt{n+1}},$ so $a_n>a_{n+1};$

$\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0.$ The alternating series converges.

Let $b_n=\frac{1}{\sqrt{n}},$ then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{2};$

$0<\frac{1}{2}<\infty.$ Thus, since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges $\left(\frac{1}{2}<1\right),$ $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$ also diverges. The series is conditionally convergent.

29. $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{n^2}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n^2};$ $\lim_{n\to\infty}\frac{3^{n+1}}{n^2}\ne 0,$ so the series diverges.

30. $a_n=\sin\frac{\pi}{n};$ for $n\ge 2$, $\sin\frac{\pi}{n}>0$ and

$\sin\frac{\pi}{n}>\sin\frac{\pi}{n+1},$ so $a_n>a_{n+1};$ $\lim_{n\to\infty}\sin\frac{\pi}{n}=0.$

The alternating series converges.

We have $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}=\lim_{n\to 0}\frac{\sin n}{n}=1$.

The series $\sum_{n=1}^{\infty}\sin\frac{\pi}{n}$ and $\sum_{n=1}^{\infty}\frac{\pi}{n}$ either both converge or both diverge. Since $\sum_{n=1}^{\infty}\frac{\pi}{n}=\pi\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, it follows that $\sum_{n=1}^{\infty}\sin\frac{\pi}{n}$ is divergent.

The series is conditionally convergent.

31. Suppose $\sum|a_n|$ converges. Thus, $\sum 2|a_n|$ converges, so $\sum\left(|a_n|+a_n\right)$ converges since $0\le|a_n|+a_n\le 2|a_n|.$ By the linearity of convergent series, $\sum a_n=\sum\left(|a_n|+a_n\right)-\sum|a_n|$ converges, which is a contradiction.

32. Let $\sum a_n=\sum(-1)^{n+1}\frac{1}{\sqrt{n}}=\sum b_n$. $\sum a_n$ and $\sum b_n$ both converge, but $\sum a_nb_n=\sum\frac{1}{n}$ diverges.

33. The positive-term series is

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots=\sum_{n=1}^{\infty}\frac{1}{2n-1}.$$

$\sum_{n=1}^{\infty}\frac{1}{2n-1}>\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges since the harmonic series diverges.

Thus, $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges.

The negative-term series is

$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\ldots=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges, since the harmonic series diverges.

34. If the positive terms and negative terms both formed convergent series then the series would be absolutely convergent. If one series was convergent and the other was divergent, the sum, which is the original series, would be divergent

35. **a.** $1+\frac{1}{3}\approx 1.33$

b. $1+\frac{1}{3}-\frac{1}{2}\approx 0.833$

c. $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}\approx 1.38$

$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}\approx 1.13$

36. $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}-\frac{1}{6}+\frac{1}{19}+\frac{1}{21}+\frac{1}{23}-\frac{1}{8}+\frac{1}{25}+\frac{1}{27}+\frac{1}{29}+\frac{1}{31}$

$S_{20}\approx 1.3265$

37. Written response. Consider the partial sum of the positive terms of the series, and the partial sum of the negative terms. If both partial sums were bounded, the series would be absolutely convergent. Therefore, at least one of the partial sums must sum to ∞ (or $-\infty$). If the series of positive terms summed to ∞ and the series of negative terms summed to a finite number, the original series would not be convergent (similarly for the series of negative terms). Therefore, the positive terms sum to ∞ and the negative terms sum to $-\infty$. We can then rearrange the terms to make the original series sum to any value we wish.

38. Possible answer: take several positive terms, add one negative term, then add positive terms whose sum is at least one greater than the negative term previously added. Add another negative term, then add positive terms whose sum is at least one greater that the negative term just added. Continue in this manner and the resulting series will diverge.

39. Consider $1-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{9}+\ldots$

It is clear that $\lim_{n\to\infty} a_n=0.$ Pairing successive terms, we obtain $\frac{1}{n}-\frac{1}{n^2}=\frac{n-1}{n^2}>0$ for $n>1$.

Let $a_n=\frac{n-1}{n^2}$ and $b_n=\frac{1}{n}$. Then

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2-n}{n^2}=1;\ 0<1<\infty.$

Thus, since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$ also diverges.

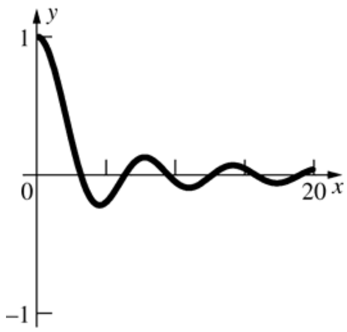
40. $\frac{1}{\sqrt{n}-1}-\frac{1}{\sqrt{n}+1}=\frac{2}{n-1}$, so

$$\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\cdots=\sum_{n=2}^{\infty}\frac{2}{n-1}$$

$=2\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges.

41. Note that $(a_k+b_k)^2 \ge 0$ and $(a_k-b_k)^2 \ge 0$ for all k. Thus, $a_k^2 \pm 2a_kb_k + b_k^2 \ge 0$, or $a_k^2+b_k^2 \ge \pm 2a_kb_k$ for all k, and $a_k^2+b_k^2 \ge 2|a_kb_k|$. Since $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ both converge, $\sum_{k=1}^{\infty}(a_k^2+b_k^2)$ also converges, and by the Comparison Test, $\sum_{k=1}^{\infty} 2|a_kb_k|$ converges. Hence, $\sum_{k=1}^{\infty}|a_kb_k| = \frac{1}{2}\sum_{k=1}^{\infty} 2|a_kb_k|$ converges, i.e., $\sum_{k=1}^{\infty} a_kb_k$ converges absolutely.

42.



$\int_0^{\infty}\frac{\sin x}{x}dx$ gives the area of the region above the x-axis minus the area of the region below.

Note that

$$\int_{2k\pi}^{(2k+1)\pi}\left(\frac{\sin x}{x}+\frac{\sin(x+\pi)}{x+\pi}\right)dx = \int_{2k\pi}^{(2k+1)\pi}\frac{\sin x}{x}dx + \int_{2k\pi}^{(2k+1)\pi}\frac{\sin(x+\pi)}{x+\pi}dx$$

$$= \int_{2k\pi}^{(2k+1)\pi}\frac{\sin x}{x}dx + \int_{(2k+1)\pi}^{(2k+2)\pi}\frac{\sin u}{u}du = \int_{2k\pi}^{(2k+2)\pi}\frac{\sin x}{x}dx$$

by using the substitution $u = x + \pi$, then changing the variable of integration back to x.

Thus, $\int_0^{\infty}\frac{\sin x}{x}dx = \sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi}\left(\frac{\sin x}{x}+\frac{\sin(x+\pi)}{x+\pi}\right)dx = \sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi}\frac{(x+\pi)\sin x + x\sin(x+\pi)}{x(x+\pi)}dx$

$$= \sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi}\frac{x\sin x+\pi\sin x - x\sin x}{x(x+\pi)}dx = \sum_{k=0}^{\infty}\int_{2k\pi}^{(2k+1)\pi}\frac{\pi\sin x}{x(x+\pi)}dx.$$

For $k > 0$, on $[2k\pi, (2k+1)\pi]$ $0 \le \sin x \le 1$ while $0 < \frac{\pi}{x(x+\pi)} \le \frac{\pi}{2k\pi(2k\pi+\pi)} = \frac{1}{(4k^2+2k)\pi}$.

Thus, $0 \le \int_{2k\pi}^{(2k+1)\pi}\frac{\pi\sin x}{x(x+\pi)}dx \le \frac{1}{(4k^2+2k)\pi}\int_{2k\pi}^{(2k+1)\pi}dx = \frac{1}{4k^2+2k}$.

Hence, $\int_{2\pi}^{\infty}\frac{\sin x}{x}dx \le \sum_{k=1}^{\infty}\frac{1}{4k^2+2k} \le \sum_{k=1}^{\infty}\frac{1}{4k^2}$ which converges.

Adding $\int_0^{2\pi}\frac{\sin x}{x}dx$ will not affect the convergence, so $\int_0^{\infty}\frac{\sin x}{x}dx$ converges.

43. Consider the graph of $\frac{|\sin x|}{x}$ on the interval $[k\pi, (k+1)\pi]$.

Note that for $k\pi+\frac{\pi}{6} \le x \le k\pi+\frac{5\pi}{6}$, $\frac{1}{2} \le |\sin x|$ while $\frac{1}{\left(k+\frac{5}{6}\right)\pi} \le \frac{1}{x}$. Thus on $\left[\left(k+\frac{1}{6}\right)\pi, \left(k+\frac{5}{6}\right)\pi\right]$

$$\frac{1}{2\left(k+\frac{5}{6}\right)\pi} = \frac{1}{\left(2k+\frac{5}{3}\right)\pi} \le \frac{|\sin x|}{x}, \text{ so } \int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{x}dx \ge \int_{(k+1/6)\pi}^{(k+5/6)\pi}\frac{|\sin x|}{x}dx \ge \frac{1}{\left(2k+\frac{5}{3}\right)\pi}\int_{(k+1/6)\pi}^{(k+5/6)\pi}dx = \frac{1}{3k+\frac{5}{2}}.$$

Hence, $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{\infty} \frac{1}{3k+\frac{5}{2}}$. Let $a_k = \frac{1}{3k+\frac{5}{2}}$ and $b_k = \frac{1}{k}$.

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k}{3k+\frac{5}{2}} = \lim_{k\to\infty} \frac{1}{3+\frac{5}{2k}} = \frac{1}{3}$; $0 < \frac{1}{3} < \infty$. Thus, since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{3k+\frac{5}{2}}$ also diverges. Hence, $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx$ also diverges and adding $\int_{0}^{\pi} \frac{|\sin x|}{x} dx$ will not affect its divergence.

44. Recall that a straight line is the shortest distance between two points. Note that $\sin\frac{\pi}{x} = 1$ when $x = \frac{2}{5}, \frac{2}{9}, \frac{2}{13}, \ldots$ and $\sin\frac{\pi}{x} = -1$ when $x = \frac{2}{3}, \frac{2}{7}, \frac{2}{11}, \ldots$. Thus, for $n \geq 1$, the curve $y = x\sin\frac{\pi}{x}$ goes from $\left(\frac{2}{4n+1}, \frac{2}{4n+1}\right)$ to $\left(\frac{2}{4n+3}, -\frac{2}{4n+3}\right)$. The distance between these two points is

$$\sqrt{\left(\frac{2}{4n+1} - \frac{2}{4n+3}\right)^2 + \left(\frac{2}{4n+1} + \frac{2}{4n+3}\right)^2} = \sqrt{2\left(\frac{2}{4n+1}\right)^2 + 2\left(\frac{2}{4n+3}\right)^2}$$

$$= \frac{2\sqrt{2(4n+3)^2 + 2(4n+1)^2}}{(4n+1)(4n+3)} = \frac{2\sqrt{64n^2+64n+20}}{16n^2+16n+3} = \frac{4\sqrt{16n^2+16n+5}}{16n^2+16n+3}$$

The length of $x\sin\frac{\pi}{x}$ on (0, 1] is greater than $\sum_{n=1}^{\infty} \frac{4\sqrt{16n^2+16n+5}}{16n^2+16n+3}$ because this sum does not even take into account the distances from $\left(\frac{2}{4n+3}, -\frac{2}{4n+3}\right)$ to $\left(\frac{2}{4(n+1)+1}, \frac{2}{4(n+1)+1}\right)$ which are still shorter than the lengths along the curve.

Let $a_n = \frac{4\sqrt{16n^2+16n+5}}{16n^2+16n+3}$ and $b_n = \frac{1}{n}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n\sqrt{16n^2+16n+5}}{16n^2+16n+3} = \lim_{n\to\infty} \frac{4\sqrt{16n^4+16n^3+5n^2}}{16n^2+16n+3} = \lim_{n\to\infty} \frac{4\sqrt{16+\frac{16}{n}+\frac{5}{n^2}}}{16+\frac{16}{n}+\frac{3}{n^2}}$

$= \frac{16}{16} = 1; 0 < 1 < \infty$

Thus, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{4\sqrt{16n^2+16n+5}}{16n^2+16n+3}$ also diverges.

Since the length of the graph is greater than $\sum_{n=1}^{\infty} a_n$, the length of the graph is infinite.

45. $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right]\left(\frac{1}{n}\right)$

This is a Riemann sum for the function $f(x) = \frac{1}{x}$ from $x = 1$ to 2 where $\Delta x = \frac{1}{n}$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{1}{1+\frac{k}{n}}\left(\frac{1}{n}\right)\right] = \int_1^2 \frac{1}{x} dx = \ln 2$$

9.6 Concepts Review

1. power series

2. where it converges

3. interval; endpoints

4. (–1, 1)

Problem Set 9.6

1. $\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}; \rho = \lim_{n\to\infty}\left|\frac{(n-1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n}\right| = 0$. Series converges for all x.

2. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}; \rho = \lim_{n\to\infty}\left|\frac{3^n x^{n+1}}{3^{n+1} x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right| = \frac{|x|}{3}$; convergence on $(-3,3)$.

For $x = 3,\ a_n = 1$ and the series diverges.

For $x = -3,\ a_n = (-1)^n$ and the series diverges.

Series converges on $(-3,3)$

3. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}; \rho = \lim_{n\to\infty}\left|\frac{n^2 x^{n+1}}{(n+1)^2 x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{(1+2/n + 1/n^2}\right| = |x|$; convergence on $(-1,1)$.

For $x = 1,\ a_n = \frac{1}{n^2}$ (p-series, p=2) and the series converges.

For $x = -1,\ a_n = \frac{(-1)^n}{n^2}$ (alternating p-series, p=2) and the series converges. by the Absolute Convergence Test.

Series converges on $[-1,1]$

4. $\sum_{n=1}^{\infty} nx^n; \rho = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right| = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)x\right| = |x|$; convergence on $(-1,1)$.

For $x = 1,\ a_n = n$ and the series diverges.

For $x = -1,\ a_n = (-1)^n n$ and the series diverges ($\lim_{n\to\infty}(-1)^n n \neq 0$)

Series converges on $(-1,1)$

5. This is the alternating series for problem 3; thus it converges on $[-1,1]$ by the Absolute Convergence Test.

6. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}; \rho = \lim_{n\to\infty}\left|\frac{nx^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\left|x\left(\frac{n}{n+1}\right)\right| = |x|$; convergence on $(-1,1)$.

For $x = 1,\ a_n = \frac{(-1)^n}{n}$ (Alternating Harmonic Series) and the series converges.

For $x = -1,\ a_n = \frac{1}{n}$ (Harmonic Series) and the series diverges.

Series converges on $(-1,1]$

7. Let $u = x-2$; then, from problem 6, the series converges when $u \in (-1,1]$; that is when $x \in (1,3]$.

8. $\sum_{n=1}^{\infty}\frac{(x+1)^n}{n!}$; $\rho=\lim_{n\to\infty}\left|\frac{n!(x+1)^{n+1}}{(n+1)!(x+1)^n}\right|=\lim_{n\to\infty}\left|\frac{x+1}{n+1}\right|=0$. Series converges for all x.

9. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n(n+1)}$; $\rho=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)(n+2)}\div\frac{x^n}{n(n+1)}\right|=\lim_{n\to\infty}|x|\left|\frac{n}{n+2}\right|=|x|$

When $x = 1$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n(n+1)}$ which converges absolutely by comparison with the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

When $x = -1$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n(n+1)}=\sum_{n=1}^{\infty}(-1)^{2n+1}\frac{1}{n(n+1)}$

$=\sum_{n=1}^{\infty}(-1)\frac{1}{n(n+1)}=(-1)\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ which converges since $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges.

The series converges on $-1 \le x \le 1$.

10. $\sum_{n=0}^{\infty}\frac{x^n}{n!}$; $\rho=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\div\frac{x^n}{n!}\right|=\lim_{n\to\infty}|x|\left|\frac{1}{n+1}\right|=0$

The series converges for all x.

11. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$; $\rho=\lim_{n\to\infty}\left|\frac{x^{2n+1}}{(2n+1)!}\div\frac{x^{2n-1}}{(2n-1)!}\right|=\lim_{n\to\infty}\left|x^2\right|\left|\frac{1}{2n(2n+1)}\right|=0$

The series converges for all x.

12. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$; $\rho=\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\div\frac{x^{2n}}{(2n)!}\right|$

$=\lim_{n\to\infty}\left|x^2\right|\left|\frac{1}{(2n+2)(2n+1)}\right|=0$

The series converges for all x.

13. $\sum_{n=1}^{\infty}nx^n$; $\rho=\lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right|$

$=\lim_{n\to\infty}|x|\left|\frac{n+1}{n}\right|=|x|$

When $x = 1$, the series is $\sum_{n=1}^{\infty}n$ which clearly diverges.

When $x = -1$, the series is $\sum_{n=1}^{\infty}n(-1)^n$; $a_n = n$;

$\lim_{n\to\infty}a_n \ne 0$, thus the series diverges.

The series converges on $-1 < x < 1$.

14. $\sum_{n=1}^{\infty}n^2x^n$; $\rho=\lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}}{n^2x^n}\right|$

$=\lim_{n\to\infty}|x|\left|\frac{(n+1)^2}{n^2}\right|=|x|$

When $x = 1$, the series is $\sum_{n=1}^{\infty}n^2$ which clearly diverges.

When $x = -1$, the series is $\sum_{n=1}^{\infty}n^2(-1)^n$;

$a_n = n^2$; $\lim_{n\to\infty}a_n \ne 0$, thus the series diverges.

The series converges on $-1 < x < 1$.

15. $1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$; $\rho=\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\div\frac{x^n}{n}\right|$

$=\lim_{n\to\infty}|x|\left|\frac{n}{n+1}\right|=|x|$

When $x = 1$, the series is $1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$, which is 1 added to the alternating harmonic series multiplied by -1, which converges.

When $x = -1$, the series is

$1+\sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n}{n}=1+\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges.

The series converges on $-1 < x \le 1$.

16. $1+\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$; $\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}} \div \frac{x^n}{\sqrt{n}}\right|$

$= \lim_{n\to\infty}|x|\left|\sqrt{\frac{n}{n+1}}\right| = |x|$

When $x = 1$, the series is $1+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which

diverges since $\frac{1}{2}<1$.

When $x = -1$, the series is $1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$;

$a_n = \frac{1}{\sqrt{n}}$; $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$, so $a_n > a_{n+1}$ and

$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, so the series converges.

The series converges on $-1 \le x < 1$.

17. $1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n(n+2)}$;

$\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)(n+3)} \div \frac{x^n}{n(n+2)}\right|$

$= \lim_{n\to\infty}|x|\left|\frac{n^2+2n}{n^2+4n+3}\right| = |x|$

When $x = 1$ the series is $1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{n(n+2)}$

which converges absolutely by comparison with

the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

When $x = -1$, the series is

$1+\sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n}{n(n+2)} = 1+\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ which

converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

The series converges on $-1 \le x \le 1$.

18. $\sum_{n=1}^{\infty}\frac{x^n}{(n+1)^2-1}$;

$\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)^2-1} \div \frac{x^n}{(n+1)^2-1}\right|$

$= \lim_{n\to\infty}|x|\left|\frac{n^2+2n}{n^2+4n+3}\right| = |x|$

When $x = 1$, the series is

$\sum_{n=1}^{\infty}\frac{1}{(n+1)^2-1} = \sum_{n=1}^{\infty}\frac{1}{n^2+2n} = \sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ which

converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

When $x = -1$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^2-1}$ which

converges absolutely by comparison with

$\sum_{n=1}^{\infty}\frac{1}{n^2}$.

The series converges on $-1 \le x \le 1$.

19. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n}$; $\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}} \div \frac{x^n}{2^n}\right| = \lim_{n\to\infty}\left|\frac{x}{2}\right|$

$= \left|\frac{x}{2}\right|$; $\left|\frac{x}{2}\right| < 1$ when $-2 < x < 2$.

When $x = 2$, the series is $\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n$

which diverges.

When $x = -2$, the series is

$\sum_{n=0}^{\infty}(-1)^n\frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n(-1)^n = \sum_{n=0}^{\infty}1$ which

diverges. The series converges on $-2 < x < 2$.

20. $\sum_{n=0}^{\infty}2^n x^n$; $\rho = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right| = \lim_{n\to\infty}|2x| = |2x|$;

$|2x| < 1$ when $-\frac{1}{2} < x < \frac{1}{2}$.

When $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty}2^n\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}1$

which diverges.

When $x = -\frac{1}{2}$, the series is

$\sum_{n=0}^{\infty}2^n\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n$ which diverges.

The series converges on $-\frac{1}{2} < x < \frac{1}{2}$.

21. $\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$; $\rho = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{(n+1)!} \div \frac{2^n x^n}{n!}\right|$

$= \lim_{n\to\infty}|2x|\left|\frac{1}{n+1}\right| = 0$.

The series converges for all x.

22. $\sum_{n=1}^{\infty} \frac{nx^n}{n+1}$; $\rho = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{n+2} \div \frac{nx^n}{n+1} \right|$

$= \lim_{n\to\infty} |x| \left| \frac{n^2+2n+1}{n^2+2n} \right| = |x|$

When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{n}{n+1}$ which diverges since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

When $x = -1$, the series is $\sum_{n=1}^{\infty} \frac{n(-1)^n}{n+1}$ which diverges since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

The series converges on $-1 < x < 1$.

23. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$; $\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{n+1} \div \frac{(x-1)^n}{n} \right|$

$= \lim_{n\to\infty} |x-1| \left| \frac{n}{n+1} \right| = |x-1|$; $|x-1| < 1$ when $0 < x < 2$.

When $x = 0$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges.

When $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

The series converges on $0 \le x < 2$.

24. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$; $\rho = \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)!} \div \frac{(x+2)^n}{n!} \right|$

$= \lim_{n\to\infty} |x+2| \left| \frac{1}{n+1} \right| = 0$

The series converges for all x.

25. $\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n}$; $\rho = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{2^{n+1}} \div \frac{(x+1)^n}{2^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x+1}{2} \right| = \left| \frac{x+1}{2} \right|$; $\left| \frac{x+1}{2} \right| < 1$ when $-3 < x < 1$.

When $x = -3$, the series is $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$ which diverges.

When $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$ which diverges.

The series converges on $-3 < x < 1$.

26. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$; $\rho = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2} \div \frac{(x-2)^n}{n^2} \right|$

$= \lim_{n\to\infty} |x-2| \left| \frac{n^2}{(n+1)^2} \right| = |x-2|$; $|x-2| < 1$ when $1 < x < 3$.

When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

When $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges. The series converges on $1 \le x \le 3$.

27. $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$; $\rho = \lim_{n\to\infty} \left| \frac{(x+5)^{n+1}}{(n+1)(n+2)} \div \frac{(x+5)^n}{n(n+1)} \right|$

$= \lim_{n\to\infty} |x+5| \left| \frac{n}{n+2} \right| = |x+5|$; $|x+5| < 1$ when $-6 < x < -4$.

When $x = -4$, the series is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ which converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

When $x = -6$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ which converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

The series converges on $-6 \le x \le -4$.

28. $\sum_{n=1}^{\infty} (-1)^{n+1} n(x+3)^n$; $\rho = \lim_{n\to\infty} \left| \frac{(n+1)(x+3)^{n+1}}{n(x+3)^n} \right|$

$= \lim_{n\to\infty} |x+3| \left| \frac{n+1}{n} \right| = |x+3|$; $|x+3| < 1$ when $-4 < x < -2$.

When $x = -2$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} n$ which diverges since $\lim_{n\to\infty} n \neq 0$.

When $x = -4$, the series is

$\sum_{n=1}^{\infty} (-1)^{n+1} n(-1)^n = \sum_{n=1}^{\infty} -n$, which diverges.

The series converges on $-4 < x < -2$.

29. If for some x_0, $\lim_{n\to\infty} \frac{x_0^n}{n!} \neq 0$, then $\sum \frac{x_0^n}{n!}$ could not converge.

30. For any number k, since $k-n < k-n+1 < \ldots < k-2 < k-1 < k$, $|(k-1)(k-2)\ldots(k-n)| < k^n$, thus

$$\lim_{n\to\infty}\left|\frac{k(k-1)(k-2)\ldots(k-n)}{n!}x^n\right| < \lim_{n\to\infty}\left|\frac{k^{n+1}}{n!}x^n\right|$$

$= |k|\lim_{n\to\infty}\left|\frac{k^n}{n!}x^n\right|$. Since $-1 < x < 1$, $\lim_{n\to\infty} x^n = 0$, and by Problem 21, $\lim_{n\to\infty}\left|\frac{k^n}{n!}\right| = 0$, hence

$$\lim_{n\to\infty}\frac{k(k-1)(k-2)\ldots(k-n)}{n!}x^n = 0.$$

31. The Absolute Ratio Test gives

$$\rho = \lim_{n\to\infty}\left|\frac{(n+1)!x^{2n+3}}{1\cdot 3\cdot 5\cdots(2n+1)} \div \frac{n!x^{2n+1}}{1\cdot 3\cdot 5\cdots(2n-1)}\right|$$

$= \lim_{n\to\infty}|x^2|\left|\frac{n+1}{2n+1}\right| = \left|\frac{x^2}{2}\right|$; $\left|\frac{x^2}{2}\right| < 1$ when $|x| < \sqrt{2}$.

The radius of convergence is $\sqrt{2}$.

32. Using the Absolute Ratio Test,

$$\lim_{n\to\infty}\left|\frac{(pn+p)!}{((n+1)!)^p}x^{n+1} \div \frac{(pn)!}{(n!)^p}x^n\right|$$

$$= \lim_{n\to\infty}|x|\left|\frac{(pn+p)(pn+p-1)\ldots(pn+p-(p-1))}{(n+1)^p}\right|$$

$$= \lim_{n\to\infty}|x|\left|p\left(p-\frac{1}{n+1}\right)\left(p-\frac{2}{n+1}\right)\ldots\left(p-\frac{p-1}{n+1}\right)\right|$$

$$= |x|\,p^p$$

The radius of convergence is p^{-p}.

33. This is a geometric series, so it converges for $|x-3| < 1$, $2 < x < 4$. For these values of x, the series converges to $\frac{1}{1-(x-3)} = \frac{1}{4-x}$.

34. $\sum_{n=0}^{\infty} a_n(x-3)^n$ converges on an interval of the form $(3-a, 3+a)$, where $a \geq 0$. If the series converges at $x = -1$, then $3 - a \leq -1$, or $a \geq 4$, since $x = -1$ could be an endpoint where the series converges. If $a \geq 4$, then $3 + a \geq 7$ so the series will converge at $x = 6$. The series may not converge at $x = 7$, since $x = 7$ may be an endpoint of the convergence intervals, where the series might or might not converge.

35. a. $\rho = \lim_{n\to\infty}\left|\frac{(3x+1)^{n+1}}{(n+1)\cdot 2^{n+1}} \div \frac{(3x+1)^n}{n\cdot 2^n}\right| = \lim_{n\to\infty}|3x+1|\left|\frac{n}{2n+2}\right| = \frac{1}{2}|3x+1|$; $\frac{1}{2}|3x+1| < 1$ when $-1 < x < \frac{1}{3}$.

When $x = -1$, the series is $\sum_{n=1}^{\infty}\frac{(-2)^n}{n\cdot 2^n} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$, which converges.

When $x = \frac{1}{3}$, the series is $\sum_{n=1}^{\infty}\frac{2^n}{n\cdot 2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which diverges. The series converges on $-1 \leq x < \frac{1}{3}$.

b. $\rho = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2x-3)^{n+1}}{4^{n+1}\sqrt{n+1}} \div \frac{(-1)^n(2x-3)^n}{4^n\sqrt{n}}\right| = \lim_{n\to\infty}|2x-3|\left|\frac{\sqrt{n}}{4\sqrt{n+1}}\right| = \frac{1}{4}|2x-3|$;

$\frac{1}{4}|2x-3| < 1$ when $-\frac{1}{2} < x < \frac{7}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{n=1}^{\infty}(-1)^n\frac{(-4)^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which diverges since $\frac{1}{2} < 1$.

When $x = \frac{7}{2}$, the series is $\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$; $a_n = \frac{1}{\sqrt{n}}$; $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$, so $a_n > a_{n+1}$;

$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, so $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ converges. The series converges on $-\frac{1}{2} < x \leq \frac{7}{2}$.

36. From Problem 52 of Section 9.1,

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = \frac{1}{2^n\sqrt{5}}\left[\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right]$$

$$\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}\sqrt{5}}\left[\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right] \div \frac{x^n}{2^n\sqrt{5}}\left[\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right]\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{2}\right|\left|\frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{x}{2}\right|\left|\frac{1+\sqrt{5} - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n\left(1-\sqrt{5}\right)}{1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}\right|$$

$$= \left|\frac{1+\sqrt{5}}{2}x\right|; \left|\frac{1+\sqrt{5}}{2}x\right| < 1 \text{ when } -\frac{2}{1+\sqrt{5}} < x < \frac{2}{1+\sqrt{5}}.$$

$$\left(\text{Note that } \lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n = 0 \text{ since } \left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right| < 1.\right)$$

$$R = \frac{2}{1+\sqrt{5}} \approx 0.618$$

37. If $a_{n+3} = a_n$, then $a_0 = a_3 = a_6 = a_{3n}, a_1 = a_4 = a_7 = a_{3n+1}$, and $a_2 = a_5 = a_8 = a_{3n+2}$. Thus,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_0 x^3 + a_1 x^4 + a_2 x^5 + \cdots = (a_0 + a_1 x + a_2 x^2)(1 + x^3 + x^6 + \cdots)$$

$$= (a_0 + a_1 x + a_2 x^2)\sum_{n=0}^{\infty} x^{3n} = (a_0 + a_1 x + a_2 x^2)\sum_{n=0}^{\infty}(x^3)^n.$$

$a_0 + a_1 x + a_2 x^2$ is a polynomial, which will converge for all *x*.

$\sum_{n=0}^{\infty}(x^3)^n$ is a geometric series which, converges for $\left|x^3\right| < 1$, or, equivalently, $|x| < 1$.

Since $\sum_{n=0}^{\infty}(x^3)^n = \frac{1}{1-x^3}$ for $|x| < 1$, $S(x) = \frac{a_0 + a_1 x + a_2 x^2}{1-x^3}$ for $|x| < 1$.

38. If $a_n = a_{n+p}$, then $a_0 = a_p = a_{2p} = a_{np}, a_1 = a_{p+1} = a_{2p+1} = a_{np+1}$, etc. Thus,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_{p-1}x^{p-1} + a_0 x^p + a_1 x^{p+1} + \cdots + a_{p-1}x^{2p-1} + \cdots$$

$$= (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})(1 + x^p + x^{2p} + \cdots) = (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})\sum_{n=0}^{\infty} x^{np}$$

$a_0 + a_1 x + \cdots + a_{p-1}x^{p-1}$ is a polynomial, which will converge for all *x*.

$\sum_{n=0}^{\infty} x^{np} = \sum_{n=0}^{\infty}(x^p)^n$ is a geometric series which converges for $\left|x^p\right| < 1$, or, equivalently, $|x| < 1$.

Since $\sum_{n=0}^{\infty}(x^p)^n = \frac{1}{1-x^p}$ for $|x| < 1$, $S(x) = (a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})\left(\frac{1}{1-x^p}\right)$ for $|x| < 1$.

9.7 Concepts Review

1. integrated; interior

2. $-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}$

3. $1+x^2+\frac{x^4}{2}+\frac{x^6}{6}$

4. $1+x+\frac{3x^2}{2}+\frac{x^3}{3}+\frac{3x^4}{4}$

Problem Set 9.7

1. From the geometric series for $\frac{1}{1-x}$ with x replaced by $-x$, we get

$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots$,

radius of convergence 1.

2. $\frac{d}{dx}\left(\frac{1}{1+x}\right)=-\frac{1}{(1+x)^2}$

$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots$ so

$\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+5x^4-\cdots$;

radius of convergence 1.

3. $\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2};\ \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right)=\frac{2}{(1-x)^3}$,

so $\frac{1}{(1-x)^3}$ is $\frac{1}{2}$ of the second derivative of $\frac{1}{1-x}$. Thus,

$\frac{1}{(1-x)^3}=1+3x+6x^2+10x^3+\cdots$;

radius of convergence 1.

4. Using the result of Problem 2,

$\frac{x}{(1+x)^2}=x-2x^2+3x^3-4x^4+5x^5-\cdots$;

radius of convergence 1.

5. From the geometric series for $\frac{1}{1-x}$ with x replaced by $\frac{3}{2}x$, we get

$\frac{1}{2-3x}=\frac{1}{2}+\frac{3x}{4}+\frac{9x^2}{8}+\frac{27x^3}{16}+\cdots$;

radius of convergence $\frac{2}{3}$.

6. $\frac{1}{3+2x}=\frac{1}{3}\left(\frac{1}{1+\frac{2}{3}x}\right)$. Since

$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots$,

$\frac{1}{3}\left(\frac{1}{1+\frac{2}{3}x}\right)=\frac{1}{3}-\frac{2x}{9}+\frac{4x^2}{27}-\frac{8x^3}{81}+\frac{16x^4}{243}-\cdots$;

radius of convergence $\frac{3}{2}$.

7. From the geometric series for $\frac{1}{1-x}$ with x replaced by x^4, we get

$\frac{x^2}{1-x^4}=x^2+x^6+x^{10}+x^{14}+\cdots$;

radius of convergence 1.

8. $\frac{x^3}{2-x^3}=\frac{x^3}{2}\left(\frac{1}{1-\frac{x^3}{2}}\right)=\frac{x^3}{2}+\frac{x^6}{4}+\frac{x^9}{8}+\frac{x^{12}}{16}+\cdots$

for $\left|\frac{x^3}{2}\right|<1$ or $-\sqrt[3]{2}<x<\sqrt[3]{2}$.

9. From the geometric series for $\ln(1+x)$ with x replaced by t, we get

$\int_0^x \ln(1+t)dt=\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{12}-\frac{x^5}{20}+\ldots$;

radius of convergence 1.

10. $\int_0^x \tan^{-1} t\, dt=\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{30}-\frac{x^8}{56}+\ldots$;

radius of convergence 1.

11. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots, -1<x\le 1$

$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\ldots, -1\le x<1$

$\ln\frac{1+x}{1-x}=\ln(1+x)-\ln(1-x)$

$=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\cdots;$ radius of convergence 1.

12. If $M=\frac{1+x}{1-x},$ then $M-Mx=1+x;$

$M-1=(M+1)x;\ x=\frac{M-1}{M+1}.$

$\left|\frac{M-1}{M+1}\right|<1$ is equivalent to $-M-1<M-1<M+1$ or $0<2M<2M+2$ which is true for $M>0$. Thus, the natural logarithm of any positive number can be found by using the series from Problem 11. For $M=8,\ x=\frac{7}{9},$ so

$$\ln 8=2\left(\frac{7}{9}\right)+\frac{2}{3}\left(\frac{7}{9}\right)^3+\frac{2}{5}\left(\frac{7}{9}\right)^5+\frac{2}{7}\left(\frac{7}{9}\right)^7+\frac{2}{9}\left(\frac{7}{9}\right)^9+\frac{2}{11}\left(\frac{7}{9}\right)^{11}+\cdots$$

$\approx$ 1.55556 + 0.31367 + 0.11385 + 0.04919 + 0.02315 + 0.01146 + 0.00586 + 0.00307 + 0.00164 + 0.00089 + 0.00049 + 0.00027 + 0.00015 + 0.00008 $\approx$ 2.079

13. Substitute $-x$ for x in the series for e^x to get:

$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots.$

14. $xe^{x^2}=x\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots\right)=x+x^3+\frac{x^5}{2!}+\frac{x^7}{3!}+\frac{x^9}{4!}+\cdots$

15. Add the result of Problem 13 to the series for e^x to get:

$e^x+e^{-x}=2+\frac{2x^2}{2!}+\frac{2x^4}{4!}+\frac{2x^6}{6!}+\cdots.$

16. $e^{2x}-1-2x=-1-2x+\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\cdots\right)=\frac{4x^2}{2!}+\frac{8x^3}{3!}+\frac{16x^4}{4!}+\frac{32x^5}{5!}+\cdots$

17. $e^{-x}\cdot\frac{1}{1-x}=\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)(1+x+x^2+\cdots)=1+\frac{x^2}{2}+\frac{x^3}{3}+\frac{3x^4}{8}+\frac{11x^5}{30}+\cdots$

18. $e^x\tan^{-1}x=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right)=x+x^2+\frac{x^3}{6}-\frac{x^4}{6}+\frac{3x^5}{40}+\cdots$

19. $\frac{\tan^{-1}x}{e^x}=e^{-x}\tan^{-1}x=\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right)=x-x^2+\frac{x^3}{6}+\frac{x^4}{6}+\frac{3x^5}{40}+\cdots$

20. $\frac{e^x}{1+\ln(1+x)}=\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}{1+x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots}=1+x^2-\frac{7x^3}{6}+\frac{47x^4}{24}-\frac{46x^5}{15}+\cdots$

21. $(\tan^{-1} x)(1+x^2+x^4) = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)(1+x^2+x^4) = x + \frac{2x^3}{3} + \frac{13x^5}{15} - \frac{29x^7}{105} + \cdots$

22. $\dfrac{\tan^{-1} x}{1+x^2+x^4} = \dfrac{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots}{1+x^2+x^4} = x - \frac{4x^3}{3} + \frac{8x^5}{15} + \frac{23x^7}{35} - \cdots$

23. The series representation of $\dfrac{e^x}{1+x}$ is $1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \frac{11x^5}{30} + \cdots$, so $\int_0^x \frac{e^t}{1+t}\,dt = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5 - \ldots$.

24. The series representation of $\dfrac{\tan^{-1} x}{x}$ is $1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots$, so $\int_0^x \frac{\tan^{-1} t}{t}\,dt = x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \cdots$

25. **a.** $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$, so $\dfrac{x}{1+x} = x - x^2 + x^3 - x^4 + x^5 - \cdots$.

b. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$, so $\dfrac{e^x - (1+x)}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots$.

c. $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots$, so $-\ln(1-2x) = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots$.

26. **a.** Since $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$, $\dfrac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$.

b. Again using $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$, $\dfrac{1}{1-\cos x} - 1 = \cos x + \cos^2 x + \cos^3 x + \cdots$.

c. $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$, so $\ln(1-x^2) = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \cdots$, and

$-\frac{1}{2}\ln(1-x^2) = \ln\dfrac{1}{\sqrt{1-x^2}} = \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \cdots$.

27. Differentiating the series for $\dfrac{1}{1-x}$ yields $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ multiplying this series by x gives

$\dfrac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$, hence $\sum_{n=1}^{\infty} nx^n = \dfrac{x}{(1-x)^2}$ for $-1 < x < 1$.

28. Differentiating the series for $\dfrac{1}{x-1}$ twice yields $\dfrac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \cdots$. Multiplying this series by x

gives $\dfrac{2x}{(1-x)^3} = 2x + 3\cdot 2x^2 + 4\cdot 3x^3 + 5\cdot 4x^4 + \cdots$, hence $\sum_{n=1}^{\infty} n(n+1)x^n = \dfrac{2x}{(1-x)^3}$ for $-1 < x < 1$.

29. **a.** $\tan^{-1}(e^x - 1) = (e^x - 1) - \dfrac{(e^x-1)^3}{3} + \dfrac{(e^x-1)^5}{5} - \cdots = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \frac{1}{3}\left(x + \frac{x^2}{2!} + \cdots\right)^3 + \cdots$

$= x + \frac{x^2}{2} - \frac{x^3}{6} - \cdots$

b. $e^{e^x-1}=1+(e^x-1)+\dfrac{(e^x-1)^2}{2!}+\dfrac{(e^x-1)^3}{3!}+\cdots$

$=1+\left(x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots\right)+\dfrac{1}{2!}\left(x+\dfrac{x^2}{2!}+\cdots\right)^2+\dfrac{1}{3!}\left(x+\dfrac{x^2}{2!}+\cdots\right)^3$

$=1+\left(x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots\right)+\dfrac{1}{2!}\left(x^2+2\dfrac{x^3}{2!}+\cdots\right)+\dfrac{1}{3!}\left(x^3+3\dfrac{x^4}{2!}+\cdots\right)=1+x+x^2+\dfrac{5x^3}{6}+\cdots$

30. $f(x)=a_0+a_1x+a_2x^2+\cdots=b_0+b_1x+b_2x^2+\cdots;$

$f(0)=a_0=b_0$, so $a_0=b_0$.

$f'(x)=a_1+2a_2x+3a_3x^2+\cdots=b_1+2b_2x+3b_3x^3+\cdots;$

$f'(0)=a_1=b_1$, so $a_1=b_1$.

The nth derivative of $f(x)$ is

$f^{(n)}(x)=n!a_n+(n+1)!a_{n+1}x+\dfrac{(n+2)!}{2}a_{n+2}x^2+\cdots=n!b_n+(n+1)!b_{n+1}x+\dfrac{(n+2)!}{2}b_{n+2}x^2+\cdots;$

$f^{(n)}(0)=n!a_n=n!b_n$, so $a_n=b_n$.

31. $\dfrac{x}{x^2-3x+2}=\dfrac{x}{(x-2)(x-1)}=\dfrac{2}{x-2}-\dfrac{1}{x-1}=-\dfrac{1}{1-\frac{x}{2}}+\dfrac{1}{1-x}=-\left(1+\dfrac{x}{2}+\dfrac{x^2}{4}+\dfrac{x^3}{8}+\cdots\right)+\left(1+x+x^2+x^3+\cdots\right)$

$=\dfrac{x}{2}+\dfrac{3x^2}{4}+\dfrac{7x^3}{8}+\cdots=\sum_{n=1}^{\infty}\dfrac{(2^n-1)x^n}{2^n}$

32. $y''=-x+\dfrac{x^3}{3!}-\dfrac{x^5}{5!}+\cdots=-y$, so $y''+y=0$. It is clear that $y(0)=0$ and $y'(0)=1$. Both the sine and cosine functions satisfy $y''+y=0$, however, only the sine function satisfies the given initial conditions. Thus, $y=\sin x$.

33. $F(x)-xF(x)-x^2F(x)=(f_0+f_1x+f_2x^2+f_3x^3+\cdots)-(f_0x+f_1x^2+f_2x^3+\cdots)-(f_0x^2+f_1x^3+f_2x^4+\cdots)$

$=f_0+(f_1-f_0)x+(f_2-f_1-f_0)x^2+(f_3-f_2-f_1)x^3+\cdots$

$=f_0+(f_1-f_0)x+\sum_{n=2}^{\infty}(f_n-f_{n-1}-f_{n-2})x^n=0+x+\sum_{n=0}^{\infty}(f_{n+2}-f_{n+1}-f_n)x^{n+2}$

Since $f_{n+2}=f_{n+1}+f_n$, $f_{n+2}-f_{n+1}-f_n=0$. Thus $F(x)-xF(x)-x^2F(x)=x$.

$F(x)=\dfrac{x}{1-x-x^2}$

34. $y(x)=\dfrac{f_0}{0!}+\dfrac{f_1}{1!}x+\dfrac{f_2}{2!}x^2+\dfrac{f_3}{3!}x^3+\dfrac{f_4}{4!}x^4+\cdots;\quad y'(x)=\dfrac{f_1}{0!}+\dfrac{f_2}{1!}x+\dfrac{f_3}{2!}x^2+\dfrac{f_4}{3!}x^3+\cdots;$

$y''(x)=\dfrac{f_2}{0!}+\dfrac{f_3}{1!}x+\dfrac{f_4}{2!}x^2+\cdots$

(Recall that 0! = 1.)

$y''(x)-y'(x)-y(x)=\left(\dfrac{f_2}{0!}+\dfrac{f_3}{1!}x+\dfrac{f_4}{2!}x^2+\cdots\right)-\left(\dfrac{f_1}{0!}+\dfrac{f_2}{1!}x+\dfrac{f_3}{2!}x^2+\cdots\right)-\left(\dfrac{f_0}{0!}+\dfrac{f_1}{1!}x+\dfrac{f_2}{2!}x^2+\cdots\right)$

$=\dfrac{1}{0!}(f_2-f_1-f_0)+\dfrac{1}{1!}(f_3-f_2-f_1)x+\dfrac{1}{2!}(f_4-f_3-f_2)x^2+\cdots$

$=\sum_{n=0}^{\infty}\dfrac{1}{n!}(f_{n+2}-f_{n+1}-f_n)x^n=0$ since $f_{n+2}=f_{n+1}+f_n$ for all $n\ge 0$.

35. $\pi \approx 16\left(\frac{1}{5}-\frac{1}{375}+\frac{1}{15,625}-\frac{1}{546,875}+\frac{1}{17,578,125}\right)-4\left(\frac{1}{239}\right) \approx 3.14159$

36. For any positive integer $k \le n$, both $\frac{n!}{k}$ and $\frac{n!}{k!}$ are positive integers. Thus, since $q < n$, $n!e = \frac{n!p}{q}$ is a positive integer and $M = n!e - n! - n! - \frac{n!}{2!} - \frac{n!}{3!} - \cdots - \frac{n!}{n!}$ is also an integer. M is positive since

$$e - 1 - 1 - \frac{1}{2!} - \cdots - \frac{1}{n!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

$M < \frac{1}{n}$ contradicts that M is a positive integer since for $n \ge 1$, $\frac{1}{n} \le 1$ and there are no positive integers less than 1.

9.8 Concepts Review

1. $\frac{f^{(k)}(0)}{k!}$

2. $\lim_{n\to\infty} R_n(x) = 0$

3. $-\infty$; ∞

4. $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3$

Problem Set 9.8

1. $\tan x = \frac{\sin x}{\cos x} = \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots} = x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots$

2. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots}{1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots} = x-\frac{x^3}{3}+\frac{2x^5}{15}+\ldots$

3. $e^x \sin x = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right) = x+x^2+\frac{x^3}{3}-\frac{x^5}{30}-\ldots$

4. $e^{-x}\cos x = \left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right) = 1-x+\frac{x^3}{3}-\frac{x^4}{6}+\frac{x^5}{30}+\ldots$

5. $\cos x \ln(1+x) = \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right) = x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{3x^5}{40}-\ldots$

6. $(\sin x)\sqrt{1+x} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots\right) = x + \frac{x^2}{2} - \frac{7x^3}{24} - \frac{x^4}{48} - \frac{19x^5}{1920} + \ldots,\ -1 < x < 1$

7. $e^x + x + \sin x = x + \left(1 + x + \frac{x^2}{2!} + \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = 1 + 3x + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{2x^5}{5!} + \ldots$

8. $\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$, so $\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{4!} - \frac{x^2}{6!} + \frac{x^4}{8!} - \ldots$

9. $\frac{1}{1-x}\cosh x = (1 + x + x^2 + x^3 + \ldots)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) = 1 + x + \frac{3x^2}{2} + \frac{3x^3}{2} + \frac{37x^4}{24} + \frac{37x^5}{24} + \ldots,\ -1 < x < 1$

10. $\frac{-\ln(1+x)}{1+x} = \frac{-\ln(1+x)}{1-(-x)} = \left(-x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \ldots\right)(1 - x + x^2 - x^3 + x^4 - \ldots)$

$= -x + \frac{3x^2}{2} - \frac{11x^3}{6} + \frac{25x^4}{12} - \frac{137x^5}{60} + \ldots,\ -1 < x < 1$

11. $\frac{1}{1+x+x^2} = \frac{1}{1+x+x^2}\cdot\frac{1-x}{1-x} = \frac{1}{1-x^3}(1-x) = (1-x)\sum_{n=0}^{\infty} x^{3n} = 1 - x + x^3 - x^4 + \cdots,\ |x| < 1$

12. $\frac{1}{1-\sin x} = 1 + \sin x + (\sin x)^2 + (\sin x)^3 + \ldots$

$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) + \left(x - \frac{x^3}{3!} + \ldots\right)^2 + \left(x - \frac{x^3}{3!} + \ldots\right)^3 + \left(x - \frac{x^3}{3!} + \ldots\right)^4 + \left(x - \frac{x^3}{3!} + \ldots\right)^5 + \cdots$

$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) + \left(x^2 - 2\frac{x^4}{3!} + \ldots\right) + \left(x^3 - 3\frac{x^5}{3!} + \ldots\right) + (x^4 - \ldots) + (x^5 - \ldots)$

$= 1 + x + x^2 + \frac{5x^3}{6} + \frac{2x^4}{3} + \frac{61x^5}{120} + \ldots,\ |x| < \frac{\pi}{2}$.

13. $\sin^3 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)^2\left(x - \frac{x}{3!} + \frac{x^5}{5!} - \ldots\right) = \left(x^2 - 2\frac{x^4}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = x^3 - \frac{x^5}{2} + \ldots$

14. $x(\sin 2x + \sin 3x) = x\left[\left(2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \ldots\right) + \left(3x - \frac{27x^3}{3!} + \frac{243x^5}{5!} - \ldots\right)\right] = x\left(5x - \frac{35x^3}{3!} + \ldots\right) = 5x^2 - \frac{35x^4}{3!} + \ldots$

15. $x\sec(x^2) + \sin x = \frac{x}{\cos(x^2)} + \sin x = \frac{x}{1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \ldots} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$

$= \left(x + \frac{x^5}{2} + \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = 2x - \frac{x^3}{3!} + \frac{61x^5}{120} + \ldots$

16. $\frac{\cos x}{\sqrt{1+x}} = (\cos x)(1+x)^{-1/2} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\left(1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \ldots\right)$

$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \frac{49x^4}{384} - \frac{85x^5}{768} + \ldots,\ -1 < x < 1$

17. $(1+x)^{3/2} = 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \ldots,\ -1 < x < 1$

18. $(1-x^2)^{2/3}=[1+(-x^2)]^{2/3} = 1+\frac{2}{3}(-x^2)-\frac{1}{9}(-x^2)^2+\frac{4}{81}(-x^2)^3+\ldots = 1-\frac{2x^2}{3}-\frac{x^4}{9}-\ldots,\ \ -1<-x^2<1$ or $-1<x<1$

19. $f^{(n)}(x)=e^x$ for all n. $f(1)=f'(1)=f''(1)=f'''(1)=e$

$$e^x \approx e+e(x-1)+\frac{e}{2}(x-1)^2+\frac{e}{6}(x-1)^3$$

20. $f\left(\frac{\pi}{6}\right)=\frac{1}{2};\ f'\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2};\ f''\left(\frac{\pi}{6}\right)=-\frac{1}{2};\ f'''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2};\ \sin x \approx \frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$

21. $f\left(\frac{\pi}{3}\right)=\frac{1}{2};\ f'\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2};\ f''\left(\frac{\pi}{3}\right)=-\frac{1}{2};\ f'''\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2};\ \cos x \approx \frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3$

22. $f\left(\frac{\pi}{4}\right)=1;\ f'\left(\frac{\pi}{4}\right)=2;\ f''\left(\frac{\pi}{4}\right)=4;\ f'''\left(\frac{\pi}{4}\right)=16$

$$\tan x \approx 1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3$$

23. $f(1)=3;\ f'(1)=2+3=5;$

$f''(1)=2+6=8;\ f'''(1)=6$

$1+x^2+x^3=3+5(x-1)+4(x-1)^2+(x-1)^3$

This is exact since $f^{(n)}(x)=0$ for $n \ge 4$.

24. $f(-1)=2+1+3+1=7;$

$f'(-1)=-1-6-3=-10;$

$f''(-1)=6+6=12;\ f'''(1)=-6$

$2-x+3x^2-x^3=7-10(x+1)+6(x+1)^2-(x+1)^3$

This is exact since $f^{(n)}(x)=0$ for $n \ge 4$.

25. The derivative of an even function is an odd function and the derivative of an odd function is an even function. (Problem 50 of Section 3.2).
Since $f(x)=\sum a_n x^n$ is an even function, $f'(x)$ is an odd function, so $f''(x)$ is an even function, hence $f'''(x)$ is an odd function, etc.
Thus $f^{(n)}(x)$ is an even function when n is even and an odd function when n is odd.
By the Uniqueness Theorem, if $f(x)=\sum a_n x^n$, then $a_n=\frac{f^{(n)}(0)}{n!}$. If $g(x)$ is an odd function, $g(0) = 0$, thence $a_n=0$ for all odd n since $f^{(n)}(x)$ is an odd function for odd n.

26. Let $f(x)=\sum a_n x^n$ be an odd function ($f(-x) = -f(x)$) for x in $(-R, R)$. Then $a_n=0$ if n is even.
The derivative of an even function is an odd function and the derivative of an odd function is an even function (Problem 50 of Section 3.2).
Since $f(x)=\sum a_n x^n$ is an odd function, $f'(x)$ is an even function, so $f''(x)$ is an odd function, hence $f'''(x)$ is an even function, etc. Thus, $f^{(n)}(x)$ is an even function when n is odd and an odd function when n is even.
By the Uniqueness Theorem, if $f(x)=\sum a_n x^n$, then $a_n=\frac{f^{(n)}(0)}{n!}$. If $g(x)$ is an odd function, $g(0) = 0$, hence $a_n=0$ for all even n since $f^{(n)}(x)$ is an odd function for all even n.

27. $\frac{1}{\sqrt{1-t^2}}=[1+(-t^2)]^{-1/2}$

$=1-\frac{1}{2}(-t^2)+\frac{3}{8}(-t^2)^2-\frac{5}{16}(-t^2)^3+\cdots$

$=1+\frac{t^2}{2}+\frac{3t^4}{8}+\frac{5t^6}{16}+\cdots$

Thus, $\sin^{-1} x=\int_0^x \frac{1}{\sqrt{1-t^2}}dt$

$=\int_0^x \left(1+\frac{t^2}{2}+\frac{3t^4}{8}+\frac{5t^6}{16}+\cdots\right)dt$

$=\left[t+\frac{t^3}{6}+\frac{3t^5}{40}+\frac{5t^7}{112}+\cdots\right]_0^x$

$=x+\frac{x^3}{6}+\frac{3x^5}{40}+\frac{5x^7}{112}+\cdots$

28. $\frac{1}{\sqrt{1+t^2}}=(1+t^2)^{-1/2}$

$=1-\frac{1}{2}t^2+\frac{3}{8}(t^2)^2-\frac{5}{16}(t^2)^3+\cdots$

$=1-\frac{t^2}{2}+\frac{3t^4}{8}-\frac{5t^6}{16}+\cdots$

Thus, $\sinh^{-1}(x)=\int_0^x \frac{1}{\sqrt{1+t^2}}dt$

$=\int_0^x\left(1-\frac{t^2}{2}+\frac{3t^4}{8}-\frac{5t^6}{16}+\cdots\right)dt$

$=\left[t-\frac{t^3}{6}+\frac{3t^5}{40}-\frac{5t^7}{112}+\cdots\right]_0^x$

$=x-\frac{x^3}{6}+\frac{3x^5}{40}-\frac{5x^7}{112}+\cdots$

29. $\cos(x^2)=1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\frac{x^{16}}{8!}-\cdots$

$\int_0^1\cos(x^2)dx=\int_0^1\left(1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\frac{x^{16}}{8!}-\cdots\right)dx$

$=\left[x-\frac{x^5}{10}+\frac{x^9}{216}-\frac{x^{13}}{9360}+\frac{x^{17}}{685,440}-\cdots\right]_0^1$

$=1-\frac{1}{10}+\frac{1}{216}-\frac{1}{9360}+\frac{1}{685,440}-\cdots\approx 0.90452$

30. $\sin\sqrt{x}=\sqrt{x}-\frac{x^{3/2}}{3!}+\frac{x^{5/2}}{5!}-\frac{x^{7/2}}{7!}+\frac{x^{9/2}}{9!}-\ldots$

$\int_0^{0.5}\sin\sqrt{x}dx=\int_0^{0.5}\left(\sqrt{x}-\frac{x^{3/2}}{3!}+\frac{x^{5/2}}{5!}-\frac{x^{7/2}}{7!}+\frac{x^{9/2}}{9!}-\ldots\right)dx$

$=\left[\frac{2}{3}x^{3/2}-\frac{2}{5}\frac{x^{5/2}}{3!}+\frac{2}{7}\frac{x^{7/2}}{5!}-\frac{2}{9}\frac{x^{9/2}}{7!}+\frac{2}{11}\frac{x^{11/2}}{9!}-\ldots\right]_0^{0.5}$

$=\frac{2}{3}(0.5)^{3/2}-\frac{1}{15}(0.5)^{5/2}+\frac{1}{420}(0.5)^{7/2}-\frac{1}{22,680}(0.5)^{9/2}+\frac{1}{1,995,840}(0.5)^{11/2}-\ldots\approx 0.22413$

31. $\frac{1}{x}=\frac{1}{1-(1-x)}=1+(1-x)+(1-x)^2+(1-x)^3+\cdots=1-(x-1)+(x-1)^2-(x-1)^3+\cdots$

for $-1<1-x<1$, or $0<x<2$.

32. $(1+x)^{1/2}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\frac{7}{256}x^5-\ldots$

$(1-x)^{1/2}=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^2-\frac{5}{128}x^4-\frac{7}{256}x^5-\ldots$

so $f(x)=2-\frac{1}{4}x^2-\frac{5}{64}x^4-\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

Note that $f^{(n)}(0)=0$ when n is odd.

Thus, $\frac{f^{(4)}(0)}{4!}=-\frac{5}{64}$ and $\frac{f^{(51)}(0)}{51!}=0$, so $f^{(4)}(0)=-\frac{5}{64}4!=-\frac{15}{8}$ and $f^{(51)}(0)=0$.

33. a. $f(x)=1+(x+x^2)+\dfrac{(x+x^2)^2}{2!}+\dfrac{(x+x^2)^3}{3!}+\dfrac{(x+x^2)^4}{4!}+\ldots$

$$=1+(x+x^2)+\frac{1}{2}(x^2+2x^3+x^4)+\frac{1}{6}(x^3+3x^4+3x^5+x^6)+\frac{1}{24}(x^4+4x^5+6x^6+4x^7+x^8)+\ldots$$

$$=1+x+\frac{3x^2}{2}+\frac{7x^3}{6}+\frac{25x^4}{24}+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Thus $\dfrac{f^{(4)}(0)}{4!}=\dfrac{25}{24}$ so $f^{(4)}(0)=\dfrac{25}{24}4!=25.$

b. $f(x)=1+\sin x+\dfrac{\sin^2 x}{2!}+\dfrac{\sin^3 x}{3!}+\dfrac{\sin^4 x}{4!}+\ldots$

$$=1+\left(x-\frac{x^3}{3!}+\ldots\right)+\frac{1}{2}\left(x-\frac{x^3}{3!}+\ldots\right)^2+\frac{1}{6}\left(x-\frac{x^3}{3!}+\ldots\right)^3+\frac{1}{24}\left(x-\frac{x^3}{3!}+\ldots\right)^4$$

$$=1+\left(x-\frac{x^3}{3!}+\ldots\right)+\frac{1}{2}\left(x^2-2\frac{x^4}{3!}+\ldots\right)+\frac{1}{6}(x^3-\ldots)+\frac{1}{24}(x^4-\ldots)\ =1+x+\frac{x^2}{2}-\frac{x^4}{8}-\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Thus, $\dfrac{f^{(4)}(0)}{4!}=-\dfrac{1}{8}$ so $f^{(4)}(0)=-\dfrac{1}{8}4!=-3.$

c. $e^{t^2}-1=-1+\left(1+t^2+\dfrac{t^4}{2!}+\dfrac{t^6}{3!}+\dfrac{t^8}{4!}+\ldots\right)=t^2+\dfrac{t^4}{2!}+\dfrac{t^6}{3!}+\dfrac{t^8}{4!}+\ldots$

so $\dfrac{e^{t^2}-1}{t^2}=1+\dfrac{t^2}{2}+\dfrac{t^4}{6}+\dfrac{t^6}{24}+\ldots$

$$f(x)=\int_0^x\left(1+\frac{t^2}{2}+\frac{t^4}{6}+\frac{t^6}{24}+\ldots\right)dt=\left[t+\frac{t^3}{6}+\frac{t^5}{30}+\ldots\right]_0^x=x+\frac{x^3}{6}+\frac{x^5}{30}+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Thus, $\dfrac{f^{(4)}(0)}{4!}=0$ so $f^{(4)}(0)=0.$

d. $e^{\cos x-1}=1+(\cos x-1)+\dfrac{(\cos x-1)^2}{2!}+\dfrac{(\cos x-1)^3}{3!}+\ldots$

$$=1+\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)+\frac{1}{2}\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)^2+\frac{1}{6}\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)^3+\ldots$$

$$=1+\left(-\frac{x^2}{2}+\frac{x^4}{24}-\ldots\right)+\frac{1}{2}\left(\frac{x^4}{4}-\ldots\right)+\frac{1}{6}\left(-\frac{x^6}{8}+\ldots\right)=1-\frac{x^2}{2}+\frac{x^4}{6}-\ldots$$

Hence $f(x)=e-\dfrac{e}{2}x^2+\dfrac{e}{6}x^4-\ldots=\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

Thus, $\dfrac{f^{(4)}(0)}{4!}=\dfrac{e}{6}$ so $f^{(4)}(0)=\dfrac{e}{6}4!=4e.$

e. Observe that $\ln(\cos^2 x) = \ln(1-\sin^2 x)$.

$$\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots$$

$$\ln(1-\sin^2 x) = -\sin^2 x - \frac{\sin^4 x}{2} - \frac{\sin^6 x}{3} - \cdots$$

$$= -\left(x^2 - \frac{x^4}{3} + \cdots\right) - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \cdots\right)^2 - \frac{1}{3}\left(x^2 - \frac{x^4}{3} + \cdots\right)^3$$

$$= -\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots\right) - \frac{1}{2}\left(x^4 - \frac{2x^6}{3} + \cdots\right) - \frac{1}{3}(x^6 - \cdots) = -x^2 - \frac{x^4}{6} - \frac{2x^6}{45} - \cdots$$

Hence $f(x) = -x^2 - \frac{1}{6}x^4 - \frac{2}{45}x^6 - \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$.

Thus, $\frac{f^{(4)}(0)}{4!} = -\frac{1}{6}$ so $f^{(4)}(0) = -\frac{1}{6}4! = -4$.

34. $\sec x = \frac{1}{\cos x} = a_0 + a_1 x + a_2 x^2 + \ldots$ so

$$1 = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right)$$

$$= a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right)x^2 + \left(a_3 - \frac{a_1}{2}\right)x^3 + \left(a_4 - \frac{a_2}{2} + \frac{a_0}{24}\right)x^4 + \ldots.$$

Thus $a_0 = 1, a_1 = 0, a_2 - \frac{a_0}{2} = 0, a_3 - \frac{a_1}{2} = 0, a_4 - \frac{a_2}{2} + \frac{a_0}{24} = 0,$ so

$a_0 = 1, a_1 = 0, a_2 = \frac{1}{2}, a_3 = 0, a_4 = \frac{5}{24}$

and therefore $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots$

35. $\tanh x = \frac{\sinh x}{\cosh x} = a_0 + a_1 x + a_2 x^2 + \ldots$

so $\sinh x = \cosh x(a_0 + a_1 x + a_2 x^2 + \ldots)$

or $x + \frac{x^3}{6} + \frac{x^5}{120} + \ldots = \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots\right)\left(a_0 + a_1 x + a_2 x^2 + \ldots\right)$

$$= a_0 + a_1 x + \left(a_2 + \frac{a_0}{2}\right)x^2 + \left(a_3 + \frac{a_1}{2}\right)x^3 + \left(a_4 + \frac{a_2}{2} + \frac{a_0}{24}\right)x^4 + \left(a_5 + \frac{a_3}{2} + \frac{a_1}{24}\right)x^5 + \ldots$$

Thus $a_0 = 0, a_1 = 1, a_2 + \frac{a_0}{2} = 0, a_3 + \frac{a_1}{2} = \frac{1}{6},$

$a_4 + \frac{a_2}{2} + \frac{a_0}{24} = 0, a_5 + \frac{a_3}{2} + \frac{a_1}{24} = \frac{1}{120},$ so

$a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -\frac{1}{3}, a_4 = 0, a_5 = \frac{2}{15}$ and therefore

$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \ldots$

36. $\operatorname{sech} x = \dfrac{1}{\cosh x} = a_0 + a_1 x + a_2 x^2 + \ldots$

so $1 = \cosh x(a_0 + a_1 x + a_2 x^2 + \ldots)$

or $1 = \left(1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \ldots\right)\left(a_0 + a_1 x + a_2 x^2 + \ldots\right)$

$= a_0 + a_1 x + \left(a_2 + \dfrac{a_0}{2}\right)x^2 + \left(a_3 + \dfrac{a_1}{2}\right)x^3 + \left(a_4 + \dfrac{a_2}{2} + \dfrac{a_0}{24}\right)x^4 + \left(a_5 + \dfrac{a_3}{2} + \dfrac{a_1}{24}\right)x^5 + \cdots$

Thus, $a_0 = 1, a_1 = 0, \left(a_2 + \dfrac{a_0}{2}\right) = 0, \left(a_3 + \dfrac{a_1}{2}\right) = 0, \left(a_4 + \dfrac{a_2}{2} + \dfrac{a_0}{24}\right) = 0, \left(a_5 + \dfrac{a_3}{2} + \dfrac{a_0}{24}\right) = 0$, so

$a_0 = 1,\ a_1 = 0,\ a_2 = -\dfrac{1}{2},\ a_3 = 0,\ a_4 = \dfrac{5}{24},\ a_5 = 0$ and therefore

$\operatorname{sech} x = 1 - \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 - \cdots$

37. a. First define $R_3(x)$ by

$$R_3(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \frac{f'''(a)}{3!}(x-a)^3$$

For any t in the interval $[a, x]$ we define

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3 - R_3(x)\frac{(x-t)^4}{(x-a)^4}$$

Next we differentiate with respect to t using the Product and Power Rules:

$$g'(t) = 0 - f'(t) - \left[-f'(t) + f''(t)(x-t)\right] - \frac{1}{2!}\left[-2f''(t)(x-t) + f'''(t)(x-t)^2\right]$$

$$-\frac{1}{3!}\left[-3f'''(t)(x-t)^2 + f^{(4)}(t)(x-t)^3\right] + R_3(x)\frac{4(x-t)^3}{(x-a)^4}$$

$$= -\frac{f^{(4)}(t)(x-t)^3}{3!} + 4R_3(x)\frac{(x-t)^3}{(x-a)^4}$$

Since $g(x) = 0$, $g(a) = R_3(x) - R_3(x) = 0$, and $g(t)$ is continuous on $[a, x]$, we can apply the Mean Value Theorem for Derivatives. There exists, therefore, a number c between a and x such that $g'(c) = 0$. Thus,

$$0 = g'(c) = -\frac{f^{(4)}(c)(x-c)^3}{3!} + 4R_3(x)\frac{(x-c)^3}{(x-a)^4}$$

which leads to:

$$R_3(x) = \frac{f^{(4)}(c)}{4!}(x-a)^4$$

b. Like the previous part, first define $R_n(x)$ by

$$R_n(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For any t in the interval $[a, x]$ we define

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}$$

Next we differentiate with respect to t using the Product and Power Rules:

$$g'(t) = 0 - f'(t) - \left[-f'(t) + f''(t)(x-t)\right] - \frac{1}{2!}\left[-2f''(t)(x-t) + f'''(t)(x-t)^2\right] - \cdots$$

$$-\frac{1}{n!}\left[-nf^{(n)}(t)(x-t)^{n-1} + f^{(n+1)}(t)(x-t)^n\right] + R_n(x)\frac{(n+1)(x-t)^n}{(x-a)^{n+1}}$$

$$= -\frac{f^{(n+1)}(t)(x-t)^n}{n!} + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

Since $g(x) = 0$, $g(a) = R_n(x) - R_n(x) = 0$, and $g(t)$ is continuous on $[a, x]$, we can apply the Mean Value Theorem for Derivatives. There exists, therefore, a number c between a and x such that $g'(c) = 0$. Thus,

$$0 = g'(c) = -\frac{f^{(n+1)}(c)(x-c)^n}{n!} + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}$$

which leads to:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

38. a. For $\sum_{n=1}^{\infty}\binom{p}{n}x^n$, $\rho = \lim_{n\to\infty}\left|\binom{p}{n+1}x^{n+1} \div \binom{p}{n}x^n\right| = \lim_{n\to\infty}|x|\left|\frac{p(p-1)\ldots(p-n+1)(p-n)}{(n+1)!} \div \frac{p(p-1)\ldots(p-n+1)}{n!}\right|$

$$= \lim_{n\to\infty}|x|\left|\frac{p-n}{n+1}\right| = |x|$$

Thus $f(x) = 1 + \sum_{n=1}^{\infty}\binom{p}{n}x^n$ converges for $|x| < 1$.

b. It is clear that $f(0) = 1$.

Since $f(x) = 1 + \sum_{n=1}^{\infty}\binom{p}{n}x^n$, $f'(x) = \sum_{n=1}^{\infty}n\binom{p}{n}x^{n-1}$ and

$$(x+1)f'(x) = \sum_{n=1}^{\infty}n(x+1)\binom{p}{n}x^{n-1} = \sum_{n=1}^{\infty}\left[nx^n\binom{p}{n} + n\binom{p}{n}x^{n-1}\right] = 1\cdot\binom{p}{1}x^0 + \sum_{n=1}^{\infty}\left[n\binom{p}{n} + (n+1)\binom{p}{n+1}\right]x^n$$

$$n\binom{p}{n} + (n+1)\binom{p}{n+1} = n\frac{p(p-1)\ldots(p-n+1)}{n!} + (n+1)\frac{p(p-1)\ldots(p-n+1)(p-n)}{(n+1)!}$$

$$= \frac{1}{n!}[np(p-1)\ldots(p-n+1) + p(p-1)\ldots(p-n+1)(p-n)] = \frac{p(p-1)\ldots(p-n+1)}{n!}[n+p-n] = \binom{p}{n}p$$

and since $\binom{p}{1} = p$, $(1+x)f'(x) = p + \sum_{n=1}^{\infty}p\binom{p}{n}x^n = pf(x)$.

c. Let $y = f(x)$, then the differential equation is $(1+x)y' = py$ or $\frac{y'}{y} = \frac{p}{1+x}$.

$$\int\frac{dy}{y} = \int\frac{p}{1+x}dx \Rightarrow \ln|y| = p\ln|1+x| + C_1 \text{ or } y = C(1+x)^p \text{ so } f(x) = C(1+x)^p.$$

Since $f(0) = C(1)^p = C$ and $f(0) = 1$, $C = 1$ and $f(x) = (1+x)^p$.

39. $f'(t)=\begin{cases}0 & \text{if } t<0\\ 4t^3 & \text{if } t\ge 0\end{cases}$,

$f''(t)=\begin{cases}0 & \text{if } t<0\\ 12t^2 & \text{if } t\ge 0\end{cases}$,

$f'''(t)=\begin{cases}0 & \text{if } t<0\\ 24t & \text{if } t\ge 0\end{cases}$,

$f^{(4)}(t)=\begin{cases}0 & \text{if } t<0\\ 24 & \text{if } t\ge 0\end{cases}$

$\lim_{t\to 0^+} f^{(4)}(t)=24$ while $\lim_{t\to 0^-} f^{(4)}(t)=0,$ thus $f^{(4)}(0)$ does not exist, and $f(t)$ cannot be represented by a Maclaurin series.
Suppose that $g(t)$ as described in the text is represented by a Maclaurin series, so

$g(t)=a_0+a_1t+a_2t^2+\ldots=\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}t^n$ for all t in $(-R, R)$ for some $R > 0$. It is clear that, for $t\le 0$, $g(t)$ is represented by $g(t)=0+0t+0t^2+\ldots.$ However, this will not represent $g(t)$ for any $t > 0$ since the car is moving for $t > 0$. Similarly, any series that represents $g(t)$ for $t > 0$ cannot be 0 everywhere, so it will not represent $g(t)$ for $t < 0$. Thus, $g(t)$ cannot be represented by a Maclaurin series.

40. a. $f'(0)=\lim_{h\to 0}\frac{e^{-1/h^2}}{h}=\lim_{h\to 0}\frac{\frac{1}{h}}{e^{1/h^2}}$

$=\lim_{h\to 0}\frac{he^{-1/h^2}}{2}=0$ (by l'Hôpital's Rule)

b. $f'(x)=\begin{cases}2x^{-3}e^{-1/x^2} & x\ne 0\\ 0 & x=0\end{cases}$

so

$f''(0)=\lim_{h\to 0}\frac{2e^{-1/h^2}}{h^4}=\lim_{h\to 0}\frac{\frac{2}{h^4}}{e^{1/h^2}}=\lim_{h\to 0}\frac{\frac{4}{h^2}}{e^{1/h^2}}$

$=\lim_{h\to 0}\frac{4}{e^{1/h^2}}=0$ (by using l'Hôpital's Rule twice)

c. If $f^{(n)}(0)=0$ for all n, then the Maclaurin series for $f(x)$ is 0.

d. No, $f(x)\ne 0$ for $x\ne 0$. It only represents $f(x)$ at $x = 0$.

e. Note that for any n and $x\ne 0$, $R(x)=e^{-1/x^2}$.

41. $\sin x=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\cdots$

42. $\exp x=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots$

43. $3\sin x-2\exp x=-2+x-x^2-\frac{5x^3}{6}-\cdots$

$3\sin x=3x-\frac{x^3}{2}+\frac{x^5}{40}-\frac{x^7}{1680}+\cdots$

$-2\exp x=-2-2x-x^2-\frac{x^3}{3}-\cdots$

Thus, $3\sin x-2\exp x=-2+x-x^2-\frac{5x^3}{6}-\cdots$

44. $\exp(x^2)=1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\cdots$

$\exp(x^2)=1+x^2+\frac{(x^2)^2}{2}+\frac{(x^2)^3}{6}+\cdots$

$=1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\cdots$

45. $\sin(\exp x - 1) = x + \frac{x^2}{2} - \frac{5x^4}{24} - \frac{23x^5}{120} - \cdots$

$\exp x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$

$\sin(\exp x - 1) = \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots\right) - \frac{1}{6}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)^3 + \frac{1}{120}\left(x + \frac{x^2}{2} + \ldots\right)^5 - \cdots$

$= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right) - \frac{1}{6}\left(x^3 + \frac{3x^4}{2} + \frac{5x^5}{4} + \cdots\right) + \frac{1}{120}(x^5 + \cdots) - \cdots$

$= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots\right) - \left(\frac{x^3}{6} + \frac{x^4}{4} + \frac{5x^5}{24} + \ldots\right) + \left(\frac{x^5}{120} + \ldots\right) - \cdots = x + \frac{x^2}{2} - \frac{5x^4}{24} - \frac{23x^5}{120} - \cdots$

46. $\exp(\sin x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \cdots$

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$

$\exp(\sin x) = 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + \cdots\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{6} + \cdots\right)^4 + \cdots$

$= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \cdots\right) + \frac{1}{6}\left(x^3 - \frac{x^5}{2} + \cdots\right) + \frac{1}{24}\left(x^4 - \frac{2x^6}{3} + \cdots\right) + \cdots$

$= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \left(\frac{x^2}{2} - \frac{x^4}{6} + \cdots\right) + \left(\frac{x^3}{6} - \frac{x^5}{12} + \cdots\right) + \left(\frac{x^4}{24} - \frac{x^6}{36} + \cdots\right) + \cdots = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \cdots$

47. $(\sin x)(\exp x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots$

$(\sin x)(\exp x) = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$

$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) + \left(x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \cdots\right) + \left(\frac{x^3}{2} - \frac{x^5}{12} + \cdots\right) + \left(\frac{x^4}{6} - \frac{x^6}{36} + \cdots\right) + \left(\frac{x^5}{24} - \frac{x^7}{144} + \cdots\right) + \cdots$

$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots$

48. $\frac{\sin x}{\exp x} = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$

$\frac{\sin x}{\exp x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots}{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots} = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$

9.9 Concepts Review

1. $f(1)$; $f'(1)$; $f''(1)$

2. $\dfrac{f^{(6)}(0)}{6!}$

3. error of the method; error of calculation

4. increase; decrease

Problem Set 9.9

1. $f(x) = e^{2x} \quad f(0) = 1$
$f'(x) = 2e^{2x} \quad f'(0) = 2$
$f''(x) = 4e^{2x} \quad f''(0) = 4$
$f^{(3)}(x) = 8e^{2x} \quad f^{(3)}(0) = 8$
$f^{(4)}(x) = 16e^{2x} \quad f^{(4)}(0) = 16$
$f(x) \approx 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$
$f(0.12) \approx 1 + 2(0.12) + 2(0.12)^2 + \frac{4}{3}(0.12)^3 + \frac{2}{3}(0.12)^4 \approx 1.2712$

2. $f(x) = e^{-3x} \quad f(0) = 1$
$f'(x) = -3e^{-3x} \quad f'(0) = -3$
$f''(x) = 9e^{-3x} \quad f''(0) = 9$
$f^{(3)}(x) = -27e^{-3x} \quad f^{(3)}(0) = -27$
$f^{(4)}(x) = 81e^{-3x} \quad f^{(4)}(0) = 81$
$f(x) \approx 1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4$
$f(0.12) \approx 1 - 3(0.12) + \frac{9}{2}(0.12)^2 - \frac{9}{2}(0.12)^3 + \frac{27}{8}(0.12)^4 \approx 0.6977$

3. $f(x) = \sin 2x \quad f(0) = 0$
$f'(x) = 2\cos 2x \quad f'(0) = 2$
$f''(x) = -4\sin 2x \quad f''(0) = 0$
$f^{(3)}(x) = -8\cos 2x \quad f^{(3)}(0) = -8$
$f^{(4)}(x) = 16\sin 2x \quad f^{(4)}(0) = 0$
$f(x) \approx 2x - \frac{8}{3!}x^3 = 2x - \frac{4}{3}x^3$
$f(0.12) \approx 2(0.12) - \frac{4}{3}(0.12)^3 \approx 0.2377$

4. $f(x) = \tan x \quad f(0) = 0$
$f'(x) = \sec^2 x \quad f'(0) = 1$
$f''(x) = 2\sec^2 x \tan x \quad f''(0) = 0$
$f^{(3)}(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$
$f^{(3)}(0) = 2$
$f^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$
$f^{(4)}(0) = 0$
$f(x) \approx x + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$
$f(0.12) \approx 0.12 + \frac{1}{3}(0.12)^3 \approx 0.1206$

5. $f(x) = \ln(1 + x) \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x} \quad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2} \quad f''(0) = -1$

$f^{(3)}(x) = \frac{2}{(1+x)^3} \quad f^{(3)}(0) = 2$

$f^{(4)}(x) = -\frac{6}{(1+x)^4} \quad f^{(4)}(0) = -6$

$f(x) \approx x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

$f(0.12) \approx 0.12 - \frac{1}{2}(0.12)^2 + \frac{1}{3}(0.12)^3 - \frac{1}{4}(0.12)^4$

≈ 0.1133

6. $f(x) = \sqrt{1+x} \qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2} \qquad f''(0) = -\frac{1}{4}$

$f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2} \qquad f^{(3)}(0) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2} \qquad f^{(4)}(0) = -\frac{15}{16}$

$f(x) \approx 1 + \frac{1}{2}x - \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3 - \frac{\frac{15}{16}}{4!}x^4$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$

$f(0.12) \approx 1 + \frac{1}{2}(0.12) - \frac{1}{8}(0.12)^2$

$+ \frac{1}{16}(0.12)^3 - \frac{5}{128}(0.12)^4 \approx 1.0583$

7. $f(x) = \tan^{-1} x \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$

$f''(x) = -\frac{2x}{(1+x^2)^2} \qquad f''(0) = 0$

$f'''(x) = \frac{6x^2 - 2}{(1+x^2)^3} \qquad f'''(0) = -2$

$f^{(4)}(x) = \frac{-24x^3 + 24x}{(1+x^2)^4} \qquad f^{(4)}(0) = 0$

$f(x) \approx x - \frac{2}{3!}x^3 = x - \frac{1}{3}x^3$

$f(0.12) \approx 0.12 - \frac{1}{3}(0.12)^3 \approx 0.1194$

8. $f(x) = \sinh x \quad f(0) = 0$

$f'(x) = \cosh x \quad f'(0) = 1$

$f''(x) = \sinh x \quad f''(0) = 0$

$f'''(x) = \cosh x \quad f'''(0) = 1$

$f^{(4)}(x) = \sinh x \quad f^{(4)}(0) = 0$

$f(x) \approx x + \frac{1}{3!}x^3 = x + \frac{1}{6}x^3$

$f(0.12) \approx 0.12 + \frac{1}{6}(0.12)^3 \approx 0.1203$

9. $f(x) = e^x \qquad f(1) = e$

$f'(x) = e^x \quad f'(1) = e$

$f''(x) = e^x \quad f''(1) = e$

$f'''(x) = e^x \quad f'''(1) = e$

$P_3(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$

10. $f(x) = \sin x \; f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x \quad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x \quad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$

$- \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$

11. $f(x) = \tan x; \quad f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$f'(x) = \sec^2 x; \quad f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$

$f''(x) = 2\sec^2 x \tan x; \quad f''\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{9}$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x; \quad f'''\left(\frac{\pi}{6}\right) = \frac{16}{3}$

$P_3(x) = \frac{\sqrt{3}}{3} + \frac{4}{3}\left(x - \frac{\pi}{6}\right) + \frac{4\sqrt{3}}{9}\left(x - \frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x - \frac{\pi}{6}\right)^3$

12. $f(x)=\sec x$; $f\left(\frac{\pi}{4}\right)=\sqrt{2}$

$f'(x)=\sec x\tan x$; $f'\left(\frac{\pi}{4}\right)=\sqrt{2}$

$f''(x)=\sec^3 x+\sec x\tan^2 x$; $f''\left(\frac{\pi}{4}\right)=3\sqrt{2}$

$f'''(x)=5\sec^3 x\tan x+\sec x\tan^3 x$;

$f'''\left(\frac{\pi}{4}\right)=11\sqrt{2}$

$$P_3(x)=\sqrt{2}+\sqrt{2}\left(x-\frac{\pi}{4}\right)+\frac{3\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2+\frac{11\sqrt{2}}{6}\left(x-\frac{\pi}{4}\right)^3$$

13. $f(x)=\cot^{-1}x$; $f(1)=\frac{\pi}{4}$

$f'(x)=-\frac{1}{1+x^2}$; $f'(1)=-\frac{1}{2}$

$f''(x)=\frac{2x}{(1+x^2)^2}$; $f''(1)=\frac{1}{2}$

$f'''(x)=\frac{-6x^2+2}{(1+x^2)^3}$; $f'''(1)=-\frac{1}{2}$

$$P_3(x)=\frac{\pi}{4}-\frac{1}{2}(x-1)+\frac{1}{4}(x-1)^2-\frac{1}{12}(x-1)^3$$

14. $f(x)=\sqrt{x}$; $f(2)=\sqrt{2}$

$f'(x)=\frac{1}{2}x^{-1/2}$; $f'(2)=\frac{\sqrt{2}}{4}$

$f''(x)=-\frac{1}{4}x^{-3/2}$; $f''(2)=-\frac{\sqrt{2}}{16}$

$f'''(x)=\frac{3}{8}x^{-5/2}$; $f'''(2)=\frac{3\sqrt{2}}{64}$

$$P_3(x)=\sqrt{2}+\frac{\sqrt{2}}{4}(x-2)-\frac{\sqrt{2}}{32}(x-2)^2+\frac{\sqrt{2}}{128}(x-2)^3$$

15. $f(x)=x^3-2x^2+3x+5$; $f(1)=7$

$f'(x)=3x^2-4x+3$; $f'(1)=2$

$f''(x)=6x-4$; $f''(1)=2$

$f^{(3)}(x)=6$; $f^{(3)}(1)=6$

$$P_3(x)=7+2(x-1)+(x-1)^2+(x-1)^3$$
$$=5+3x-2x^2+x^3=f(x)$$

16. $f(x)=x^4$; $f(2)=16$

$f'(x)=4x^3$; $f'(2)=32$

$f''(x)=12x^2$; $f''(2)=48$

$f^{(3)}(x)=24x$; $f^{(3)}(2)=48$

$f^{(4)}(x)=24$; $f^{(4)}(2)=24$

$$P_4(x)=16+32(x-2)+24(x-2)^2+8(x-2)^3+(x-2)^4$$
$$=x^4=f(x)$$

17. $f(x)=\frac{1}{1-x}$; $f(0)=1$

$f'(x)=\frac{1}{(1-x)^2}$; $f'(0)=1$

$f''(x)=\frac{2}{(1-x)^3}$; $f''(0)=2$

$f^{(3)}(x)=\frac{6}{(1-x)^4}$; $f^{(3)}(0)=6$

$f^{(4)}(x)=\frac{24}{(1-x)^5}$; $f^{(4)}(0)=24$

$f^{(n)}(x)=\frac{n!}{(1-x)^{n+1}}$; $f^{(n)}(0)=n!$

$$f(x)\approx 1+x+\frac{2}{2!}x^2+\frac{6}{3!}x^3+\ldots+\frac{n!}{n!}x^n$$
$$=1+x+x^2+x^3+\ldots+x^n$$

Using $n=4$, $f(x)\approx 1+x+x^2+x^3+x^4$

a. $f(0.1)\approx 1.1111$

b. $f(0.5)\approx 1.9375$

c. $f(0.9)\approx 4.0951$

d. $f(2)\approx 31$

18. $f(x)=\sin x$; $f(0)=0$

$f'(x)=\cos x$; $f'(0)=1$

$f''(x)=-\sin x$; $f''(0)=0$

$f^{(3)}(x)=-\cos x$; $f^{(3)}(0)=-1$

$f^{(4)}(x)=\sin x$; $f^{(4)}(0)=0$

When n is odd,

$$\sin x\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^{(n-1)/2}x^n}{n!}$$

Using $n=5$, $\sin x\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}$.

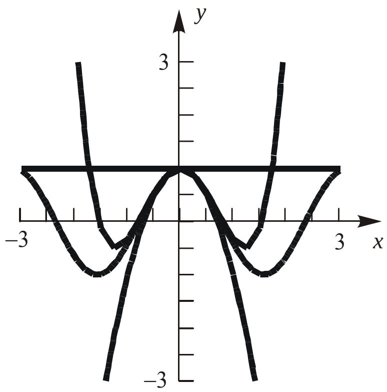
a. $\sin(0.1)\approx 0.0998$

b. $\sin(0.5) \approx 0.4794$

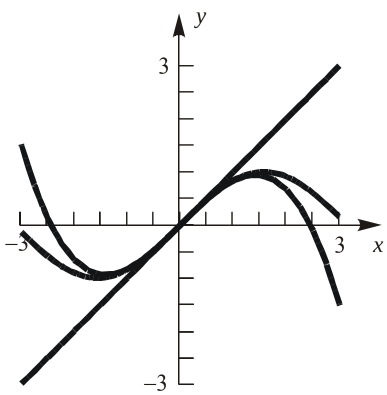
c. $\sin(1) \approx 0.8417$

d. $\sin(10) \approx 676.67$

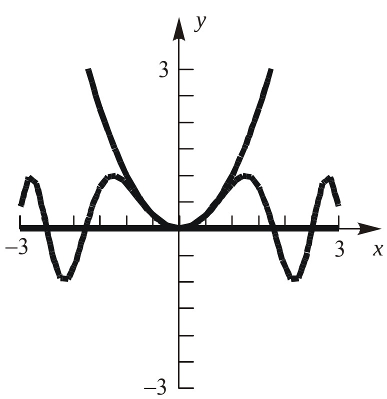
19.



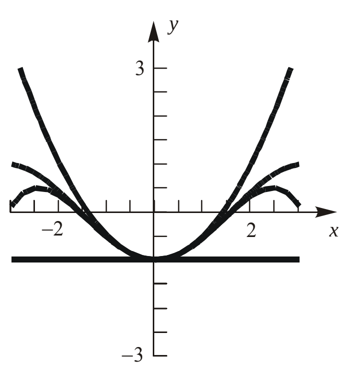
20.



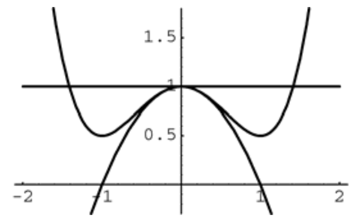
21.



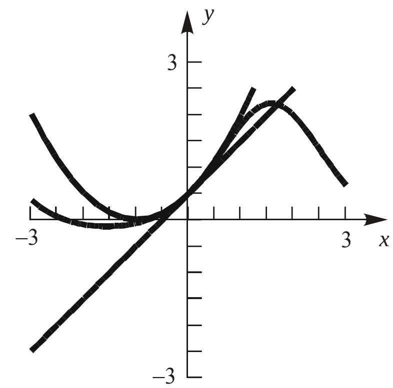
22.



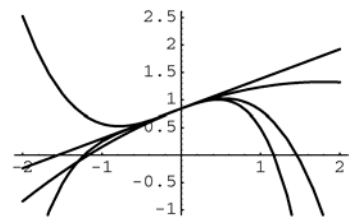
23.



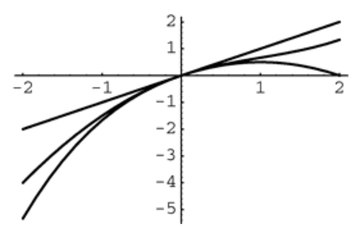
24.



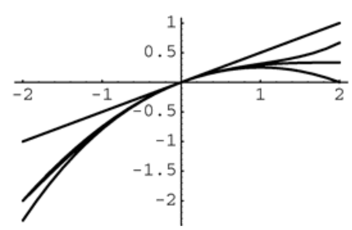
25.



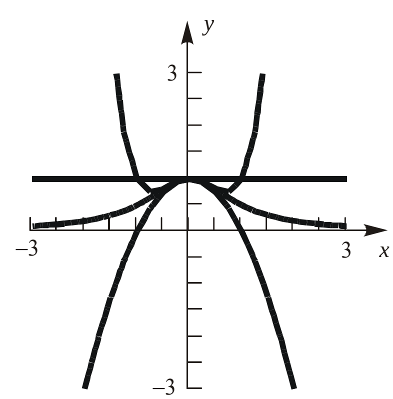
26.



27.



28.



29. $\left|e^{2c}+e^{-2c}\right| \le \left|e^{2c}\right| + \left|\frac{1}{e^{2c}}\right| \le e^6 + 1$

30. $\left|\tan c + \sec c\right| \le \left|\tan c\right| + \left|\sec c\right| \le 1+\sqrt{2}$

31. $\left|\frac{4c}{\sin c}\right| = \frac{|4c|}{|\sin c|} \le \frac{2\pi}{\frac{1}{\sqrt{2}}} = 2\sqrt{2}\pi$

32. $\left|\frac{4c}{c+4}\right| = \frac{|4c|}{|c+4|} \le \frac{4}{4} = 1$

33. $\left|\frac{e^c}{c+5}\right| = \frac{\left|e^c\right|}{|c+5|} \le \frac{e^4}{3}$

34. $\left|\frac{\cos c}{c+2}\right| = \frac{|\cos c|}{|c+2|} \le \frac{1}{2}$

35. $\left|\frac{c^2+\sin c}{10\ln c}\right| = \frac{\left|c^2+\sin c\right|}{|10\ln c|} \le \frac{\left|c^2\right|+|\sin c|}{|10\ln c|}$

$\le \frac{16+1}{10\ln 2} = \frac{17}{10\ln 2}$

36. $\left|\frac{c^2-c}{\cos c}\right| = \frac{\left|c^2-c\right|}{|\cos c|} \le \sqrt{2}\left|c^2-c\right| \le \sqrt{2}\left|\left(\frac{1}{2}\right)^2 - \frac{1}{2}\right|$

$= \frac{\sqrt{2}}{4}$ (Note that $\left|x^2-x\right|$ is maximum at $\frac{1}{2}$ in $\left[0, \pi/4\right]$.)

37. $f(x) = \ln(2+x)$; $f'(x) = \frac{1}{2+x}$;

$f''(x) = -\frac{1}{(2+x)^2}$; $f^{(3)}(x) = \frac{2}{(2+x)^3}$;

$f^{(4)}(x) = -\frac{6}{(2+x)^4}$; $f^{(5)}(x) = \frac{24}{(2+x)^5}$;

$f^{(6)}(x) = -\frac{120}{(2+x)^6}$; $f^{(7)}(x) = \frac{720}{(2+x)^7}$

$R_6(x) = \frac{1}{7!}\cdot\frac{720}{(2+c)^7}x^7 = \frac{x^7}{7(2+c)^7}$

$\left|R_6(0.5)\right| \le \left|\frac{0.5^7}{7\cdot 2^7}\right| \approx 8.719\times 10^{-6}$

38. $f(x) = e^{-x}$; $f'(x) = -e^{-x}$;

$f^{(n)}(x) = \begin{cases} e^{-x} & \text{if } n \text{ is even} \\ -e^{-x} & \text{if } n \text{ is odd} \end{cases}$

$R_6(x) = \frac{-e^{-c}}{7!}(x-1)^7 = -\frac{(x-1)^7}{5040e^c}$

$\left|R_6(0.5)\right| \le \left|\frac{(-0.5)^7}{5040e^{0.5}}\right| \approx 9.402\times 10^{-7}$

39. $f(x) = \sin x$; $f^{(7)}(x) = -\cos x$

$R_6(x) = \frac{-\cos c}{7!}\left(x-\frac{\pi}{4}\right)^7 = \frac{-\cos c\left(x-\frac{\pi}{4}\right)^7}{5040}$

$\left|R_6(0.5)\right| \le \left|\frac{\cos 0.5\left(0.5-\frac{\pi}{4}\right)^7}{5040}\right| \approx 2.685\times 10^{-8}$

40. $f(x) = \frac{1}{x-3}$; $f'(x) = -\frac{1}{(x-3)^2}$; $f''(x) = \frac{2}{(x-3)^3}$; $f^{(3)}(x) = -\frac{6}{(x-3)^4}$; $f^{(4)}(x) = \frac{24}{(x-3)^5}$;

$f^{(5)}(x) = -\frac{120}{(x-3)^6}$; $f^{(6)}(x) = \frac{720}{(x-3)^7}$; $f^{(7)}(x) = -\frac{5040}{(x-3)^8}$

$R_6(x) = \frac{1}{7!}\cdot -\frac{5040}{(c-3)^8}(x-1)^7 = -\frac{(x-1)^7}{(c-3)^8}$

$\left|R_6(0.5)\right| \le \left|\frac{(0.5-1)^7}{(1-3)^8}\right| = \left|\frac{0.5^7}{2^8}\right| \approx 3.052\times 10^{-5}$

41. If $f(x)=\frac{1}{x}$, it is easily verified that

$f^{(n)}(x)=\frac{(-1)^n n!}{x^{(n+1)}}$. Thus for $a=1$

$R_6(x)=\frac{-(x-1)^7}{c^8}$, where c is between x and 1 .

Thus, $|R_6(0.5)|=\frac{(0.5)^7}{c^8}$, where $c\in(0.5,1)$.

Therefore, $|R_6(0.5)|\le\frac{(0.5)^7}{(0.5)^8}=2$

42. If $f(x)=\frac{1}{x^2}$, it is easily verified that

$f^{(n)}(x)=\frac{(-1)^n(n+1)!}{x^{(n+2)}}$. Thus for $a=1$

$R_6(x)=\frac{(-1)^7 8!(x-1)^7}{(7!)c^9}=\frac{-8(x-1)^7}{c^9}$, where c is between x and 1 . Thus,

$|R_6(0.5)|=\frac{8(0.5)^7}{c^9}$, where $c\in(0.5,1)$.

Therefore, $|R_6(0.5)|\le\frac{8(0.5)^7}{(0.5)^9}=32$

43. $R_n(x)=\frac{e^c}{(n+1)!}x^{n+1}$

Note that $e^1<3$.

$|R_n(1)|<\frac{3}{(n+1)!}$

$\frac{3}{(n+1)!}<0.000005$ or $600000<(n+1)!$ when $n\ge 9$.

44. To find a formula for $f^{(n)}(x)$ (and thus for $R_n(x)$), is difficult, but we can use another approach: From section 9.8 we know that

$4(\arctan x)=4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{2k-1}}{(2k-1)}$, which is an alternating series.(because of the $(-1)^{k+1}$ <u>and</u> the fact that all powers are odd) for all $x\in[-1,1]$. Thus, by the Alternating Series Test,

$|R_n(x)|\le\frac{4|x|^{2(n+1)-1}}{2(n+1)-1}$ and so $|R_n(1)|\le\frac{4}{2n+1}$. Since we want $|R_n(1)|\le 0.000005$, we set

$\frac{4}{2n+1}\le 0.000005$, which yields $n>399{,}999$.

45. This is a Binomial Series ($p=\frac{1}{2}$), so the third-order Maclaurin polynomial is (see section 9.8, Thm D and example 6) $P_3(x)=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$; further,

$R_3(x)=\frac{-5x^4}{128(1+c)^{7/2}}$. Now if $x\in[-0.5,0.5]$ and c is between 0 and x, then

$1+c>0.5$ and $x^4\le\left(\frac{1}{2}\right)^4=\frac{1}{16}$ so that, for all x,

$|R_3(x)|\le\frac{5\left(\frac{1}{16}\right)}{128(0.5)^{7/2}}=\frac{5\sqrt{2}}{256}\approx 0.0276$

46. $f(x)=(1+x)^{3/2}$ $\quad f(0)=1$

$f'(x)=\frac{3}{2}(1+x)^{1/2}$ $\quad f'(0)=\frac{3}{2}$

$f''(x)=\frac{3}{4}(1+x)^{-1/2}$ $\quad f''(0)=\frac{3}{4}$

$f^{(3)}(x)=-\frac{3}{8}(1+x)^{-3/2}$ $\quad f'''(0)=-\frac{3}{8}$

$f^{(4)}(x)=\frac{9}{16}(1+x)^{-5/2}$ $\quad f^{(4)}(c)=\frac{9}{16}(1+c)^{-5/2}$

$(1+x)^{3/2}\approx 1+\frac{3}{2}x+\frac{3}{8}x^2-\frac{1}{16}x^3$

$R_3(x)=\frac{3}{128}(1+c)^{-5/2}x^4$

$|R_3(x)|\le\left|\frac{3}{128}(0.9)^{-5/2}(-0.1)^4\right|\approx 3.05\times10^{-6}$

47. $f(x)=(1+x)^{-1/2}$ $\quad f(0)=1$

$f'(x)=-\frac{1}{2}(1+x)^{-3/2}$ $\quad f'(0)=-\frac{1}{2}$

$f''(x)=\frac{3}{4}(1+x)^{-5/2}$ $\quad f''(0)=\frac{3}{4}$

$f^{(3)}(x)=-\frac{15}{8}(1+x)^{-7/2}$ $\quad f^{(3)}(0)=-\frac{15}{8}$

$f^{(4)}(x)=\frac{105}{16}(1+x)^{-9/2}$ $\quad f^{(4)}(c)=\frac{105}{16}(1+c)^{-9/2}$

$(1+x)^{-1/2}\approx 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3$

$R_3(x)=\frac{35}{128}(1+c)^{-9/2}x^4$

$|R_3(x)|\le\left|\frac{35}{128}(0.95)^{-9/2}(0.05)^4\right|\approx 2.15\times10^{-6}$

48. $f(x)=\ln\left(\frac{1+x}{1-x}\right) \quad f(0)=0$

$f'(x)=\frac{2}{1-x^2} \quad f'(0)=2$

$f''(x)=\frac{4x}{(1-x^2)^2} \quad f''(0)=0$

$f^{(3)}(x)=\frac{4(1+3x^2)}{(1-x^2)^3} \quad f^{(3)}(0)=4$

$f^{(4)}(x)=\frac{48x(1+x^2)}{(1-x^2)^4} \quad f^{(4)}(0)=0$

$f^{(5)}(x)=\frac{48(1+10x^2+5x^4)}{(1-x^2)^5}$

$f^{(5)}(c)=\frac{48(1+10c^2+5c^4)}{(1-c^2)^5}$

$\ln\left(\frac{1+x}{1-x}\right)\approx 2x+\frac{2}{3}x^3$

$R_4(x)=\frac{2}{5}\left[\frac{1+10c^2+5c^4}{(1-c^2)^5}\right]x^5$

$|R_4(x)|<\frac{2}{5}\left[\frac{1+10(0.5)^2+5(0.5)^4}{(1-(0.5)^2)^5}\right](0.5)^5$

≈ 0.201

49. $R_4(x)=\frac{\cos c}{5!}x^5$

$|R_4(x)|\le\frac{(0.5)^5}{5!}\approx 0.00026042\le 0.0002605$

$\int_0^{0.5}\sin x\,dx\approx\int_0^{0.5}\left(x-\frac{1}{6}x^3\right)dx$

$=\left[\frac{1}{2}x^2-\frac{1}{24}x^4\right]_0^{0.5}\approx 0.1224$

Error $\le 0.0002605(0.5-0)=0.00013025$

50. $R_5(x)=-\frac{\cos c}{6!}x^6$

$|R_5(x)|\le\frac{1}{6!}\approx 0.001389$

$\int_0^1\cos x\,dx\approx\int_0^1\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)dx$

$=\left[x-\frac{x^3}{6}+\frac{x^5}{120}\right]_0^1\approx 0.8417$

Error $\le 0.001389(1-0)=0.001389$

51. Assume n is odd; that is $n=2m+1$ for $m\ge 0$.

Then, $R_{n+1}(x)=R_{2m+2}(x)=\frac{f^{(2m+3)}(c)}{(2m+3)!}x^{2m+3}$.

Note that, for all m,

$f^{(4m)}(x)=\sin x,\ f^{(4m+1)}(x)=\cos x$

$f^{(4m+2)}(x)=-\sin x,\ f^{(4m+3)}(x)=-\cos x;$

therefore, $|R_{n+1}(x)|=\frac{\cos c}{(2m+3)!}x^{2m+3}$ where c is between 0 and x. For $x\in[0,\pi/2]$, $c\in(0,x)$ so that $\cos c<1$ and $x\le\frac{\pi}{2}$; hence

$|R_{n+1}(x)|\le\frac{\left(\frac{\pi}{2}\right)^{2m+3}}{(2m+3)!}$. Now, for

$k=2,3,\dots,2m+3,\ \frac{\frac{\pi}{2}}{k}\le\frac{\pi}{4}$ so that

$|R_{n+1}(x)|\le\frac{\left(\frac{\pi}{2}\right)^{2m+3}}{(2m+3)!}\le\frac{\pi}{2}\left(\frac{\pi}{4}\right)^{2m+2}$ for all

$x\in[0,\frac{\pi}{2}]$. Now

$\frac{\pi}{2}\left(\frac{\pi}{4}\right)^{2m+2}\le 0.00005\Rightarrow$

$(2m+2)\ln\left(\frac{\pi}{4}\right)\le\ln\left(\frac{2(0.00005)}{\pi}\right)\Rightarrow$

$2m+2\ge 42.8666\Rightarrow n=2m+1>42$

52. Assume n is even; that is $n=2m$ for $m\ge 0$.

Then, $R_{n+1}(x)=R_{2m+1}(x)=\frac{f^{(2m+2)}(c)}{(2m+2)!}x^{2m+2}$.

Note that, for all m,

$f^{(4m)}(x)=\cos x,\ f^{(4m+1)}(x)=-\sin x$

$f^{(4m+2)}(x)=-\cos x,\ f^{(4m+3)}(x)=\sin x;$

therefore, $|R_{n+1}(x)|=\frac{\cos c}{(2m+2)!}x^{2m+2}$ where c is between 0 and x. For $x\in[0,\pi/2]$, $c\in(0,x)$ so that $\cos c<1$ and $x\le\frac{\pi}{2}$; hence

$|R_{n+1}(x)|\le\frac{\left(\frac{\pi}{2}\right)^{2m+2}}{(2m+2)!}$. Now, for

$k=2,3,\dots,2m+2,\ \frac{\frac{\pi}{2}}{k}\le\frac{\pi}{4}$ so that

$|R_{n+1}(x)|\le\frac{\left(\frac{\pi}{2}\right)^{2m+2}}{(2m+2)!}\le\frac{\pi}{2}\left(\frac{\pi}{4}\right)^{2m+1}$ for all

$x\in[0,\frac{\pi}{2}]$. Now

$$\frac{\pi}{2}\left(\frac{\pi}{4}\right)^{2m+1} \le 0.00005 \Rightarrow$$

$$(2m+1)\ln\left(\frac{\pi}{4}\right) \le \ln\left(\frac{2(0.00005)}{\pi}\right) \Rightarrow$$

$$2m+1 \ge 42.8666 \Rightarrow n = 2m > 42$$

53. The area of the sector with angle t is $\frac{1}{2}tr^2$. The area of the triangle is

$$\frac{1}{2}\left(r\sin\frac{t}{2}\right)\left(2r\cos\frac{t}{2}\right) = r^2\sin\frac{t}{2}\cos\frac{t}{2} = \frac{1}{2}r^2\sin t$$

$$A = \frac{1}{2}tr^2 - \frac{1}{2}r^2\sin t$$

Using $n = 3$, $\sin t \approx t - \frac{1}{6}t^3$.

$$A \approx \frac{1}{2}tr^2 - \frac{1}{2}r^2\left(t - \frac{1}{6}t^3\right) = \frac{1}{12}r^2t^3$$

54. $m(v) = \dfrac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$; $m(0) = m_0$

$$m'(v) = \frac{m_0 v}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}};\quad m'(0) = 0$$

$$m''(v) = \frac{2m_0v^2 + m_0c^2}{c^4\left(1-\frac{v^2}{c^2}\right)^{5/2}};\quad m''(0) = \frac{m_0}{c^2}$$

The Maclaurin polynomial of order 2 is:

$$m(v) \approx m_0 + \frac{1}{2}\frac{m_0}{c^2}v^2 = m_0 + \frac{m_0}{2}\left(\frac{v}{c}\right)^2.$$

55. a. $\ln\left(1+\frac{r}{12}\right)^{12n} = \ln 2$

$$12n\ln\left(1+\frac{r}{12}\right) = \ln 2$$

$$n = \frac{\ln 2}{12\ln\left(1+\frac{r}{12}\right)}$$

b. $f(x) = \ln(1+x)$; $f(0) = 0$

$$f'(x) = \frac{1}{1+x};\quad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2};\quad f''(0) = -1$$

$$\ln(1+x) \approx x - \frac{x^2}{2}$$

$$n \approx \frac{\ln 2}{r - \frac{r^2}{24}} = \left[\frac{24}{r(24-r)}\right]\ln 2$$

$$= \frac{\ln 2}{r} + \frac{\ln 2}{24-r}$$

$$\approx \frac{\ln 2}{r} + \frac{\ln 2}{24} \approx \frac{0.693}{r} + 0.029$$

We let $24 - r \approx 24$ since the interest rate r is going to be close to 0.

c.

r	n (exact)	n (approx.)	n (rule 72)
0.05	13.8918	13.889	14.4
0.10	6.9603	6.959	7.2
0.15	4.6498	4.649	4.8
0.20	3.4945	3.494	3.6

56. $f(x) = 1 - e^{-(1+k)x}$; $f(0) = 0$

$$f'(x) = (1+k)e^{-(1+k)x};\quad f'(0) = (1+k)$$

$$f''(x) = -(1+k)^2e^{-(1+k)x};\quad f''(0) = -(1+k)^2$$

$$1 - e^{-(1+k)x} \approx (1+k)x - \frac{(1+k)^2}{2}x^2$$

For $x = 2k$, the polynomial is

$2k - 4k^3 - 2k^4 \approx 2k$ when k is very small.

$$1 - e^{-(1+0.01)(0.02)} \approx 0.019997 \approx 0.02$$

57. $f(x) = x^4 - 3x^3 + 2x^2 + x - 2$; $f(1) = -1$

$$f'(x) = 4x^3 - 9x^2 + 4x + 1;\quad f'(1) = 0$$

$$f''(x) = 12x^2 - 18x + 4;\quad f''(1) = -2$$

$$f^{(3)}(x) = 24x - 18;\quad f^{(3)}(1) = 6$$

$$f^{(4)}(x) = 24;\quad f^{(4)}(1) = 24$$

$$f^{(5)}(x) = 0$$

Since $f^{(5)}(x) = 0$, $R_5(x) = 0$.

$$x^4 - 3x^3 + 2x^2 + x - 2$$

$$= -1 - (x-1)^2 + (x-1)^3 + (x-1)^4$$

58. $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$P_n'(x) = f'(a) + \frac{f''(a)}{2!}2(x-a) + \frac{f'''(a)}{3!}3(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}n(x-a)^{n-1}$

$= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$

$P_n'(a) = f'(a) + 0 + 0 + \cdots + 0 = f'(a)$

$P_n'' = 0 + f''(a) + \frac{f'''(a)}{2!}2(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(n-1)(x-a)^{n-2}$

$= f''(a) + f'''(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2}$

$P_n'' = f''(a) + 0 + 0 + \cdots + 0 = f''(a)$

$\vdots \quad \vdots \quad \vdots$

$P_n^{(n)}(x) = \frac{f^{(n)}(a)}{0!}(x-a)^0 = f^{(n)}(a)$

$P_n^{(n)}(a) = f^{(n)}(a)$

59. $f(x) = \sin x$; $\quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$; $\quad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$; $\quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(3)}(x) = -\cos x$; $\quad f^{(3)}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$; $\quad f^{(4)}(c) = \sin c$

$43° = \frac{\pi}{4} - \frac{\pi}{90}$ radians

$$\sin x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + R_3(x)$$

$$\sin\left(\frac{\pi}{4} - \frac{\pi}{90}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(-\frac{\pi}{90}\right) - \frac{\sqrt{2}}{4}\left(-\frac{\pi}{90}\right)^2 - \frac{\sqrt{2}}{12}\left(-\frac{\pi}{90}\right)^3 + R_3\left(\frac{\pi}{4} - \frac{\pi}{90}\right)$$

$\approx 0.681998 + R_3$

$$|R_3| = \left|\frac{\sin c}{4!}\left(-\frac{\pi}{90}\right)^4\right| < \frac{1}{24}\left(\frac{\pi}{90}\right)^4 \approx 6.19 \times 10^{-8}$$

60. $63° = \frac{\pi}{3} + \frac{\pi}{60}$ radians

Since $f^{(n)}(x)$ is $\pm\sin x$ or $\pm\cos x$,

$|R_n(x)| \le \frac{1}{(n+1)!}\left(x - \frac{\pi}{3}\right)^{n+1}$

$\left|R_n\left(\frac{\pi}{3} + \frac{\pi}{60}\right)\right| \le \frac{1}{(n+1)!}\left(\frac{\pi}{60}\right)^{n+1}$

$\frac{1}{(n+1)!}\left(\frac{\pi}{60}\right)^{n+1} \le 0.0005$ when $n \ge 2$.

$f(x) = \cos x$; $\quad f\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'(x) = -\sin x$; $\quad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x$; $\quad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + R_3(x)$$

$$\cos 63° \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{60}\right) - \frac{1}{4}\left(\frac{\pi}{60}\right)^2 \approx 0.45397$$

61. $|R_9(x)| \le \frac{1}{10!}x^{10} \le \frac{1}{10!}\left(\frac{\pi}{2}\right)^{10} \approx 2.5202 \times 10^{-5}$

62. **a.** $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{\sin c}{720}x^6$

$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0}\left(\frac{1}{120} - \frac{\sin c}{720}x\right) = \frac{1}{120}$$

b. $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{\sin c}{5040}x^7$

$$\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{x^6} =$$

$$\lim_{x\to 0}\left(-\frac{1}{720} + \frac{\sin c}{5040}x\right) = -\frac{1}{720}$$

63. The kth derivative of $h(x)f(x)$ is

$\sum_{i=0}^{k}\binom{k}{i}h^{(i)}(x)f^{(k-i)}(x)$. If $h(x) = x^{n+1}$,

$h^{(i)}(x) = \frac{(n+1)!}{(n+1-i)!}x^{n+1-i}$.

Thus for $i \le n + 1$, $h^{(i)}(0) = 0$. Let

$q(x) = x^{n+1}f(x)$. Then

$q^{(k)}(0) = \sum_{i=0}^{k}\binom{k}{i}h^{(i)}(0)f^{(k-i)}(0) = 0$

for $k \le n + 1$.

$g^{(k)}(x) = p^{(k)}(x) + q^{(k)}(x)$, so

$g^{(k)}(0) = p^{(k)}(0) + q^{(k)}(0) = p^{(k)}(0)$

for $k \le n + 1$.

The Maclaurin polynomial of order n for g is

$p(0) + p'(0)x + \frac{p''(0)}{2!}x^2 + \ldots + \frac{p^{(n)}(0)}{n!}x^n$ which is the Maclaurin polynomial of order n for $p(x)$. Since $p(x)$ is a polynomial of degree at most n, the remainder $R_n(x)$ of Maclaurin's Formula for $p(x)$ is 0, so the Maclaurin polynomial of order n for $g(x)$ is $p(x)$.

64. Using Taylor's formula,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

Since $f'(c) = f''(c) = f'''(c) = \ldots = f^{(n)}(c) = 0$,

$f(x) = f(c) + R_n(x)$

$R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(a)(x-c)^{n+1}$ where a is between x and c.

(i) Since $f^{(n+1)}(x)$ is continuous near c, then $f^{(n+1)}(a) < 0$ when a is near c.
Thus $R_n(x) < 0$ when x is near c, so $f(x) < f(c)$ when x is near c. $f(c)$ is a local maximum.

(ii) Since $f^{(n+1)}(x)$ is continuous near c, then $f^{(n+1)}(a) > 0$ when a is near c.
Thus $R_n(x) > 0$ when x is near c, so $f(x) > f(c)$ when x is near c.
$f(c)$ is a local minimum.

Suppose $f(x) = x^4$. $f(x) > 0$ when $x > 0$ and $f(x) < 0$ when $x > 0$. Thus $x = 0$ is a local minimum.

$f'(0) = f''(0) = f'''(0) = 0$, $f^{(4)}(0) = 24 > 0$

9.10 Chapter Review

Concepts Test

1. False: If $b_n = 100$ and $a_n = 50 + (-1)^n$ then since $a_n = \begin{cases} 51 & \text{if } n \text{ is even} \\ 49 & \text{if } n \text{ is odd} \end{cases}$, $0 \le a_n \le b_n$ for all n and $\lim_{n\to\infty} b_n = 100$ while $\lim_{n\to\infty} a_n$ does not exist.

2. True: It is clear that $n! \le n^n$. The inequality $n! \le n^n \le (2n-1)!$ is equivalent to $1 \le \frac{n^n}{n!} \le \frac{(2n-1)!}{n!}$.

Expanding the terms gives

$\frac{n^n}{n!} = \frac{n}{1}\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot\cdots\cdot\frac{n}{n-1}\cdot\frac{n}{n}$

$\le (n+1)(n+2)\cdot\cdots\cdot(n+n-1)$ or

$n\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot\cdots\cdot\frac{n}{n-1} \le (n+1)(n+2)\cdot\cdots\cdot(n+n-1)$

The left-hand side consists of $n - 1$ terms, each of which is less than or equal to n, while the right-hand side consists of $n - 1$ terms, each of which is greater than n. Thus, the inequality is true so $n! \le n^n \le (2n-1)!$

3. True: If $\lim_{n\to\infty} a_n = L$ then for any $\varepsilon > 0$ there is a number $M > 0$ such that $|a_n - L| < \varepsilon$ for all $n \ge M$. Thus, for the same ε, $|a_{3n+4} - L| < \varepsilon$ for $3n + 4 \ge M$ or $n \ge \frac{M-4}{3}$. Since ε was arbitrary, $\lim_{n\to\infty} a_{3n+4} = L$.

4. False: Suppose $a_n = 1$ if $n = 2k$ or $n = 3k$ where k is any positive integer and $a_n = 0$ if n is not a multiple of 2 or 3. Then $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{3n} = 1$ but $\lim_{n\to\infty} a_n$ does not exist.

5. False: Let a_n be given by

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{if } n \text{ is composite} \end{cases}$$

Then $a_{mn} = 0$ for all mn since $m \ge 2$, hence $\lim_{n\to\infty} a_{mn} = 0$ for $m \ge 2$.

$\lim_{n\to\infty} a_n$ does not exist since for any $M > 0$ there will be a_n's with $a_n = 1$ since there are infinitely many prime numbers.

6. True: Given $\varepsilon > 0$ there are numbers M_1 and M_2 such that $|a_{2n} - L| < \varepsilon$ when $n \ge M_1$ and $|a_{2n+1} - L| < \varepsilon$ when $n \ge M_2$. Let $M = \max\{M_1, M_2\}$, then when $n \ge M$ we have $|a_{2n} - L| < \varepsilon$ and $|a_{2n+1} - L| < \varepsilon$ so $|a_n - L| < \varepsilon$ for all $n \ge 2M + 1$ since every $k \ge 2M + 1$ is either even ($k = 2n$) or odd ($k = 2n + 1$).

7. False: Let $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Then

$a_n - a_{n+1} = -\frac{1}{n+1}$ so

$\lim_{n\to\infty}(a_n - a_{n+1}) = 0$ but $\lim_{n\to\infty} a_n$ is not finite since $\lim_{n\to\infty} a_n = \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges.

8. False: $\left\{(-1)^n\right\}$ and $\left\{(-1)^{n+1}\right\}$ both diverge but $\left\{(-1)^n + (-1)^{n+1}\right\} = \left\{(-1)^n(1-1)\right\} = \{0\}$ converges.

9. True: If $\{a_n\}$ converges, then for some N, there are numbers m and M with $m \le a_n \le M$ for all $n \ge N$. Thus $\frac{m}{n} \le \frac{a_n}{n} \le \frac{M}{n}$ for all $n \ge N$. Since $\left\{\frac{m}{n}\right\}$ and $\left\{\frac{M}{n}\right\}$ both converge to 0, $\left\{\frac{a_n}{n}\right\}$ must also converge to 0.

10. False: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges.

$a_n = (-1)^n \frac{1}{\sqrt{n}}$ so $a_n^{\,2} = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

11. True: The series converges by the Alternating Series Test.

$S_1 = a_1, S_2 = a_1 - a_2, S_3 = a_1 - a_2 + a_3,$
$S_4 = a_1 - a_2 + a_3 - a_4,$ etc.

$0 < a_2 < a_1 \Rightarrow 0 < a_1 - a_2 = S_2 < a_1;$
$0 < a_3 < a_2 \Rightarrow -a_2 < -a_2 + a_3 < 0$ so
$0 < a_1 - a_2 < a_1 - a_2 + a_3 = S_3 < a_1;$
$0 < a_4 < a_3 \Rightarrow 0 < a_3 - a_4 < a_3,$ so
$-a_2 < -a_2 + a_3 - a_4 < -a_2 + a_3 < 0,$
hence
$0 < a_1 - a_2 < a_1 - a_2 + a_3 - a_4 = S_4$
$< a_1 - a_2 + a_3 < a_1;$ etc.

For each even n, $0 < S_{n-1} - a_n$ while for each odd n, $n > 1$, $S_{n-1} + a_n < a_1$.

12. True: For $n \ge 2$, $\frac{1}{n} \le \frac{1}{2}$ so

$\sum_{n=2}^{\infty} \left(\frac{1}{n}\right)^n \le \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$ which converges since it is a geometric series with $r = \frac{1}{2}$.

$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n = 1 + \frac{1}{4} + \frac{1}{27} + \ldots > 1$ since all terms are positive.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n = 1+\sum_{n=2}^{\infty}\left(\frac{1}{n}\right)^n \le 1+\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n$$

$$= 1+\frac{1}{4}+\frac{1}{8}+\ldots$$

$$= -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1} = -\frac{1}{2}+\frac{1}{1-\frac{1}{2}}$$

$$= -\frac{1}{2}+2 = \frac{3}{2}$$

Thus, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n = S$ with

$1 < S \le \frac{3}{2} < 2.$

13. False: $\sum_{n=1}^{\infty}(-1)^n$ diverges but the partial sums are bounded ($S_n = -1$ for odd n and $S_n = 0$ for even n.)

14. False: $0 < \frac{1}{n^2} \le \frac{1}{n}$ for all n in N but $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges while $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

15. True: $\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1,$ Ratio Test is inconclusive. (See the discussion before Example 5 in Section 9.4.)

16. False: $\frac{1}{n^2} > 0$ for all n in N and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, but

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1.$$

17. False: $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e} \ne 0$ so the series cannot converge.

18. False: Since $\lim_{n\to\infty}\frac{n^4+1}{e^n} = 0,$ there is some number M such that $e^n > n^4+1$ for all $n \ge M$, thus $n > \ln(n^4+1)$ and $\frac{1}{n} < \frac{1}{\ln(n^4+1)}$ for $n \ge M$. Hence, $\sum_{n=M}^{\infty}\frac{1}{n} < \sum_{n=M}^{\infty}\frac{1}{\ln(n^4+1)}$ and so $\sum_{n=1}^{\infty}\frac{1}{\ln(n^4+1)}$ diverges by the Comparison Test.

19. True: $\sum_{n=2}^{\infty}\frac{n+1}{(n\ln n)^2}$

$$= \sum_{n=2}^{\infty}\left[\frac{n}{(n\ln n)^2}+\frac{1}{(n\ln n)^2}\right]$$

$$= \sum_{n=2}^{\infty}\left[\frac{1}{n(\ln n)^2}+\frac{1}{n^2(\ln n)^2}\right]$$

$\frac{1}{x(\ln x)^2}$ is continuous, positive, and nonincreasing on $[2, \infty)$. Using $u = \ln x,\ du = \frac{1}{x}dx,$

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}dx = \int_{\ln 2}^{\infty}\frac{1}{u^2}du$$

$$= \left[-\frac{1}{u}\right]_{\ln 2}^{\infty} = 0+\frac{1}{\ln 2} < \infty$$ so

$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges.

For $n \ge 3$, $\ln n > 1$, so $(\ln n)^2 > 1$ and $\frac{1}{n^2(\ln n)^2} < \frac{1}{n^2}.$ Thus

$\sum_{n=3}^{\infty}\frac{1}{n^2(\ln n)^2} < \sum_{n=3}^{\infty}\frac{1}{n^2}$ so

$\sum_{n=3}^{\infty}\frac{1}{n^2(\ln n)^2}$ converges by the Comparison Test. Since both series converge, so does their sum.

20. False: This series is

$\sum_{n=1}^{\infty}\frac{1}{2n-1} = 1+\frac{1}{3}+\frac{1}{5}+\cdots$ which diverges.

21. True: If $0 \le a_{n+100} \le b_n$ for all n in N, then $\sum_{n=101}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$ so $\sum_{n=1}^{\infty} a_n$ also converges, since adding a finite number of terms does not affect the convergence or divergence of a series.

22. True: If $ca_n \ge \frac{1}{n}$ for all n in N with $c > 0$, then $a_n \ge \frac{1}{cn}$ for all n in N so $\sum_{n=1}^{\infty} a_n \ge \sum_{n=1}^{\infty} \frac{1}{cn} = \frac{1}{c}\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. Thus, $\sum_{n=1}^{\infty} a_n$ diverges by the Comparison Test.

23. True: $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$

$= \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$, so the sum of the first thousand terms is less than $\frac{1}{2}$.

24. False: Consider the series with $a_n = \frac{(-1)^{n+1}}{n}$. Then $(-1)^n a_n = \frac{(-1)^{2n+1}}{n} = \frac{-1}{n}$ so $\sum_{n=1}^{\infty} (-1)^n a_n = -1 - \frac{1}{2} - \frac{1}{3} - \cdots$ which diverges.

25. True: If $b_n \le a_n \le 0$ for all n in N then $0 \le -a_n \le -b_n$ for all n in N.

$\sum_{n=1}^{\infty} -b_n = (-1)\sum_{n=1}^{\infty} b_n$ which converges since $\sum_{n=1}^{\infty} b_n$ converges.

Thus, by the Comparison Test, $\sum_{n=1}^{\infty} -a_n$ converges, hence $\sum_{n=1}^{\infty} a_n = (-1)\sum_{n=1}^{\infty}(-a_n)$ also converges.

26. True Since $a_n \ge 0$ for all n, $\sum_{n=1}^{\infty} \left|(-1)^n a_n\right| = \sum_{n=1}^{\infty} a_n$ so the series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely.

27. True: $\left|\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} - \sum_{n=1}^{99} (-1)^{n+1}\frac{1}{n}\right|$

$= \left|-\frac{1}{100} + \frac{1}{101} - \frac{1}{102} - \cdots\right| < \frac{1}{100} = 0.01$

28. True: Suppose $\sum |a_n|$ converges. Thus, $\sum 2|a_n|$ converges, so $\sum\left(|a_n| + a_n\right)$ converges since $0 \le |a_n| + a_n \le 2|a_n|$. But by the linearity of convergent series $\sum a_n = \sum\left(|a_n| + a_n\right) - \sum |a_n|$ converges, which is a contradiction.

29. True: $|3-(-1.1)| = 4.1$, so the radius of convergence of the series is at least 4.1.

$|3-7| = 4 < 4.1$ so $x = 7$ is within the interval of convergence.

30. False: If the radius of convergence is 2, then the convergence at $x = 2$ is independent of the convergence at $x = -2$.

31. True: The radius of convergence is at least 1.5, so 1 is within the interval of convergence.

Thus $\int_0^1 f(x)dx = \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}\right]_0^1$

$= \sum_{n=0}^{\infty} \frac{a_n}{n+1}$.

32. False: The convergence set of a power series may consist of a single point.

33. False: Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \ne 0 \\ 0 & x = 0 \end{cases}.$$

The Maclaurin series for this function represents the function only at $x = 0$.

34. True: On (–1, 1), $f(x) = \frac{1}{1-x}$.

$f'(x) = \frac{1}{(1-x)^2} = [f(x)]^2$.

35. True: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} = e^{-x}$, $\frac{d}{dx}e^{-x} + e^{-x} = 0$

36. True: $P(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

37. True: If $p(x)$ and $q(x)$ are polynomials of degree less than or equal to n, satisfying $p(a) = q(a) = f(a)$ and $p^{(k)}(a) = q^{(k)}(a) = f^{(k)}(a)$ for $k \le n$, then $p(x) = q(x)$.

38. True: $f(0) = f'(0) = f''(0) = 0$, its second order Maclaurin polynomial is 0.

39. True: After simplifying, $P_3(x) = f(x)$.

40. True: Any Maclaurin polynomial for cos x involves only even powers of x.

41. True: The Maclaurin polynomial of an even function involves only even powers of x, so $f'(0) = 0$ if $f(x)$ is an even function.

42. True: Taylor's Formula with Remainder for $n = 0$ is $f(x) = f(a) + f'(c)(x-a)$ which is equivalent to the Mean Value Theorem.

Sample Test Problems

1. $\displaystyle\lim_{n\to\infty}\frac{9n}{\sqrt{9n^2+1}} = \lim_{n\to\infty}\frac{9}{\sqrt{9+\frac{1}{n^2}}} = 3$

The sequence converges to 3.

2. Using l'Hôpital's Rule,

$$\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty}\frac{2\sqrt{n}}{n}$$

$$= \lim_{n\to\infty}\frac{2}{\sqrt{n}} = 0.$$

3. $\displaystyle\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{4}{n}\right)^{n/4}\right)^4 = e^4$

The sequence converges to e^4.

4. $a_{n+1} = \frac{n+1}{3}a_n$ thus for $n > 3$, since $\frac{n+1}{3} > 1$, $a_{n+1} > a_n$ and the sequence diverges.

5. Let $y = \sqrt[n]{n} = n^{1/n}$ then $\ln y = \frac{1}{n}\ln n.$

$\displaystyle\lim_{n\to\infty}\frac{1}{n}\ln n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = \lim_{n\to\infty}\frac{1}{n} = 0$ by using l'Hôpital's Rule. Thus, $\displaystyle\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty}e^{\ln y} = 1.$ The sequence converges to 1.

6. $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt[3]{n}} = 0$ while $\frac{1}{\sqrt[n]{3}} = \left(\frac{1}{3}\right)^{1/n}$. As $n \to \infty$, $\frac{1}{n} \to 0$ so

$$\lim_{n\to\infty}\left(\frac{1}{3}\right)^{1/n} = \lim_{\frac{1}{n}\to 0}\left(\frac{1}{3}\right)^{1/n} = \left(\frac{1}{3}\right)^0 = 1.$$

The sequence converges to 1.

7. $a_n \ge 0$; $\displaystyle\lim_{n\to\infty}\frac{\sin^2 n}{\sqrt{n}} \le \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

The sequence converges to 0.

8. The sequence does not converge, since whenever n is an even multiple of 6, $a_n = 1$, while whenever n is an odd multiple of 6, $a_n = -1$.

9. $S_n = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \ldots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}$, so

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$. The series converges to 1.

10. $S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$, so

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}$. The series converges to $\frac{3}{2}$.

11. $\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots = \sum_{n=1}^{\infty} \ln\frac{n}{n+1} = \sum_{n=1}^{\infty}[\ln n - \ln(n+1)]$

$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \ldots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) = \ln 1 - \ln(n+1) = \ln\frac{1}{n+1}$

As $n \to \infty, \frac{1}{n+1} \to 0$ so $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\frac{1}{n+1} = -\infty$.
The series diverges.

12. $\cos k\pi = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$ so

$\sum_{k=0}^{\infty} \cos k\pi = \sum_{k=0}^{\infty} (-1)^k$ which diverges since

$S_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ so $\{S_n\}$ does not converge.

13. $\sum_{k=0}^{\infty} e^{-2k} = \sum_{k=0}^{\infty}\left(\frac{1}{e^2}\right)^k = \frac{1}{1 - \frac{1}{e^2}} = \frac{e^2}{e^2 - 1} \approx 1.1565$

since $\frac{1}{e^2} < 1$.

14. $\sum_{k=0}^{\infty} \frac{3}{2^k} = 3\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{3}{1 - \frac{1}{2}} = 6$

$\sum_{k=0}^{\infty} \frac{4}{3^k} = 4\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k = \frac{4}{1 - \frac{1}{3}} = 6$

Since both series converge, their sum converges to 6 + 6 = 12.

15. $\sum_{k=1}^{\infty} 91\left(\frac{1}{100}\right)^k = \frac{91}{1 - \frac{1}{100}} - 91 = \frac{9100}{99} - 91 = \frac{91}{99}$

The series converges since $\left|\frac{1}{100}\right| < 1$.

16. $\sum_{k=1}^{\infty} \left(\frac{1}{\ln 2}\right)^k$ diverges since $\left|\frac{1}{\ln 2}\right| > 1$.

17. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, so

$1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \cdots$ converges to $\cos 2 \approx -0.41615$.

18. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, so

$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = e^{-1} \approx 0.3679$.

19. Let $a_n = \frac{n}{1+n^2}$ and $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2} + 1} = 1;$

$0 < 1 < \infty$.
By the Limit Comparison Test, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{1+n^2}$ also diverges.

20. Let $a_n = \frac{n+5}{1+n^3}$ and $b_n = \frac{1}{n^2}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3 + 5n^2}{1+n^3} = \lim_{n\to\infty} \frac{1 + \frac{5}{n}}{\frac{1}{n^3} + 1} = 1;$

$0 < 1 < \infty$.
By the Limit Comparison Test, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+5}{1+n^2}$ also converges.

21. Since the series alternates, $\frac{1}{\sqrt[3]{n}} > \frac{1}{\sqrt[3]{n+1}} > 0$, and

$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$, the series converges by the Alternating Series Test.

22. The series diverges since

$\lim_{n\to\infty} \frac{1}{\sqrt[n]{3}} = \lim_{n\to\infty} 3^{-1/n} = 1$.

23. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n \right)$

$= \left(\frac{1}{1-\frac{1}{2}} - 1 \right) + \left(\frac{1}{1-\frac{3}{4}} - 1 \right) = 1 + 3 = 4$

The series converges to 4. The 1's must be subtracted since the index starts with $n = 1$.

24. $\rho = \lim_{n\to\infty} \left(\frac{n+1}{e^{(n+1)^2}} \div \frac{n}{e^{n^2}} \right) = \lim_{n\to\infty} \left(\frac{n+1}{ne^{2n+1}} \right)$

$= \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{e^{2n+1}} \right) = 0 < 1,$ so the series converges.

25. $\lim_{n\to\infty} \frac{n+1}{10n+12} = \frac{1}{10} \neq 0$, so the series diverges.

26. Let $a_n = \frac{\sqrt{n}}{n^2+7}$ and $b_n = \frac{1}{n^{3/2}}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+7} = \lim_{n\to\infty} \frac{1}{1+\frac{7}{n^2}} = 1;$

$0 < 1 < \infty$.

By the Limit Comparison Test, since

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $\left(\frac{3}{2} > 1\right)$,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+7}$ also converges.

27. $\rho = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)!} \div \frac{n^2}{n!} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n^2} \right| = 0 < 1$, so the series converges.

28. $\rho = \lim_{n\to\infty} \left| \frac{(n+1)^3 3^{n+1}}{(n+2)!} \div \frac{n^3 3^n}{(n+1)!} \right|$

$= \lim_{n\to\infty} \left| \frac{3(n+1)^3}{n^3(n+2)} \right| = \lim_{n\to\infty} \left| \frac{\frac{3}{n}\left(1+\frac{1}{n}\right)^3}{1+\frac{2}{n^4}} \right| = 0 < 1$

The series converges.

29. $\rho = \lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)!}{(n+3)!} \div \frac{2^n n!}{(n+2)!} \right|$

$= \lim_{n\to\infty} \left| \frac{2(n+1)}{n+3} \right| = 2 > 1$

The series diverges.

30. $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \neq 0$, so the series does not converge.

31. $\rho = \lim_{n\to\infty} \left| \frac{(n+1)^2 \left(\frac{2}{3}\right)^{n+1}}{n^2 \left(\frac{2}{3}\right)^n} \right| = \lim_{n\to\infty} \left| \frac{2}{3} \right| \left| \frac{(n+1)^2}{n^2} \right|$

$= \frac{2}{3} < 1$, so the series converges.

32. Since the series alternates, $\frac{1}{1+\ln n} > \frac{1}{1+\ln(n+1)}$,

and $\lim_{n\to\infty} \frac{1}{1+\ln n} = 0,$ the series converges by the Alternating Series Test.

33. $a_n = \frac{1}{3n-1}; \frac{1}{3n-1} > \frac{1}{3n+2}$ so $a_n > a_{n+1};$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3n-1} = 0,$ so the series

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{3n-1}$ converges by the Alternating Series Test.

Let $b_n = \frac{1}{n},$ then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3};$

$0 < \frac{1}{3} < \infty.$ By the Limit Comparison Test, since

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3n-1}$ also diverges.

The series is conditionally convergent.

34. $\rho = \lim_{n\to\infty} \left| \frac{(n+1)^3}{2^{n+1}} \div \frac{n^3}{2^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{2n^3} \right|$

$= \lim_{n\to\infty} \left| \frac{\left(1+\frac{1}{n}\right)^3}{2} \right| = \frac{1}{2} < 1$

The series is absolutely convergent.

35. $\frac{3^n}{2^{n+8}} = \frac{1}{2^8}\left(\frac{3}{2}\right)^n$;

$\lim_{n\to\infty} \frac{1}{2^8}\left(\frac{3}{2}\right)^n = \frac{1}{2^8} \lim_{n\to\infty}\left(\frac{3}{2}\right)^n = \infty$ since $\frac{3}{2} > 1$.

The series is divergent.

36. Let $f(x) = \frac{\sqrt[x]{x}}{\ln x}$, then

$$f'(x) = \frac{1}{(\ln x)^2}\left[\frac{x^{1/x}}{x^2}(1-\ln x)\ln x - \frac{x^{1/x}}{x}\right]$$

$$= \frac{x^{1/x}}{(x\ln x)^2}[\ln x - (\ln x)^2 - x],$$ for $x \ge 3$, $\ln x > 1$

so $(\ln x)^2 > \ln x$ hence $f(x)$ is decreasing on

$[3, \infty)$. Thus, if $a_n = \frac{\sqrt[n]{n}}{\ln n}, a_n > a_{n+1}$.

Let $y = \sqrt[n]{n} = n^{1/n}$, so $\ln y = \frac{1}{n}\ln n$.

Using l'Hôpital's Rule,

$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$, thus

$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\ln y} = e^0 = 1$. Hence, $\lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n}$

is of the form $\frac{1}{\infty}$ so $\lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = 0$.

Thus, by the Alternating Series Test,

$\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt[n]{n}}{\ln n}$ converges.

$\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$ hence $\sum_{n=2}^{\infty} \frac{\sqrt[n]{n}}{n} < \sum_{n=2}^{\infty} \frac{\sqrt[n]{n}}{\ln n}$.

Thus if $\sum_{n=2}^{\infty} \frac{\sqrt[n]{n}}{n}$ diverges, $\sum_{n=2}^{\infty} \frac{\sqrt[n]{n}}{\ln n}$ also diverges.

Let $a_n = \frac{\sqrt[n]{n}}{n}$ and $b_n = \frac{1}{n}$. Then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt[n]{n} = 1$ as shown above;

$0 < 1 < \infty$. Since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=2}^{\infty} \frac{\sqrt[n]{n}}{n}$ also diverges by the Limit Comparison

Test, hence $\sum_{n=2}^{\infty} \frac{\sqrt[n]{n}}{\ln n}$ also diverges.

The series is conditionally convergent.

37. $\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^3+1} \div \frac{x^n}{n^3+1} \right|$

$= \lim_{n\to\infty} |x| \left| \frac{n^3+1}{(n+1)^3+1} \right| = |x|$

When $x = 1$, the series is

$\sum_{n=0}^{\infty} \frac{1}{n^3+1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^3+1} \le 1 + \sum_{n=1}^{\infty} \frac{1}{n^3}$, which converges.

When $x = -1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3+1}$ which

converges absolutely since $\sum_{n=0}^{\infty} \frac{1}{n^3+1}$ converges.

The series converges on $-1 \le x \le 1$.

38. $\rho = \lim_{n\to\infty} \left| \frac{(-2)^{n+2} x^{n+1}}{2n+5} \div \frac{(-2)^{n+1} x^n}{2n+3} \right|$

$= \lim_{n\to\infty} |2x| \left| \frac{2n+3}{2n+5} \right| = |2x|$; $|2x| < 1$ when

$-\frac{1}{2} < x < \frac{1}{2}$.

When $x = \frac{1}{2}$, the series is

$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}\left(\frac{1}{2}\right)^n}{2n+3} = \sum_{n=0}^{\infty} \frac{(-2)\left(\frac{-2}{2}\right)^n}{2n+3}$

$= \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{2n+3}.$ $a_n = \frac{2}{2n+3}$; $\frac{2}{2n+3} > \frac{2}{2n+5}$, so

$a_n > a_{n+1}$; $\lim_{n\to\infty} \frac{2}{2n+3} = 0$ so $\sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{2n+3}$

converges by the Alternating Series Test.

When $x = -\frac{1}{2}$, the series is

$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}\left(-\frac{1}{2}\right)^n}{2n+3} = \sum_{n=0}^{\infty} \frac{(-2)\left(\frac{-2}{-2}\right)^n}{2n+3} = -\sum_{n=0}^{\infty} \frac{2}{2n+3}.$

$a_n = \frac{2}{2n+3}$, let $b_n = \frac{1}{n}$ then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n}{2n+3} = \lim_{n\to\infty} \frac{2}{2+\frac{3}{n}} = 1;$

$0 < 1 < \infty$ hence since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=0}^{\infty} \frac{2}{2n+3}$ and also $-\sum_{n=0}^{\infty} \frac{2}{2n+3}$ diverges.

The series converges on $-\frac{1}{2} < x \le \frac{1}{2}$.

39. $\rho = \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{n+2} \div \frac{(x-4)^n}{n+1} \right|$

$= \lim_{n\to\infty} |x-4| \left| \frac{n+1}{n+2} \right| = |x-4|$; $|x-4| < 1$ when $3 < x < 5$.

When $x = 5$, the series is

$\sum_{n=0}^{\infty} \frac{(-1)^n (1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

$a_n = \frac{1}{n+1}; \frac{1}{n+1} > \frac{1}{n+2}$, so $a_n > a_{n+1}$;

$\lim_{n\to\infty} \frac{1}{n+1} = 0$ so $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by the Alternating Series Test.

When $x = 3$, the series is

$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$. $a_n = \frac{1}{n+1}$, let

$b_n = \frac{1}{n}$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$; $0 < 1 < \infty$

hence since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=0}^{\infty} \frac{1}{n+1}$ also diverges.

The series converges on $3 < x \le 5$.

40. $\rho = \lim_{n\to\infty} \left| \frac{3^{n+1} x^{3n+3}}{(3n+3)!} \div \frac{3^n x^{3n}}{(3n)!} \right|$

$= \lim_{n\to\infty} |3x^3| \left| \frac{1}{(3n+3)(3n+2)(3n+1)} \right| = 0$

The series converges for all x.

41. $\rho = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{2^{n+1}+1} \div \frac{(x-3)^n}{2^n+1} \right|$

$= \lim_{n\to\infty} |x-3| \left| \frac{1+\frac{1}{2^n}}{2+\frac{1}{2^n}} \right| = \frac{|x-3|}{2}$; $\frac{|x-3|}{2} < 1$

when $1 < x < 5$.

When $x = 5$, the series is

$\sum_{n=0}^{\infty} \frac{2^n}{2^n+1} = \sum_{n=0}^{\infty} \frac{1}{1+\left(\frac{1}{2}\right)^n}$; $\lim_{n\to\infty} \frac{1}{1+\left(\frac{1}{2}\right)^n} = 1 \ne 0$

so the series diverges.

When $x = 1$, the series is

$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+\left(\frac{1}{2}\right)^n}$; $\lim_{n\to\infty} \frac{1}{1+\left(\frac{1}{2}\right)^n} = 1 \ne 0$

so the series diverges.

The series converges on $1 < x < 5$.

42. $\rho = \lim_{n\to\infty} \left| \frac{(n+1)!(x+1)^{n+1}}{3^{n+1}} \div \frac{n!(x+1)^n}{3^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x+1}{3} \right| |n+1| = \infty$ unless $x = -1$.

The series converges only for $x = -1$.

43. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ for $-1 < x < 1$.

If $f(x) = \frac{1}{1+x}$, then $f'(x) = -\frac{1}{(1+x)^2}$. Thus, differentiating the series for $\frac{1}{1+x}$ and multiplying by –1 yields

$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$. The series converges on $-1 < x < 1$.

44. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $-1 < x < 1$. If

$f(x) = \frac{1}{1+x}$, then $f''(x) = \frac{2}{(1+x)^3}$.

Differentiating the series for $\frac{1}{1+x}$ twice and dividing by 2 gives

$\frac{1}{(1+x)^3} = 1 - 3x + \frac{1}{2}(4\cdot 3)x^2 - \frac{1}{2}(5\cdot 4)x^3 + \cdots$

$= 1 - 3x + 6x^2 - 10x^3 + \cdots$.

The series converges on $-1 < x < 1$.

45. $\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)^2$

$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots$

Since the series for $\sin x$ converges for all x, so does the series for $\sin^2 x$.

46. If $f(x) = e^x$, then $f^{(n)}(x) = e^x$. Thus, $e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^3}{3!}(x-2)^3 + \frac{e^4}{4!}(x-2)^4 + \cdots$.

47. $\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$

Since the series for $\sin x$ and $\cos x$ converge for all x, so does the series for $\sin x + \cos x$.

48. Let $a_k = \frac{1}{9+k^2}$ and define $f(x) = \frac{1}{9+x^2}$; then $f(k) = a_k$ and f is positive, continuous and non-increasing (since $f'(x) = \frac{-2x}{(9+x^2)^2} < 0$) on $[1,\infty)$. Thus, by the Integral Test, $E_n < \int_n^\infty \frac{1}{9+x^2}dx = \lim_{A\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_n^A =$ $\frac{1}{3}\left[\lim_{A\to\infty}\tan^{-1}\frac{A}{3} - \tan^{-1}\frac{n}{3}\right] = \frac{\pi}{6} - \frac{1}{3}\tan^{-1}\frac{n}{3}$. Now $\frac{\pi}{6} - \frac{1}{3}\tan^{-1}\frac{n}{3} \le 0.00005 \Rightarrow n \ge 3\left[\tan\left(3\left(\frac{\pi}{6} - 0.00005\right)\right)\right] \approx 20,000$.

49. Let $a_k = \frac{k}{e^{k^2}}$ and define $f(x) = \frac{x}{e^{x^2}}$; then $f(k) = a_k$ and f is positive, continuous and non-increasing (since $f'(x) = \frac{1-2x^2}{e^{x^2}} < 0$) on $[1,\infty)$. Thus, by the Integral Test, $E_n < \int_n^\infty \frac{x}{e^{x^2}}dx = \lim_{A\to\infty}\left[-\frac{1}{2e^{x^2}}\right]_n^A =$ $\frac{1}{2}\left[-\lim_{A\to\infty}\frac{1}{2e^{A^2}} + \frac{1}{2e^{n^2}}\right] = \frac{1}{4e^{n^2}}$. Now $\frac{1}{4e^{n^2}} \le 0.000005 \Rightarrow e^{n^2} \ge 50,000 \Rightarrow n^2 \ge \ln(50,000) \approx 10.82 \Rightarrow \ n > 3$.

50. One million terms are needed to approximate the sum to within 0.001 since $\frac{1}{\sqrt{n+1}} < 0.001$ is equivalent to $999,999 < n$.

51. **a.** From the Maclaurin series for $\frac{1}{1-x}$, we have $\frac{1}{1-x^3} = 1 + x^3 + x^6 + \cdots$.

b. In Example 6 of Section 9.8 it is shown that $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$ so $\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots$.

c. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$, so $e^{-x} - 1 + x = \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$.

d. Using division with the Maclaurin series for cos x, we get $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$.

Thus, $x\sec x = x + \frac{x^3}{2} + \frac{5x^5}{4!} + \cdots$.

e. $e^{-x}\sin x = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = x - x^2 + \frac{x^3}{3} - \cdots$

f. $1 + \sin x = 1 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$; Using division, we get $\frac{1}{1+\sin x} = 1 - x + x^2 - \cdots$.

52. $f(x) = \cos x \qquad f(0) = 1$

$f^{(1)}(x) = -\sin x \qquad f^{(1)}(0) = 0$

$f^{(2)}(x) = -\cos x \qquad f^{(2)}(0) = -1$

$\therefore\ P_2(x) = 1 - \frac{x^2}{2}$

Thus, $\cos(0.1) \approx 1 - \frac{(0.1)^2}{2} = 1 - 0.005 = 0.995$

53. $f(0) = 0$

$f'(x) = \cos^2 x - 2x^2\sin^2 x$

$f'(0) = 1$

$p(x) = x$; $p(0.2) = 0.2$; $f(0.2) = 0.1998$

54. a. $f(x) = xe^x \quad f(0) = 0$

$f'(x) = e^x + xe^x \quad f'(0) = 1$

$f''(x) = 2e^x + xe^x \quad f''(0) = 2$

$f^{(3)}(x) = 3e^x + xe^x \quad f^{(3)}(0) = 3$

$f^{(4)}(x) = 4e^x + xe^x \quad f^{(4)}(0) = 4$

$f(x) \approx x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

$f(0.1) \approx 0.11052$

b. $f(x) = \cosh x \quad f(0) = 1$

$f'(x) = \sinh x \quad f'(0) = 0$

$f''(x) = \cosh x \quad f''(0) = 1$

$f^{(3)}(x) = \sinh x \quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cosh x \quad f^{(4)}(0) = 1$

$f(x) \approx 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$

$f(0.1) \approx 1.0050042$

55. $g(x) = x^3 - 2x^2 + 5x - 7 \quad g(2) = 3$

$g'(x) = 3x^2 - 4x + 5 \quad g'(2) = 9$

$g''(x) = 6x - 4 \quad g''(2) = 8$

$g^{(3)}(x) = 6 \quad g^{(3)}(2) = 6$

Since $g^{(4)}(x) = 0, R_3(x) = 0,$ so the Taylor polynomial of order 3 based at 2 is an exact representation.

$g(x) = P_4(x) = 3 + 9(x-2) + 4(x-2)^2 + (x-2)^3$

56. $g(2.1) = 3 + 9(0.1) + 4(0.1)^2 + (0.1)^3 = 3.941$

57. $f(x) = \frac{1}{x+1} \quad f(1) = \frac{1}{2}$

$f'(x) = -\frac{1}{(x+1)^2} \quad f'(1) = -\frac{1}{4}$

$f''(x) = \frac{2}{(x+1)^3} \quad f''(1) = \frac{1}{4}$

$f^{(3)}(x) = -\frac{6}{(x+1)^4} \quad f^{(3)}(1) = -\frac{3}{8}$

$f^{(4)}(x) = \frac{24}{(x+1)^5} \quad f^{(4)}(1) = \frac{3}{4}$

$$f(x) \approx \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4$$

58. $f^{(5)}(x) = -\frac{120}{(x+1)^6}$, so $R_4(x) = -\frac{(x-1)^5}{(c+1)^6}$.

$$|R_4(1.2)| = \frac{(0.2)^5}{(c+1)^6} \le \frac{(0.2)^5}{(2)^6} = 0.000005$$

59. $f(x) = \frac{1}{2}(1 - \cos 2x) \quad f(0) = 0$

$f'(x) = \sin 2x \quad f'(0) = 0$

$f''(x) = 2\cos 2x \quad f''(0) = 2$

$f^{(3)}(x) = -4\sin 2x \quad f^{(3)}(0) = 0$

$f^{(4)}(x) = -8\cos 2x \quad f^{(4)}(0) = -8$

$f^{(5)}(x) = 16\sin 2x \quad f^{(5)}(0) = 0$

$f^{(6)}(x) = 32\cos 2x \quad f^{(6)}(c) = 32\cos 2c$

$\sin^2 x \approx \frac{2}{2!}x^2 - \frac{8}{4!}x^4 = x^2 - \frac{1}{3}x^4$

$$|R_4(x)| = |R_5(x)| = \left|\frac{32}{6!}(\cos 2c)x^6\right| \le \frac{2}{45}(0.2)^6 < 2.85 \times 10^{-6}$$

60. $f^{(n+1)}(x) = \frac{(-1)^n n!}{x^{n+1}}$

$$|R_n(x)| = \left|\frac{(-1)^n}{(n+1)c^{n+1}}(x-1)^{n+1}\right| \le \frac{0.2^{n+1}}{(n+1)0.8^{n+1}} = \frac{(0.25)^{n+1}}{(n+1)}$$

$\frac{(0.25)^{n+1}}{(n+1)} < 0.00005$ when $n \ge 5$.

61. From Problem 60,

$$\ln x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5.$$

$$|R_5(x)| = \left|\frac{1}{6c^6}(x-1)^6\right| \le \frac{0.2^6}{6 \cdot 0.8^6} < 4.07 \times 10^{-5}$$

$$\int_{0.8}^{1.2} \ln x\,dx \approx \int_{0.8}^{1.2} \left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5\right]dx$$

$$= \left[\frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{30}(x-1)^6\right]_{0.8}^{1.2}$$

≈ -0.00269867

Error $\le (1.2 - 0.8)4.07 \times 10^{-5} < 1.63 \times 10^{-5}$

Review and Preview Problems

1. $f(x) = \frac{x^2}{4}$ so that $f'(x) = \frac{x}{2}$ and $f'(2) = 1$

a. The tangent line will be the line through the point $(2,1)$ having slope $m = 1$. Using the point slope formula: $(y-1) = 1(x-2)$ or $y = x-1$.

b. The normal line will be the line through the point $(2,1)$ having slope $m = -\frac{1}{1} = -1$. Using the point slope formula: $(y-1) = -1(x-2)$ or $y = -x+3$ or $x + y = 3$.

2. $f(x) = y = \frac{x^2}{4}$, $f'(x) = \frac{x}{2}$

a. The line $y = x$ has slope = 1, so we seek x such that $f'(x) = 1$ or $x = 2$. The point is $(2,1)$.

b. Since $f'(x) = \frac{x}{2}$ is the slope of the tangent line at the point $(x, x^2/4)$, $-\frac{2}{x}$ will be the slope of the normal line at the same point. Since $y = x$ has slope 1, we seek x such that $-\frac{2}{x} = 1$ or $x = -2$. The point is $(-2,1)$.

3. Solving equation 1 for y^2: $y^2 = 9 - \frac{9}{16}x^2$ and putting this result into equation 2:

$$\frac{x^2}{9} + \frac{1}{16}\left(9 - \frac{9}{16}x^2\right) = 1$$

$175x^2 = 1008 \quad x^2 = 5.76 \quad x = \pm 2.4$

Putting these values into equation 1 we get

$y^2 = 9 - \frac{9}{16}(5.76) = 9 - 3.24 = 5.76$ so $y = \pm 2.4$

also. Thus the points of intersection are $(2.4, 2.4)$, $(-2.4, 2.4)$, $(2.4, -2.4)$, $(-2.4, -2.4)$

4. Solving equation 2 for y^2: $y^2 = 9 - x^2$ and putting this result into equation 1: $\frac{x^2}{16} - \frac{x^2}{9} = 0$

Thus $x = 0$ and $y^2 = 9$, $y = \pm 3$. Thus the points of intersection are $(0,3)$, $(0,-3)$

5. Since we are given a point, all we need is the slope to determine the equation of our tangent line.

$$\frac{d}{dx}\left(x^2 + \frac{y^2}{4}\right) = \frac{d}{dx}(1)$$

$$2x + \frac{y}{2}\frac{dy}{dx} = 0$$

$$\frac{y}{2}\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-4x}{y}$$

At the point $\left(-\frac{\sqrt{3}}{2}, 1\right)$, we get

$$\frac{dy}{dx} = \frac{-4\left(-\sqrt{3}/2\right)}{1} = 2\sqrt{3} = m_{\tan}$$

Therefore, the equation of the tangent line to the curve at $\left(-\frac{\sqrt{3}}{2}, 1\right)$ is given by

$$y - 1 = 2\sqrt{3}\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$y - 1 = 2\sqrt{3}\,x + 3$$

$$y = 2\sqrt{3}\,x + 4$$

6. Since we are given a point, all we need is the slope to determine the equation of our tangent line.

$$\frac{d}{dx}\left(\frac{x^2}{9} - \frac{y^2}{16}\right) = \frac{d}{dx}(1)$$

$$\frac{2x}{9} - \frac{y}{8}\frac{dy}{dx} = 0$$

$$-\frac{y}{8}\frac{dy}{dx} = -\frac{2x}{9}$$

$$\frac{dy}{dx} = \frac{16x}{9y}$$

At the point $\left(9, 8\sqrt{2}\right)$, we get

$$\frac{dy}{dx} = \frac{16(9)}{9\left(8\sqrt{2}\right)} = \frac{2}{\sqrt{2}} = \sqrt{2} = m_{\tan}$$

Therefore, the equation of the tangent line to the curve at $\left(9, 8\sqrt{2}\right)$ is given by

$$y - 8\sqrt{2} = \sqrt{2}(x - 9)$$

$$y - 8\sqrt{2} = \sqrt{2}\,x - 9\sqrt{2}$$

$$y = \sqrt{2}\,x - \sqrt{2}$$

7. Denote the curves as

$$C_1: \frac{x^2}{100}+\frac{y^2}{64}=1 \quad \text{and} \quad C_2: \frac{x^2}{9}-\frac{y^2}{27}=1$$

a. From C_2, $3x^2-27=y^2$ and so, from C_1,

$$16x^2+25\left(3x^2-27\right)=1600$$

$$91x^2=2275,\ x^2=25,\ x=\pm 5$$

$$y^2=3(25)-27,\ y^2=48,\ y=\pm 4\sqrt{3}$$

Thus the point of intersection in the first quadrant is $P=(5,4\sqrt{3})$.

b. Slope m_1 of the line tangent to C_1 at P:

$$C_1':\ \frac{x}{50}+\frac{y}{32}y'=0,\ \ y'=-\frac{16x}{25y} \text{ so } m_1=-\frac{4\sqrt{3}}{15}$$

The line T_1 tangent to C_1 at P is

$$T_1:\ (y-4\sqrt{3})=-\frac{4\sqrt{3}}{15}(x-5) \text{ or}$$

$$4\sqrt{3}\,x+15y=80\sqrt{3}$$

Slope m_2 of the line tangent to C_2 at P:

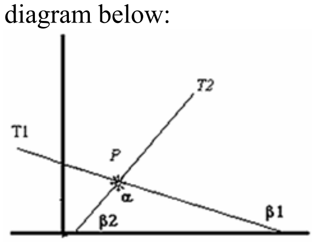
$$C_2':\ \frac{2}{9}x-\frac{2}{27}yy'=0, \qquad y'=\frac{3x}{y} \text{ so}$$

$$m_2=\frac{15}{4\sqrt{3}}=\frac{5\sqrt{3}}{4}.$$

The line T_2 tangent to C_2 at P is

$$T_2:\ (y-4\sqrt{3})=\frac{5\sqrt{3}}{4}(x-5) \text{ or } 5\sqrt{3}\,x-4y=9\sqrt{3}$$

c. To find the angles between the tangent lines, you can use problem 40 of section 0.7 or consider the diagram below:



Note that

$\alpha+\beta 2+(180-\beta 1)=180$ or $\alpha=\beta 1-\beta 2$; further

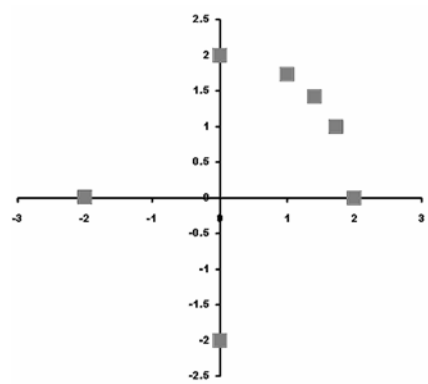
$$\beta 1=180-\tan^{-1}|m_1|=180-\tan^{-1}\left(\frac{4\sqrt{3}}{15}\right)=155.2^\circ$$

$$\beta 2=\tan^{-1}m_2=\tan^{-1}\left(\frac{5\sqrt{3}}{4}\right)=65.2$$

Thus $\alpha=155.2-65.2=90^\circ$ so the tangent lines are perpendicular. This can be verified by noting that $-\frac{1}{m_1}=\frac{15}{4\sqrt{3}}=\frac{15\sqrt{3}}{12}=\frac{5\sqrt{3}}{4}=m_2$

8.

t	$x=2\cos t$	$y=2\sin t$
0	2	0
$\pi/6$	$\sqrt{3}$	1
$\pi/4$	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	1	$\sqrt{3}$
$\pi/2$	0	2
π	-2	0
$3\pi/2$	0	-2
2π	2	0



Note that

$x^2+y^2=(2\cos t)^2+(2\sin t)^2=$

$4(\cos^2 t+\sin^2 t)=4$

so all the points will lie on the circle of radius 2 that is centered at the origin.

9. By the Pythagorean Theorem, $r^2=3^2+4^2$ or $r=\sqrt{9+16}=5$. Since

$$\sin\theta=\frac{3}{r}=\frac{3}{5},\quad \theta=\sin^{-1}(0.6)=36.9^\circ$$

10. By the Pythagorean Theorem, $r^2=2^2+5^2$ or $r=\sqrt{4+25}=\sqrt{29}$. Since

$$\sin\theta=\frac{2}{r}=\frac{2}{\sqrt{29}},\quad \theta=\sin^{-1}\left(\frac{2\sqrt{29}}{29}\right)=21.8^\circ$$

11. Since the triangle is an isosceles right triangle, $x=y$ and $x^2+y^2=8^2$. Thus $2x^2=64$ and $x=y=\sqrt{32}=4\sqrt{2}$

12. Since $\sin\frac{\pi}{6}=\frac{1}{2},\ \frac{y}{12}=\frac{1}{2}$ or $y=6$. Further $x^2+y^2=12^2$ or $x^2=144-36=108$. Hence $x=\sqrt{108}=6\sqrt{3}$.