CHAPTER 10 Conics and Polar Coordinates

10.1 Concepts Review

1. $e < 1$; $e = 1$; $e > 1$

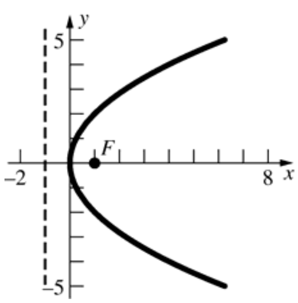
2. $y^2 = 4px$

3. (0, 1); $y = -1$

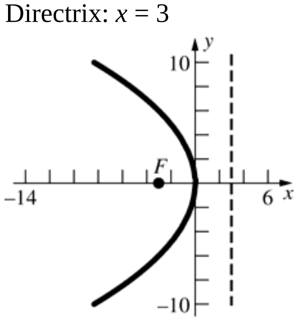
4. parallel to the axis

Problem Set 10.1

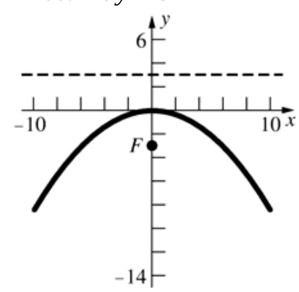
1. $y^2 = 4(1)x$
   Focus at (1, 0)
   Directrix: $x = -1$



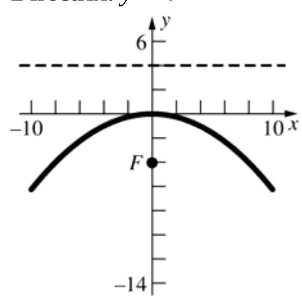
2. $y^2 = -4(3)x$
   Focus at (–3, 0)
   Directrix: $x = 3$



3. $x^2 = -4(3)y$
   Focus at (0, –3)
   Directrix: $y = 3$



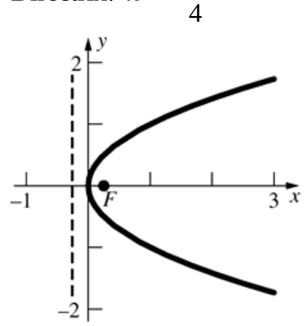
4. $x^2 = -4(4)y$
   Focus at (0, –4)
   Directrix: $y = 4$



5. $y^2 = 4\left(\frac{1}{4}\right)x$

   Focus at $\left(\frac{1}{4}, 0\right)$

   Directrix: $x = -\frac{1}{4}$

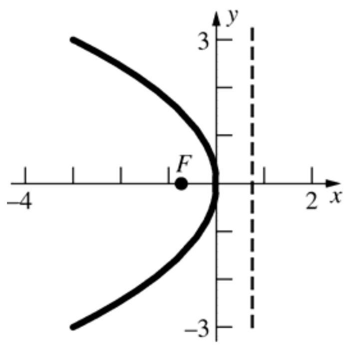


6. $y^2 = -3x$

$y^2 = -4\left(\frac{3}{4}\right)x$

Focus at $\left(-\frac{3}{4}, 0\right)$

Directrix: $x = \frac{3}{4}$

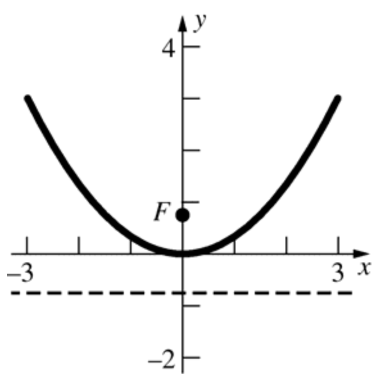


7. $2x^2 = 6y$

$x^2 = 4\left(\frac{3}{4}\right)y$

Focus at $\left(0, \frac{3}{4}\right)$

Directrix: $y = -\frac{3}{4}$

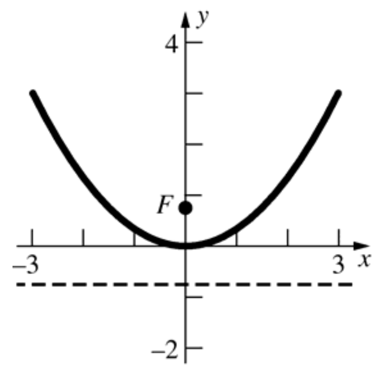


8. $3x^2 = 9y$

$x^2 = 4\left(\frac{3}{4}\right)y$

Focus at $\left(0, \frac{3}{4}\right)$

Directrix: $y = -\frac{3}{4}$



9. The parabola opens to the right, and $p = 2$.
$y^2 = 8x$

10. The parabola opens to the left, and $p = 3$.
$y^2 = -12x$

11. The parabola opens downward, and $p = 2$.
$x^2 = -8y$

12. The parabola opens downward, and $p = \frac{1}{9}$.

$x^2 = -\frac{4}{9}y$

13. The parabola opens to the left, and $p = 4$.
$y^2 = -16x$

14. The parabola opens downward, and $p = \frac{7}{2}$.

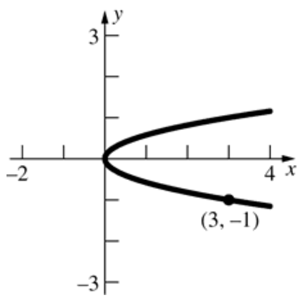
$x^2 = -14y$

15. The equation has the form $y^2 = cx$, so

$(-1)^2 = 3c.$

$c = \dfrac{1}{3}$

$y^2 = \dfrac{1}{3}x$

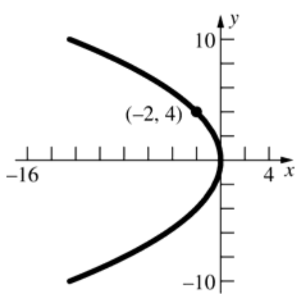


16. The equation has the form $y^2 = cx$, so

$(4)^2 = -2c.$

$c = -8$

$y^2 = -8x$

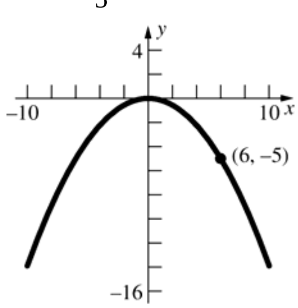


17. The equation has the form $x^2 = cy$, so

$(6)^2 = -5c.$

$c = -\dfrac{36}{5}$

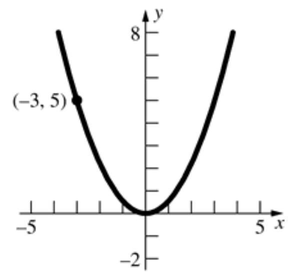
$x^2 = -\dfrac{36}{5}y$



18. The equation has the form $x^2 = cy$, so

$(-3)^2 = 5c$.

$c = \dfrac{9}{5} \quad \Rightarrow \quad x^2 = \dfrac{9}{5}y$



19. $y^2 = 16x$

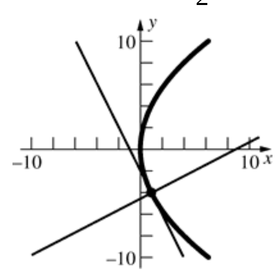
$2yy' = 16$

$y' = \dfrac{16}{2y}$

At $(1, -4)$, $y' = -2.$

Tangent: $y + 4 = -2(x - 1)$ or $y = -2x - 2$

Normal: $y + 4 = \dfrac{1}{2}(x - 1)$ or $y = \dfrac{1}{2}x - \dfrac{9}{2}$



20. $x^2 = -10y$

$2x = -10y'$

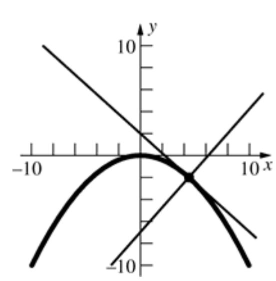
$y' = -\dfrac{x}{5}$

At $\left(2\sqrt{5}, -2\right)$, $y' = -\dfrac{2\sqrt{5}}{5}.$

Tangent: $y + 2 = -\dfrac{2\sqrt{5}}{5}\left(x - 2\sqrt{5}\right)$ or

$y = -\dfrac{2\sqrt{5}}{5}x + 2$

Normal: $y + 2 = \dfrac{\sqrt{5}}{2}\left(x - 2\sqrt{5}\right)$ or $y = \dfrac{\sqrt{5}}{2}x - 7$



21. $x^2 = 2y$

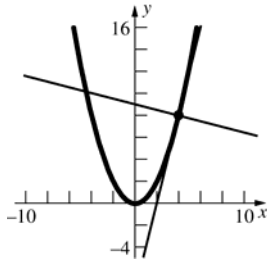
$2x = 2y'$

$y' = x$

At (4, 8), $y' = 4$.

Tangent: $y - 8 = 4(x - 4)$ or $y = 4x - 8$

Normal: $y - 8 = -\frac{1}{4}(x - 4)$ or $y = -\frac{1}{4}x + 9$



22. $y^2 = -9x$

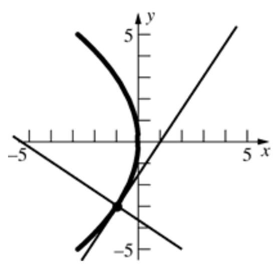
$2yy' = -9$

$y' = -\frac{9}{2y}$

At (–1, –3), $y' = \frac{3}{2}$

Tangent: $y + 3 = \frac{3}{2}(x + 1)$ or $y = \frac{3}{2}x - \frac{3}{2}$

Normal: $y + 3 = -\frac{2}{3}(x + 1)$ or $y = -\frac{2}{3}x - \frac{11}{3}$



23. $y^2 = -15x$

$2yy' = -15$

$y' = -\frac{15}{2y}$

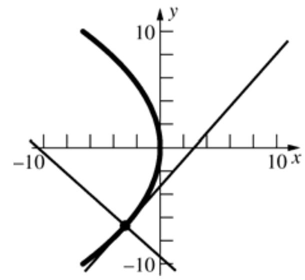
At $\left(-3, -3\sqrt{5}\right)$, $y' = \frac{\sqrt{5}}{2}$.

Tangent: $y + 3\sqrt{5} = \frac{\sqrt{5}}{2}(x + 3)$ or

$y = \frac{\sqrt{5}}{2}x - \frac{3\sqrt{5}}{2}$

Normal: $y + 3\sqrt{5} = -\frac{2\sqrt{5}}{5}(x + 3)$ or

$y = -\frac{2\sqrt{5}}{5}x - \frac{21\sqrt{5}}{5}$



24. $x^2 = 4y$

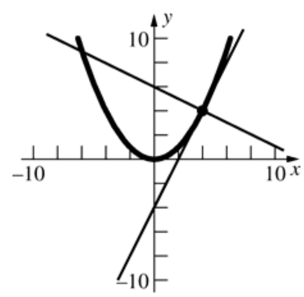
$2x = 4y'$

$y' = \frac{x}{2}$

At (4, 4), $y' = 2$.

Tangent: $y - 4 = 2(x - 4)$ or $y = 2x - 4$

Normal: $y - 4 = -\frac{1}{2}(x - 4)$ or $y = -\frac{1}{2}x + 6$



25. $x^2 = -6y$

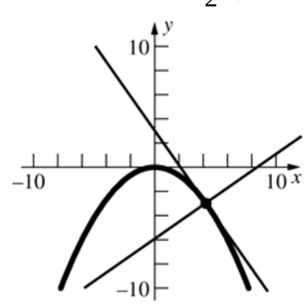
$2x = -6y'$

$y' = -\frac{x}{3}$

At $\left(3\sqrt{2}, -3\right)$, $y' = -\sqrt{2}$.

Tangent: $y + 3 = -\sqrt{2}\left(x - 3\sqrt{2}\right)$ or $y = -\sqrt{2}x + 3$

Normal: $y + 3 = \frac{\sqrt{2}}{2}\left(x - 3\sqrt{2}\right)$ or $y = \frac{\sqrt{2}}{2}x - 6$



26. $y^2 = 20x$

$2yy' = 20$

$y' = \frac{10}{y}$

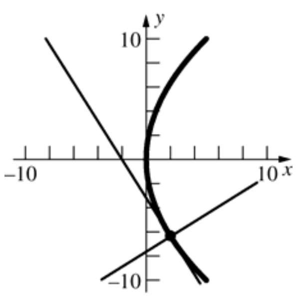
At $\left(2, -2\sqrt{10}\right)$, $y' = -\frac{\sqrt{10}}{2}$.

Tangent: $y + 2\sqrt{10} = -\frac{\sqrt{10}}{2}(x-2)$ or

$y = -\frac{\sqrt{10}}{2}x - \sqrt{10}$

Normal: $y + 2\sqrt{10} = \frac{\sqrt{10}}{5}(x-2)$ or

$y = \frac{\sqrt{10}}{5}x - \frac{12\sqrt{10}}{5}$



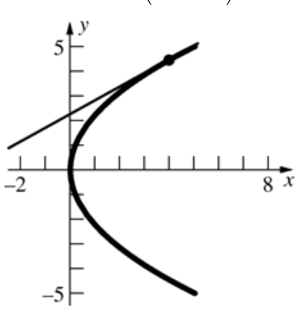
27. $y^2 = 5x$

$2yy' = 5$

$y' = \frac{5}{2y}$

$y' = \frac{\sqrt{5}}{4}$ when $y = 2\sqrt{5}$, so $x = 4$.

The point is $\left(4, 2\sqrt{5}\right)$.



28. $x^2 = -14y$

$2x = -14y'$

$y' = -\frac{x}{7}$

$y' = -\frac{2\sqrt{7}}{7}$ when $x = 2\sqrt{7}$, so $y = -2$.

The point is $\left(2\sqrt{7}, -2\right)$.

29. The slope of the line is $\frac{3}{2}$.

$y^2 = -18x;\ 2yy' = -18$

$2y\left(\frac{3}{2}\right) = -18;\ y = -6$

$(-6)^2 = -18x;\ x = -2$

The equation of the tangent line is

$y + 6 = \frac{3}{2}(x+2)$ or $y = \frac{3}{2}x - 3$.

30. Place the x-axis along the axis of the parabola such that the equation $y^2 = 4px$ describes the parabola. Let $\left(\frac{y_0^2}{4p}, y_0\right)$ be one of the extremities and $\left(\frac{y_1^2}{4p}, y_1\right)$ be the other.

First solve for y_1 in terms of y_0 and p. Since the focal chord passes through the focus $(p, 0)$, we have the following relation.

$$\frac{y_1}{\frac{y_1^2}{4p} - p} = \frac{y_0}{\frac{y_0^2}{4p} - p}$$

$$y_1(y_0^2 - 4p^2) = y_0(y_1^2 - 4p^2)$$

$$y_0 y_1^2 - (y_0^2 - 4p^2)y_1 - 4p^2 y_0 = 0$$

$$(y_1 - y_0)(y_0 y_1 + 4p^2) = 0$$

$$y_1 = y_0 \text{ or } y_1 = -\frac{4p^2}{y_0}$$

Thus, the other extremity is $\left(\frac{4p^3}{y_0^2}, -\frac{4p^2}{y_0}\right)$.

Implicitly differentiate $y^2 = 4px$ to get $2yy' = 4p$, so $y' = \frac{2p}{y}$.

At $\left(\frac{y_0^2}{4p}, y_0\right)$, $y' = \frac{2p}{y_0}$. The equation of the tangent line is $y - y_0 = \frac{2p}{y_0}\left(x - \frac{y_0^2}{4p}\right)$. When $x = -p$, $y = -\frac{2p^2}{y_0} + \frac{y_0}{2}$.

At $\left(\frac{4p^3}{y_0^2}, -\frac{4p^2}{y_0}\right)$, $y' = -\frac{y_0}{2p}$. The equation of the tangent line is $y + \frac{4p^2}{y_0} = -\frac{y_0}{2p}\left(x - \frac{4p^3}{y_0^2}\right)$.

When $x = -p$, $y = \frac{y_0}{2} - \frac{2p^2}{y_0}$.

Thus, the two tangent lines intersect on the directrix at $\left(-p, \frac{y_0}{2} - \frac{2p^2}{y_0}\right)$.

31. From Problem 30, if the parabola is described by the equation $y^2 = 4px$, the slopes of the tangent lines are $\frac{2p}{y_0}$ and $-\frac{y_0}{2p}$. Since they are negative reciprocals, the tangent lines are perpendicular.

32. Place the x-axis along the axis of the parabola such that the equation $y^2 = 4px$ describes the parabola. The endpoints of the chord are $\left(1, \frac{1}{2}\right)$ and $\left(1, -\frac{1}{2}\right)$, so $\left(\frac{1}{2}\right)^2 = 4(1)p$ or $p = \frac{1}{16}$. The distance from the vertex to the focus is $\frac{1}{16}$.

33. Assume that the x- and y-axes are positioned such that the axis of the parabola is the y-axis with the vertex at the origin and opening upward. Then the equation of the parabola is $x^2 = 4py$ and $(0, p)$ is the focus. Let D be the distance from a point on the parabola to the focus.

$$D = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + \left(\frac{x^2}{4p} - p\right)^2}$$

$$= \sqrt{\frac{x^4}{16p^2} + \frac{x^2}{2} + p^2} = \frac{x^2}{4p} + p$$

$$D' = \frac{x}{2p}; \frac{x}{2p} = 0, x = 0$$

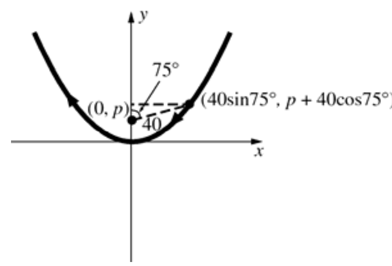
$D'' > 0$ so at $x = 0$, D is minimum. $y = 0$
Therefore, the vertex (0, 0) is the point closest to the focus.

34. Let the y-axis be the axis of the parabola, so Earth's coordinates are $(0, p)$ and the equation of the path is $x^2 = 4py$, where the coordinates are in millions of miles. When the line from Earth to the asteroid makes an angle of 90° with the axis of the parabola, the asteroid is at $(40, p)$.

$(40)^2 = 4p(p),\ p = 20$

The closest point to Earth is (0, 0), so the asteroid will come within 20 million miles of Earth.

35. Let the y-axis be the axis of the parabola, so the Earth's coordinates are $(0, p)$ and the equation of the path is $x^2 = 4py$, where the coordinates are in millions of miles. When the line from Earth to the asteroid makes an angle of 75° with the axis of the parabola, the asteroid is at $(40\sin 75^\circ, p + 40\cos 75^\circ)$. (See figure.)



$(40\sin 75^\circ)^2 = 4p(p + 40\cos 75^\circ)$

$p^2 + 40p\cos 75^\circ - 400\sin^2 75^\circ = 0$

$$p = \frac{-40\cos 75^\circ \pm \sqrt{1600\cos^2 75^\circ + 1600\sin^2 75^\circ}}{2}$$

$= -20\cos 75^\circ \pm 20$

$p = 20 - 20\cos 75^\circ \approx 14.8\ (p > 0)$

The closest point to Earth is (0, 0), so the asteroid will come within 14.8 million miles of Earth.

36. Let the equation $x^2 = 4py$ describe the cables. The cables are attached to the towers at $(\pm 400, 400)$.

$(400)^2 = 4p(400),\ p = 100$

The vertical struts are at $x = \pm 300$.

$(300)^2 = 4(100)y,\ y = 225$

The struts must be 225 m long.

37. Let $|RL|$ be the distance from R to the directrix. Observe that the distance from the latus rectum to the directrix is $2p$ so $|RG| = 2p - |RL|$. From the definition of a parabola, $|RL| = |FR|$. Thus, $|FR| + |RG| = |RL| + 2p - |RL| = 2p$.

38. Let the coordinates of P be (x_0, y_0).

$2yy' = 4p$, so $y' = \frac{2p}{y}$. Thus the slope of the normal line at P is $-\frac{y_0}{2p}$.

The equation of the normal line is

$y - y_0 = -\frac{y_0}{2p}(x - x_0)$. When $y = 0$,

$x = 2p + x_0$, so B is at $(2p + x_0, 0)$. A is at $(x_0, 0)$. Thus, $|AB| = 2p + x_0 - x_0 = 2p$.

39. Let P_1 and P_2 denote (x_1, y_1) and (x_2, y_2), respectively. $|P_1P_2| = |P_1F| + |P_2F|$ since the focal chord passes through the focus. By definition of a parabola, $|P_1F| = p + x_1$ and $|P_2F| = p + x_2$.

Thus, the length of the chord is

$|P_1P_2| = x_1 + x_2 + 2p$.

$L = p + p + 2p = 4p$

40. Let C denote the center of the circle and r the radius. Observe that the distance from a point P to the circle is $|PC| - r$. Let ℓ be the line and $|PL|$ the distance from the point to the line. Thus, $|PC| - r = |PL|$. Let l' be the line parallel to ℓ, r units away and on the side opposite from the circle. Then $|PL'|$, the distance from P to l', is $|PL| + r$; so $|PL| = |PL'| - r$. Therefore, $|PC| - r = |PL'| - r$ or $|PC| = |PL'|$. The set of points is a parabola by definition.

41. $\frac{dy}{dx} = \frac{\delta x}{H}$

$y = \frac{\delta x^2}{2H} + C$

$y(0) = 0$ implies that $C = 0$. $y = \frac{\delta x^2}{2H}$

This is an equation for a parabola.

42. a. $A(T_1)$ is the area of the trapezoid formed by $(a, 0)$, P, Q, $(b, 0)$ minus the area the two trapezoids formed by $(a, 0)$, P, (c, c^2), $(c, 0)$ and by $(c, 0)$, (c, c^2), Q, $(b, 0)$. Observe that since $c = \frac{a+b}{2}, \frac{b-c}{2} = \frac{c-a}{2} = \frac{b-a}{4}$.

$$A(T_1) = \frac{b-a}{2}[a^2+b^2] - \frac{c-a}{2}[a^2+c^2] - \frac{b-c}{2}[c^2+b^2] = \frac{b-a}{2}[a^2+b^2] - \frac{b-a}{4}[a^2+2c^2+b^2]$$

$$= \frac{b-a}{2}[a^2+b^2] - \frac{b-a}{4}\left[a^2 + 2\left(\frac{a+b}{2}\right)^2 + b^2\right] = \frac{b-a}{4}\left[a^2+b^2-\frac{a^2}{2} - ab - \frac{b^2}{2}\right]$$

$$= \frac{b-a}{4}\left(\frac{a^2}{2} - ab + \frac{b^2}{2}\right) = \frac{(b-a)^3}{8}$$

b. $A(T_2) = \frac{(c-a)^3}{8} + \frac{(b-c)^3}{8} = \frac{\left(\frac{b-a}{2}\right)^3}{8} + \frac{\left(\frac{b-a}{2}\right)^3}{8} = \frac{(b-a)^3}{32} = \frac{A(T_1)}{4}$

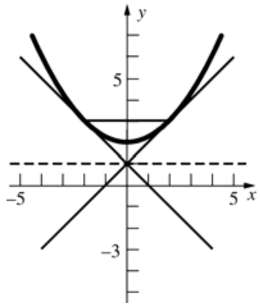
c. Using reasoning similar to part b, $A(T_n) = \frac{A(T_{n-1})}{4}$, so $A(T_n) = \frac{A(T_1)}{4^{n-1}}$.

$$A(S) = A(T_1) + A(T_2) + A(T_3) + \ldots = \sum_{n=1}^{\infty} \frac{A(T_1)}{4^{n-1}} = A(T_1)\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{4}{3}A(T_1)$$

d. Area $= \frac{b-a}{2}[a^2+b^2] - A(S) = \frac{(b-a)(a^2+b^2)}{2} - \frac{(b-a)^3}{6}$

$$= \frac{b-a}{6}[3a^2+3b^2-(b^2-2ab+a^2)] = \frac{(b-a)}{6}[2a^2+2ab+2b^2] = \frac{1}{3}(b^3-a^3) = \frac{b^3}{3} - \frac{a^3}{3}$$

43.



44. a. Since the vertex is on the positive y-axis and the parabola crosses the x-axis, its equation is of the form: $x^2 = 4p(y-k)$, where k is the y-coordinate of the vertex; that is $x^2 = 4p(y-630)$. Since the point $(315, 0)$ is on the parabola, we have

$$(315)^2 = 4p(0-630) \text{ or } 4p = \frac{-315}{2} = -157.5$$

Thus the parabola has the equation

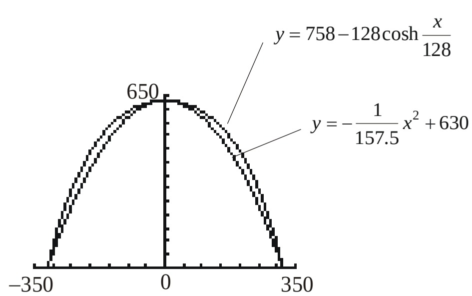
$$x^2 = -157.5(y-630)$$

b. Solving for y, we get

$$y = -\frac{1}{157.5}x^2 + 630$$

The catenary for the Gateway Arch is

$$y = 758 - 128\cosh\frac{x}{128}.$$

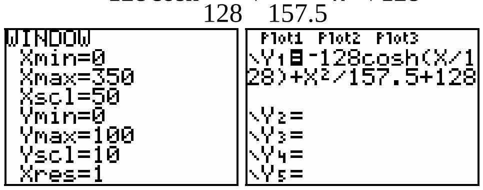


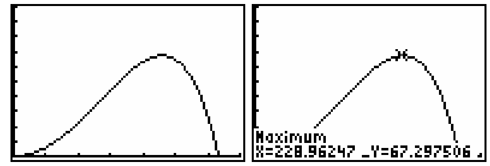
c. Because of symmetry, we can focus on the largest vertical distance between the graphs for positive x.

Let $f(x) = y_{\text{Arch}} - y_{\text{parabola}}$. That is,

$$f(x) = \left(758 - 128\cosh\frac{x}{128}\right) - \left(-\frac{1}{157.5}x^2 + 630\right)$$

$$= -128\cosh\frac{x}{128} + \frac{1}{157.5}x^2 + 128$$





Using a CAS, we find that the largest vertical distance between the catenary and the parabola is roughly 67 feet.

10.2 Concepts Review

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

2. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

3. foci

4. to the other focus; directly away from the other focus

Problem Set 10.2

1. Horizontal ellipse

2. Horizontal hyperbola

3. Vertical hyperbola

4. Horizontal hyperbola

5. Vertical parabola

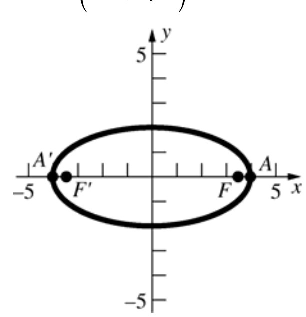
6. Vertical parabola

7. Vertical ellipse

8. Horizontal hyperbola

9. $\frac{x^2}{16} + \frac{y^2}{4} = 1$; horizontal ellipse

   $a = 4$, $b = 2$, $c = 2\sqrt{3}$
   Vertices: $(\pm 4, 0)$
   Foci: $\left(\pm 2\sqrt{3}, 0\right)$



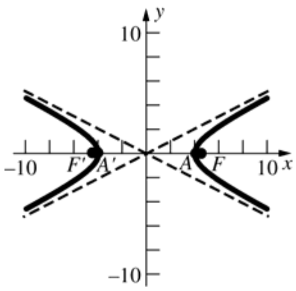
10. $\frac{x^2}{16}-\frac{y^2}{4}=1$; horizontal hyperbola

$a = 4, b = 2, \ c = 2\sqrt{5}$

Vertices: $(\pm4, 0)$

Foci: $\left(\pm2\sqrt{5}, 0\right)$

Asymptotes: $y=\pm\frac{1}{2}x$



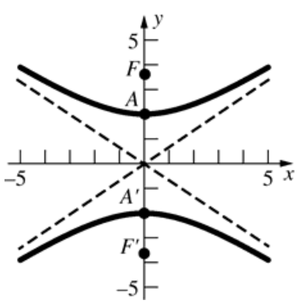
11. $\frac{y^2}{4}-\frac{x^2}{9}=1$; vertical hyperbola

$a = 2, b = 3, \ c = \sqrt{13}$

Vertices: $(0, \pm2)$

Foci: $\left(0, \pm\sqrt{13}\right)$

Asymptotes: $y=\pm\frac{2}{3}x$

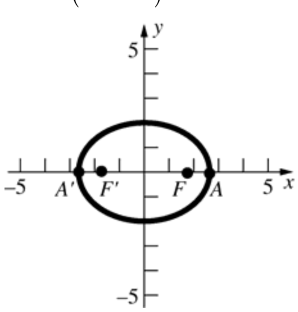


12. $\frac{x^2}{7}+\frac{y^2}{4}=1$; horizontal ellipse

$a=\sqrt{7}, b=2, c=\sqrt{3}$

Vertices: $\left(\pm\sqrt{7}, 0\right)$

Foci: $\left(\pm\sqrt{3}, 0\right)$

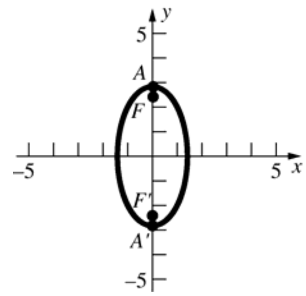


13. $\frac{x^2}{2}+\frac{y^2}{8}=1$; vertical ellipse

$a=2\sqrt{2}, b=\sqrt{2}, c=\sqrt{6}$

Vertices: $(0, \pm2\sqrt{2})$

Foci: $\left(0, \pm\sqrt{6}\right)$

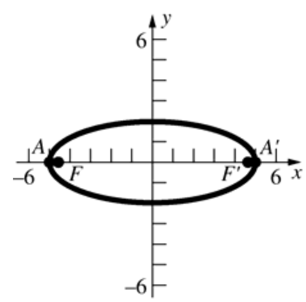


14. $\frac{x^2}{25}+\frac{y^2}{4}=1$; horizontal ellipse

$a = 5, b = 2, \ c = \sqrt{21}$

Vertices: $(\pm5, 0)$

Foci: $\left(\pm\sqrt{21}, 0\right)$



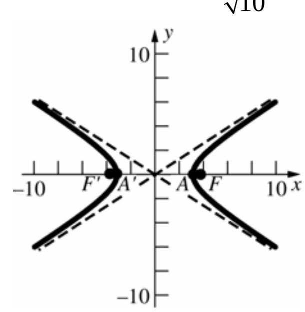
15. $\frac{x^2}{10}-\frac{y^2}{4}=1$; horizontal hyperbola

$a=\sqrt{10}, b=2, c=\sqrt{14}$

Vertices: $\left(\pm\sqrt{10}, 0\right)$

Foci: $\left(\pm\sqrt{14}, 0\right)$

Asymptotes: $y=\pm\frac{2}{\sqrt{10}}x$



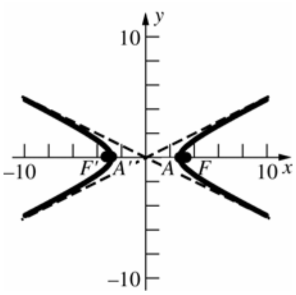
16. $\frac{x^2}{8}-\frac{y^2}{2}=1$; horizontal hyperbola

$a=2\sqrt{2}, b=\sqrt{2}, c=\sqrt{10}$

Vertices: $\left(\pm 2\sqrt{2}, 0\right)$

Foci: $\left(\pm\sqrt{10}, 0\right)$

Asymptotes: $y=\pm\frac{1}{2}x$



17. This is a horizontal ellipse with $a = 6$ and $c = 3$.

$b=\sqrt{36-9}=\sqrt{27}$

$\frac{x^2}{36}+\frac{y^2}{27}=1$

18. This is a horizontal ellipse with $c = 6$.

$a=\frac{c}{e}=\frac{6}{\frac{2}{3}}=9, b=\sqrt{81-36}=\sqrt{45}$

$\frac{x^2}{81}+\frac{y^2}{45}=1$

19. This is a vertical ellipse with $c = 5$.

$a=\frac{c}{e}=\frac{5}{\frac{1}{3}}=15, b=\sqrt{225-25}=\sqrt{200}$

$\frac{x^2}{200}+\frac{y^2}{225}=1$

20. This is a vertical ellipse with $b = 4$ and $c = 3$.

$a=\sqrt{16+9}=5$

$\frac{x^2}{16}+\frac{y^2}{25}=1$

21. This is a horizontal ellipse with $a = 5$.

$\frac{x^2}{25}+\frac{y^2}{b^2}=1$

$\frac{4}{25}+\frac{9}{b^2}=1$

$b^2=\frac{225}{21}$

$\frac{x^2}{25}+\frac{y^2}{\frac{225}{21}}=1$

22. This is a horizontal hyperbola with $a = 4$ and $c = 5$.

$b=\sqrt{25-16}=3$

$\frac{x^2}{16}-\frac{y^2}{9}=1$

23. This is a vertical hyperbola with $a = 4$ and $c = 5$.

$b=\sqrt{25-16}=3$

$\frac{y^2}{16}-\frac{x^2}{9}=1$

24. This is a vertical hyperbola with $a = 3$.

$c=ae=3\left(\frac{3}{2}\right)=\frac{9}{2}, b=\sqrt{\frac{81}{4}-9}=\frac{\sqrt{45}}{2}$

$\frac{y^2}{9}-\frac{x^2}{\frac{45}{4}}=1$

25. This is a horizontal hyperbola with $a = 8$.

The asymptotes are $y=\pm\frac{1}{2}x$, so $\frac{b}{8}=\frac{1}{2}$ or $b = 4$.

$\frac{x^2}{64}-\frac{y^2}{16}=1$

26. $c=ae=\frac{\sqrt{6}}{2}a, b^2=c^2-a^2=\frac{3}{2}a^2-a^2=\frac{1}{2}a^2$

$\frac{y^2}{a^2}-\frac{x^2}{\frac{1}{2}a^2}=1$

$\frac{16}{a^2}-\frac{4}{\frac{1}{2}a^2}=1$

$a^2=8$

$\frac{y^2}{8}-\frac{x^2}{4}=1$

27. This is a horizontal ellipse with $c = 2$.

$8=\frac{a}{e}, 8=\frac{a}{\frac{c}{a}}$, so $a^2=8c=16$.

$b=\sqrt{16-4}=\sqrt{12}$

$\frac{x^2}{16}+\frac{y^2}{12}=1$

28. This is a horizontal hyperbola with $c = 4$.

$1=\frac{a}{e}, 1=\frac{a}{\frac{c}{a}}$, so $a^2=c=4$.

$b=\sqrt{16-4}=\sqrt{12}$

$\frac{x^2}{4}-\frac{y^2}{12}=1$

29. The asymptotes are $y = \pm\frac{1}{2}x$. If the hyperbola is horizontal $\frac{b}{a} = \frac{1}{2}$ or $a = 2b$. If the hyperbola is vertical, $\frac{a}{b} = \frac{1}{2}$ or $b = 2a$.

Suppose the hyperbola is horizontal.

$$\frac{x^2}{4b^2} - \frac{y^2}{b^2} = 1$$

$$\frac{16}{4b^2} - \frac{9}{b^2} = 1$$

$$b^2 = -5$$

This is not possible.

Suppose the hyperbola is vertical.

$$\frac{y^2}{a^2} - \frac{x^2}{4a^2} = 1$$

$$\frac{9}{a^2} - \frac{16}{4a^2} = 1$$

$$a^2 = 5$$

$$\frac{y^2}{5} - \frac{x^2}{20} = 1$$

30. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\left.\begin{aligned}\frac{25}{a^2} + \frac{1}{b^2} = 1\\ \frac{16}{a^2} + \frac{4}{b^2} = 1\end{aligned}\right\} -\frac{84}{a^2} = -3$$

$$a^2 = 28$$

$$b^2 = \frac{28}{3}$$

$$\frac{x^2}{28} + \frac{y^2}{\frac{28}{3}} = 1$$

31. This is an ellipse whose foci are (0, 9) and (0, - 9) and whose major diameter has length $2a = 26$. Since the foci are on the y-axis, it is the major axis of the ellipse so the equation has the form $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$. Since $2a = 26,\ a^2 = 169$ and since $a^2 = b^2 + c^2,\ b^2 = 169 - (9)^2 = 88$. Thus the equation is $\frac{y^2}{169} + \frac{x^2}{88} = 1$

32. This is an ellipse whose foci are (4, 0) and (- 4, 0) and whose major diameter has length $2a = 14$. Since the foci are on the x-axis, it is the major axis of the ellipse so the equation has the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since $2a = 14,\ a^2 = 49$ and since $a^2 = b^2 + c^2,\ b^2 = 49 - (4)^2 = 33$. Thus the equation is $\frac{x^2}{49} + \frac{y^2}{33} = 1$

33. This is an hyperbola whose foci are (7, 0) and (- 7, 0) and whose axis is the x-axis. So the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since $2a = 12,\ a^2 = 36$ and $b^2 = c^2 - a^2 = (7^2) - 36 = 13$

Thus the equation is $\frac{x^2}{36} - \frac{y^2}{13} = 1$

34. This is an hyperbola whose foci are (0, 6) and (0, - 6) and whose axis is the y-axis. So the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Since $2a = 10,\ a^2 = 25$ and $b^2 = c^2 - a^2 = (6^2) - 25 = 11$

Thus the equation is $\frac{y^2}{25} - \frac{x^2}{11} = 1$

35. Use implicit differentiation to find the slope: $\frac{2}{27}x + \frac{2}{9}y\,y' = 0$. At the point $(3, \sqrt{6}),\ \frac{2}{9} + \frac{2\sqrt{6}}{9}y' = 0$, or $y' = -\frac{\sqrt{6}}{6}$ so the equation of the tangent line is $(y - \sqrt{6}) = -\frac{\sqrt{6}}{6}(x - 3)$ or $x + \sqrt{6}y = 9$.

36. Use implicit differentiation to find the slope: $\frac{1}{12}x + \frac{y}{8}y' = 0$. At the point $(3\sqrt{2}, -2),\ \frac{\sqrt{2}}{4} - \frac{1}{4}y' = 0$, or $y' = \sqrt{2}$ so the equation of the tangent line is $(y + 2) = \sqrt{2}(x - 3\sqrt{2})$ or $\sqrt{2}x - y = 8$.

37. Use implicit differentiation to find the slope: $\frac{2}{27}x+\frac{2}{9}y\,y'=0$. At the point $(3,-\sqrt{6})$, $\frac{2}{9}-\frac{2\sqrt{6}}{9}y'=0$, or $y'=\frac{\sqrt{6}}{6}$ so the equation of the tangent line is

$(y+\sqrt{6})=-\frac{\sqrt{6}}{6}(x-3)$ or $x-\sqrt{6}y=9$.

38. Use implicit differentiation to find the slope: $x-\frac{1}{2}y\,y'=0$. At the point $(\sqrt{3},\sqrt{2})$

$\sqrt{3}-\frac{\sqrt{2}}{2}y'=0$, or $y'=\sqrt{6}$ so the equation of the tangent line is $(y-\sqrt{2})=\sqrt{6}(x-\sqrt{3})$ or $6x-\sqrt{6}y=4\sqrt{3}$.

39. Use implicit differentiation to find the slope: $2x+2y\,y'=0$. At the point $(5,12)$

$10+24y'=0$, or $y'=-\frac{5}{12}$ so the equation of the tangent line is $(y-12)=-\frac{5}{12}(x-5)$ or $5x+12y=169$.

40. Use implicit differentiation to find the slope: $2x-2y\,y'=0$. At the point $(\sqrt{2},\sqrt{3})$

$2\sqrt{2}-2\sqrt{3}y'=0$, or $y'=\frac{\sqrt{6}}{3}$ so the equation of the tangent line is $(y-\sqrt{3})=\frac{\sqrt{6}}{3}(x-\sqrt{2})$ or $3y-\sqrt{6}\,x=\sqrt{3}$.

41. Use implicit differentiation to find the slope: $\frac{1}{44}x+\frac{2}{169}y\,y'=0$. At the point $(0,13)$, $\frac{2}{13}y'=0$, or $y'=0$. The tangent line is horizontal and thus has equation $y=13$.

42. Use implicit differentiation to find the slope: $\frac{2}{49}x+\frac{2}{33}y\,y'=0$. At the point $(7,0)$

$\frac{2}{7}+0y'=0$, or y' is undefined.. The tangent line is vertical and thus has equation $x=7$.

43. Let the y-axis run through the center of the arch and the x-axis lie on the floor. Thus a = 5 and b = 4 and the equation of the arch is $\frac{x^2}{25}+\frac{y^2}{16}=1$.

When y = 2, $\frac{x^2}{25}+\frac{(2)^2}{16}=1$, so $x=\pm\frac{5\sqrt{3}}{2}$.

The width of the box can at most be $5\sqrt{3}\approx 8.66$ ft.

44. Let the y-axis run through the center of the arch and the x-axis lie on the floor.

The equation of the arch is $\frac{x^2}{25}+\frac{y^2}{16}=1$.

When x = 2, $\frac{(2)^2}{25}+\frac{y^2}{16}=1$, so $y=\pm\frac{4\sqrt{21}}{5}$.

The height at a distance of 2 feet to the right of the center is $\frac{4\sqrt{21}}{5}\approx 3.67$ ft.

45. The foci are at ($\pm c$, 0).

$$c=\sqrt{a^2-b^2}$$
$$\frac{a^2-b^2}{a^2}+\frac{y^2}{b^2}=1$$
$$y^2=\frac{b^4}{a^2},\ y=\pm\frac{b^2}{a}$$

Thus, the length of the latus rectum is $\frac{2b^2}{a}$.

46. The foci are at ($\pm c$, 0)

$$c=\sqrt{a^2+b^2}$$
$$\frac{a^2+b^2}{a^2}-\frac{y^2}{b^2}=1$$
$$y^2=\frac{b^4}{a^2},\ y=\pm\frac{b^2}{a}$$

Thus, the length of the latus rectum is $\frac{2b^2}{a}$.

47. a = 18.09, b = 4.56,

$c=\sqrt{(18.09)^2-(4.56)^2}\approx 17.51$

The comet's minimum distance from the sun is 18.09 – 17.51 $\approx$ 0.58 AU.

48. $a-c=0.13, c=ae, a(1-e)=0.13,$

$$a=\frac{0.13}{1-0.999925}\approx 1733$$
$$a+c=a(1+e)\approx 1733(1+0.999925)\approx 3466\ \text{AU}$$

49. $a - c = 4132$; $a + c = 4583$
$2a = 8715$; $a = 4357.5$
$c = 4357.5 - 4132 = 225.5$

$$e = \frac{c}{a} = \frac{225.5}{4357.5} \approx 0.05175$$

50. (See Example 5) Since $a + c = 49.31$ and $a - c = 29.65$, we conclude that $2a = 78.96$, $2c = 19.66$ and so $a = 39.48$, $c = 9.83$. Thus $b = \sqrt{a^2 - c^2} = \sqrt{1462.0415} \approx 38.24$. So the major diameter $= 2a = 78.96$ and the minor diameter $= 2b = 76.48$.

51. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Equation of tangent line at (x_0, y_0):

$$\frac{xx_0}{4} + \frac{yy_0}{9} = 1$$

At (0, 6), $y_0 = \frac{3}{2}$.

When $y = \frac{3}{2}$, $\frac{x^2}{4} + \frac{1}{4} = 1$, $x = \pm\sqrt{3}$.

The points of tangency are $\left(\sqrt{3}, \frac{3}{2}\right)$ and $\left(-\sqrt{3}, \frac{3}{2}\right)$

52. $\frac{x^2}{4} - \frac{y^2}{36} = 1$

Equation of tangent line at (x_0, y_0):

$$\frac{xx_0}{4} - \frac{yy_0}{36} = 1$$

At (0, 6), $y_0 = -6$.

When $y = -6$, $\frac{x^2}{4} - \frac{36}{36} = 1$, $x = \pm 2\sqrt{2}$.

The points of tangency are $\left(2\sqrt{2}, -6\right)$ and $\left(-2\sqrt{2}, -6\right)$.

53. $2x^2 - 7y^2 - 35 = 0$; $4x - 14yy' = 0$

$$y' = \frac{2x}{7y}; -\frac{2}{3} = \frac{2x}{7y}; x = -\frac{7y}{3}$$

Substitute $x = -\frac{7y}{3}$ into the equation of the hyperbola.

$$\frac{98}{9}y^2 - 7y^2 - 35 = 0, y = \pm 3$$

The coordinates of the points of tangency are $\left(-7, 3\right)$ and $\left(7, -3\right)$.

54. The slope of the line is $\frac{1}{\sqrt{2}}$.

$$x^2 + 2y^2 - 2 = 0; 2x + 4yy' = 0$$

$$y' = -\frac{x}{2y}; \frac{1}{\sqrt{2}} = -\frac{x}{2y}; x = -\sqrt{2}y$$

Substitute $x = -\sqrt{2}y$ into the equation of the ellipse.

$$2y^2 + 2y^2 - 2 = 0; y = \pm\frac{1}{\sqrt{2}}$$

The tangent lines are tangent at $\left(-1, \frac{1}{\sqrt{2}}\right)$ and $\left(1, -\frac{1}{\sqrt{2}}\right)$. The equations of the tangent lines are

$y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x+1)$ and $y + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x-1)$ or $x - \sqrt{2}y + 2 = 0$ and $x - \sqrt{2}\,y - 2 = 0$.

55. $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

$$A = 4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\,dx$$

Let $x = a \sin t$ then $dx = a \cos t\, d\, t$. Then the limits are 0 and $\frac{\pi}{2}$.

$$A = 4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\,dx = 4ab\int_0^{\pi/2} \cos^2 t\,dt$$

$$= 2ab\int_0^{\pi/2} (1 + \cos 2t)dt = 2ab\left[t + \frac{\sin 2t}{2}\right]_0^{\pi/2}$$

$$= \pi ab$$

56. $x = \pm a\sqrt{1 - \frac{y^2}{b^2}}$

$$V = 2 \cdot \pi\int_0^b a^2\left(1 - \frac{y^2}{b^2}\right)dy = 2\pi a^2\left[y - \frac{y^3}{3b^2}\right]_0^b$$

$$= \frac{4\pi a^2 b}{3}$$

57. $y = \pm b\sqrt{\frac{x^2}{a^2} - 1}$

The vertical line at one focus is $x = \sqrt{a^2 + b^2}$.

$$V = \pi \int_a^{\sqrt{a^2+b^2}} \left(b\sqrt{\frac{x^2}{a^2} - 1} \right)^2 dx$$

$$= b^2 \pi \int_a^{\sqrt{a^2+b^2}} \left(\frac{x^2}{a^2} - 1 \right) dx = b^2 \pi \left[\frac{x^3}{3a^2} - x \right]_a^{\sqrt{a^2+b^2}}$$

$$= b^2 \pi \left[\frac{(a^2+b^2)^{3/2}}{3a^2} - \sqrt{a^2+b^2} + \frac{2}{3}a \right]$$

$$= \frac{\pi b^2}{3a^2} \left[(a^2+b^2)^{3/2} - 3a^2\sqrt{a^2+b^2} + 2a^3 \right]$$

58. $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

$$V = 2 \cdot \pi \int_0^a \left(b\sqrt{1 - \frac{x^2}{a^2}} \right)^2 dx$$

$$= 2\pi b^2 \int_0^a \left(1 - \frac{x^2}{a^2} \right) dx = 2\pi b^2 \left[x - \frac{x^3}{3a^2} \right]_0^a = \frac{4}{3}\pi a b^2$$

59. If one corner of the rectangle is at (x, y) the sides have length $2x$ and $2y$.

$$x = \pm a\sqrt{1 - \frac{y^2}{b^2}}$$

$$A = 4xy = 4ya\sqrt{1 - \frac{y^2}{b^2}} = 4a\sqrt{y^2 - \frac{y^4}{b^2}}$$

$$\frac{dA}{dy} = \frac{2a\left(2y - \frac{4y^3}{b^2}\right)}{\sqrt{y^2 - \frac{y^4}{b^2}}}; \frac{dA}{dy} = 0 \text{ when}$$

$$y - \frac{2y^3}{b^2} = 0$$

$$y\left(1 - \frac{2y^2}{b^2}\right) = 0$$

$y = 0$ or $y = \pm \frac{b}{\sqrt{2}}$

The Second Derivative Test shows that $y = \frac{b}{\sqrt{2}}$ is a maximum.

$$x = a\sqrt{1 - \frac{\left(\frac{b}{\sqrt{2}}\right)^2}{b^2}} = \frac{a}{\sqrt{2}}$$

Therefore, the rectangle is $a\sqrt{2}$ by $b\sqrt{2}$.

60. Position the x-axis on the axis of the hyperbola such that the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ describes the hyperbola. The equation of the tangent line at (x_0, y_0) is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$. The equations of the asymptotes are $y = \pm \frac{b}{a}x$.

Substitute $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ into the equation of the tangent line.

$$\frac{x_0 x}{a^2} - \frac{y_0 x}{ab} = 1 \qquad \frac{x_0 x}{a^2} + \frac{y_0 x}{ab} = 1$$

$$x\left(\frac{bx_0 - ay_0}{a^2 b} \right) = 1 \qquad x\left(\frac{bx_0 + ay_0}{a^2 b} \right) = 1$$

$$x = \frac{a^2 b}{bx_0 - ay_0} \qquad x = \frac{a^2 b}{bx_0 + ay_0}$$

Thus the tangent line intersects the asymptotes at $\left(\frac{a^2 b}{bx_0 - ay_0}, \frac{ab^2}{bx_0 - ay_0} \right)$ and $\left(\frac{a^2 b}{bx_0 + ay_0}, -\frac{ab^2}{bx_0 + ay_0} \right)$.

Observe that $b^2 x_0 - a^2 y_0 = a^2 b^2$.

$$\frac{1}{2}\left(\frac{a^2 b}{bx_0 - ay_0} + \frac{a^2 b}{bx_0 + ay_0} \right)$$

$$= \frac{a^2 b^2 x_0}{b^2 {x_0}^2 - a^2 {y_0}^2} = x_0$$

$$\frac{1}{2}\left(\frac{ab^2}{bx_0 - ay_0} - \frac{ab^2}{bx_0 + ay_0} \right) = \frac{a^2 b^2 y_0}{b^2 {x_0}^2 - a^2 {y_0}^2} = y_0$$

Thus, the point of contact is midway between the two points of intersection.

61. Add the two equations to get $9y^2 = 675$.

$y = \pm 5\sqrt{3}$

Substitute $y = 5\sqrt{3}$ into either of the two equations and solve for $x \Rightarrow \quad x = \pm 6$

The point in the first quadrant is $\left(6, 5\sqrt{3}\right)$.

62. Substitute $x = 6 - 2y$ into $x^2 + 4y^2 = 20$.

$(6-2y)^2 + 4y^2 = 20$

$8y^2 - 24y + 16 = 0$

$y^2 - 3y + 2 = 0$

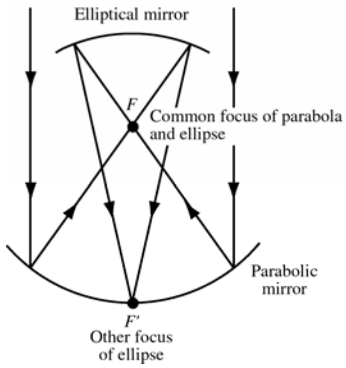
$(y - 1)(y - 2) = 0$

$y = 1$ or $y = 2$

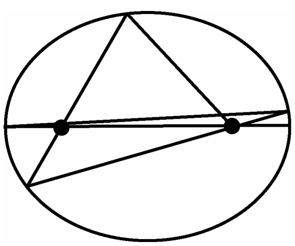
$x = 4$ or $x = 2$

The points of intersection are (4, 1) and (2, 2).

63.

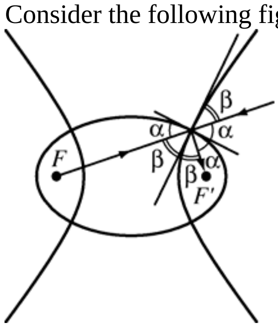


64. If the original path is not along the major axis, the ultimate path will approach the major axis.



65. Written response. Possible answer: the ball will follow a path that does not go between the foci.

66. Consider the following figure.



Observe that $2(\alpha + \beta) = 180°$, so $\alpha + \beta = 90°$. The ellipse and hyperbola meet at right angles.

67. Possible answer: Attach one end of a string to F and attach one end of another string to F'. Place a spool at a vertex. Tightly wrap both strings in the same direction around the spool. Insert a pencil through the spool. Then trace out a branch of the hyperbola by unspooling the strings while keeping both strings taut.

68. $\dfrac{|AP|}{u} = \dfrac{|AB|}{v} + \dfrac{|BP|}{u}$

$|AP| - |BP| = \dfrac{2uc}{v}$

Thus the curve is the right branch of the horizontal hyperbola with $a = \dfrac{uc}{v}$, so $b = \sqrt{\left(1 - \dfrac{u^2}{v^2}\right)}c$.

The equation of the curve is

$$\frac{x^2}{\frac{u^2c^2}{v^2}} - \frac{y^2}{\left(1-\frac{u^2}{v^2}\right)c^2} = 1\left(x \ge \frac{uc}{v}\right).$$

69. Let $P(x, y)$ be the location of the explosion.

$3|AP| = 3|BP| + 12$

$|AP| - |BP| = 4$

Thus, P lies on the right branch of the horizontal hyperbola with $a = 2$ and $c = 8$, so $b = 2\sqrt{15}$.

$\dfrac{x^2}{4} - \dfrac{y^2}{60} = 1$

Since $|BP| = |CP|$, the y-coordinate of P is 5.

$\dfrac{x^2}{4} - \dfrac{25}{60} = 1, x = \pm\sqrt{\dfrac{17}{3}}$

P is at $\left(\sqrt{\dfrac{17}{3}}, 5\right)$.

70. $\lim\limits_{x\to\infty}\left(\sqrt{x^2-a^2} - x\right)$

$$= \lim_{x\to\infty}\left[\frac{\left(\sqrt{x^2-a^2}-x\right)}{1}\cdot\frac{\left(\sqrt{x^2-a^2}+x\right)}{\left(\sqrt{x^2-a^2}+x\right)}\right]$$

$$= \lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0$$

71. $2a = p + q,\ 2c = |p - q|$

$b^2 = a^2 - c^2 = \dfrac{(p+q)^2}{4} - \dfrac{(p-q)^2}{4} = pq$

$b = \sqrt{pq}$

72. $x = a\cos t,\ y = a\sin t - b\sin t = (a-b)\sin t$

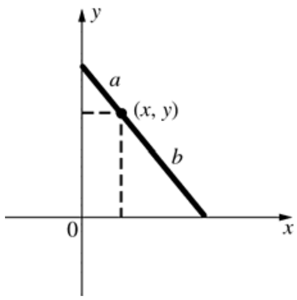
$\cos t = \dfrac{x}{a}, \sin t = \dfrac{y}{a-b}$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{(a-b)^2} = 1$

Thus the coordinates of R at time t lie on an ellipse.

73. Let (x, y) be the coordinates of P as the ladder slides. Using a property of similar triangles,

$$\frac{x}{a}=\frac{\sqrt{b^2-y^2}}{b}.$$



Square both sides to get

$\frac{x^2}{a^2}=\frac{b^2-y^2}{b^2}$ or $b^2x^2+a^2y^2=a^2b^2$ or

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

74. Place the x-axis on the axis of the hyperbola such that the equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ One focus is at $(c, 0)$ and the asymptotes are $y=\pm\frac{b}{a}x.$ The equations of the lines through the focus, perpendicular to the asymptotes, are

$y=\pm\frac{a}{b}(x-c).$ Then solve for x in

$$\frac{b}{a}x=-\frac{a}{b}(x-c).$$

$$\frac{a^2+b^2}{ab}x=\frac{ac}{b}$$

$$x=\frac{a^2c}{a^2+b^2}$$

Since $c^2=a^2+b^2, x=\frac{a^2}{c}.$ The equation of the directrix nearest the focus is $x=\frac{a^2}{c},$ so the line through a focus and perpendicular to an asymptote intersects that asymptote on the directrix nearest the focus.

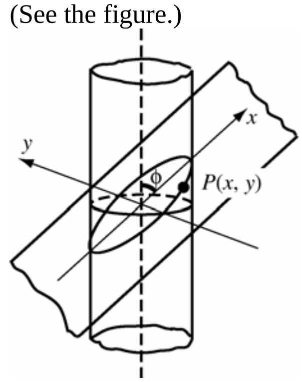
75. The equations of the hyperbolas are $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1.$

$$e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}$$

$$E=\frac{c}{b}=\frac{\sqrt{a^2+b^2}}{b}$$

$$e^{-2}+E^{-2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=1$$

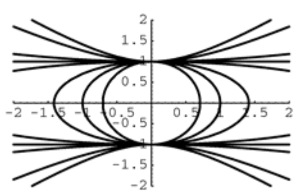
76. Position the x-axis on the plane so that it makes the angle ϕ with the axis of the cylinder and the y-axis is perpendicular to the axis of the cylinder. (See the figure.)



If $P(x, y)$ is a point on C, $(x\sin\phi)^2+y^2=r^2$ where r is the radius of the cylinder. Then

$$\frac{x^2}{\frac{r^2}{\sin^2\phi}}+\frac{y^2}{r^2}=1.$$

77.



When $a < 0$, the conic is an ellipse. When $a > 0$, the conic is a hyperbola. When $a = 0$, the graph is two parallel lines.

10.3 Concepts Review

1. $\dfrac{a^2}{4}$

2. 14; ellipse

3. $\dfrac{A-C}{B}$

4. rotation, translation

Problem Set 10.3

1. $x^2+y^2-2x+2y+1=0$
$(x^2-2x+1)+(y^2+2y+1)=-1+1+1$
$(x-1)^2+(y+1)^2=1$
This is a circle.

2. $x^2+y^2+6x-2y+6=0$
$(x^2+6x+9)+(y^2-2y+1)=-6+9+1$
$(x+3)^2+(y-1)^2=4$
This is a circle.

3. $9x^2+4y^2+72x-16y+124=0$
$9(x^2+8x+16)+4(y^2-4y+4)=-124+144+16$
$9(x+4)^2+4(y-2)^2=36$
This is an ellipse.

4. $16x^2-9y^2+192x+90y-495=0$
$16(x^2+12x+36)-9(y^2-10y+25)$
$=495+576-225$
$16(x+6)^2-9(y-5)^2=846$
This is a hyperbola.

5. $9x^2+4y^2+72x-16y+160=0$
$9(x^2+8x+16)+4(y^2-4y+4)=-160+144+16$
$9(x+4)^2+4(y-2)^2=0$
This is a point.

6. $16x^2+9y^2+192x+90y+1000=0$
$16(x^2+12x+36)+9(y^2+10y+25)$
$=-1000+576+225$
$16(x+6)^2+9(y+5)^2=-199$
This is the empty set.

7. $y^2-5x-4y-6=0$
$(y^2-4y+4)=5x+6+4$
$(y-2)^2=5(x+2)$
This is a parabola.

8. $4x^2+4y^2+8x-28y-11=0$
$4(x^2+2x+1)+4\left(y^2-7y+\dfrac{49}{4}\right)=11+4+49$
$4(x+1)^2+4\left(y-\dfrac{7}{2}\right)^2=64$
This is a circle.

9. $3x^2+3y^2-6x+12y+60=0$
$3(x^2-2x+1)+3(y^2+4y+4)=-60+3+12$
$3(x-1)^2+3(y+2)^2=-45$
This is the empty set.

10. $4x^2-4y^2-2x+2y+1=0$
$4\left(x^2-\dfrac{1}{2}x+\dfrac{1}{16}\right)-4\left(y^2-\dfrac{1}{2}y+\dfrac{1}{16}\right)=-1+\dfrac{1}{4}-\dfrac{1}{4}$
$4\left(x-\dfrac{1}{4}\right)^2-4\left(y-\dfrac{1}{4}\right)^2=-1$
$4\left(y-\dfrac{1}{4}\right)^2-4\left(x-\dfrac{1}{4}\right)^2=1$
This is a hyperbola.

11. $4x^2-4y^2+8x+12y-5=0$
$4(x^2+2x+1)-4\left(y^2-3y+\dfrac{9}{4}\right)=5+4-9$
$4(x+1)^2-4\left(y-\dfrac{3}{2}\right)^2=0$
This is two intersecting lines.

12. $4x^2-4y^2+8x+12y-6=0$
$4(x^2+2x+1)-4\left(y^2-3y+\dfrac{9}{4}\right)=6+4-9$
$4(x+1)^2-4\left(y-\dfrac{3}{2}\right)^2=1$
This is a hyperbola.

13. $4x^2-24x+36=0$
$4(x^2-6x+9)=-36+36$
$4(x-3)^2=0$
This is a line.

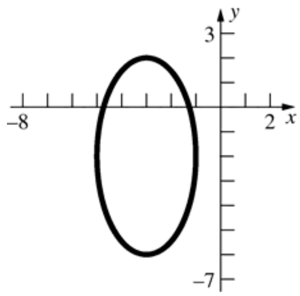
14. $4x^2 - 24x + 35 = 0$

$4(x^2 - 6x + 9) = -35 + 36$

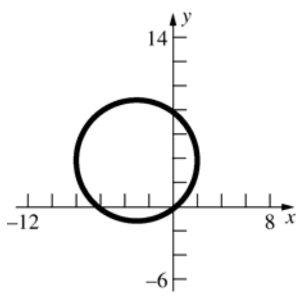
$4(x-3)^2 = 1$

This is two parallel lines.

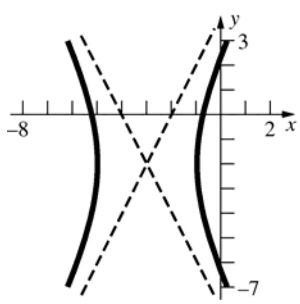
15. $\dfrac{(x+3)^2}{4} + \dfrac{(y+2)^2}{16} = 1$



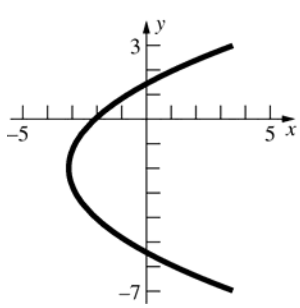
16. $(x+3)^2 + (y-4)^2 = 25$



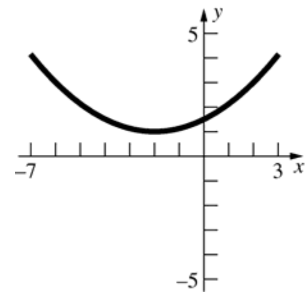
17. $\dfrac{(x+3)^2}{4} - \dfrac{(y+2)^2}{16} = 1$



18. $4(x+3) = (y+2)^2$



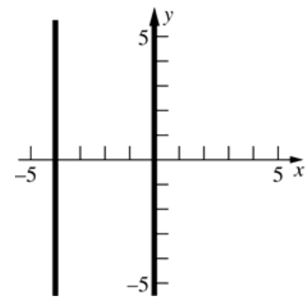
19. $(x+2)^2 = 8(y-1)$



20. $(x+2)^2 = 4$

$x + 2 = \pm 2$

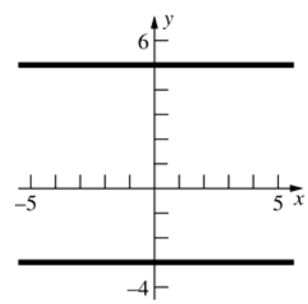
$x = -4, x = 0$



21. $(y-1)^2 = 16$

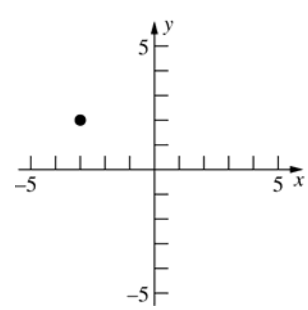
$y - 1 = \pm 4$

$y = 5, y = -3$



22. $\dfrac{(x+3)^2}{4} + \dfrac{(y-2)^2}{8} = 0$

$(-3, 2)$

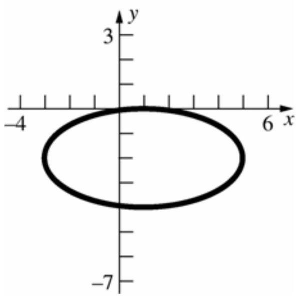


23. $x^2+4y^2-2x+16y+1=0$

$(x^2-2x+1)+4(y^2+4y+4)=-1+1+16$

$(x-1)^2+4(y+2)^2=16$

$\dfrac{(x-1)^2}{16}+\dfrac{(y+2)^2}{4}=1$

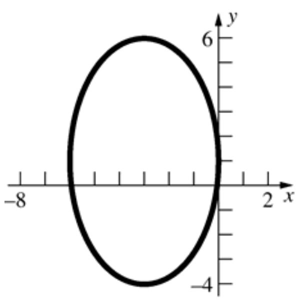


24. $25x^2+9y^2+150x-18y+9=0$

$25(x^2+6x+9)+9(y^2-2y+1)=-9+225+9$

$25(x+3)^2+9(y-1)^2=225$

$\dfrac{(x+3)^2}{9}+\dfrac{(y-1)^2}{25}=1$

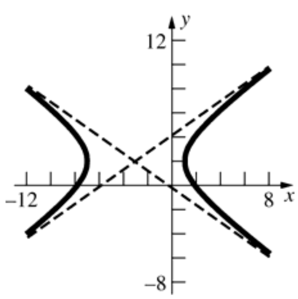


25. $9x^2-16y^2+54x+64y-127=0$

$9(x^2+6x+9)-16(y^2-4y+4)=127+81-64$

$9(x+3)^2-16(y-2)^2=144$

$\dfrac{(x+3)^2}{16}-\dfrac{(y-2)^2}{9}=1$

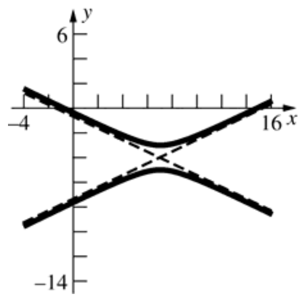


26. $x^2-4y^2-14x-32y-11=0$

$(x^2-14x+49)-4(y^2+8y+16)=11+49-64$

$4(y+4)^2-(x-7)^2=4$

$(y+4)^2-\dfrac{(x-7)^2}{4}=1$

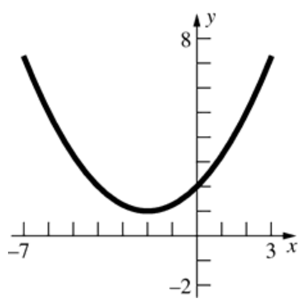


27. $4x^2+16x-16y+32=0$

$4(x^2+4x+4)=16y-32+16$

$4(x+2)^2=16(y-1)$

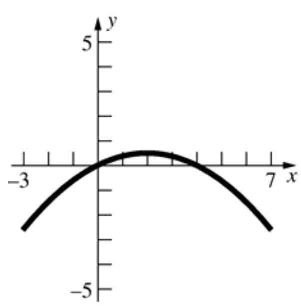
$(x+2)^2=4(y-1)$



28. $x^2-4x+8y=0$

$x^2-4x+4=-8y+4$

$(x-2)^2=-8\left(y-\dfrac{1}{2}\right)$



29. $2y^2 - 4y - 10x = 0$

$2(y^2 - 2y + 1) = 10x + 2$

$(y-1)^2 = 5\left(x + \frac{1}{5}\right)$

$(y-1)^2 = 4\left(\frac{5}{4}\right)\left(x + \frac{1}{5}\right)$

Horizontal parabola, $p = \frac{5}{4}$

Vertex $\left(-\frac{1}{5}, 1\right)$; Focus is at $\left(\frac{21}{20}, 1\right)$ and directrix is at $x = -\frac{29}{20}$.

30. $-9x^2 + 18x + 4y^2 + 24y = 9$

$-9(x^2 - 2x + 1) + 4(y^2 + 6y + 9) = 9 - 9 + 36$

$4(y+3)^2 - 9(x-1)^2 = 36$

$\frac{(y+3)^2}{9} - \frac{(x-1)^2}{4} = 1$

$a^2 = 9, a = 3$

The distance between the vertices is $2a = 6$.

31. $16(x-1)^2 + 25(y+2)^2 = 400$

$\frac{(x-1)^2}{25} + \frac{(y+2)^2}{16} = 1$

Horizontal ellipse, center (1, –2), $a = 5$, $b = 4$,
$c = \sqrt{25-16} = 3$

Foci are at (–2, –2) and (4, –2).

32. $x^2 - 6x + 4y + 3 = 0$

$x^2 - 6x + 9 = -4y - 3 + 9$

$(x-3)^2 = -4\left(y - \frac{3}{2}\right)$

Vertical parabola, opens downward, vertex $\left(3, \frac{3}{2}\right)$, $p = 1$

Focus is at $\left(3, \frac{1}{2}\right)$ and directrix is $y = \frac{5}{2}$.

33. $a = 5$, $b = 4$

$\frac{(x-5)^2}{25} + \frac{(y-1)^2}{16} = 1$

34. Horizontal hyperbola, $a = 2$, $c = 3$,
$b = \sqrt{9-4} = \sqrt{5}$

$\frac{(x-2)^2}{4} - \frac{(y+1)^2}{5} = 1$

35. Vertical parabola, opens upward, $p = 5 - 3 = 2$

$(x-2)^2 = 4(2)(y-3)$

$(x-2)^2 = 8(y-3)$

36. An equation for the ellipse can be written in the form $\frac{(x-2)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$.

Substitute the points into the equation.

$\frac{16}{a^2} = 1, \frac{4}{b^2} = 1$

Therefore, $a = 4$ and $b = 2$.

$\frac{(x-2)^2}{16} + \frac{(y-3)^2}{4} = 1$

37. Vertical hyperbola, center (0, 3), $2a = 6$, $a = 3$,
$c = 5$, $b = \sqrt{25-9} = 4$

$\frac{(y-3)^2}{9} - \frac{x^2}{16} = 1$

38. Vertical ellipse; center (2, 6), $a = 8$, $c = 6$,
$b = \sqrt{64-36} = \sqrt{28}$

$\frac{(x-2)^2}{28} + \frac{(y-6)^2}{64} = 1$

39. Horizontal parabola, opens to the left

Vertex (6, 5), $p = \frac{10-2}{2} = 4$

$(y-5)^2 = -4(4)(x-6)$

$(y-5)^2 = -16(x-6)$

40. Vertical parabola, opens downward, $p = 1$

$(x-2)^2 = -4(y-6)$

41. Horizontal ellipse, center (0, 2), $c = 2$

Since it passes through the origin and center is at (0, 2), $b = 2$.

$a = \sqrt{4+4} = \sqrt{8}$

$\frac{x^2}{8} + \frac{(y-2)^2}{4} = 1$

42. Vertical hyperbola, center (0, 2), $c = 2$,
$b^2 = 4 - a^2$

An equation for the hyperbola can be written in the form $\frac{(y-2)^2}{a^2} - \frac{x^2}{4-a^2} = 1$.

Substitute (12, 9) into the equation.

$\frac{49}{a^2} - \frac{144}{4-a^2} = 1$

$49(4-a^2) - 144a^2 = a^2(4-a^2)$

$a^4 - 197a^2 + 196 = 0$

$(a^2 - 196)(a^2 - 1) = 0$

$a^2 = 196, a^2 = 1$

Since $a < c$, $a = 1$, $b = \sqrt{4-1} = \sqrt{3}$

$(y-2)^2 - \frac{x^2}{3} = 1$

43. $x^2 + xy + y^2 = 6$

$\cot 2\theta = 0$

$2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

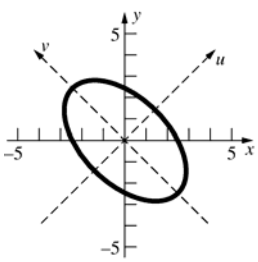
$x = u\frac{\sqrt{2}}{2} - v\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(u - v)$

$y = u\frac{\sqrt{2}}{2} + v\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(u + v)$

$\frac{1}{2}(u - v)^2 + \frac{1}{2}(u - v)(u + v) + \frac{1}{2}(u + v)^2 = 6$

$\frac{3}{2}u^2 + \frac{1}{2}v^2 = 6$

$\frac{u^2}{4} + \frac{v^2}{12} = 1$



44. $3x^2 + 10xy + 3y^2 + 10 = 0$

$\cot 2\theta = 0,\ 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

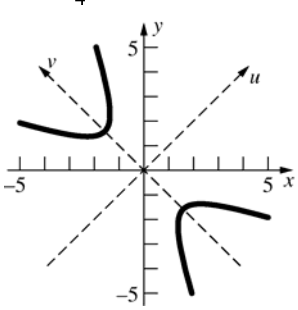
$x = \frac{\sqrt{2}}{2}(u - v)$

$y = \frac{\sqrt{2}}{2}(u + v)$

$\frac{3}{2}(u - v)^2 + 5(u - v)(u + v) + \frac{3}{2}(u + v)^2 + 10 = 0$

$8u^2 - 2v^2 = -10$

$\frac{v^2}{5} - \frac{u^2}{\frac{5}{4}} = 1$



45. $4x^2 + xy + 4y^2 = 56$

$\cot 2\theta = 0,\ 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

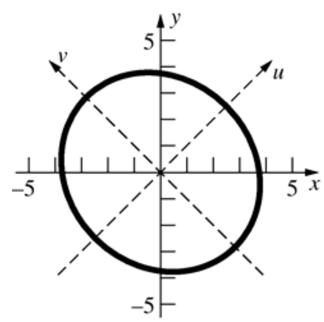
$x = \frac{\sqrt{2}}{2}(u - v)$

$y = \frac{\sqrt{2}}{2}(u + v)$

$2(u - v)^2 + \frac{1}{2}(u - v)(u + v) + 2(u + v)^2 = 56$

$\frac{9}{2}u^2 + \frac{7}{2}v^2 = 56$

$\frac{u^2}{\frac{112}{9}} + \frac{v^2}{16} = 1$



46. $4xy - 3y^2 = 64$

$\cot 2\theta = \frac{3}{4},\ r = 5$

$\cos 2\theta = \frac{3}{5}$

$\cos\theta = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}$

$\sin\theta = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$

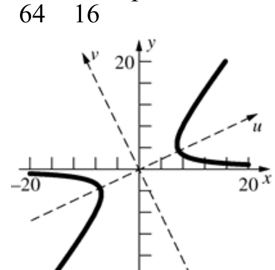
$x = \frac{1}{\sqrt{5}}(2u - v)$

$y = \frac{1}{\sqrt{5}}(u + 2v)$

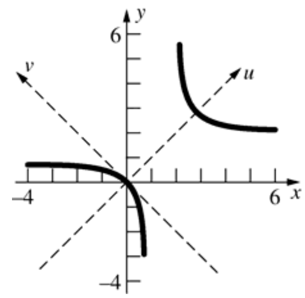
$\frac{4}{5}(2u - v)(u + 2v) - \frac{3}{5}(u + 2v)^2 = 64$

$u^2 - 4v^2 = 64$

$\frac{u^2}{64} - \frac{v^2}{16} = 1$



47. $-\frac{1}{2}x^2 + 7xy - \frac{1}{2}y^2 - 6\sqrt{2}x - 6\sqrt{2}y = 0$



$\cot 2\theta = 0, 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

$x = \frac{\sqrt{2}}{2}(u - v)$; $\quad y = \frac{\sqrt{2}}{2}(u + v)$

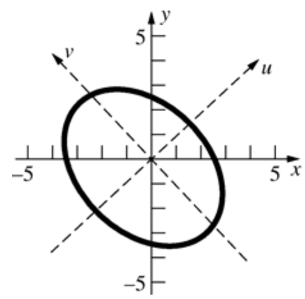
$-\frac{1}{4}(u-v)^2 + \frac{7}{2}(u-v)(u+v) - \frac{1}{4}(u+v)^2 - 6(u-v) - 6(u+v) = 0$

$3u^2 - 4v^2 - 12u = 0$

$3(u^2 - 4u + 4) - 4v^2 = 12$

$\frac{(u-2)^2}{4} - \frac{v^2}{3} = 1$

48. $\frac{3}{2}x^2 + xy + \frac{3}{2}y^2 + \sqrt{2}x + \sqrt{2}y = 13$



$\cot 2\theta = 0, 2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$

$x = \frac{\sqrt{2}}{2}(u - v)$; $\quad y = \frac{\sqrt{2}}{2}(u + v)$

$\frac{3}{4}(u-v)^2 + \frac{1}{2}(u-v)(u+v) + \frac{3}{4}(u+v)^2 + (u-v) + (u+v) = 13$

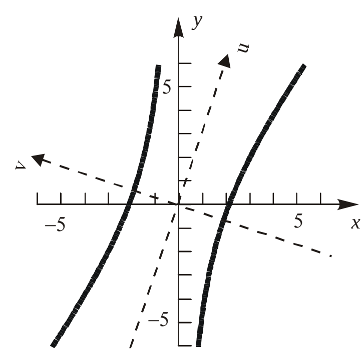
$2u^2 + v^2 + 2u = 13$

$2\left(u^2 + u + \frac{1}{4}\right) + v^2 = 13 + \frac{1}{2}$

$2\left(u + \frac{1}{2}\right)^2 + v^2 = \frac{27}{2}$

$\frac{\left(u + \frac{1}{2}\right)^2}{\frac{27}{4}} + \frac{v^2}{\frac{27}{2}} = 1$

49. $A = 4,\ B = -3,\ C = D = E = 0,\ F = -18$



$\cot 2\theta = \frac{4-0}{-3} = -\frac{4}{3}$

Since $0 \le 2\theta \le \pi$, $\sin 2\theta$ is positive, so $\cos 2\theta$ is negative; using a 3-4-5 right triangle, we conclude $\cos 2\theta = -\frac{4}{5}$. Thus

$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-(-4/5)}{2}} = \frac{3\sqrt{10}}{10}$ and

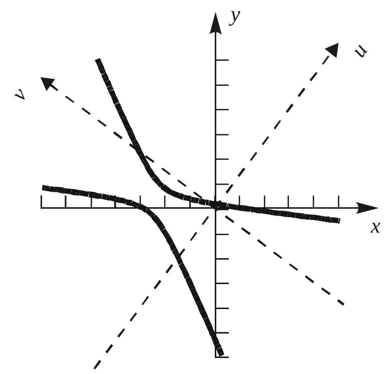
$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+(-4/5)}{2}} = \frac{\sqrt{10}}{10}$. Rotating through the angle

$\theta = \frac{1}{2}\cos^{-1}(-0.8) = 71.6^\circ$, we have

$4\left(\frac{\sqrt{10}}{10}u - \frac{3\sqrt{10}}{10}v\right)^2 - 3\left(\frac{\sqrt{10}}{10}u - \frac{3\sqrt{10}}{10}v\right)\left(\frac{3\sqrt{10}}{10}u + \frac{\sqrt{10}}{10}v\right) = 18$ or

$45v^2 - 5u^2 = 180$ or $\frac{v^2}{4} - \frac{u^2}{36} = 1$. This is a hyperbola in standard position in the uv-system; its axis is the v-axis, and $a = 2,\ b = 6$.

50. $A = 11,\ B = 96,\ C = 39,\ D = 240,\ E = 570,\ F = 875$



$\cot 2\theta = \dfrac{11-39}{96} = -\dfrac{7}{24}$

Since $0 \le 2\theta \le \pi$, $\cos 2\theta$ is negative; using a 7-24-25 right triangle, we conclude $\cos 2\theta = -\dfrac{7}{25}$.

Thus $\sin\theta = \sqrt{\dfrac{1-\cos 2\theta}{2}} = \sqrt{\dfrac{1-(-7/25)}{2}} = \dfrac{4}{5}$ and

$\cos\theta = \sqrt{\dfrac{1+\cos 2\theta}{2}} = \sqrt{\dfrac{1+(-7/25)}{2}} = \dfrac{3}{5}$.

Rotating through the angle $\theta = \dfrac{1}{2}\cos^{-1}(-0.28) = 53.13^\circ$, we have

$11\left(\frac{3}{5}u - \frac{4}{5}v\right)^2 + 96\left(\frac{3}{5}u - \frac{4}{5}v\right)\left(\frac{4}{5}u + \frac{3}{5}v\right) +$

$39\left(\frac{4}{5}u + \frac{3}{5}v\right)^2 + 240\left(\frac{3}{5}u - \frac{4}{5}v\right) + 570\left(\frac{4}{5}u + \frac{3}{5}v\right) = -875$

or

$3u^2 - v^2 + 24u + 6v = -35$

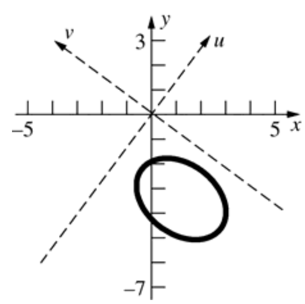
$3(u^2 + 8u + 16) - (v^2 - 6v + 9) = -35 + 48 - 9$

$3(u+4)^2 - (v-3)^2 = 4$

$\dfrac{(u+4)^2}{4/3} - \dfrac{(v-3)^2}{4} = 1$

This is a hyperbola in standard position in the *uv*-system; its axis is the *u*-axis, its center is $(u, v) = (-4, 3)$ and $a = \dfrac{2\sqrt{3}}{3}$, $b = 2$.

51. $34x^2 + 24xy + 41y^2 + 250y = -325$



$\cot 2\theta = -\dfrac{7}{24}, r = 25$

$\cos 2\theta = -\dfrac{7}{25}$

$\cos\theta = \sqrt{\dfrac{1-\frac{7}{25}}{2}} = \dfrac{3}{5}$; $\quad \sin\theta = \sqrt{\dfrac{1+\frac{7}{25}}{2}} = \dfrac{4}{5}$

$x = \dfrac{1}{5}(3u - 4v)$; $\quad y = \dfrac{1}{5}(4u + 3v)$

$\dfrac{34}{25}(3u-4v)^2 + \dfrac{24}{25}(3u-4v)(4u+3v) + \dfrac{41}{25}(4u+3v)^2 + 50(4u+3v) = -325$

$50u^2 + 25v^2 + 200u + 150v = -325$

$2u^2 + v^2 + 8u + 6v = -13$

$2(u^2 + 4u + 4) + (v^2 + 6v + 9) = -13 + 8 + 9$

$2(u+2)^2 + (v+3)^2 = 4$

$\dfrac{(u+2)^2}{2} + \dfrac{(v+3)^2}{4} = 1$

This is an ellipse in standard position in the *uv*-system, with major axis parallel to the *v*-axis. Its center is $(u, v) = (-2, -3)$ and $a = 2$, $b = \sqrt{2}$.

52. $16x^2 + 24xy + 9y^2 - 20x - 15y - 150 = 0$

$\cot 2\theta = \frac{7}{24}, \ r = 25$

$\cos 2\theta = \frac{7}{25}$

$\cos\theta = \sqrt{\frac{1+\frac{7}{25}}{2}} = \frac{4}{5}; \quad \sin\theta = \sqrt{\frac{1-\frac{7}{25}}{2}} = \frac{3}{5}$

$x = \frac{1}{5}(4u - 3v); \quad y = \frac{1}{5}(3u + 4v)$

$\frac{16}{25}(4u-3v)^2 + \frac{24}{25}(4u-3v)(3u+4v) + \frac{9}{25}(3u+4v)^2 \ 25u^2 - 25u = 150$

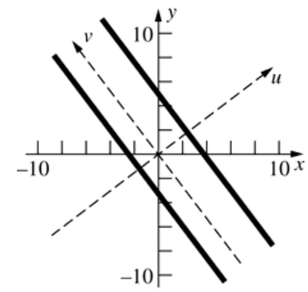
$-4(4u-3v) - 3(3u+4v) = 150$

$u^2 - u = 6$

$u^2 - u + \frac{1}{4} = 6 + \frac{1}{4}$

$\left(u - \frac{1}{2}\right)^2 = \frac{25}{4}$

The graph consists of the two parallel lines $u = -2$ and $u = 3$.



53. a. If C is a vertical parabola, the equation for C can be written in the form $y = ax^2 + bx + c$. Substitute the three points into the equation.

$2 = a - b + c$
$0 = c$
$6 = 9a + 3b + c$

Solve the system to get $a = 1$, $b = -1$, $c = 0$.

$y = x^2 - x$

b. If C is a horizontal parabola, an equation for C can be written in the form $x = ay^2 + by + c$. Substitute the three points into the equation.

$-1 = 4a + 2b + c$
$0 = c$
$3 = 36a + 6b + c$

Solve the system to get $a = \frac{1}{4}$, $b = -1$, $c = 0$.

$x = \frac{1}{4}y^2 - y$

c. If C is a circle, an equation for C can be written in the form $(x-h)^2 + (y-k)^2 = r^2$. Substitute the three points into the equation.

$(-1-h)^2 + (2-k)^2 = r^2$

$h^2 + k^2 = r^2$

$(3-h)^2 + (6-k)^2 = r^2$

Solve the system to get $h = \frac{5}{2}, k = \frac{5}{2}$, and $r^2 = \frac{25}{2}$.

$\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{2}$

54. Let (p, q) be the coordinates of P. By properties of similar triangles and since $\alpha = \frac{|KP|}{|AP|}, \alpha = \frac{x-p}{a-p}$ and $\alpha = \frac{y-q}{b-q}$. Solve for p and q to get $p = \frac{x - \alpha a}{1-\alpha}$ and $q = \frac{y - \alpha b}{1-\alpha}$. Since $P(p, q)$ is a point on a circle of radius r centered at (0, 0), $p^2 + q^2 = r^2$

Therefore, $\left(\frac{x-\alpha a}{1-\alpha}\right)^2 + \left(\frac{y-\alpha b}{1-\alpha}\right)^2 = r^2$ or $(x-\alpha a)^2 + (y-\alpha b)^2 = (1-\alpha)^2 r^2$ is the equation for C.

55. $y^2 = Lx + Kx^2$

$$K\left(x^2 + \frac{L}{K}x + \frac{L^2}{4K^2}\right) - y^2 = \frac{L^2}{4K}$$

$$K\left(x + \frac{L}{2K}\right)^2 - y^2 = \frac{L^2}{4K}$$

$$\frac{(x+\frac{L}{2K})^2}{\frac{L^2}{4K^2}} - \frac{y^2}{\frac{L^2}{4K}} = 1$$

If $K < -1$, the conic is a vertical ellipse. If $K = -1$, the conic is a circle. If $-1 < K < 0$, the conic is a horizontal ellipse. If $K = 0$, the original equation is $y^2 = Lx$, so the conic is a horizontal parabola. If $K > 0$, the conic is a horizontal hyperbola.

If $-1 < K < 0$ (a horizontal ellipse) the length of the latus rectum is (see problem 45, Section 10.2)

$$\frac{2b^2}{a} = 2\frac{L^2}{4|K|}\frac{1}{\frac{|L|}{2|K|}} = |L|$$

From general considerations, the result for a vertical ellipse is the same as the one just obtained.

For $K = -1$ (a circle) we have

$$\left(x - \frac{L}{2}\right)^2 + y^2 = \frac{L^2}{4} \Rightarrow 2\frac{|L|}{2} = |L|$$

If $K = 0$ (a horizontal parabola) we have

$y^2 = Lx; y^2 = 4\frac{L}{4}x; p = \frac{L}{4}$, and the latus rectum is

$$2\sqrt{Lp} = 2\sqrt{L\frac{L}{4}} = |L|.$$

If $K > 0$ (a horizontal hyperbola) we can use the result of Problem 46, Section 10.2. The length of the latus rectum is $\frac{2b^2}{a}$, which is equal to $|L|$.

56. Parabola: horizontal parabola, opens to the right, $p = c - a,\ y^2 = 4(c-a)(x-a)$

Hyperbola: horizontal hyperbola, $b^2 = c^2 - a^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$ Now show that y^2 (hyperbola) is greater than y^2 (parabola).

$$\frac{b^2}{a^2}(x^2 - a^2) = \frac{c^2 - a^2}{a^2}(x^2 - a^2)$$

$$= \frac{(c+a)(c-a)}{a^2}(x+a)(x-a)$$

$$\frac{(c+a)(x+a)}{a^2}(c-a)(x-a) > \frac{(2a)(2a)}{a^2}(c-a)(x-a)$$

$$\frac{(2a)(2a)}{a^2}(c-a)(x-a) = 4(c-a)(x-a)$$

$c + a > 2a$ and $x + a > 2a$ since $c > a$ and $x > a$ except at the vertex.

57. $x = u\cos\alpha - v\sin\alpha$

$y = u\sin\alpha + v\cos\alpha$

$(u\cos\alpha - v\sin\alpha)\cos\alpha + (u\sin\alpha + v\cos\alpha)\sin\alpha = d$

$u(\cos^2\alpha + \sin^2\alpha) = d$

$u = d$

Thus, the perpendicular distance from the origin is d.

58. $x = \frac{\sqrt{2}}{2}(u - v)$; $y = \frac{\sqrt{2}}{2}(u + v)$

$$\left[\frac{\sqrt{2}}{2}(u - v)\right]^{1/2} + \left[\frac{\sqrt{2}}{2}(u + v)\right]^{1/2} = a^{1/2}$$

$$\frac{\sqrt{2}}{2}(u - v) + 2\left[\frac{1}{2}(u - v)(u + v)\right]^{1/2} + \frac{\sqrt{2}}{2}(u + v) = a$$

$$\sqrt{2}u + \sqrt{2}(u^2 - v^2)^{1/2} = a$$

$$\sqrt{2}(u^2 - v^2)^{1/2} = a - \sqrt{2}u$$

$$2(u^2 - v^2) = a^2 - 2\sqrt{2}au + 2u^2$$

$$v^2 = \sqrt{2}au - \frac{1}{2}a^2$$

The corresponding curve is a parabola with $x > 0$ and $y > 0$.

59. $x = u\cos\theta - v\sin\theta$; $y = u\sin\theta + v\cos\theta$

$$x(\cos\theta) + y(\sin\theta) = (u\cos^2\theta - v\cos\theta\sin\theta) + (u\sin^2\theta + v\cos\theta\sin\theta) = u$$

$$x(-\sin\theta) + y(\cos\theta) = (-u\cos\theta\sin\theta + v\sin^2\theta) + (u\cos\theta\sin\theta + v\cos^2\theta) = v$$

Thus, $u = x\cos\theta + y\sin\theta$ and $v = -x\sin\theta + y\cos\theta$.

60. $u = 5\cos 60° - 3\sin 60° = \frac{5}{2} - \frac{3\sqrt{3}}{2}$; $v = -5\sin 60° - 3\cos 60° = -\frac{5\sqrt{3}}{2} - \frac{3}{2}$

$$(u, v) = \left(\frac{5}{2} - \frac{3\sqrt{3}}{2}, -\frac{5\sqrt{3}}{2} - \frac{3}{2}\right)$$

61. Rotate to eliminate the *xy*-term.

$$x^2 + 14xy + 49y^2 = 100$$

$$\cot 2\theta = -\frac{24}{7}$$

$$\cos 2\theta = -\frac{24}{25}$$

$$\cos\theta = \sqrt{\frac{1 - \frac{24}{25}}{2}} = \frac{1}{5\sqrt{2}}; \quad \sin\theta = \sqrt{\frac{1 + \frac{24}{25}}{2}} = \frac{7}{5\sqrt{2}}$$

$$x = \frac{1}{5\sqrt{2}}(u - 7v); \quad y = \frac{1}{5\sqrt{2}}(7u + v)$$

$$\frac{1}{50}(u - 7v)^2 + \frac{14}{50}(u - 7v)(7u + v) + \frac{49}{50}(7u + v)^2 = 100$$

$$50u^2 = 100$$

$$u^2 = 2$$

$$u = \pm\sqrt{2}$$

Thus the points closest to the origin in *uv*-coordinates are $\left(\sqrt{2}, 0\right)$ and $\left(-\sqrt{2}, 0\right)$.

$$x = \frac{1}{5\sqrt{2}}\left(\sqrt{2}\right) = \frac{1}{5} \text{ or } x = \frac{1}{5\sqrt{2}}\left(-\sqrt{2}\right) = -\frac{1}{5}$$

$$y = \frac{1}{5\sqrt{2}}\left(7\sqrt{2}\right) = \frac{7}{5} \text{ or } y = \frac{1}{5\sqrt{2}}\left(-7\sqrt{2}\right) = -\frac{7}{5}$$

The points closest to the origin in *xy*-coordinates are $\left(\frac{1}{5}, \frac{7}{5}\right)$ and $\left(-\frac{1}{5}, -\frac{7}{5}\right)$.

62. $x = u\cos\theta - v\sin\theta$
$y = u\sin\theta + v\cos\theta$

$$Ax^2 = A(u\cos\theta - v\sin\theta)^2 = A(u^2\cos^2\theta - 2uv\cos\theta\sin\theta + v^2\sin^2\theta)$$
$$Bxy = B(u\cos\theta - v\sin\theta)(u\sin\theta + v\cos\theta) = B(u^2\cos\theta\sin\theta + uv(\cos^2\theta - \sin^2\theta) - v^2\cos\theta\sin\theta)$$
$$Cy^2 = C(u\sin\theta + v\cos\theta)^2 = C(u^2\sin^2\theta + 2uv\cos\theta\sin\theta + v^2\cos^2\theta)$$
$$Ax^2 + Bxy + Cy^2 = (A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta)u^2 + (-2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\cos\theta\sin\theta)uv$$
$$+(A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta)v^2$$

Thus, $a = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$ and $c = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta.$

$$a + c = A(\cos^2\theta + \sin^2\theta) + B(\cos\theta\sin\theta - \cos\theta\sin\theta) + C(\sin^2\theta + \cos^2\theta) = A + C$$

63. From Problem 62, $a = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta,$

$b = -2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\cos\theta\sin\theta,$ and

$c = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta.$

$$b^2 = B^2\cos^4\theta + 4(-AB + BC)\cos^3\theta\sin\theta + 2(2A^2 - B^2 - 4AC + 2C^2)\cos^2\theta\sin^2\theta$$
$$+4(AB - BC)\cos\theta\sin^3\theta + B^2\sin^4\theta$$
$$4ac = 4AC\cos^4\theta + 4(-AB + BC)\cos^3\theta\sin\theta + 4(A^2 - B^2 + C^2)\cos^2\theta\sin^2\theta + 4(AB - BC)\cos\theta\sin^3\theta + 4AC\sin^4\theta$$
$$b^2 - 4ac = (B^2 - 4AC)\cos^4\theta + 2(B^2 - 4AC)\cos^2\theta\sin^2\theta + (B^2 - 4AC)\sin^4\theta$$
$$= (B^2 - 4AC)(\cos^2\theta)(\cos^2\theta + \sin^2\theta) + (B^2 - 4AC)(\sin^2\theta)(\cos^2\theta + \sin^2\theta)$$
$$= (B^2 - 4AC)(\cos^2\theta + \sin^2\theta) = B^2 - 4AC$$

64. By choosing an appropriate angle of rotation, the second-degree equation can be written in the form $au^2 + cv^2 + du + ev + f = 0.$ From Problem 63, $-4ac = B^2 - 4AC$.

a. If $B^2 - 4AC = 0,$ then $4ac = 0,$ so the graph is a parabola or limiting form.

b. If $B^2 - 4AC < 0,$ then $4ac > 0,$ so the graph is an ellipse or limiting form.

c. If $B^2 - 4AC > 0,$ then $4ac < 0,$ so the graph is a hyperbola or limiting form.

65. a. From Problem 63, $-4ac = B^2 - 4AC = -\Delta$ or $\frac{1}{ac} = \frac{4}{\Delta}.$

b. From Problem 62, $a + c = A + C.$
$$\frac{1}{a} + \frac{1}{c} = \frac{a + c}{ac} = \frac{4(A + C)}{\Delta}$$

c.
$$\frac{2}{\Delta}\left(A + C \pm \sqrt{(A - C)^2 + B^2}\right)$$
$$= \frac{2}{\Delta}\left[\frac{\Delta}{4}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \sqrt{A^2 + 2AC + C^2 + B^2 - 4AC}\right]$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \frac{2}{\Delta}\sqrt{(A + C)^2 - \Delta}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \frac{2}{\Delta}\sqrt{\frac{\Delta^2}{16}\left(\frac{1}{a} + \frac{1}{c}\right)^2 - \Delta}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{a} + \frac{1}{c}\right)^2 - 4\left(\frac{4}{\Delta}\right)}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \sqrt{\frac{1}{a^2} + \frac{2}{ac} + \frac{1}{c^2} - 4\left(\frac{1}{ac}\right)}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{a} - \frac{1}{c}\right)^2}$$
$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c} \pm \left|\frac{1}{a} - \frac{1}{c}\right|\right)$$

The two values are $\frac{1}{a}$ and $\frac{1}{c}.$

66. $Ax^2 + Bxy + Cy^2 = 1$ can be transformed to $au^2 + cv^2 = 1$. Since $4ac = \Delta > 0$, the graph is an ellipse or a limiting form. $\frac{1}{a} + \frac{1}{c} = \frac{4}{\Delta}(A+C) > 0$, so $a > 0$ and $c > 0$. Thus, the graph is an ellipse (or circle).

The area of $au^2 + cv^2 = 1$ is

$$\pi\frac{1}{\sqrt{ac}} = \pi\sqrt{\frac{4}{\Delta}} = \frac{2\pi}{\sqrt{\Delta}}.$$

67. $\cot 2\theta = 0,\ \theta = \frac{\pi}{4}$

$$x = \frac{\sqrt{2}}{2}(u - v)$$

$$y = \frac{\sqrt{2}}{2}(u + v)$$

$$\frac{1}{2}(u-v)^2 + \frac{B}{2}(u-v)(u+v) + \frac{1}{2}(u+v)^2 = 1$$

$$\frac{2+B}{2}u^2 + \frac{2-B}{2}v^2 = 1$$

a. The graph is an ellipse if $\frac{2+B}{2} > 0$ and $\frac{2-B}{2} > 0$, so $-2 < B < 2$.

b. The graph is a circle if $\frac{2+B}{2} = \frac{2-B}{2}$, so $B = 0$.

c. The graph is a hyperbola if $\frac{2+B}{2} > 0$ and $\frac{2-B}{2} < 0$ or if $\frac{2+B}{2} < 0$ and $\frac{2-B}{2} > 0$, so $B < -2$ or $B > 2$.

d. The graph is two parallel lines if $\frac{2+B}{2} = 0$ or $\frac{2-B}{2} = 0$, so $B = \pm 2$.

68. $\Delta = 4(25)(1) - 8^2 = 36$

Since $c < a$, $\frac{1}{c} = \frac{2}{\Delta}\left(A + C + \sqrt{(A-C)^2 + B^2}\right)$

$$\frac{1}{c} = \frac{1}{18}\left(25 + 1 + \sqrt{24^2 + 8^2}\right) = \frac{1}{9}\left(13 + 4\sqrt{10}\right)$$

$$c = \left(\frac{9}{13 + 4\sqrt{10}}\right)\left(\frac{13 - 4\sqrt{10}}{13 - 4\sqrt{10}}\right) = 13 - 4\sqrt{10}$$

$$\frac{2\pi}{\sqrt{\Delta}} = \frac{2\pi}{6} = \frac{\pi}{3}$$

Thus, the distance between the foci is $26 - 8\sqrt{10}$ and the area is $\frac{\pi}{3}$.

69. From Figure 6 it is clear that $v = r\sin\phi$ and $u = r\cos\phi$.

Also noting that $y = r\sin(\theta + \phi)$ leads us to

$$y = r\sin(\theta+\phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi$$

$$= (r\cos\phi)(\sin\theta) + (r\sin\phi)(\cos\theta)$$

$$= u\sin\theta + v\cos\theta$$

10.4 Concepts Review

1. simple; closed; simple

2. parametric; parameter

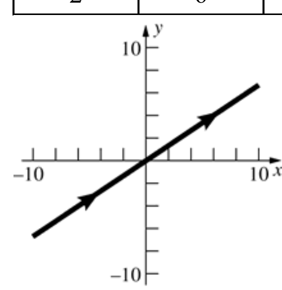
3. cycloid

4. $(dy/dt)/(dx/dt) = g'(t)/f'(t)$

Problem Set 10.4

1. a.

t	x	y
–2	–6	–4
–1	–3	–2
0	0	0
1	3	2
2	6	4

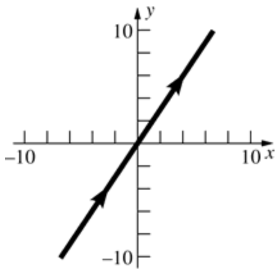


b. Simple; not closed

c. $t = \frac{x}{3} \Rightarrow y = \frac{2}{3}x$

2. a.

t	x	y
–2	–4	–6
–1	–2	–3
0	0	0
1	2	3
2	4	6

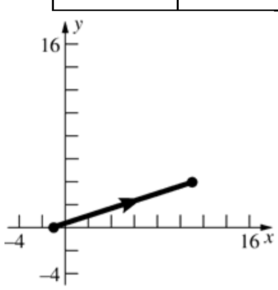


b. Simple; not closed

c. $t = \frac{x}{2} \Rightarrow y = \frac{3}{2}x$

3. a.

t	x	y
0	–1	0
1	2	1
2	5	2
3	8	3
4	11	4

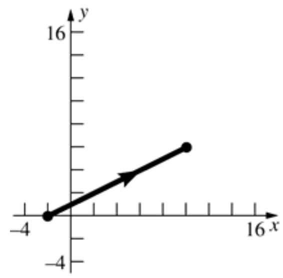


b. Simple; not closed

c. $t = \frac{1}{3}(x+1) \Rightarrow y = \frac{1}{3}(x+1)$

4. a.

t	x	y
0	–2	0
1	2	2
2	6	4
3	10	6

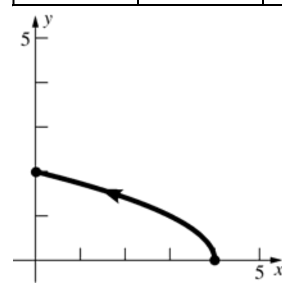


b. Simple; not closed

c. $t = \frac{1}{4}(x+2) \Rightarrow y = \frac{1}{2}(x+2)$

5. a.

t	x	y
0	4	0
1	3	1
2	2	$\sqrt{2}$
3	1	$\sqrt{3}$
4	0	2

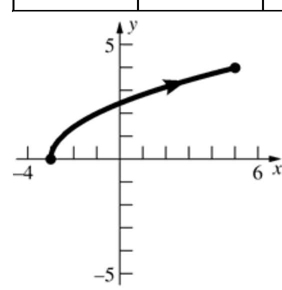


b. Simple; not closed

c. $t = 4 - x$
$y = \sqrt{4 - x}$

6. a.

t	x	y
0	–3	0
2	–1	2
4	1	$2\sqrt{2}$
6	3	$2\sqrt{3}$
8	5	4

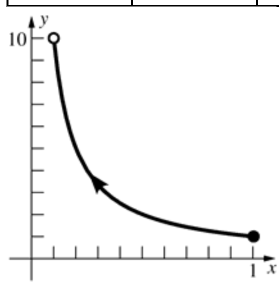


b. Simple; not closed

c. $t = x + 3$
$y = \sqrt{2x + 6}$

7. a.

s	x	y
1	1	1
3	$\frac{1}{3}$	3
5	$\frac{1}{5}$	5
7	$\frac{1}{7}$	7
9	$\frac{1}{9}$	9



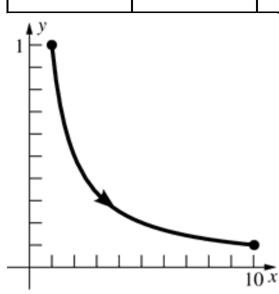
b. Simple; not closed

c. $s = \dfrac{1}{x}$

$y = \dfrac{1}{x}$

8. a.

s	x	y
1	1	1
3	3	$\frac{1}{3}$
5	5	$\frac{1}{5}$
7	7	$\frac{1}{7}$
9	9	$\frac{1}{9}$



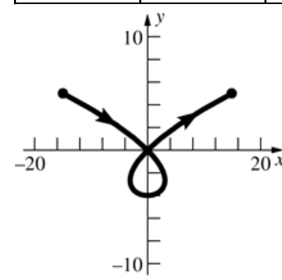
b. Simple; not closed

c. $s = x$

$y = \dfrac{1}{x}$

9. a.

t	x	y
–3	–15	5
–2	0	0
–1	3	–3
0	0	–4
1	–3	–3
2	0	0
3	15	5



b. Not simple; not closed

c. $x^2 = t^6 - 8t^4 + 16t^2$

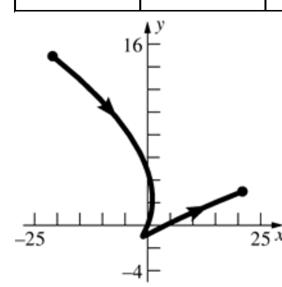
$t^2 = y + 4$

$x^2 = (y+4)^3 - 8(y+4)^2 + 16(y+4)$

$x^2 = y^3 + 4y^2$

10. a.

t	x	y
–3	–21	15
–2	–4	8
–1	1	3
0	0	0
1	–1	–1
2	4	0
3	21	3



b. Simple; not closed

c. $t^2 - 2t - y = 0$

$t = 1 \pm \sqrt{1+y}$

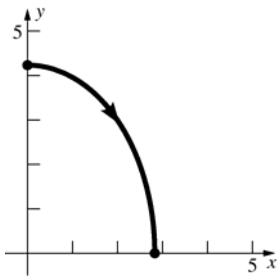
$x = \left(1 \pm \sqrt{1+y}\right)^3 - 2\left(1 \pm \sqrt{1+y}\right)$

$x = 2 + 3y \pm (y+2)\sqrt{1+y}$

$(x - 3y - 2)^2 = (y+1)(y+2)^2$

11. a.

t	x	y
2	0	$3\sqrt{2}$
3	2	3
4	$2\sqrt{2}$	0



b. Simple; not closed

c. $t = \frac{1}{4}x^2 + 2$

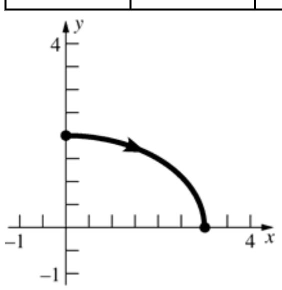
$t = 4 - \frac{1}{9}y^2$

$\frac{1}{4}x^2 + 2 = 4 - \frac{1}{9}y^2$

$\frac{x^2}{8} + \frac{y^2}{18} = 1$

12. a.

t	x	y
3	0	2
$\frac{7}{2}$	$\frac{3}{\sqrt{2}}$	$\sqrt{2}$
4	3	0



b. Simple; not closed

c. $t = \frac{1}{9}x^2 + 3$

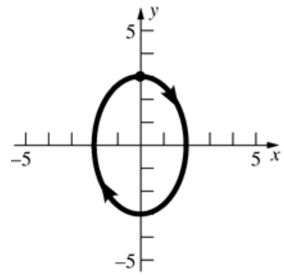
$t = 4 - \frac{1}{4}y^2$

$\frac{1}{9}x^2 + 3 = 4 - \frac{1}{4}y^2$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

13. a.

t	x	y
0	0	3
$\frac{\pi}{2}$	2	0
π	0	–3
$\frac{3\pi}{2}$	–2	0
2π	0	3



b. Simple; closed

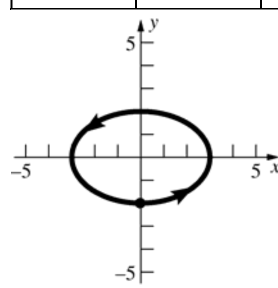
c. $\sin^2 t = \frac{x^2}{4}$

$\cos^2 t = \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

14. a.

r	x	y
0	0	–2
$\frac{\pi}{2}$	3	0
π	0	2
$\frac{3\pi}{2}$	–3	0
2π	0	–2



b. Simple; closed

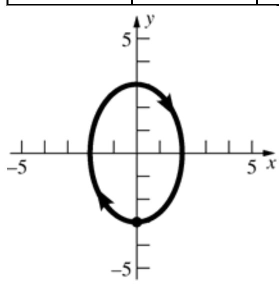
c. $\sin^2 r = \frac{x^2}{9}$

$\cos^2 r = \frac{y^2}{4}$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

15. **a.**

r	x	y
0	0	–3
$\frac{\pi}{2}$	–2	0
π	0	3
$\frac{3\pi}{2}$	2	0
2π	0	–3
$\frac{5\pi}{2}$	–2	0
3π	0	3
$\frac{7\pi}{2}$	2	0
4π	0	–3



b. Not simple; closed

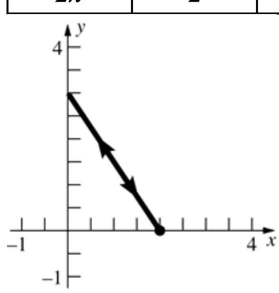
c. $\sin^2 r = \frac{x^2}{4}$

$\cos^2 r = \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

16. **a.**

r	x	y
0	2	0
$\frac{\pi}{2}$	0	3
π	2	0
$\frac{3\pi}{2}$	0	3
2π	2	0



b. Not simple; closed

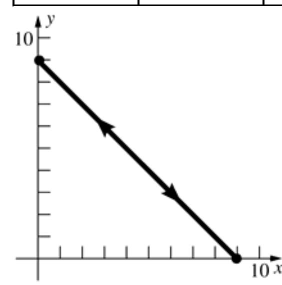
c. $\cos^2 r = \frac{x}{2}$

$\sin^2 r = \frac{y}{3}$

$\frac{x}{2} + \frac{y}{3} = 1$

17. **a.**

θ	x	y
0	0	9
$\frac{\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
$\frac{\pi}{2}$	0	9
$\frac{3\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
π	0	9



b. Not simple; closed

c. $\sin^2 \theta = \frac{x}{9}$

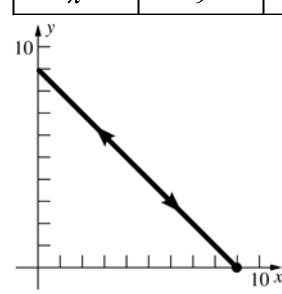
$\cos^2 \theta = \frac{y}{9}$

$\frac{x}{9} + \frac{y}{9} = 1$

$x + y = 9$

18. **a.**

θ	x	y
0	9	0
$\frac{\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
$\frac{\pi}{2}$	0	9
$\frac{3\pi}{4}$	$\frac{9}{2}$	$\frac{9}{2}$
π	9	0



b. Not simple; closed

c. $\cos^2\theta = \frac{x}{9}$

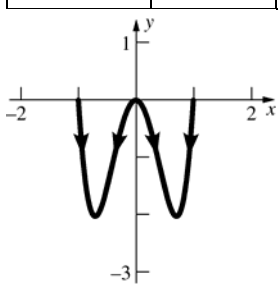
$\sin^2\theta = \frac{y}{9}$

$\frac{x}{9} + \frac{y}{9} = 1$

$x + y = 9$

19. a.

θ	x	y
$0 + 2\pi n$	1	0
$\frac{\pi}{3} + 2\pi n$	$\frac{1}{2}$	$-\frac{3}{2}$
$\frac{\pi}{2} + 2\pi n$	0	0
$\frac{2\pi}{3} + 2\pi n$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\pi + 2\pi n$	-1	0
$\frac{4\pi}{3} + 2\pi n$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\frac{3\pi}{2} + 2\pi n$	0	0
$\frac{5\pi}{3} + 2\pi n$	$\frac{1}{2}$	$-\frac{3}{2}$



b. Not simple; not closed

c. $\cos\theta = x$

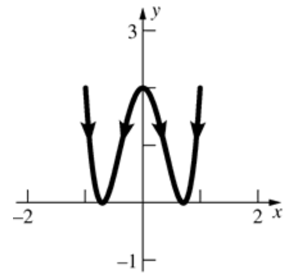
$\sin\theta = \sqrt{1-x^2}$

$y = -8\sin^2\theta\cos^2\theta$

$y = -8x^2(1-x^2)$

20. a.

θ	x	y
$0 + 2\pi n$	0	2
$\frac{\pi}{6} + 2\pi n$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{2} + 2\pi n$	1	2
$\frac{5\pi}{6} + 2\pi n$	$\frac{1}{2}$	$\frac{1}{2}$
$\pi + 2\pi n$	0	2
$\frac{7\pi}{6} + 2\pi n$	$-\frac{1}{2}$	$\frac{1}{2}$
$\frac{3\pi}{2} + 2\pi n$	-1	2
$\frac{11\pi}{6} + 2\pi n$	$-\frac{1}{2}$	$\frac{1}{2}$



b. Not simple; not closed

c. $\sin\theta = x$

$\cos\theta = \sqrt{1-x^2}$

$y = 2(\cos^2\theta - \sin^2\theta)^2$

$y = 2(2x^2-1)^2$

21. $\frac{dx}{d\tau} = 6\tau$

$\frac{dy}{d\tau} = 12\tau^2$

$\frac{dy}{dx} = 2\tau$

$\frac{dy'}{d\tau} = 2$

$\frac{d^2y}{dx^2} = \frac{1}{3\tau}$

22. $\frac{dx}{ds} = 12s$

$\frac{dy}{ds} = -6s^2$

$\frac{dy}{dx} = -\frac{1}{2}s$

$\frac{dy'}{ds} = -\frac{1}{2}$

$\frac{d^2y}{dx^2} = -\frac{1}{24s}$

23. $\frac{dx}{d\theta} = 4\theta$

$\frac{dy}{d\theta} = 3\sqrt{5}\theta^2$

$\frac{dy}{dx} = \frac{3\sqrt{5}}{4}\theta$

$\frac{dy'}{d\theta} = \frac{3\sqrt{5}}{4}$

$\frac{d^2y}{dx^2} = \frac{3\sqrt{5}}{16\theta}$

24. $\dfrac{dx}{d\theta} = 2\sqrt{3}\theta$

$\dfrac{dy}{d\theta} = -3\sqrt{3}\theta^2$

$\dfrac{dy}{dx} = -\dfrac{3}{2}\theta$

$\dfrac{dy'}{d\theta} = -\dfrac{3}{2}$

$\dfrac{d^2y}{dx^2} = -\dfrac{\sqrt{3}}{4\theta}$

25. $\dfrac{dx}{dt} = \sin t$

$\dfrac{dy}{dt} = \cos t$

$\dfrac{dy}{dx} = \cot t$

$\dfrac{dy'}{dt} = -\csc^2 t$

$\dfrac{d^2y}{dx^2} = -\csc^3 t$

26. $\dfrac{dx}{dt} = 2\sin t$

$\dfrac{dy}{dt} = 5\cos t$

$\dfrac{dy}{dx} = \dfrac{5}{2}\cot t$

$\dfrac{dy'}{dt} = -\dfrac{5}{2}\csc^2 t$

$\dfrac{d^2y}{dx^2} = -\dfrac{5}{4}\csc^3 t$

27. $\dfrac{dx}{dt} = 3\sec^2 t$

$\dfrac{dy}{dt} = 5\sec t\tan t$

$\dfrac{dy}{dx} = \dfrac{5}{3}\sin t$

$\dfrac{dy'}{dt} = \dfrac{5}{3}\cos t$

$\dfrac{d^2y}{dx^2} = \dfrac{5}{9}\cos^3 t$

28. $\dfrac{dx}{dt} = -\csc^2 t$

$\dfrac{dy}{dt} = 2\csc t\cot t$

$\dfrac{dy}{dx} = -2\cos t$

$\dfrac{dy'}{dt} = 2\sin t$

$\dfrac{d^2y}{dx^2} = -2\sin^3 t$

29. $\dfrac{dx}{dt} = -\dfrac{2t}{(1+t^2)^2}$

$\dfrac{dy}{dt} = \dfrac{2t-1}{t^2(1-t)^2}$

$\dfrac{dy}{dx} = \dfrac{(1-2t)(1+t^2)^2}{2t^3(1-t)^2}$

$\dfrac{dy'}{dt} = -\dfrac{3t^5+7t^4-6t^3+10t^2-9t+3}{2t^4(1-t)^3}$

$\dfrac{d^2y}{dx^2} = \dfrac{(3t^5+7t^4-6t^3+10t^2-9t+3)(1+t^2)^2}{4t^5(1-t)^3}$

30. $\dfrac{dx}{dt} = -\dfrac{4t}{(1+t^2)^2}$

$\dfrac{dy}{dt} = -\dfrac{2(3t^2+1)}{t^2(1+t^2)^2}$

$\dfrac{dy}{dx} = \dfrac{3t^2+1}{2t^3}$

$\dfrac{dy'}{dt} = -\dfrac{3(t^2+1)}{2t^4}$

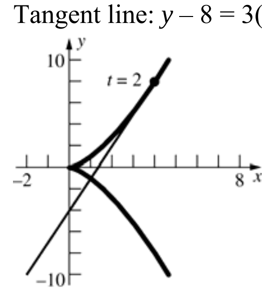
$\dfrac{d^2y}{dx^2} = \dfrac{3(t^2+1)^3}{8t^5}$

31. $\dfrac{dx}{dt} = 2t, \dfrac{dy}{dt} = 3t^2$

$\dfrac{dy}{dx} = \dfrac{3}{2}t$

At $t = 2$, $x = 4$, $y = 8$, and $\dfrac{dy}{dx} = 3$.

Tangent line: $y - 8 = 3(x - 4)$ or $3x - y - 4 = 0$

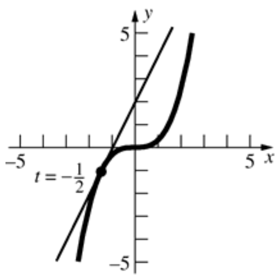


32. $\frac{dx}{dt} = 3, \frac{dy}{dt} = 24t^2$

$\frac{dy}{dx} = 8t^2$

At $t = -\frac{1}{2}, x = -\frac{3}{2}, y = -1$, and $\frac{dy}{dx} = 2$.

Tangent line: $y + 1 = 2\left(x + \frac{3}{2}\right)$ or $2x - y + 2 = 0$



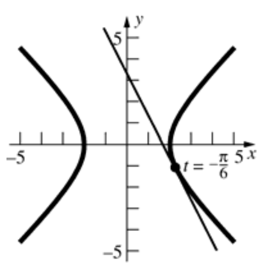
33. $\frac{dx}{dt} = 2\sec t \tan t, \frac{dy}{dt} = 2\sec^2 t$

$\frac{dy}{dx} = \csc t$

At $t = -\frac{\pi}{6}, x = \frac{4}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}$, and $\frac{dy}{dx} = -2.$

Tangent line: $y + \frac{2}{\sqrt{3}} = -2\left(x - \frac{4}{\sqrt{3}}\right)$ or

$2\sqrt{3}x + \sqrt{3}y - 6 = 0$



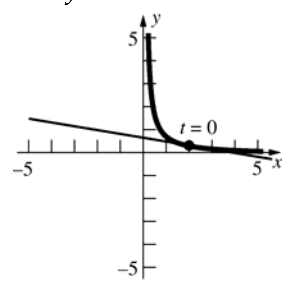
34. $\frac{dx}{dt} = 2e^t, \frac{dy}{dt} = -\frac{1}{3}e^{-t}$

$\frac{dy}{dx} = -\frac{1}{6}e^{-2t}$

At $t = 0$, $x = 2$, $y = \frac{1}{3}$, and $\frac{dy}{dx} = -\frac{1}{6}$.

Tangent line: $y - \frac{1}{3} = -\frac{1}{6}(x - 2)$ or

$x + 6y - 4 = 0$



35. $\frac{dx}{dt} = 2, \frac{dy}{dt} = 3$

$L = \int_0^3 \sqrt{4+9}dt = \sqrt{13}\int_0^3 dt = \sqrt{13}[t]_0^3 = 3\sqrt{13}$

36. $\frac{dx}{dt} = -1, \frac{dy}{dt} = 2$

$L = \int_{-3}^3 \sqrt{1+4}dt = \sqrt{5}\int_{-3}^3 dt = \sqrt{5}[t]_{-3}^3 = 6\sqrt{5}$

37. $\frac{dx}{dt} = 1, \frac{dy}{dt} = \frac{3}{2}t^{1/2}$

$L = \int_0^3 \sqrt{1 + \frac{9}{4}t}\,dt$

$= \frac{1}{2}\int_0^3 \sqrt{4+9t}\,dt$

$= \frac{1}{18}\left[\frac{2}{3}(4+9t)^{3/2}\right]_0^3$

$= \frac{1}{27}(31^{3/2} - 8) = \frac{1}{27}(31\sqrt{31} - 8)$

38. $\frac{dx}{dt} = 2\cos t, \frac{dy}{dt} = -2\sin t$

$L = \int_0^\pi \sqrt{4\cos^2 t + 4\sin^2 t}\,dt = 2\int_0^\pi dt = 2[t]_0^\pi = 2\pi$

39. $\frac{dx}{dt} = 6t, \frac{dy}{dt} = 3t^2$

$L = \int_0^2 \sqrt{36t^2 + 9t^4}\,dt = 3\int_0^2 t\sqrt{4+t^2}\,dt$

$3\left[\frac{1}{3}(4+t^2)^{3/2}\right]_0^2 = 16\sqrt{2} - 8$

40. $\frac{dx}{dt}=1-\frac{1}{t^2}, \frac{dy}{dt}=\frac{2}{t}$

$$L=\int_1^4\sqrt{\left(1-\frac{2}{t^2}+\frac{1}{t^4}\right)+\frac{4}{t^2}}dt$$

$$=\int_1^4\sqrt{1+\frac{2}{t^2}+\frac{1}{t^4}}dt=\int_1^4\sqrt{\left(1+\frac{1}{t^2}\right)^2}dt$$

$$=\int_1^4\left(1+\frac{1}{t^2}\right)dt=\left[t-\frac{1}{t}\right]_1^4=\frac{15}{4}$$

41. $\frac{dx}{dt}=2e^t, \frac{dy}{dt}=\frac{9}{2}e^{3t/2}$

$$L=\int_{\ln 3}^{2\ln 3}\sqrt{4e^{2t}+\frac{81}{4}e^{3t}}dt=\int_{\ln 3}^{2\ln 3}e^t\sqrt{4+\frac{81}{4}e^t}dt$$

$$=\left[\frac{8}{243}\left(4+\frac{81}{4}e^t\right)^{3/2}\right]_{\ln 3}^{2\ln 3}$$

$$=\frac{745\sqrt{745}-259\sqrt{259}}{243}$$

42. $\frac{dx}{dt}=-\frac{t}{\sqrt{1-t^2}}, \frac{dy}{dt}=-1$

$$L=\int_0^{1/4}\sqrt{\frac{t^2}{1-t^2}+1}dt=\int_0^{1/4}\frac{1}{\sqrt{1-t^2}}dt$$

$$=[\sin^{-1}t]_0^{1/4}=\sin^{-1}\frac{1}{4}$$

43. $\frac{dx}{dt}=\frac{2}{\sqrt{t}}, \frac{dy}{dt}=2t-\frac{1}{2t^2}$

$$L=\int_{1/4}^1\sqrt{\frac{4}{t}+\left(4t^2-\frac{2}{t}+\frac{1}{4t^4}\right)}dt$$

$$=\int_{1/4}^1\sqrt{4t^2+\frac{2}{t}+\frac{1}{4t^4}}dt$$

$$=\int_{1/4}^1\sqrt{\left(2t+\frac{1}{2t^2}\right)^2}dt$$

$$=\int_{1/4}^1\left(2t+\frac{1}{2t^2}\right)dt=\left[t^2-\frac{1}{2t}\right]_{1/4}^1=\frac{39}{16}$$

44. $\frac{dx}{dt}=\text{sech}^2 t, \frac{dy}{dt}=2\tanh t$

$$L=\int_{-3}^3\sqrt{\text{sech}^4 t+4\tanh^2 t}\,dt$$

$$=\int_{-3}^3\sqrt{4-4\text{sech}^2 t+\text{sech}^4 t}\,dt$$

$$=\int_{-3}^3\sqrt{(2-\text{sech}^2 t)^2}\,dt=\int_{-3}^3(2-\text{sech}^2 t)dt$$

$$=[2t-\tanh t]_{-3}^3=12-2\tanh 3$$

45. $\frac{dx}{dt}=-\sin t,$

$$\frac{dy}{dt}=\frac{\sec t\tan t+\sec^2 t}{\sec t+\tan t}-\cos t=\sec t-\cos t$$

$$L=\int_0^{\pi/4}\sqrt{\sin^2 t+(\sec^2 t-2+\cos^2 t)}dt$$

$$=\int_0^{\pi/4}\tan t\,dt$$

$$=\left[-\ln|\cos t|\right]_0^{\pi/4}=-\ln\frac{1}{\sqrt{2}}=\frac{1}{2}\ln 2$$

46. $\frac{dx}{dt}=t\sin t, \frac{dy}{dt}=t\cos t$

$$L=\int_{\pi/4}^{\pi/2}\sqrt{t^2\sin^2 t+t^2\cos^2 t}\,dt$$

$$=\int_{\pi/4}^{\pi/2}t\,dt=\left[\frac{1}{2}t^2\right]_{\pi/4}^{\pi/2}=\frac{3\pi^2}{32}$$

47. a. $\frac{dx}{d\theta}=\cos\theta, \frac{dy}{d\theta}=-\sin\theta$

$$L=\int_0^{2\pi}\sqrt{\cos^2\theta+\sin^2\theta}\,d\theta=\int_0^{2\pi}d\theta$$

$$=[\theta]_0^{2\pi}=2\pi$$

b. $\frac{dx}{d\theta}=3\cos 3\theta, \frac{dy}{d\theta}=-3\sin 3\theta$

$$L=\int_0^{2\pi}\sqrt{9\cos^2 3\theta+9\sin^2 3\theta}\,d\theta$$

$$=3\int_0^{2\pi}d\theta=3[\theta]_0^{2\pi}=6\pi$$

c. The curve in part a goes around the unit circle once, while the curve in part b goes around the unit circle three times.

48. $\Delta S=2\pi x\Delta s$

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$

$$S=\int_a^b 2\pi x\,ds=\int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$

See Section 5.4 of the text

49. $\frac{dx}{dt}=-\sin t, \frac{dy}{dt}=\cos t$

$$S=\int_0^{2\pi}2\pi(1+\cos t)\sqrt{\sin^2 t+\cos^2 t}\,dt$$

$$=2\pi\int_0^{2\pi}(1+\cos t)dt=2\pi[t+\sin t]_0^{2\pi}=4\pi^2$$

50. $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$

$$S = \int_0^{2\pi} 2\pi(3+\sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt$$

$$= 2\pi\int_0^{2\pi}(3+\sin t)dt = 2\pi[3t - \cos t]_0^{2\pi} = 12\pi^2$$

51. $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$

$$S = \int_0^{2\pi} 2\pi(1+\sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt$$

$$= 2\pi\int_0^{2\pi}(1+\sin t)dt = 2\pi[t - \cos t]_0^{2\pi} = 4\pi^2$$

52. $\frac{dx}{dt} = \sqrt{t}, \frac{dy}{dt} = \frac{1}{\sqrt{t}}$

$$S = \int_0^{2\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{t + \frac{1}{t}}dt$$

$$= \frac{4\pi}{3}\int_0^{2\sqrt{3}} t\sqrt{t^2+1}dt$$

$$= \frac{4\pi}{3}\left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^{2\sqrt{3}} = \frac{4\pi}{9}(13\sqrt{13}-1)$$

53. $\frac{dx}{dt} = 1, \frac{dy}{dt} = t + \sqrt{7}$

$$S = \int_{-\sqrt{7}}^{\sqrt{7}} 2\pi\left(t+\sqrt{7}\right)\sqrt{1+\left(t+\sqrt{7}\right)^2}\,dt$$

$$= 2\pi\left[\frac{1}{3}\left(1+\left(t+\sqrt{7}\right)^2\right)^{3/2}\right]_{-\sqrt{7}}^{\sqrt{7}}$$

$$= \frac{2\pi}{3}\left(29\sqrt{29}-1\right)$$

54. $\frac{dx}{dt} = t + a, \frac{dy}{dt} = 1$

$$S = \int_{-\sqrt{a}}^{\sqrt{a}} 2\pi(t+a)\sqrt{(t+a)^2+1}\,dt$$

$$= 2\pi\left[\frac{1}{3}((t+a)^2+1)^{3/2}\right]_{-\sqrt{a}}^{\sqrt{a}}$$

$$= \frac{2\pi}{3}\left[\left(a^2+2a\sqrt{a}+a+1\right)^{3/2} - \left(a^2-2a\sqrt{a}+a+1\right)^{3/2}\right]$$

55. $dx = dt$; when $x = 0$, $t = -1$; when $x = 1$, $t = 0$.

$$\int_0^1 (x^2 - 4y)dx = \int_{-1}^0 [(t+1)^2 - 4(t^3+4)]dt$$

$$= \int_{-1}^0 (-4t^3 + t^2 + 2t - 15)dt$$

$$= \left[-t^4 + \frac{1}{3}t^3 + t^2 - 15t\right]_{-1}^0 = -\frac{44}{3}$$

56. $dy = \sec^2 t\,dt$; when $y = 1, t = \frac{\pi}{4}$;

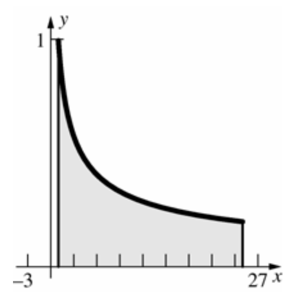
when $y = \sqrt{3}, t = \frac{\pi}{3}$.

$$\int_1^{\sqrt{3}} xy\,dy = \int_{\pi/4}^{\pi/3}(\sec t)(\tan t)\sec^2 t\,dt$$

$$= \left[\frac{1}{3}\sec^3 t\right]_{\pi/4}^{\pi/3} = \frac{8}{3} - \frac{2\sqrt{2}}{3}$$

57. $dx = 2e^{2t}dt$

$$A = \int_1^{25} y\,dx = \int_0^{\ln 5} 2e^t dt = [2e^t]_0^{\ln 5} = 8$$



58. a. $t = \frac{x}{v_0 \cos\alpha}$

$$y = -16\left(\frac{x}{v_0\cos\alpha}\right)^2 + (v_0\sin\alpha)\left(\frac{x}{v_0\cos\alpha}\right)$$

$$y = -\left(\frac{16}{v_0{}^2\cos^2\alpha}\right)x^2 + (\tan\alpha)x$$

This is an equation for a parabola.

b. Solve for t when $y = 0$.

$$-16t^2 + (v_0\sin\alpha)t = 0$$

$$t(-16t + v_0\sin\alpha) = 0$$

$$t = 0, \frac{v_0\sin\alpha}{16}$$

The time of flight is $\frac{v_0\sin\alpha}{16}$ seconds.

c. At $t = \frac{v_0\sin\alpha}{16}, x = (v_0\cos\alpha)\left(\frac{v_0\sin\alpha}{16}\right)$

$$= \frac{v_0^2\sin\alpha\cos\alpha}{16} = \frac{v_0^2\sin 2\alpha}{32}.$$

d. Let R be the range as a function of α.

$$R = \frac{v_0^2 \sin 2\alpha}{32}$$

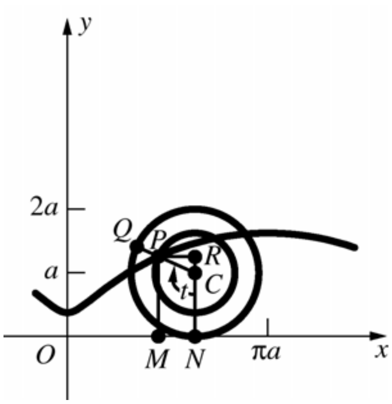
$$\frac{dR}{d\alpha} = \frac{v_0^2 \cos 2\alpha}{16}$$

$$\frac{v_0^2 \cos 2\alpha}{16} = 0, \cos 2\alpha = 0, \alpha = \frac{\pi}{4}$$

$$\frac{d^2R}{d\alpha^2} = -\frac{v_0^2 \sin 2\alpha}{8}; \frac{d^2R}{d\alpha^2} < 0 \text{ at } \alpha = \frac{\pi}{4}.$$

The range is the largest possible when $\alpha = \frac{\pi}{4}$.

59.

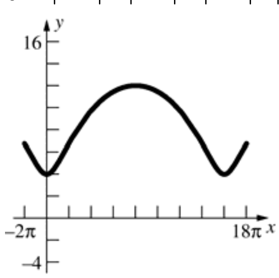


Let the wheel roll along the x-axis with P initially at $(0, a - b)$.

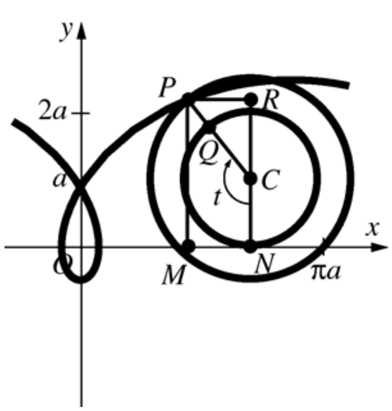
$|ON| = \text{arc } NQ = at$

$x = |OM| = |ON| - |MN| = at - b\sin t$

$y = |MP| = |RN| = |NC| + |CR| = a - b\cos t$



60.

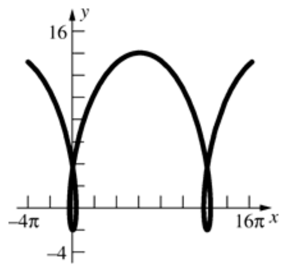


Let the wheel roll along the x-axis with P initially at $(0, a - b)$.

$|ON| = \text{arc } NQ = at$

$x = |OM| = |ON| - |MN| = at - b\sin t$

$y = |MP| = |RN| = |NC| + |CR| = a - b\cos t$



61. The x- and y-coordinates of the center of the circle of radius b are $(a - b)\cos t$ and $(a - b)\sin t$, respectively. The angle measure (in a clockwise direction) of arc BP is $\frac{a}{b}t$. The horizontal change from the center of the circle of radius b to P is $b\cos\left(-\left(\frac{a}{b}t - t\right)\right) = b\cos\left(\frac{a-b}{b}t\right)$ and the vertical change is $b\sin\left(-\left(\frac{a}{b}t - t\right)\right) = -b\sin\left(\frac{a-b}{b}t\right)$.

Therefore, $x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right)$ and

$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right)$.

62. From Problem 61,

$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right)$ and

$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right)$.

Substitute $b = \frac{a}{4}$.

$$x = \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos(3t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos(2t + t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)(\cos 2t \cos t - \sin 2t \sin t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)(\cos^3 t - \sin^2 t \cos t - 2\sin^2 t \cos t)$$

$$= \left(\frac{3a}{4}\right)\cos t + \left(\frac{a}{4}\right)\cos^3 t - \left(\frac{3a}{4}\right)\cos t \sin^2 t$$

$$= \left(\frac{3a}{4}\right)(\cos t)(1 - \sin^2 t) + \left(\frac{a}{4}\right)\cos^3 t = a\cos^3 t$$

$$y = \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)\sin(3t)$$

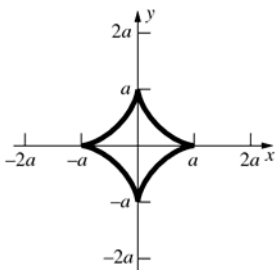
$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)\sin(2t+t)$

$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)(\sin 2t \cos t + \cos 2t \sin t)$

$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{a}{4}\right)(2\sin t \cos^2 t + \cos^2 t \sin t - \sin^3 t)$

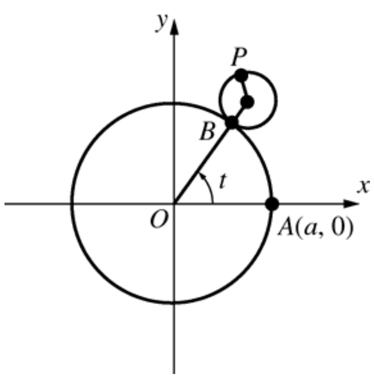
$= \left(\frac{3a}{4}\right)\sin t - \left(\frac{3a}{4}\right)\sin t \cos^2 t + \left(\frac{a}{4}\right)\sin^3 t$

$= \left(\frac{3a}{4}\right)(\sin t)(1-\cos^2 t) + \left(\frac{a}{4}\right)\sin^3 t = a\sin^3 t$



$x^{2/3} + y^{2/3} = a^{2/3}\cos^2 t + a^{2/3}\sin^2 t = a^{2/3}$

63. Consider the following figure similar to the one in the text for Problem 61.



The x- and y-coordinates of the center of the circle of radius b are $(a + b)\cos t$ and $(a + b)\sin t$ respectively. The angle measure (in a counter-clockwise direction) of arc PB is $\frac{a}{b}t$. The horizontal change from the center of the circle of radius b to P is

$b\cos\left(\frac{a}{b}t + t + \pi\right) = -b\cos\left(\frac{a+b}{b}t\right)$ and the vertical change is

$b\sin\left(\frac{a}{b}t + t + \pi\right) = -b\sin\left(\frac{a+b}{b}t\right)$. Therefore,

$x = (a+b)\cos t - b\cos\left(\frac{a+b}{b}t\right)$ and

$y = (a+b)\sin t - b\sin\left(\frac{a+b}{b}t\right)$.

64. $x = 2a\cos t - a\cos 2t = 2a\cos t - 2a\cos^2 t + a$

$= 2a\cos t(1-\cos t) + a$

$y = 2a\sin t - a\sin 2t = 2a\sin t - 2a\sin t\cos t$

$= 2a\sin t(1-\cos t)$

$x - a = 2a\cos t(1-\cos t)$

$(x-a)^2 + y^2 = 4a^2(1-\cos t)^2$

$(x-a)^2 + y^2 + 2a(x-a) =$

$4a^2(1-\cos t)^2 + 4a^2(1-\cos t)\cos t$

$(x-a)^2 + y^2 + 2a(x-a) = 4a^2(1-\cos t)$

$[(x-a)^2 + y^2 + 2a(x-a)]^2 = 4a^2 \cdot 4a^2(1-\cos t)^2$

$[(x-a)^2 + y^2 + 2a(x-a)]^2 = 4a^2[(x-a)^2 + y^2]$

65. $\frac{dx}{dt} = \left(\frac{a}{3}\right)(-2\sin t - 2\sin 2t),$

$\frac{dy}{dt} = \left(\frac{a}{3}\right)(2\cos t - 2\cos 2t)$

$\left(\frac{dx}{dt}\right)^2 = \left(\frac{a}{3}\right)^2(4\sin^2 t + 8\sin t\sin 2t + 4\sin^2 2t)$

$\left(\frac{dy}{dt}\right)^2 = \left(\frac{a}{3}\right)^2(4\cos^2 t - 8\cos t\cos 2t + 4\cos^2 2t)$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{a}{3}\right)^2(8 + 8\sin t\sin 2t - 8\cos t\cos 2t)$

$= \left(\frac{a}{3}\right)^2(8 + 16\sin^2 t\cos t - 8\cos^3 t + 8\sin^2 t\cos t)$

$= \left(\frac{a}{3}\right)^2(8 + 24\cos t\sin^2 t - 8\cos^3 t)$

$= \left(\frac{a}{3}\right)^2(8 + 24\cos t - 32\cos^3 t)$

$L = 3\int_0^{2\pi/3}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= a\int_0^{2\pi/3}\sqrt{8 + 24\cos t - 32\cos^3 t}\,dt$

Using a CAS to evaluate the length, $L = \frac{16a}{3}$.

66. a. Let $x = a\cos t$ and $y = b\sin t$.

$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = b\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\sin^2 t + b^2\cos^2 t$$

$$= a^2 + (b^2 - a^2)\cos^2 t$$

$$= a^2 - c^2\cos^2 t = a^2\left(1 - \frac{c^2}{a^2}\cos^2 t\right)$$

$$= a^2\left(1 - \left(\frac{c}{a}\right)^2\cos^2 t\right) = a^2(1 - e^2\cos^2 t)$$

$$P = 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= 4a\int_0^{\pi/2}\sqrt{1 - e^2\cos^2 t}\,dt$$

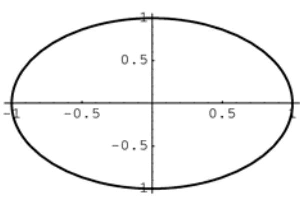
b. $P = 4\int_0^{\pi/2}\sqrt{1 - \frac{\cos^2 t}{16}}\,dt$

$= \int_0^{\pi/2}\sqrt{16 - \cos^2 t}\,dt \approx 6.1838$

(The answer is near 2π because it is slightly smaller than a circle of radius 1 whose perimeter is 2π).

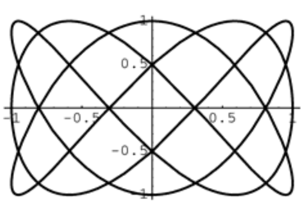
c. $P = \int_0^{\pi/2}\sqrt{16 - \cos^2 t}\,dt \approx 6.1838$

67. a.



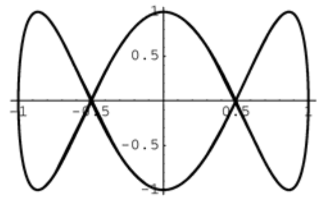
The curve touches a horizontal border twice and touches a vertical border twice.

b.



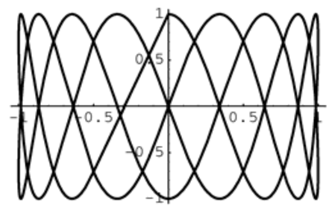
The curve touches a horizontal border five times and touches a vertical border three times.

c.



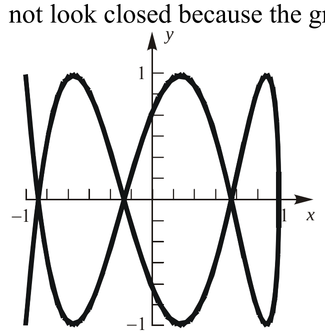
The curve touches a horizontal border six times and touches a vertical border twice. Note that the curve is traced out five times.

d.

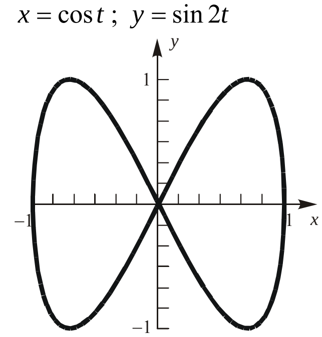


The curve touches a horizontal border 18 times and touches a vertical border four times.

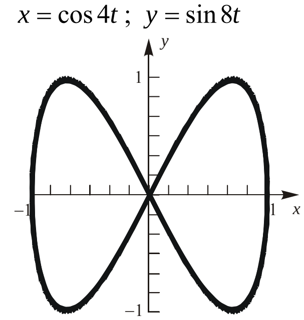
68. This is a closed curve even thought the graph does not look closed because the graph retraces itself.



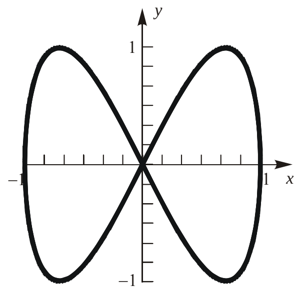
69. a. $x = \cos t$; $y = \sin 2t$



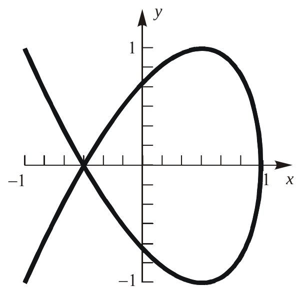
b. $x = \cos 4t$; $y = \sin 8t$



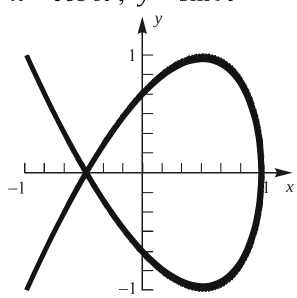
c. $x = \cos 5t$; $y = \sin 10t$



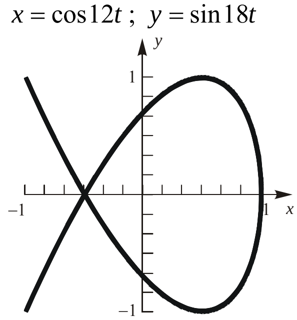
d. $x = \cos 2t$; $y = \sin 3t$



e. $x = \cos 6t$; $y = \sin 9t$



f. $x = \cos 12t$; $y = \sin 18t$



70. Consider the curve defined parametrically by $x = \cos at$, $y = \sin bt$, $t \in [0, 2\pi)$; we assume a and b are integers. This graph will be contained in the box with sides $x = \pm 1$, $y = \pm 1$. Let H be the number of times the graph touches a horizontal side, V the number of times it touches a vertical side, and C the number of times it touches a corner (right now, C is included in H and V). Finally, let w = the *greatest common divisor* of a and b; write $a = u \cdot w$, $b = v \cdot w$. Note: $\frac{a}{b}$ in lowest terms is $\frac{u}{v}$.

a. The graph touches a horizontal side if $\sin bt = \pm 1$ or $bt = \frac{k}{2}\pi$ (k odd) ; that is,

$t = \frac{k}{2b}\pi$, where $k = 1, 3, \ldots, 4b - 1$.

Hence, H = 2b.

b. The graph touches a vertical side if $\cos at = \pm 1$ or $at = n\pi$ (n an integer) ; that is,

$t = \frac{n}{a}\pi$, where $n = 0, 1, \ldots, 2a - 1$.

Hence, V = 2a.

c. If t_0 yields a corner, then (see **a.** and **b.**) then

$at_0 = n\pi$, $bt_0 = \frac{k}{2}\pi$ so that

$\frac{u}{v} = \frac{a}{b} = \frac{n}{k/2} = \frac{2n}{k}$. Thus corners can only occur if u is even and v is odd. Assume that is the case, write $u = 2r$, and assume we have a corner at t_0 ; then $t_0 = \frac{k}{2b}\pi$ and

$at_0 = \frac{ak}{2b}\pi = n\pi$, ($n$ an integer). Thus

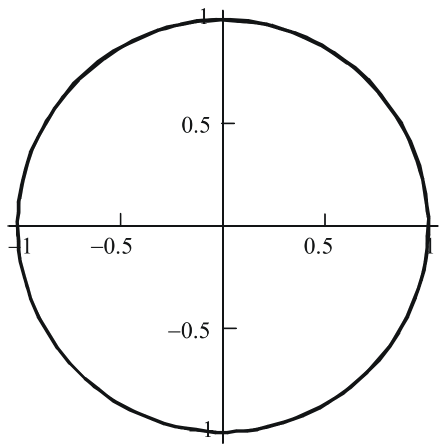
$\frac{ak}{2b} = \frac{2rwk}{2vw} = \frac{rk}{v}$ is an integer; hence v is a factor of rk. But v and r have no factors in common, so v must be a factor of k. Conclusion: k is an odd multiple of v. Thus corners occur at $\frac{m}{2b}\pi$ where

$m = v, 3v, 5v, \ldots, (4w - 1)v$; therefore C = 2w.

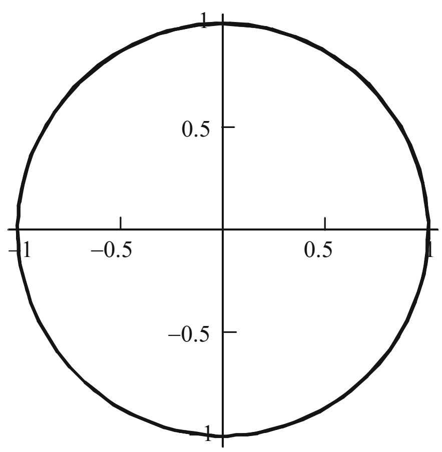
Now if we count corner contacts as half horizontal and half vertical, the ratio of vertical contacts to horizontal contacts is given by

$$\frac{\text{V-}\frac{1}{2}\text{C}}{\text{H-}\frac{1}{2}\text{C}} = \begin{cases} \frac{2a}{2b} = \frac{a}{b} = \frac{u}{v} & \textit{if } \text{C} = 0 \\ \frac{2a - w}{2b - w} = \frac{2u - 1}{2v - 1} & \textit{if } \text{C} \neq 0 \end{cases}$$

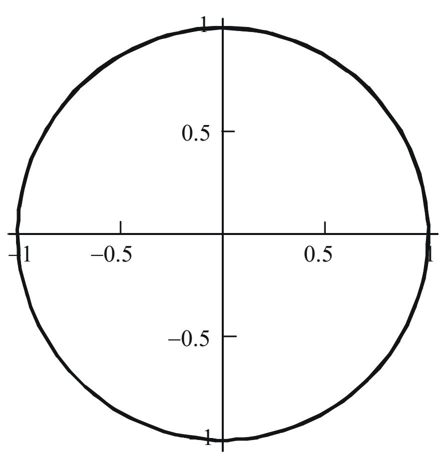
71. a.



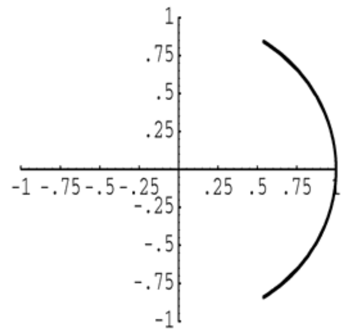
b.



c.

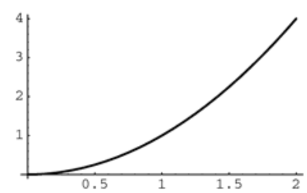


d.



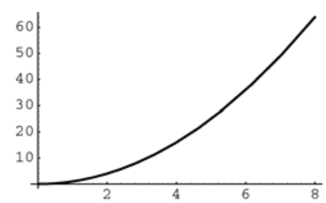
Given a parameterization of the form $x = \cos f(t)$ and $y = \sin f(t)$, the point moves around the curve (which is a circle of radius 1) at a speed of $|f'(t)|$. The point travels clockwise around the circle when $f(t)$ is decreasing and counterclockwise when $f(t)$ is increasing. Note that in part d, only part of the circle will be traced out since the range of $f(t) = \sin t$ is $[-1, 1]$.

72. a.



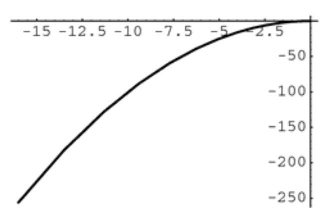
The curve traced out is the graph of $y = x^2$ for $0 \le x \le 2$

b.



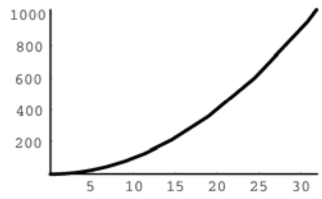
The curve traced out is the graph of $y = x^2$ for $0 \le x \le 8$.

c.



The curve traced out is the graph of $y = -x^2$ for $-16 \le x \le 0$.

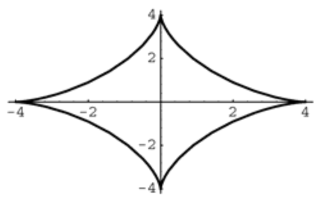
d.



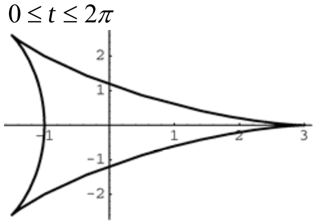
The curve traced out is the graph of $y = x^2$ for $0 \le x \le 32$

All of the curves lie on the graph of $y = \pm x^2$, but trace out different parts because of the parameterization

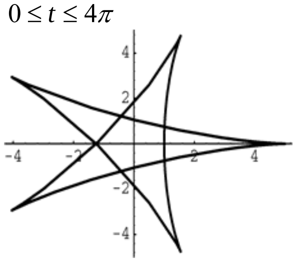
73. a. $0 \le t \le 2\pi$



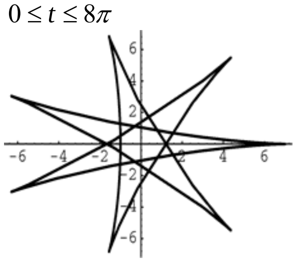
b. $0 \le t \le 2\pi$



c. $0 \le t \le 4\pi$

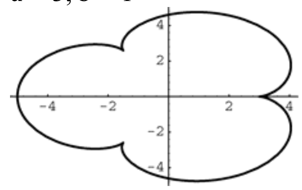


d. $0 \le t \le 8\pi$

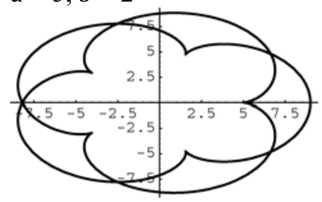


Let $\frac{p}{q} = \frac{a}{b}$ where $\frac{p}{q}$ is the reduced fraction of $\frac{a}{b}$. The length of the t-interval is $2q\pi$. The number of times the graph would touch the circle of radius a during the t-interval is p. If $\frac{a}{b}$ is irrational, the curve is not periodic.

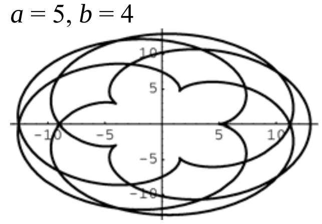
74. Some possible graphs for different a and b are shown below.
$a = 3$, $b = 1$



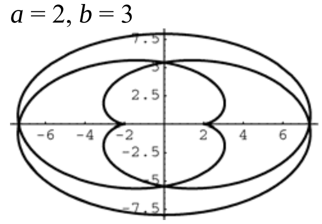
$a = 5$, $b = 2$



$a = 5$, $b = 4$

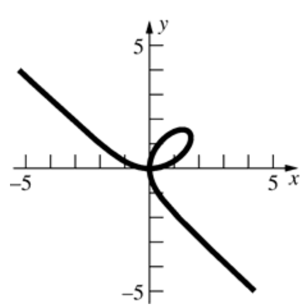


$a = 2$, $b = 3$



Let $\frac{p}{q} = \frac{a}{b}$ where $\frac{p}{q}$ is the reduced fraction of $\frac{a}{b}$. The length of the t-interval is $2q\pi$. The number of times the graph would touch the circle of radius a during the t-interval is p. If $\frac{a}{b}$ is irrational, the curve is not periodic.

75. $x = \frac{3t}{t^3+1}, y = \frac{3t^2}{t^3+1}$



When $x > 0$, $t > 0$ or $t < -1$.
When $x < 0$, $-1 < t < 0$.
When $y > 0$, $t > -1$. When $y < 0$, $t < -1$.
Therefore the graph is in quadrant I for $t > 0$, quadrant II for $-1 < t < 0$, quadrant III for no t, and quadrant IV for $t < -1$.

10.5 Concepts Review

1. infinitely many

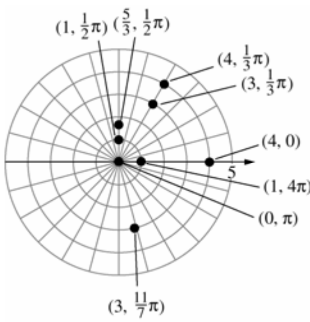
2. $r\cos\theta$; $r\sin\theta$; r^2

3. circle; line

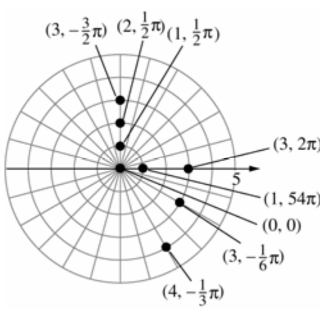
4. conic

Problem Set 10.5

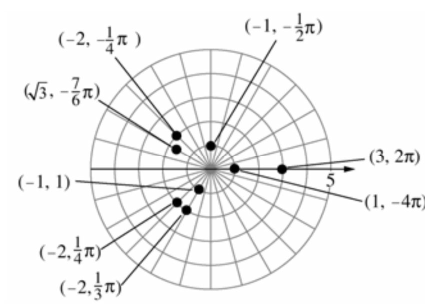
1.



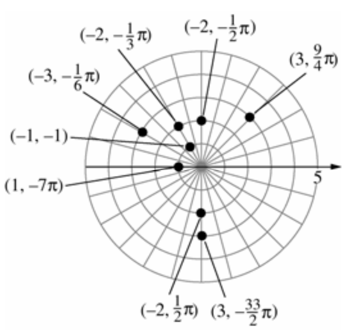
2.



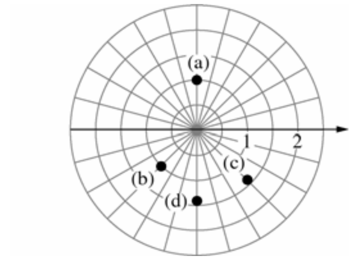
3.



4.



5.



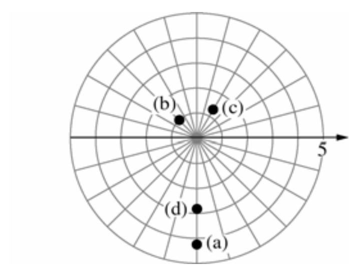
a. $\left(1, -\frac{3}{2}\pi\right), \left(1, \frac{5}{2}\pi\right), \left(-1, -\frac{1}{2}\pi\right), \left(-1, \frac{3}{2}\pi\right)$

b. $\left(1, -\frac{3}{4}\pi\right), \left(1, \frac{5}{4}\pi\right), \left(-1, -\frac{7}{4}\pi\right), \left(-1, \frac{9}{4}\pi\right)$

c. $\left(\sqrt{2}, -\frac{7}{3}\pi\right), \left(\sqrt{2}, \frac{5}{3}\pi\right), \left(-\sqrt{2}, -\frac{4}{3}\pi\right),$
$\left(-\sqrt{2}, \frac{2}{3}\pi\right)$

d. $\left(\sqrt{2}, -\frac{1}{2}\pi\right), \left(\sqrt{2}, \frac{3}{2}\pi\right), \left(-\sqrt{2}, -\frac{3}{2}\pi\right),$
$\left(-\sqrt{2}, \frac{1}{2}\pi\right)$

6.



a. $\left(3\sqrt{2}, -\frac{1}{2}\pi\right), \left(3\sqrt{2}, \frac{3}{2}\pi\right), \left(-3\sqrt{2}, -\frac{3}{2}\pi\right),$
$\left(-3\sqrt{2}, \frac{1}{2}\pi\right)$

b. $\left(1, -\frac{5}{4}\pi\right), \left(1, \frac{3}{4}\pi\right), \left(-1, -\frac{1}{4}\pi\right), \left(-1, \frac{7}{4}\pi\right)$

c. $\left(\sqrt{2}, -\frac{5}{3}\pi\right), \left(\sqrt{2}, \frac{1}{3}\pi\right), \left(-\sqrt{2}, -\frac{8}{3}\pi\right),$
$\left(-\sqrt{2}, \frac{4}{3}\pi\right)$

d. $\left(2\sqrt{2}, -\frac{1}{2}\pi\right), \left(2\sqrt{2}, \frac{3}{2}\pi\right), \left(-2\sqrt{2}, -\frac{3}{2}\pi\right),$
$\left(-2\sqrt{2}, \frac{1}{2}\pi\right)$

7. a. $x = 1\cos\frac{1}{2}\pi = 0$

$y = 1\sin\frac{1}{2}\pi = 1$

(0, 1)

b. $x = -1\cos\frac{1}{4}\pi = -\frac{\sqrt{2}}{2}$

$y = -1\sin\frac{1}{4}\pi = -\frac{\sqrt{2}}{2}$

$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

c. $x = \sqrt{2}\cos\left(-\frac{1}{3}\pi\right) = \frac{\sqrt{2}}{2}$

$y = \sqrt{2}\sin\left(-\frac{1}{3}\pi\right) = -\frac{\sqrt{6}}{2}$

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{2}\right)$

d. $x = -\sqrt{2}\cos\frac{5}{2}\pi = 0$

$y = -\sqrt{2}\sin\frac{5}{2}\pi = -\sqrt{2}$

$\left(0, -\sqrt{2}\right)$

8. a. $x = 3\sqrt{2}\cos\frac{7}{2}\pi = 0$

$y = 3\sqrt{2}\sin\frac{7}{2}\pi = -3\sqrt{2}$

$\left(0, -3\sqrt{2}\right)$

b. $x = -1\cos\frac{15}{4}\pi = -\frac{\sqrt{2}}{2}$

$y = -1\sin\frac{15}{4}\pi = \frac{\sqrt{2}}{2}$

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

c. $x = -\sqrt{2}\cos\left(-\frac{2}{3}\pi\right) = \frac{\sqrt{2}}{2}$

$y = -\sqrt{2}\sin\left(-\frac{2}{3}\pi\right) = \frac{\sqrt{6}}{2}$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$

d. $x = -2\sqrt{2}\cos\frac{29}{2}\pi = 0$

$y = -2\sqrt{2}\sin\frac{29}{2}\pi = -2\sqrt{2}$

$\left(0, -2\sqrt{2}\right)$

9. a. $r^2 = \left(3\sqrt{3}\right)^2 + 3^2 = 36,\ r = 6$

$\tan\theta = \frac{1}{\sqrt{3}}, \theta = \frac{\pi}{6}$

$\left(6, \frac{\pi}{6}\right)$

b. $r^2 = \left(-2\sqrt{3}\right)^2 + 2^2 = 16,\ r = 4$

$\tan\theta = \frac{2}{-2\sqrt{3}}, \theta = \frac{5\pi}{6}$

$\left(4, \frac{5\pi}{6}\right)$

c. $r^2 = \left(-\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2 = 4,\ r = 2$

$\tan\theta = \frac{-\sqrt{2}}{-\sqrt{2}}, \theta = \frac{5\pi}{4}$

$\left(2, \frac{5\pi}{4}\right)$

d. $r^2 = 0^2 + 0^2 = 0,\ r = 0$

$\tan\ \theta = 0,\ \theta = 0$

(0, 0)

10. a. $r^2 = \left(-\frac{3}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{10}{3}, r = \frac{\sqrt{10}}{\sqrt{3}}$

$\tan\theta = \frac{\frac{1}{\sqrt{3}}}{\frac{-3}{\sqrt{3}}}, \theta = \pi + \tan^{-1}\left(-\frac{1}{3}\right)$

$\left(\frac{\sqrt{10}}{\sqrt{3}}, \pi + \tan^{-1}\left(-\frac{1}{3}\right)\right)$

b. $r^2 = \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{6}{4}, r = \frac{\sqrt{6}}{2}$

$\tan\theta = \frac{\frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2}}, \theta = \frac{3\pi}{4}$

$\left(\frac{\sqrt{6}}{2}, \frac{3\pi}{4}\right)$

c. $r^2 = 0^2 + (-2)^2 = 4,\ r = 2$

$\tan\theta = -\frac{2}{0}, \theta = \frac{3\pi}{2}$

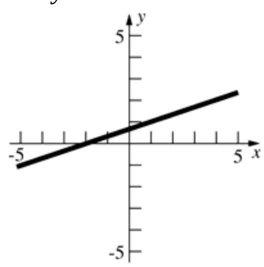
$\left(2, \frac{3\pi}{2}\right)$

d. $r^2 = 3^2 + (-4)^2 = 25,\ r = 5$

$\tan\theta = -\frac{4}{3}, \theta = \tan^{-1}\left(-\frac{4}{3}\right)$

$\left(2, \tan^{-1}\left(-\frac{4}{3}\right)\right)$

11. $x - 3y + 2 = 0$

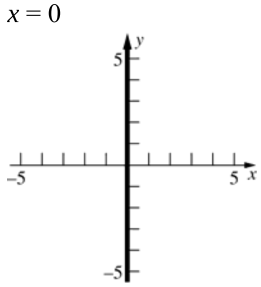


$r\cos\theta - 3r\sin\theta + 2 = 0$

$r = -\frac{2}{\cos\theta - 3\sin\theta}$

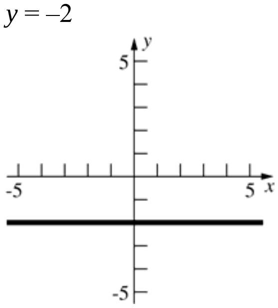
$r = \frac{2}{3\sin\theta - \cos\theta}$

12. $x = 0$



$\theta = \frac{\pi}{2}$

13. $y = -2$

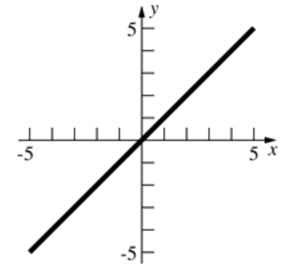


$r\sin\theta = -2$

$r = -\frac{2}{\sin\theta}$

$r = -2\csc\theta$

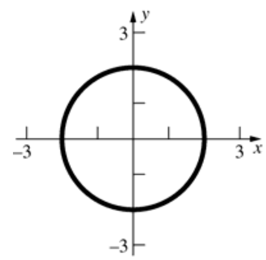
14. $x - y = 0$



$r\cos\theta - r\sin\theta = 0$
$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

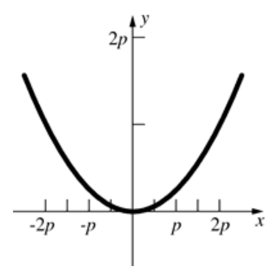
15. $x^2 + y^2 = 4$



$(r\cos\theta)^2 + (r\sin\theta)^2 = 4$

$r^2 = 4$
$r = 2$

16. $x^2 = 4py$



$(r\cos\theta)^2 = 4p(r\sin\theta)$

$r = \frac{4p\sin\theta}{\cos^2\theta}$

$r = 4p\sec\theta\tan\theta$

17. $\theta = \frac{\pi}{2}$

$\cot\theta = 0$

$\frac{x}{y} = 0$

$x = 0$

18. $r = 3$

$r^2 = 9$

$x^2 + y^2 = 9$

19. $r \cos \theta + 3 = 0$
$x + 3 = 0$
$x = -3$

20. $r - 5 \cos \theta = 0$

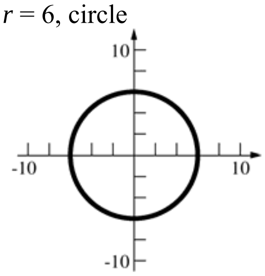
$$r^2 - 5r\cos\theta = 0$$
$$x^2 + y^2 - 5x = 0$$
$$\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = \frac{25}{4}$$
$$\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$$

21. $r \sin \theta - 1 = 0$
$y - 1 = 0$
$y = 1$

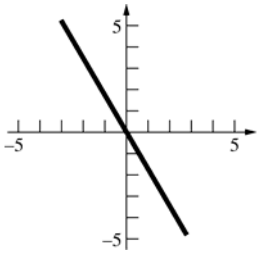
22. $r^2 - 6r\cos\theta - 4r\sin\theta + 9 = 0$

$$x^2 + y^2 - 6x - 4y + 9 = 0$$
$$(x^2 - 6x + 9) + (y^2 - 4y + 4) = -9 + 9 + 4$$
$$(x-3)^2 + (y-2)^2 = 4$$

23. $r = 6$, circle

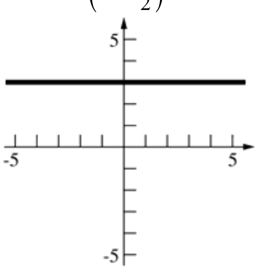


24. $\theta = \frac{2\pi}{3}$, line



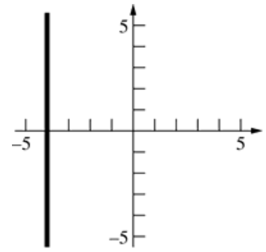
25. $r = \frac{3}{\sin\theta}$

$r = \frac{3}{\cos\left(\theta - \frac{\pi}{2}\right)}$, line



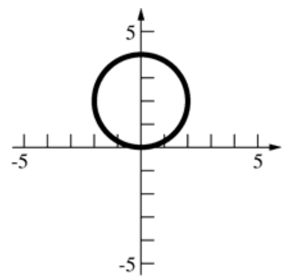
26. $r = -\frac{4}{\cos\theta}$

$r = \frac{4}{\cos(\theta - \pi)}$, line



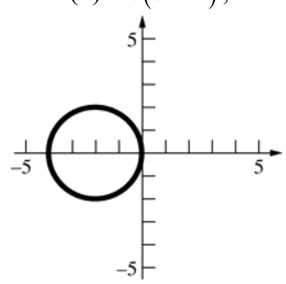
27. $r = 4\sin\theta$

$r = 2(2)\cos\left(\theta - \frac{\pi}{2}\right)$, circle



28. $r = -4\cos\theta$

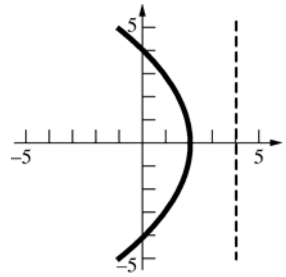
$r = 2(2)\cos(\theta - \pi)$, circle



29. $r = \frac{4}{1 + \cos\theta}$

$r = \frac{(1)(4)}{1 + (1)\cos\theta}$, parabola

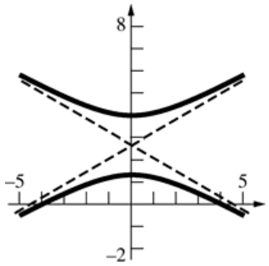
$e = 1$



30. $r = \dfrac{4}{1 + 2\sin\theta}$

$r = \dfrac{(2)(2)}{1 + 2\cos\left(\theta - \frac{\pi}{2}\right)}$, hyperbola

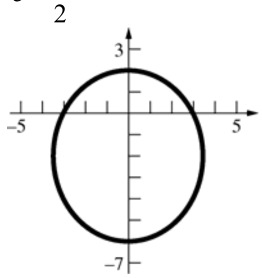
$e = 2$



31. $r = \dfrac{6}{2 + \sin\theta}$

$r = \dfrac{\left(\frac{1}{2}\right)6}{1 + \left(\frac{1}{2}\right)\cos\left(\theta - \frac{\pi}{2}\right)}$, ellipse

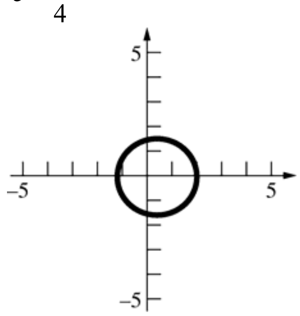
$e = \dfrac{1}{2}$



32. $r = \dfrac{6}{4 - \cos\theta}$

$r = \dfrac{6\left(\frac{1}{4}\right)}{1 + \left(\frac{1}{4}\right)\cos(\theta - \pi)}$, ellipse

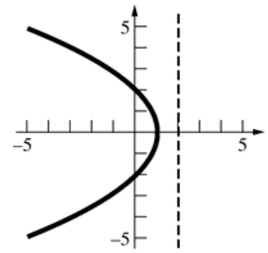
$e = \dfrac{1}{4}$



33. $r = \dfrac{4}{2 + 2\cos\theta}$

$r = \dfrac{(1)(2)}{1 + (1)\cos\theta}$, parabola

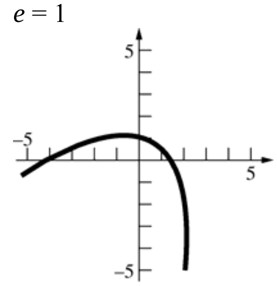
$e = 1$



34. $r = \dfrac{4}{2 + 2\cos\left(\theta - \frac{\pi}{3}\right)}$

$r = \dfrac{2(1)}{1 + (1)\cos\left(\theta - \frac{\pi}{3}\right)}$, parabola

$e = 1$

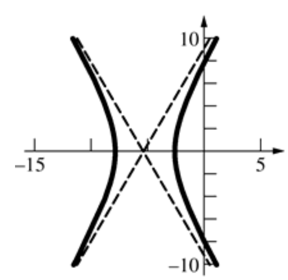


35. $r = \dfrac{4}{\frac{1}{2} + \cos(\theta - \pi)}$

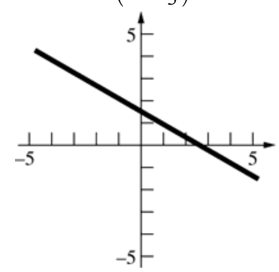
$r = \dfrac{4(2)}{1 + 2\cos(\theta - \pi)}$,

hyperbola

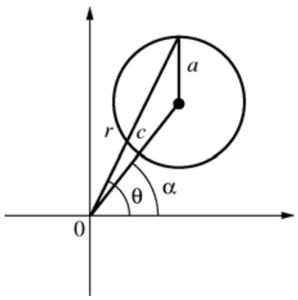
$e = 2$



36. $r = \dfrac{4}{3\cos\left(\theta - \frac{\pi}{3}\right)}$, line



37. By the Law of Cosines,
$a^2 = r^2 + c^2 - 2rc\cos(\theta - \alpha)$ (see figure below).



38. $r = a\sin\theta + b\cos\theta$
$r^2 = ar\sin\theta + br\cos\theta$
$x^2 + y^2 = ay + bx$
$x^2 - bx + y^2 - ay = 0$
$$\left(x^2 - bx + \frac{b^2}{4}\right) + \left(y^2 - ay + \frac{a^2}{4}\right) = \frac{a^2 + b^2}{4}$$
$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}$$
This is an equation of a circle with radius $\frac{\sqrt{a^2 + b^2}}{2}$ and center $\left(\frac{b}{2}, \frac{a}{2}\right)$.

39. Recall that the latus rectum is perpendicular to the axis of the conic through a focus.
$$r\left(\theta_0 + \frac{\pi}{2}\right) = \frac{ed}{1 + e\cos\frac{\pi}{2}} = ed$$
Thus the length of the latus rectum is $2ed$.

40. a. The point closest to the pole is at θ_0.
$$r_1 = r_1(\theta_0) = \frac{ed}{1 + e\cos(0)} = \frac{ed}{1 + e}$$
The point furthest from the pole is at $\theta_0 + \pi$.
$$r_2 = r(\theta_0 + \pi) = \frac{ed}{1 + e\cos\pi} = \frac{ed}{1 - e}$$

b. The length of the major diameter is
$$r_1 + r_2 = \frac{ed}{1+e} + \frac{ed}{1-e} = \frac{ed - e^2d}{1-e^2} + \frac{ed + e^2d}{1-e^2}$$
$$= \frac{2ed}{1-e^2}.$$
$$a = \frac{ed}{1-e^2}$$
$$c = ea = \frac{e^2d}{1-e^2}$$
$$b^2 = a^2 - c^2 = \left(\frac{ed}{1-e^2}\right)^2 - \left(\frac{e^2d}{1-e^2}\right)^2$$
$$= \frac{e^2d^2(1-e^2)}{(1-e^2)^2} = \frac{e^2d^2}{1-e^2}$$
$$b = \frac{ed}{\sqrt{1-e^2}}$$
The length of the minor diameter is $\frac{2ed}{\sqrt{1-e^2}}$.

41. $a + c = 183$, $a - c = 17$
$2a = 200$, $a = 100$
$2c = 166$, $c = 83$
$$e = \frac{c}{a} = 0.83$$

42. $a = \frac{185.8}{2} = 92.9$,
$c = ea = (0.0167)92.9 = 1.55143$
Perihelion $= a - c \approx 91.3$ million miles

43. Let sun lie at the pole and the axis of the parabola lie on the pole so that the parabola opens to the left. Then the path is described by the equation $r = \frac{d}{1 + \cos\theta}$. Substitute $(100, 120^\circ)$ into the equation and solve for d.
$$100 = \frac{d}{1 + \cos 120^\circ}$$
$d = 50$
The closest distance occurs when $\theta = 0^\circ$.
$$r = \frac{50}{1 + \cos 0^\circ} = 25 \text{ million miles}$$

44. a. $4 = \dfrac{d}{1+\cos\left(\frac{\pi}{2}-\theta_0\right)}$ $\qquad 3 = \dfrac{d}{1+\cos\left(\frac{\pi}{4}-\theta_0\right)}$

$$4+4\left(\cos\frac{\pi}{2}\cos\theta_0+\sin\frac{\pi}{2}\sin\theta_0\right)=d \qquad 3+3\left(\cos\frac{\pi}{4}\cos\theta_0+\sin\frac{\pi}{4}\sin\theta_0\right)=d$$

$$d=4+4\sin\theta_0 \qquad d=3+\frac{3\sqrt{2}}{2}\cos\theta_0+\frac{3\sqrt{2}}{2}\sin\theta_0$$

$$4+4\sin\theta_0=3+\frac{3\sqrt{2}}{2}\cos\theta_0+\frac{3\sqrt{2}}{2}\sin\theta_0$$

$$\frac{3\sqrt{2}}{2}\cos\theta_0+\left(\frac{3\sqrt{2}}{2}-4\right)\sin\theta_0-1=0$$

$$3\sqrt{2}\cos\theta_0+\left(3\sqrt{2}-8\right)\sin\theta_0-2=0$$

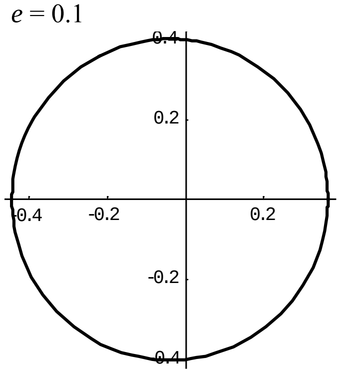
$$4.24\cos\theta_0-3.76\sin\theta_0-2=0$$

b. A graph shows that a root lies near 0.5. Using Newton's Method, $\theta_0 \approx 0.485$.
$d=4+4\sin\theta_0 \approx 5.86$

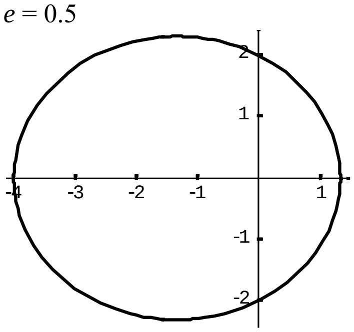
c. The closest the comet gets to the sun is $r=\dfrac{d}{1+\cos(\theta_0-\theta_0)}=\dfrac{d}{2}\approx 2.93$ AU

45. $x=\dfrac{4e}{1+e\cos t}\cos t,\ y=\dfrac{4e}{1+e\cos t}\sin t$

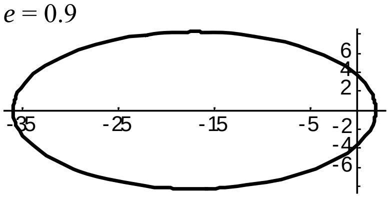
$e = 0.1$



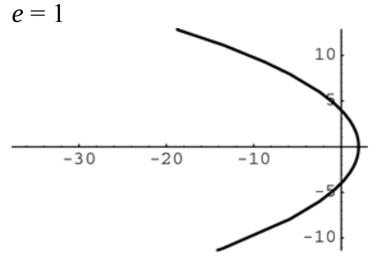
$e = 0.5$



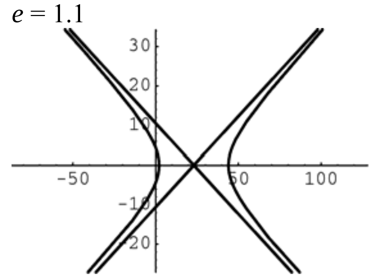
$e = 0.9$



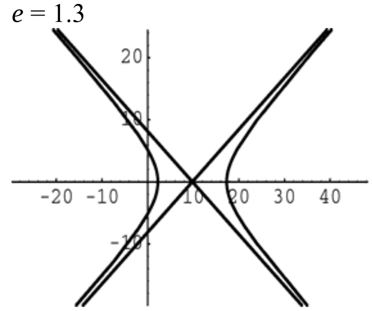
$e = 1$



$e = 1.1$



$e = 1.3$



10.6 Concepts Review

1. limaçon

2. cardioid

3. rose; odd; even

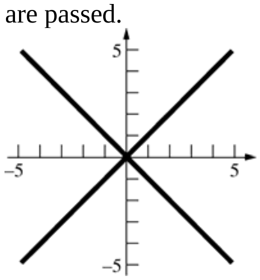
4. spiral

Problem Set 10.6

1. $\theta^2 - \frac{\pi^2}{16} = 0$

$\theta = \pm\frac{\pi}{4}$

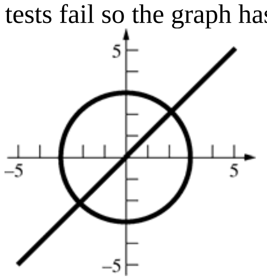
Changing $\theta \to -\theta$ or $r \to -r$ yields an equivalent set of equations. Therefore all 3 tests are passed.



2. $(r-3)\left(\theta - \frac{\pi}{4}\right) = 0$

$r = 3$ or $\theta = \frac{\pi}{4}$

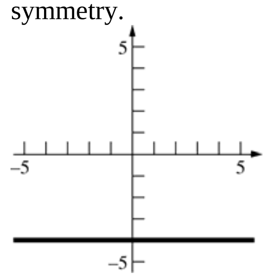
$\theta = \theta_0$ defines a line through the pole. Since a line forms an angle of π radians, changing $\theta \to \pi + \theta$ results in an equivalent set of equations, thus passing test 3. The other two tests fail so the graph has only origin symmetry.



3. $r \sin\theta + 4 = 0$

$r = -\frac{4}{\sin\theta}$

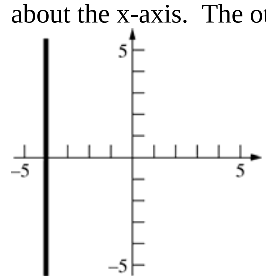
Since $\sin(-\theta) = -\sin\theta$, test 2 is passed. The other two tests fail so the graph has only y-axis symmetry.



4. $r = -4 \sec\theta$

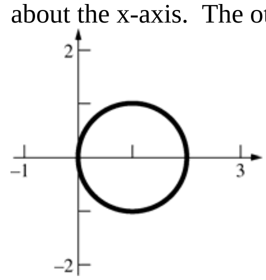
$r = -\frac{4}{\cos\theta}$

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



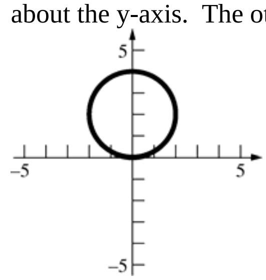
5. $r = 2\cos\theta$

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



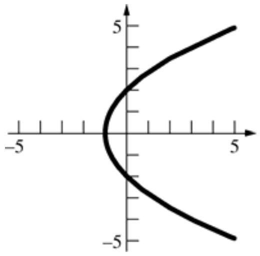
6. $r = 4\sin\theta$

Since $\sin(-\theta) = -\sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



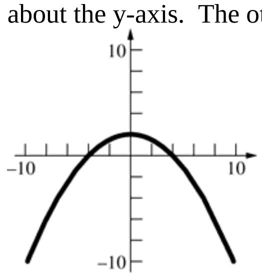
7. $r = \dfrac{2}{1-\cos\theta}$

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



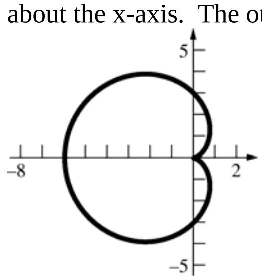
8. $r = \dfrac{4}{1+\sin\theta}$

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



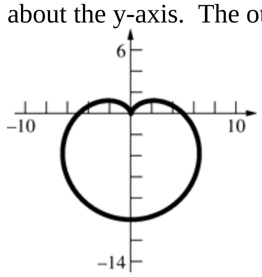
9. $r = 3 - 3\cos\theta$ (cardioid)

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



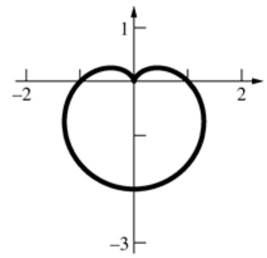
10. $r = 5 - 5\sin\theta$ (cardioid)

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



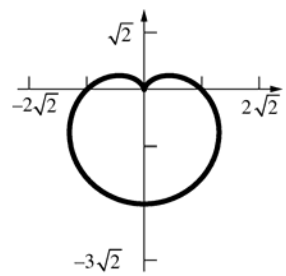
11. $r = 1 - 1\sin\theta$ (cardioid)

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



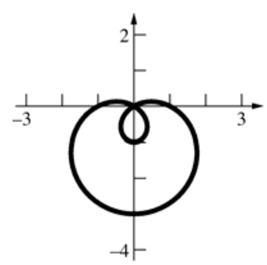
12. $r = \sqrt{2} - \sqrt{2}\sin\theta$ (cardioid)

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



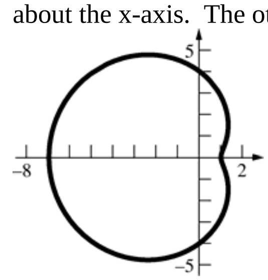
13. $r = 1 - 2\sin\theta$ (limaçon)

Since $\sin(\pi-\theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



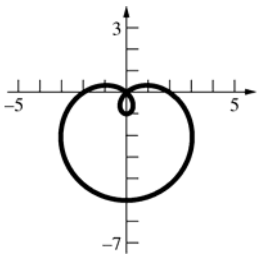
14. $r = 4 - 3\cos\theta$ (limaçon)

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



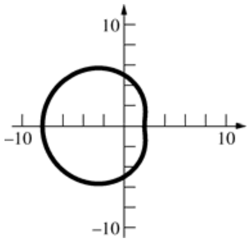
15. $r = 2 - 3\sin\theta$ (limaçon)

Since $\sin(\pi - \theta) = \sin\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



16. $r = 5 - 3\cos\theta$ (limaçon)

Since $\cos(-\theta) = \cos\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



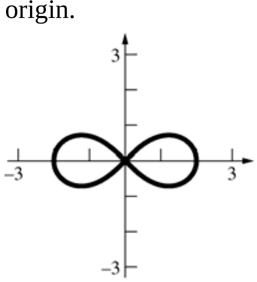
17. $r^2 = 4\cos 2\theta$ (lemniscate)

$r = \pm 2\sqrt{\cos 2\theta}$

Since $\cos(-2\theta) = \cos 2\theta$ and

$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$

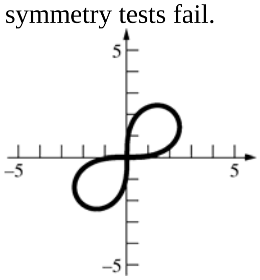
the graph is symmetric about both axes and the origin.



18. $r^2 = 9\sin 2\theta$ (lemniscate)

$r = \pm 3\sqrt{\sin(2\theta)}$

Since $\sin(2(\pi + \theta)) = \sin(2\pi + 2\theta) = \sin 2\theta$, the graph is symmetric about the origin. The other symmetry tests fail.



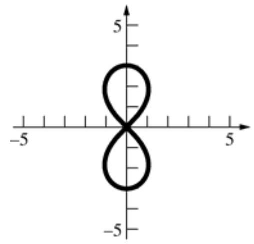
19. $r^2 = -9\cos 2\theta$ (lemniscate)

$r = \pm 3\sqrt{-\cos 2\theta}$

Since $\cos(-2\theta) = \cos 2\theta$ and

$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$

the graph is symmetric about both axes and the origin.



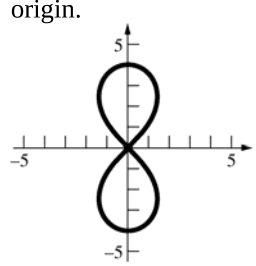
20. $r^2 = -16\cos 2\theta$ (lemniscate)

$r = \pm 4\sqrt{-\cos 2\theta}$

Since $\cos(-2\theta) = \cos 2\theta$ and

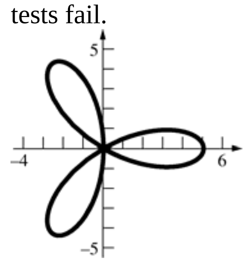
$\cos(2(\pi - \theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$

the graph is symmetric about both axes and the origin.



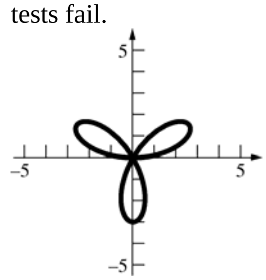
21. $r = 5\cos 3\theta$ (three-leaved rose)

Since $\cos(-3\theta) = \cos(3\theta)$, the graph is symmetric about the x-axis. The other symmetry tests fail.

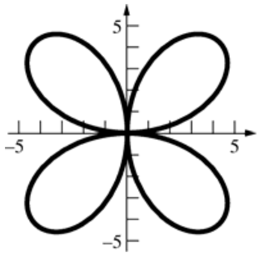


22. $r = 3\sin 3\theta$ (three-leaved rose)

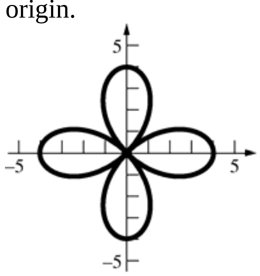
Since $\sin(-3\theta) = -\sin(3\theta)$, the graph is symmetric about the y-axis. The other symmetry tests fail.



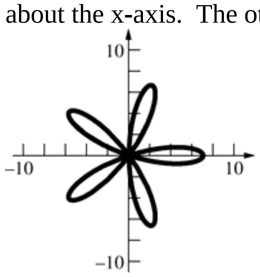
23. $r = 6\sin 2\theta$ (four-leaved rose)
Since
$$\sin(2(\pi-\theta)) = \sin(2\pi - 2\theta) = \sin(-2\theta) = -\sin(2\theta)$$
and $\sin(-2\theta) = -\sin(2\theta)$, the graph is symmetric about both axes and the origin.



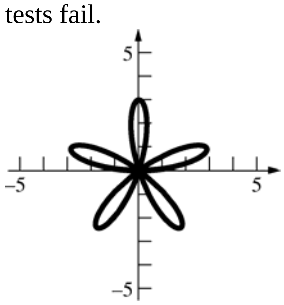
24. $r = 4\cos 2\theta$ (four-leaved rose)
Since $\cos(-2\theta) = \cos 2\theta$ and
$\cos(2(\pi-\theta)) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$
the graph is symmetric about both axes and the origin.



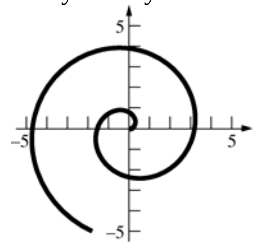
25. $r = 7\cos 5\theta$ (five-leaved rose)
Since $\cos(-5\theta) = \cos 5\theta$, the graph is symmetric about the x-axis. The other symmetry tests fail.



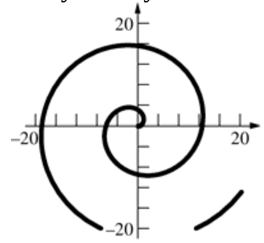
26. $r = 3\sin 5\theta$ (five-leaved rose)
Since $\sin(-5\theta) = -\sin 5\theta$, the graph is symmetric about the y-axis. The other symmetry tests fail.



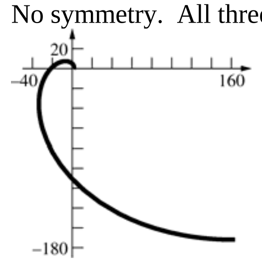
27. $r = \frac{1}{2}\theta, \theta \ge 0$ (spiral of Archimedes)
No symmetry. All three tests fail.



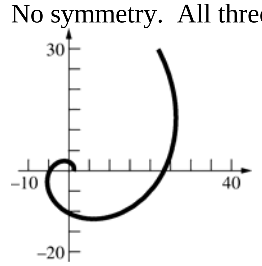
28. $r = 2\theta, \theta \ge 0$ (spiral of Archimedes)
No symmetry. All three tests fail.



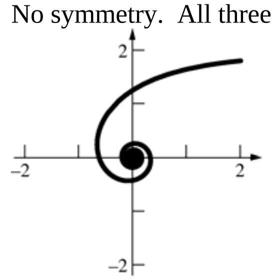
29. $r = e^{\theta}, \theta \ge 0$ (logarithmic spiral)
No symmetry. All three tests fail.



30. $r = e^{\theta/2}, \theta \ge 0$ (logarithmic spiral)
No symmetry. All three tests fail.

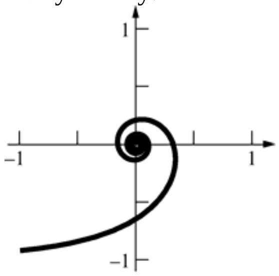


31. $r = \frac{2}{\theta}, \theta > 0$ (reciprocal spiral)
No symmetry. All three tests fail.

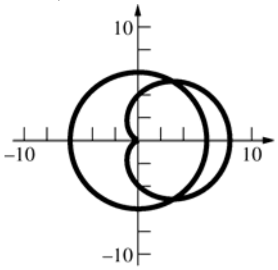


32. $r = -\dfrac{1}{\theta}, \theta > 0$ (reciprocal spiral)

No symmetry. All three tests fail.



33. $r = 6,\ r = 4 + 4\cos\theta$



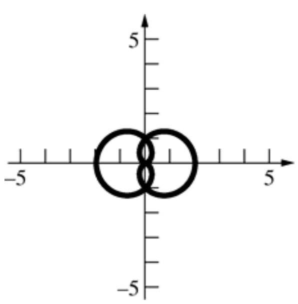
$6 = 4 + 4\cos\theta$

$\cos\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{3}, \theta = \dfrac{5\pi}{3}$

$\left(6, \dfrac{\pi}{3}\right), \left(6, \dfrac{5\pi}{3}\right)$

34. $r = 1 - \cos\theta, r = 1 + \cos\theta$



$1 - \cos\theta = 1 + \cos\theta$

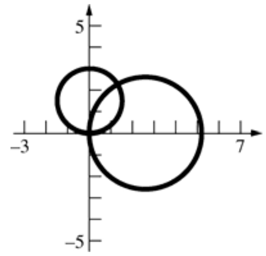
$\cos\theta = 0$

$\theta = \dfrac{\pi}{2}, \theta = \dfrac{3\pi}{2}$

$\left(1, \dfrac{\pi}{2}\right), \left(1, \dfrac{3\pi}{2}\right)$

(0, 0) is also a solution since both graphs include the pole.

35. $r = 3\sqrt{3}\cos\theta, r = 3\sin\theta$



$3\sqrt{3}\cos\theta = 3\sin\theta$

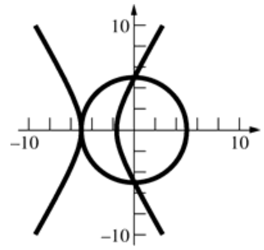
$\tan\theta = \sqrt{3}$

$\theta = \dfrac{\pi}{3}, \theta = \dfrac{4\pi}{3}$

$\left(\dfrac{3\sqrt{3}}{2}, \dfrac{\pi}{3}\right) = \left(-\dfrac{3\sqrt{3}}{2}, \dfrac{4\pi}{3}\right)$

(0, 0) is also a solution since both graphs include the pole.

36. $r = 5, r = \dfrac{5}{1 - 2\cos\theta}$



$5 = \dfrac{5}{1 - 2\cos\theta}$

$\cos\theta = 0$

$\theta = \dfrac{\pi}{2}, \theta = \dfrac{3\pi}{2}$; $\left(5, \dfrac{\pi}{2}\right), \left(5, \dfrac{3\pi}{2}\right)$

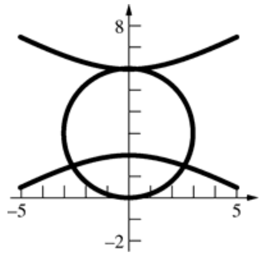
Note that r = –5 is equivalent to r = 5.

$-5 = \dfrac{5}{1 - 2\cos\theta}$

$\cos\theta = 1$

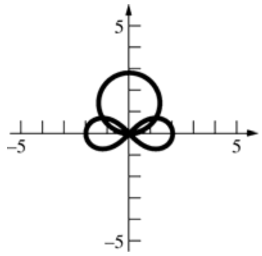
$\theta = 0$; (–5, 0)

37. $r = 6\sin\theta, r = \dfrac{6}{1 + 2\sin\theta}$



$$6\sin\theta = \frac{6}{1+2\sin\theta}$$
$$12\sin^2\theta + 6\sin\theta - 6 = 0$$
$$6(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\sin\theta = \frac{1}{2}, \sin\theta = -1$$
$$\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}, \theta = \frac{3\pi}{2}$$
$$\left(3, \frac{\pi}{6}\right), \left(3, \frac{5\pi}{6}\right), \left(-6, \frac{3\pi}{2}\right) \text{ or } \left(6, \frac{\pi}{2}\right)$$

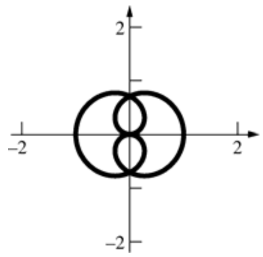
38. $r^2 = 4\cos 2\theta, r = 2\sqrt{2}\sin\theta$



$$4\cos 2\theta = \left(2\sqrt{2}\sin\theta\right)^2$$
$$4 - 8\sin^2\theta = 8\sin^2\theta$$
$$\sin^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \pm\frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}, \theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$$
$$\left(\sqrt{2}, \frac{\pi}{6}\right) = \left(-\sqrt{2}, \frac{7\pi}{6}\right),$$
$$\left(\sqrt{2}, \frac{5\pi}{6}\right) = \left(-\sqrt{2}, \frac{11\pi}{6}\right)$$

(0, 0) is also a solution since both graphs includes the pole.

39. Consider $r = \cos\frac{1}{2}\theta$.



The graph is clearly symmetric with respect to the y-axis.

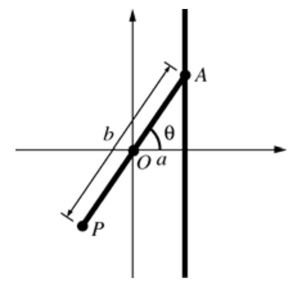
Substitute (r, θ) by $(-r, -\theta)$.

$$-r = \cos\left(-\frac{1}{2}\theta\right) = \cos\frac{1}{2}\theta$$
$$r = -\cos\frac{1}{2}\theta$$

Substitute (r, θ) by $(r, \pi - \theta)$

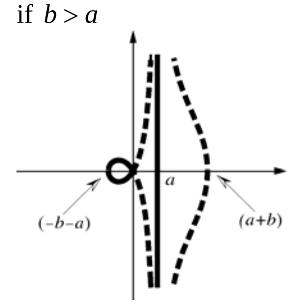
$$r = \cos\frac{1}{2}(\pi - \theta) = \cos\frac{1}{2}\pi\cos\frac{1}{2}\theta + \sin\frac{1}{2}\pi\sin\frac{1}{2}\theta$$
$$= \sin\frac{1}{2}\theta$$
$$r = \sin\frac{1}{2}\theta$$

40. Consider the following figure.

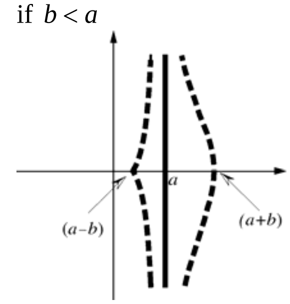


$$r = \frac{a}{\cos\theta} - b$$
$$r\cos\theta = a - b\cos\theta$$
$$x = a - b\cos\theta$$
$$xr = ar - br\cos\theta$$
$$(x-a)r = -bx$$
$$(x-a)^2 r^2 = b^2 x^2$$
$$(x-a)^2(x^2 + y^2) = b^2 x^2$$
$$y^2 = \frac{b^2 x^2}{(x-a)^2} - x^2$$

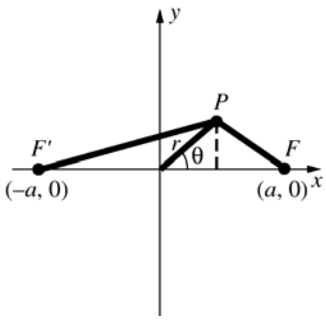
if $b > a$



if $b < a$



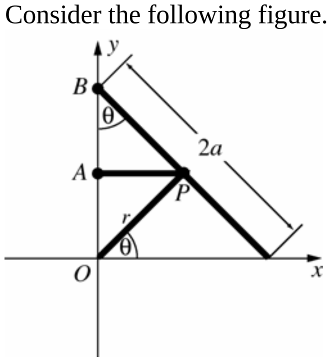
41.



$$|PF| = \sqrt{(a - r\cos\theta)^2 + (r\sin\theta)^2}$$
$$= \sqrt{r^2 + a^2 - 2ar\cos\theta}$$
$$|PF'| = \sqrt{(a + r\cos\theta)^2 + (r\sin\theta)^2}$$
$$= \sqrt{r^2 + a^2 + 2ar\cos\theta}$$
$$|PF||PF'| = \sqrt{(r^2 + a^2)^2 - 4a^2r^2\cos^2\theta} = a^2$$
$$(r^4 + 2a^2r^2 + a^4) - 4a^2r^2\cos^2\theta = a^4$$
$$r^4 - 4a^2r^2\cos^2\theta + 2a^2r^2 = 0$$
$$r^2(r^2 - 2a^2(2\cos^2\theta - 1)) = 0$$
$$r^2 - 2a^2(2\cos^2\theta - 1) = 0$$
$$r^2 = 2a^2(1 + \cos 2\theta - 1)$$
$$r^2 = 2a^2\cos 2\theta$$

This is the equation of a lemniscate.

42. Consider the following figure.



Then $\tan\theta = \dfrac{\overline{AP}}{\overline{BA}} = \dfrac{r\cos\theta}{2a\cos\theta - r\sin\theta}$

$$\frac{\sin\theta}{\cos\theta} = \frac{r\cos\theta}{2a\cos\theta - r\sin\theta}$$
$$2a\sin\theta\cos\theta - r\sin^2\theta = r\cos^2\theta$$
$$r\cos^2\theta + r\sin^2\theta = 2a\sin\theta\cos\theta$$
$$r = a\sin 2\theta$$

This is a polar equation for a four-leaved rose.

43. a. $y = 45$

$r\sin\theta = 45$

$$r = \frac{45}{\sin\theta}$$

b. $x^2 + y^2 = 36$

$r^2 = 36$

$r = 6$

c. $x^2 - y^2 = 1$

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2 = \frac{1}{\cos 2\theta}$$
$$r = \pm\frac{1}{\sqrt{\cos 2\theta}}$$

d. $4xy = 1$

$$4r^2\cos\theta\sin\theta = 1$$
$$r^2 = \frac{1}{2\sin 2\theta}$$
$$r = \pm\frac{1}{\sqrt{2\sin 2\theta}}$$

e. $y = 3x + 2$

$r\sin\theta = 3r\cos\theta + 2$

$r(\sin\theta - 3\cos\theta) = 2$

$$r = \frac{2}{\sin\theta - 3\cos\theta}$$

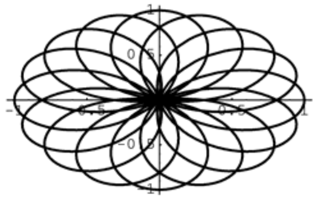
f. $3x^2 + 4y = 2$

$$3r^2\cos^2\theta + 4r\sin\theta = 2$$
$$(3\cos^2\theta)r^2 + (4\sin\theta)r - 2 = 0$$
$$r = \frac{-4\sin\theta \pm \sqrt{16\sin^2\theta + 24\cos^2\theta}}{6\cos^2\theta}$$
$$r = \frac{-2\sin\theta \pm \sqrt{4\sin^2\theta + 6\cos^2\theta}}{3\cos^2\theta}$$

g. $x^2 + 2x + y^2 - 4y - 25 = 0$

$$r^2 + 2r\cos\theta - 4r\sin\theta - 25 = 0$$
$$r^2 + (2\cos\theta - 4\sin\theta)r - 25 = 0$$
$$r = \frac{-2\cos\theta + 4\sin\theta \pm \sqrt{(2\cos\theta - 4\sin\theta)^2 + 100}}{2}$$
$$r = -\cos\theta + 2\sin\theta \pm \sqrt{(\cos\theta - 2\sin\theta)^2 + 25}$$

44.



The curve repeats itself after period p if $f(\theta + p) = f(\theta)$.

$$\cos\left(\frac{8(\theta+p)}{5}\right) = \cos\left(\frac{8\theta}{5} + \frac{p}{5}\right)$$

We need $\frac{p}{5} = 2\pi$.

45. **a.** VII

b. I

c. VIII

d. III

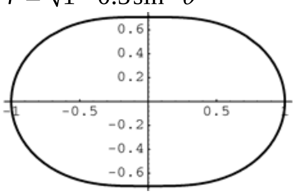
e. V

f. II

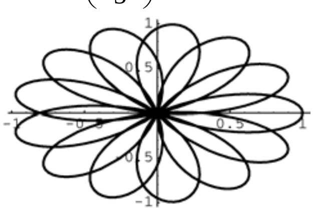
g. VI

h. IV

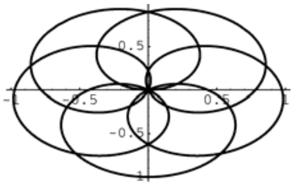
46. $r = \sqrt{1 - 0.5\sin^2\theta}$



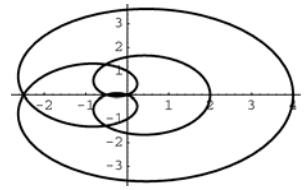
47. $r = \cos\left(\frac{13\theta}{5}\right)$



48. $r = \sin\left(\frac{5\theta}{7}\right)$



49. $r = 1 + 3\cos\left(\frac{\theta}{3}\right)$



50. **a.** The graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is the rotation of the graph of $r = 1 + \sin\theta$ by $\frac{\pi}{3}$ counter-clockwise about the pole. The graph of $r = 1 + \sin\left(\theta + \frac{\pi}{3}\right)$ is the rotation of the graph of $r = 1 + \sin\theta$ by $\frac{\pi}{3}$ clockwise about the pole.

b. $r = 1 - \sin\theta = 1 + \sin(\theta - \pi)$
The graph of $r = 1 + \sin\theta$ is the rotation of the graph of $r = 1 - \sin\theta$ by π about the pole.

c. $r = 1 + \cos\theta = 1 + \sin\left(\theta + \frac{\pi}{2}\right)$
The graph of $r = 1 + \sin\theta$ is the rotation of the graph of $r = 1 + \cos\theta$ by $\frac{\pi}{2}$ counter-clockwise about the pole.

d. The graph of $r = f(\theta)$ is the rotation of the graph of $r = f(\theta - \alpha)$ by a clockwise about the pole.

51. **a.** The graph for $\phi = 0$ is the graph for $\phi \neq 0$ rotated by ϕ counterclockwise about the pole.

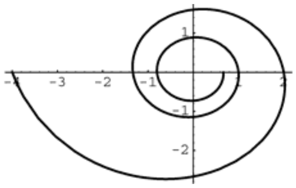
b. As n increases, the number of "leaves" increases.

c. If $|a| > |b|$, the graph will not pass through the pole and will not "loop." If $|b| < |a|$, the graph will pass through the pole and will have $2n$ "loops" (n small "loops" and n large "loops"). If $|a| = |b|$, the graph passes through the pole and will have n "loops." If $ab \neq 0, n > 1$, and $\phi = 0$, the graph will be symmetric about $\theta = \frac{\pi}{n}k$, where $k = 0, n-1$.

52. The number of loops is $2n$.

53. The spiral will unwind clockwise for $c < 0$. The spiral will unwind counter-clockwise for $c > 0$.

54. This is for $c = 4\pi$.



The spiral will wind in the counter-clockwise direction.

55. **a.** III

b. IV

c. I

d. II

e. VI

f. V

10.7 Concepts Review

1. $\frac{1}{2}r^2\theta$

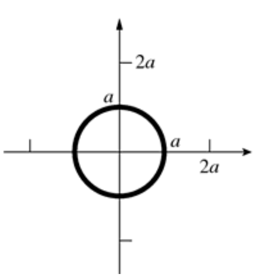
2. $\frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta$

3. $\frac{1}{2}\int_{0}^{2\pi}(2+2\cos\theta)^2\,d\theta$

4. $f(\theta) = 0$

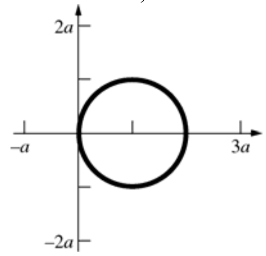
Problem Set 10.7

1. $r = a,\ a > 0$



$$A = \frac{1}{2}\int_0^{2\pi} a^2\,d\theta = \pi a^2$$

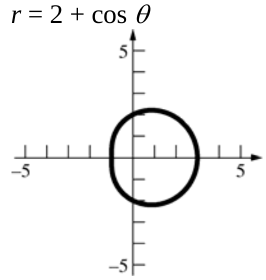
2. $r = 2a\cos\theta,\ a > 0$



$$A = \frac{1}{2}\int_0^{\pi}(2a\cos\theta)^2\,d\theta = 2a^2\int_0^{\pi}\cos^2\theta\,d\theta$$

$$= a^2\int_0^{\pi}(1+\cos 2\theta)d\theta = a^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \pi a^2$$

3. $r = 2 + \cos\theta$



$$A = \frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^2\,d\theta$$

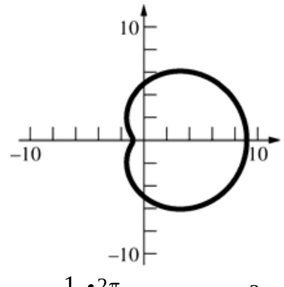
$$= \frac{1}{2}\int_0^{2\pi}(4+4\cos\theta+\cos^2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[4+4\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{9}{2}+4\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\left[\frac{9}{2}\theta+4\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{9}{2}\pi$$

4. $r = 5 + 4\cos\theta$



$$A = \frac{1}{2}\int_0^{2\pi}(5+4\cos\theta)^2\,d\theta$$

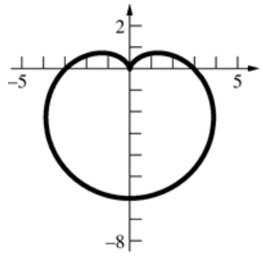
$$= \frac{1}{2}\int_0^{2\pi}(25+40\cos\theta+16\cos^2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}[25+40\cos\theta+8(1+\cos 2\theta)]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(33+40\cos\theta+8\cos 2\theta)d\theta$$

$$= \frac{1}{2}[33\theta+40\sin\theta+4\sin 2\theta]_0^{2\pi} = 33\pi$$

5. $r = 3 - 3\sin\theta$



$$A = \frac{1}{2}\int_0^{2\pi}(3 - 3\sin\theta)^2\,d\theta$$

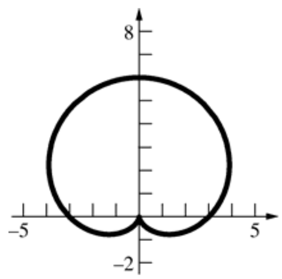
$$= \frac{1}{2}\int_0^{2\pi}(9 - 18\sin\theta + 9\sin^2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[9 - 18\sin\theta + \frac{9}{2}(1 - \cos 2\theta)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{27}{2} - 18\sin\theta - \frac{9}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\left[\frac{27}{2}\theta + 18\cos\theta - \frac{9}{4}\sin 2\theta\right]_0^{2\pi} = \frac{27}{2}\pi$$

6.



$$A = \frac{1}{2}\int_0^{2\pi}(3 + 3\sin\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(9 + 18\sin\theta + 9\sin^2\theta)\,d\theta$$

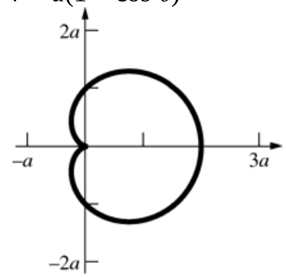
$$= \frac{1}{2}\int_0^{2\pi}\left[9 + 18\sin\theta + \frac{9}{2}(1 - \cos 2\theta)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{27}{2} + 18\sin\theta - \frac{9}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\left[\frac{27}{2}\theta - 18\cos\theta - \frac{9}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{27}{2}\pi$$

7. $r = a(1 + \cos\theta)$



$$A = \frac{1}{2}\int_0^{2\pi}[a(1 + \cos\theta)]^2\,d\theta$$

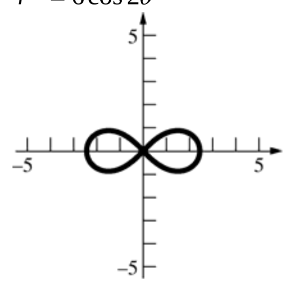
$$= \frac{a^2}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left[1 + 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi a^2}{2}$$

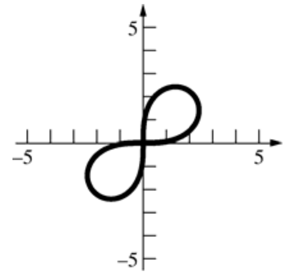
8. $r^2 = 6\cos 2\theta$



$$A = 2\cdot\frac{1}{2}\int_{-\pi/4}^{\pi/4} 6\cos 2\theta\,d\theta = 6\int_{-\pi/4}^{\pi/4}\cos 2\theta\,d\theta$$

$$= 3[\sin 2\theta]_{-\pi/4}^{\pi/4} = 6$$

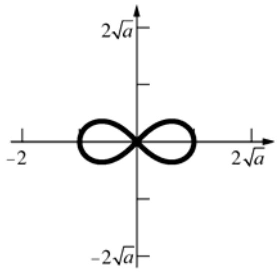
9. $r^2 = 9\sin 2\theta$



$$A = 2\cdot\frac{1}{2}\int_0^{\pi/2} 9\sin 2\theta\,d\theta = 9\int_0^{\pi/2}\sin 2\theta\,d\theta$$

$$= \frac{9}{2}[-\cos 2\theta]_0^{\pi/2} = 9$$

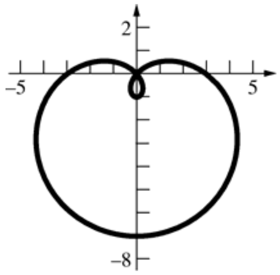
10. $r^2 = a\cos 2\theta$



$$A = 2\cdot\frac{1}{2}\int_{-\pi/4}^{\pi/4} a\cos 2\theta\, d\theta = a\int_{-\pi/4}^{\pi/4} \cos 2\theta\, d\theta$$

$$= \frac{a}{2}[\sin 2\theta]_{-\pi/4}^{\pi/4} = a$$

11. $r = 3 - 4\sin\theta$



$3 - 4\sin\theta = 0,\ \theta = \sin^{-1}\frac{3}{4}$

$$A = 2\cdot\frac{1}{2}\int_{\sin^{-1}\frac{3}{4}}^{\pi/2}(3-4\sin\theta)^2\,d\theta$$

$$= \int_{\sin^{-1}\frac{3}{4}}^{\pi/2}(9-24\sin\theta+16\sin^2\theta)d\theta$$

$$= \int_{\sin^{-1}\frac{3}{4}}^{\pi/2}[9-24\sin\theta+8(1-\cos 2\theta)]d\theta$$

$$= \int_{\sin^{-1}\frac{3}{4}}^{\pi/2}(17-24\sin\theta-8\cos 2\theta)d\theta$$

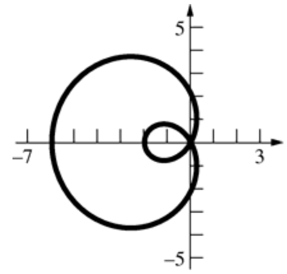
$$= [17\theta+24\cos\theta-4\sin 2\theta]_{\sin^{-1}\frac{3}{4}}^{\pi/2}$$

$$= [17\theta+24\cos\theta-8\sin\theta\cos\theta]_{\sin^{-1}\frac{3}{4}}^{\pi/2}$$

$$= \frac{17\pi}{2} - \left[17\sin^{-1}\frac{3}{4} + 24\left(\frac{\sqrt{7}}{4}\right) - 8\left(\frac{3}{4}\right)\left(\frac{\sqrt{7}}{4}\right)\right]$$

$$= \frac{17\pi}{2} - 17\sin^{-1}\frac{3}{4} - \frac{9\sqrt{7}}{2}$$

12. $r = 2 - 4\cos\theta$



$2 - 4\cos\theta = 0,\ \theta = \frac{\pi}{3}$

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/3}(2-4\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/3}(4-16\cos\theta+16\cos^2\theta)d\theta$$

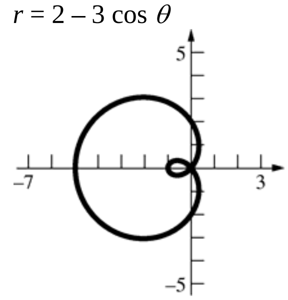
$$= \int_0^{\pi/3}[4-16\cos\theta+8(1+\cos 2\theta)]d\theta$$

$$= \int_0^{\pi/3}(12-16\cos\theta+8\cos 2\theta)d\theta$$

$$= \left[12\theta-16\sin\theta+4\sin 2\theta\right]_0^{\pi/3}$$

$$= 4\pi - 6\sqrt{3}$$

13. $r = 2 - 3\cos\theta$



$2 - 3\cos\theta = 0,\ \theta = \cos^{-1}\frac{2}{3}$

$$A = 2\cdot\frac{1}{2}\int_{\cos^{-1}\frac{2}{3}}^{\pi}(2-3\cos\theta)^2\,d\theta$$

$$= \int_{\cos^{-1}\frac{2}{3}}^{\pi}(4-12\cos\theta+9\cos^2\theta)d\theta$$

$$= \int_{\cos^{-1}\frac{2}{3}}^{\pi}\left[4-12\cos\theta+\frac{9}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \int_{\cos^{-1}\frac{2}{3}}^{\pi}\left(\frac{17}{2}-12\cos\theta+\frac{9}{2}\cos 2\theta\right)d\theta$$

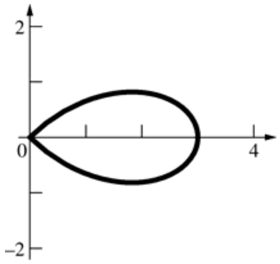
$$= \left[\frac{17}{2}\theta-12\sin\theta+\frac{9}{4}\sin 2\theta\right]_{\cos^{-1}\frac{2}{3}}^{\pi}$$

$$= \left[\frac{17\theta}{2}-12\sin\theta+\frac{9}{2}\sin\theta\cos\theta\right]_{\cos^{-1}\frac{2}{3}}^{\pi}$$

$$= \frac{17\pi}{2} - \left[\frac{17}{2}\cos^{-1}\frac{2}{3} - 12\left(\frac{\sqrt{5}}{3}\right) + \frac{9}{2}\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)\right]$$

$$= \frac{17\pi}{2} - \frac{17}{2}\cos^{-1}\frac{2}{3} + 3\sqrt{5}$$

14. $r = 3\cos 2\theta$

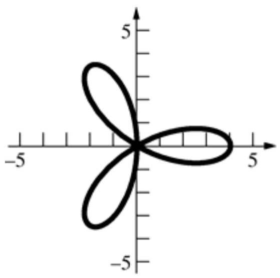


$$A = 2\cdot\frac{1}{2}\int_0^{\pi/4}(3\cos 2\theta)^2\,d\theta = 9\int_0^{\pi/4}\cos^2 2\theta\,d\theta$$

$$= 9\int_0^{\pi/4}\frac{1}{2}(1+\cos 4\theta)d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{9\pi}{8}$$

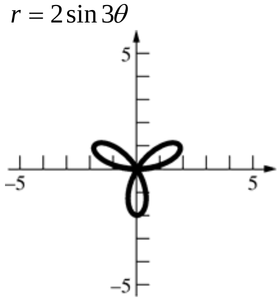
15. $r = 4\cos 3\theta$



$$A = 6\cdot\frac{1}{2}\int_0^{\pi/6}(4\cos 3\theta)^2\,d\theta = 48\int_0^{\pi/6}\cos^2 3\theta\,d\theta$$

$$= 24\int_0^{\pi/6}(1+\cos 6\theta)d\theta = 24\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 4\pi$$

16. $r = 2\sin 3\theta$



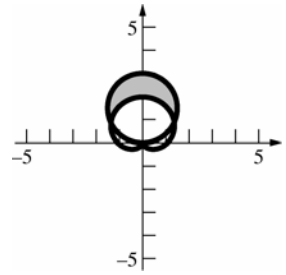
$$A = 3\cdot\frac{1}{2}\int_0^{\pi/3}(2\sin 3\theta)^2\,d\theta = 6\int_0^{\pi/3}\sin^2 3\theta\,d\theta$$

$$= 3\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta$$

$$= 3\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \pi$$

17. $A = \frac{1}{2}\int_0^{2\pi}100\,d\theta - \frac{1}{2}\int_0^{2\pi}49\,d\theta = 51\pi$

18. $r = 3\sin\theta, r = 1+\sin\theta$



Solve for the θ-coordinate of the first intersection point.

$3\sin\theta = 1+\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

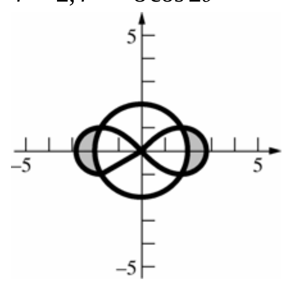
$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}[(3\sin\theta)^2 - (1+\sin\theta)^2]d\theta$$

$$= \int_{\pi/6}^{\pi/2}(8\sin^2\theta - 2\sin\theta - 1)d\theta$$

$$= \int_{\pi/6}^{\pi/2}(3 - 4\cos 2\theta - 2\sin\theta)d\theta$$

$$= [3\theta - 2\sin 2\theta + 2\cos\theta]_{\pi/6}^{\pi/2} = \pi$$

19. $r = 2, r^2 = 8\cos 2\theta$



Solve for the θ-coordinate of the first intersection point.

$4 = 8\cos 2\theta$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}$

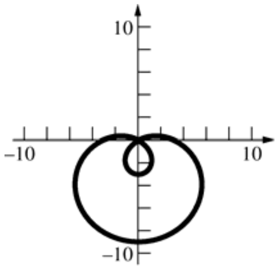
$\theta = \frac{\pi}{6}$

$$A = 4\cdot\frac{1}{2}\int_0^{\pi/6}(8\cos 2\theta - 4)d\theta$$

$$= 2[4\sin 2\theta - 4\theta]_0^{\pi/6}$$

$$= 4\sqrt{3} - \frac{4\pi}{3}$$

20. $r = 3 - 6\sin\theta$



Let A_1 be the area inside the large loop and let A_2 be the area inside the small loop.

$$A_1 = 2 \cdot \frac{1}{2}\int_{-\pi/2}^{\pi/6}(3-6\sin\theta)^2\,d\theta$$

$$= \int_{-\pi/2}^{\pi/6}(9-36\sin\theta+36\sin^2\theta)d\theta$$

$$= \int_{-\pi/2}^{\pi/6}(27-36\sin\theta-18\cos 2\theta)d\theta$$

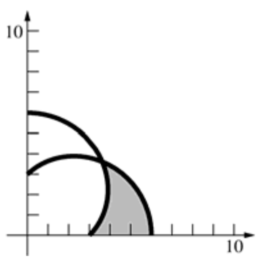
$$= \left[27\theta+36\cos\theta-9\sin 2\theta\right]_{-\pi/2}^{\pi/6} = 18\pi + \frac{27\sqrt{3}}{2}$$

$$A_2 = 2 \cdot \frac{1}{2}\int_{\pi/6}^{\pi/2}(3-6\sin\theta)^2\,d\theta$$

$$= [27\theta+36\cos\theta-9\sin 2\theta]_{\pi/6}^{\pi/2} = 9\pi - \frac{27\sqrt{3}}{2}$$

$$A = A_1 - A_2 = 9\pi + 27\sqrt{3}$$

21. $r = 3+3\cos\theta,\ r = 3+3\sin\theta$



Solve for the θ-coordinate of the intersection point.

$3+3\cos\theta = 3+3\sin\theta$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$$A = \frac{1}{2}\int_0^{\pi/4}[(3+3\cos\theta)^2-(3+3\sin\theta)^2]d\theta$$

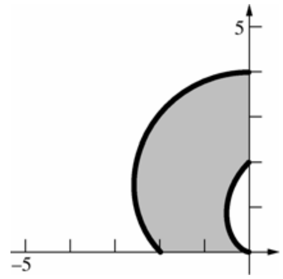
$$= \frac{1}{2}\int_0^{\pi/4}(18\cos\theta+9\cos^2\theta-18\sin\theta-9\sin^2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}(18\cos\theta-18\sin\theta+9\cos 2\theta)d\theta$$

$$= \frac{1}{2}\left[18\sin\theta+18\cos\theta+\frac{9}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= 9\sqrt{2} - \frac{27}{4}$$

22. $r = 2+2\sin\theta,\ r = 2+2\cos\theta$



$$A = \frac{1}{2}\int_{\pi/2}^{\pi}[(2+2\sin\theta)^2-(2+2\cos\theta)^2]d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi}(8\sin\theta+4\sin^2\theta-8\cos\theta-4\cos^2\theta)d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi}(8\sin\theta-8\cos\theta-4\cos 2\theta)d\theta$$

$$= \frac{1}{2}\left[-8\cos\theta-8\sin\theta-2\sin 2\theta\right]_{\pi/2}^{\pi} = 8$$

23. a. $f(\theta) = 2\cos\theta,\ f'(\theta) = -2\sin\theta$

$$m = \frac{(2\cos\theta)\cos\theta+(-2\sin\theta)\sin\theta}{-(2\cos\theta)\sin\theta+(-2\sin\theta)\cos\theta}$$

$$= \frac{2\cos^2\theta-2\sin^2\theta}{-4\cos\theta\sin\theta} = \frac{\cos 2\theta}{-\sin 2\theta}$$

At $\theta = \frac{\pi}{3}$, $m = \dfrac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}}$.

b. $f(\theta) = 1+\sin\theta,\ f'(\theta) = \cos\theta$

$$m = \frac{(1+\sin\theta)\cos\theta+(\cos\theta)\sin\theta}{-(1+\sin\theta)\sin\theta+(\cos\theta)\cos\theta}$$

$$= \frac{\cos\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta-\sin\theta} = \frac{\cos\theta+\sin 2\theta}{\cos 2\theta-\sin\theta}$$

At $\theta = \frac{\pi}{3}$, $m = \dfrac{\frac{1}{2}+\frac{\sqrt{3}}{2}}{-\frac{1}{2}-\frac{\sqrt{3}}{2}} = -1$.

c. $f(\theta) = \sin 2\theta,\ f'(\theta) = 2\cos 2\theta$

$$m = \frac{(\sin 2\theta)\cos\theta+(2\cos 2\theta)\sin\theta}{-(\sin 2\theta)\sin\theta+(2\cos 2\theta)\cos\theta}$$

At $\theta = \frac{\pi}{3}$, .

$$m = \frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)+(-1)\left(\frac{\sqrt{3}}{2}\right)}{-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+(-1)\left(\frac{1}{2}\right)} = \frac{-\frac{\sqrt{3}}{4}}{-\frac{5}{4}} = \frac{\sqrt{3}}{5}$$

d. $f(\theta)=4-3\cos\theta,\ f'(\theta)=3\sin\theta$

$$m=\frac{(4-3\cos\theta)\cos\theta+(3\sin\theta)\sin\theta}{-(4-3\cos\theta)\sin\theta+(3\sin\theta)\cos\theta}$$

$$=\frac{4\cos\theta-3\cos^2\theta+3\sin^2\theta}{-4\sin\theta+6\sin\theta\cos\theta}$$

$$=\frac{4\cos\theta-3\cos 2\theta}{-4\sin\theta+3\sin 2\theta}$$

At $\theta=\frac{\pi}{3}$,

$$m=\frac{4\left(\frac{1}{2}\right)-3\left(-\frac{1}{2}\right)}{-4\left(\frac{\sqrt{3}}{2}\right)+3\left(\frac{\sqrt{3}}{2}\right)}=\frac{\frac{7}{2}}{-\frac{\sqrt{3}}{2}}=-\frac{7}{\sqrt{3}}.$$

24. $f(\theta)=a(1+\cos\theta),\ f'(\theta)=-a\sin\theta$

$$m=\frac{a(1+\cos\theta)\cos\theta+(-a\sin\theta)\sin\theta}{-a(1+\cos\theta)\sin\theta+(-a\sin\theta)\cos\theta}$$

$$=\frac{\cos\theta+\cos^2\theta-\sin^2\theta}{-\sin\theta-2\sin\theta\cos\theta}=\frac{2\cos^2\theta+\cos\theta-1}{-\sin\theta(1+2\cos\theta)}$$

a. $m=0$ when $2\cos^2\theta+\cos\theta-1=0$

$(2\cos\theta-1)(\cos\theta+1)=0.$

$\cos\theta=\frac{1}{2},\cos\theta=-1$

$\theta=\frac{\pi}{3},-\frac{\pi}{3},\theta=\pi$; when

$\theta=\pi,\ f(\theta)=0,$ so $\theta=\pi$ is the tangent line.

$\left(\frac{3a}{2},\frac{\pi}{3}\right),\left(\frac{3a}{2},-\frac{\pi}{3}\right),(0,\pi)$

b. m is undefined when $\sin\theta(1+2\cos\theta)=0$ and $2\cos^2\theta+\cos\theta-1\neq 0$.

$\theta=0,\frac{2\pi}{3},\frac{4\pi}{3}$

$(2a,0),\left(\frac{a}{2},\frac{2\pi}{3}\right),\left(\frac{a}{2},\frac{4\pi}{3}\right)$

There is no vertical tangent at $\theta=\pi$ since $\lim_{\theta\to 0}m(\theta)=0$ (see part (a)).

25. $f(\theta)=1-2\sin\theta,\ f'(\theta)=-2\cos\theta$

$$m=\frac{(1-2\sin\theta)\cos\theta+(-2\cos\theta)\sin\theta}{-(1-2\sin\theta)\sin\theta+(-2\cos\theta)\cos\theta}$$

$$=\frac{\cos\theta-4\sin\theta\cos\theta}{-\sin\theta+2\sin^2\theta-2\cos^2\theta}$$

$$=\frac{\cos\theta(1-4\sin\theta)}{-\sin\theta+2\sin^2\theta-2\cos^2\theta}$$

$m=0$ when $\cos\theta(1-4\sin\theta)=0$

$\cos\theta=0,$ or $1-4\sin\theta=0$

$\theta=\frac{\pi}{2},\theta=\frac{3\pi}{2},\theta=\sin^{-1}\left(\frac{1}{4}\right)\approx 0.25,$

$\theta=\pi-\sin^{-1}\left(\frac{1}{4}\right)\approx 2.89$

$f\left(\frac{\pi}{2}\right)=-1,\ f\left(\frac{3\pi}{2}\right)=3,\ f\left(\sin^{-1}\left(\frac{1}{4}\right)\right)=\frac{1}{2},$

$f\left(\pi-\sin^{-1}\left(\frac{1}{4}\right)\right)=\frac{1}{2}$

$\left(-1,\frac{\pi}{2}\right),\left(3,\frac{3\pi}{2}\right),\left(\frac{1}{2},0.25\right),\left(\frac{1}{2},2.89\right)$

26. Recall from Chapter 5 that $L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ for x and y functions of t and $a\le t\le b$.

$x=r\cos\theta=f(\theta)\cos\theta,\ y=r\sin\theta=f(\theta)\sin\theta$

$\frac{dx}{d\theta}=f'(\theta)\cos\theta-f(\theta)\sin\theta,\ \frac{dy}{d\theta}=f'(\theta)\sin\theta+f(\theta)\cos\theta$

$$L=\int_\alpha^\beta\sqrt{(f'(\theta)\cos\theta-f(\theta)\sin\theta)^2+(f'(\theta)\sin\theta+f(\theta)\cos\theta)^2}\,d\theta$$

$$=\int_\alpha^\beta\sqrt{[f(\theta)]^2(\sin^2\theta+\cos^2\theta)+[f'(\theta)]^2(\sin^2\theta+\cos^2\theta)}\,d\theta=\int_\alpha^\beta\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta$$

27. $f(\theta)=a(1+\cos\theta),\ f'(\theta)=-a\sin\theta$

$$L=\int_0^{2\pi}\sqrt{[a(1+\cos\theta)]^2+[-a\sin\theta]^2}\,d\theta=a\int_0^{2\pi}\sqrt{2+2\cos\theta}\,d\theta=2a\int_0^{2\pi}\sqrt{\frac{1+\cos\theta}{2}}\,d\theta=2a\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|d\theta$$

$$=2a\left[\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta-\int_\pi^{2\pi}\cos\frac{\theta}{2}\,d\theta\right]=2a\left(\left[2\sin\frac{\theta}{2}\right]_0^{\pi}-\left[2\sin\frac{\theta}{2}\right]_\pi^{2\pi}\right)=8a$$

28. $f(\theta)=e^{\theta/2},\ f'(\theta)=\frac{1}{2}e^{\theta/2}$

$$L=\int_0^{2\pi}\sqrt{[e^{\theta/2}]^2+\left[\frac{1}{2}e^{\theta/2}\right]^2}\,d\theta=\int_0^{2\pi}\sqrt{\frac{5}{4}e^{\theta}}\,d\theta=\int_0^{2\pi}\frac{\sqrt{5}}{2}e^{\theta/2}d\theta=\left[\sqrt{5}e^{\theta/2}\right]_0^{2\pi}=\sqrt{5}(e^{\pi}-1)\approx 49.51$$

29. If n is even, there are $2n$ leaves.

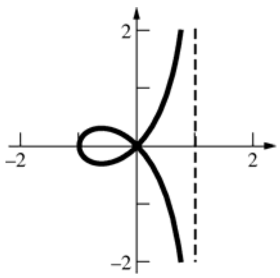
$$A=2n\frac{1}{2}\int_{-\pi/2n}^{\pi/2n}(a\cos n\theta)^2d\theta=na^2\int_{-\pi/2n}^{\pi/2n}\cos^2 n\theta\,d\theta=na^2\int_{-\pi/2n}^{\pi/2n}\frac{1+\cos 2n\theta}{2}d\theta$$

$$=na^2\left[\frac{1}{2}\theta+\frac{\sin 2n\theta}{4n}\right]_{-\pi/2n}^{\pi/2n}=\frac{1}{2}a^2\pi$$

If n is odd, there are n leaves.

$$A=n\cdot\frac{1}{2}\int_{-\pi/2n}^{\pi/2n}(a\cos n\theta)^2d\theta=\frac{na^2}{2}\int_{-\pi/2n}^{\pi/2n}\cos^2 n\theta\,d\theta=\frac{na^2}{2}\left[\frac{1}{2}\theta+\frac{\sin 2n\theta}{4n}\right]_{-\pi/2n}^{\pi/2n}=\frac{1}{4}a^2\pi$$

30. $r=\sec\theta-2\cos\theta$



Solve for the θ-coordinate when $r=0$.

$\sec\theta-2\cos\theta=0$

$\cos^2\theta=\frac{1}{2}$

$\cos\theta=\pm\frac{1}{\sqrt{2}}$

$\theta=\pm\frac{\pi}{4},\pm\frac{3\pi}{4}$

Notice that the loop is produced for $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$.

$$A=\frac{1}{2}\int_{-\pi/4}^{\pi/4}(\sec\theta-2\cos\theta)^2d\theta$$

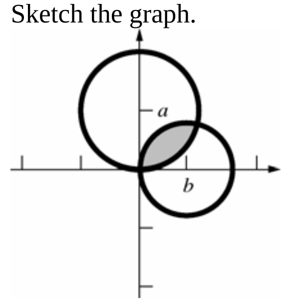
$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}(\sec^2\theta-4+4\cos^2\theta)d\theta$$

$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}(\sec^2\theta-2+2\cos 2\theta)d\theta$$

$$=\frac{1}{2}\left[\tan\theta-2\theta+\sin 2\theta\right]_{-\pi/4}^{\pi/4}$$

$$=\frac{1}{2}\left[\left(1-\frac{\pi}{2}+1\right)-\left(-1+\frac{\pi}{2}-1\right)\right]=2-\frac{\pi}{2}$$

31. a. Sketch the graph.



Solve for the θ-coordinate of the intersection.

$2a\sin\theta=2b\cos\theta$

$\tan\theta=\frac{b}{a};\quad \theta=\tan^{-1}\left(\frac{b}{a}\right)$

Let $\theta_0=\tan^{-1}\left(\frac{b}{a}\right)$.

$$A=\frac{1}{2}\int_0^{\theta_0}(2a\sin\theta)^2d\theta+\frac{1}{2}\int_{\theta_0}^{\pi/2}(2b\cos\theta)^2d\theta$$

$$=2a^2\int_0^{\theta_0}\sin^2\theta\,d\theta+2b^2\int_{\theta_0}^{\pi/2}\cos^2\theta\,d\theta$$

$$=a^2\int_0^{\theta_0}(1-\cos 2\theta)d\theta+b^2\int_{\theta_0}^{\pi/2}(1+\cos 2\theta)d\theta$$

$$=a^2\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\theta_0}+b^2\left[\theta+\frac{\sin 2\theta}{2}\right]_{\theta_0}^{\pi/2}$$

$$=a^2\theta_0+b^2\left(\frac{\pi}{2}-\theta_0\right)-\frac{a^2+b^2}{2}\sin 2\theta_0$$

$$=a^2\theta_0+b^2\left(\frac{\pi}{2}-\theta_0\right)-(a^2+b^2)\sin\theta_0\cos\theta_0$$

$$=a^2\tan^{-1}\left(\frac{b}{a}\right)+b^2\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{b}{a}\right)\right)-ab.$$

Note that since $\tan\theta=\frac{b}{a},\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$

and $\sin\theta=\frac{b}{\sqrt{a^2+b^2}}$.

b. Let m_1 be the slope of $r = 2a\sin\theta$.

$$m_1 = \frac{2a\sin\theta\cos\theta + 2a\cos\theta\sin\theta}{-2a\sin\theta\sin\theta + 2a\cos\theta\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

At $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, $m_1 = \frac{2ab}{a^2 - b^2}$.

At $\theta = 0$ (the pole), $m_1 = 0$.

Let m_2 be the slope of $r = 2b\cos\theta$.

$$m_2 = \frac{2b\cos\theta\cos\theta - 2b\sin\theta\sin\theta}{-2b\cos\theta\sin\theta - 2b\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{-2\sin\theta\cos\theta}$$

At $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, $m_2 = -\frac{a^2 - b^2}{2ab}$.

At $\theta = \frac{\pi}{2}$ (the pole), m_2 is undefined.

Therefore the two circles intersect at right angles.

32. The area swept from time t_0 to t_1 is

$$A = \int_{\theta(t_0)}^{\theta(t_1)} \frac{1}{2} r^2 d\theta.$$

By the Fundamental Theorem of Calculus,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

So $\frac{dA}{dt} = \frac{k}{2m}$ where k is the constant angular momentum.

Therefore, $\frac{dA}{dt}$ is a constant so $A = \frac{k}{2m}(t_1 - t_0)$.

Equal areas will be swept out in equal time.

33. The edge of the pond is described by the equation $r = 2a\cos\theta$.

Solve for intersection points of the circles $r = ak$ and $r = 2a\cos\theta$.

$$ak = 2a\cos\theta$$

$$\cos\theta = \frac{k}{2}, \theta = \cos^{-1}\left(\frac{k}{2}\right)$$

Let A be the grazing area.

$$A = \frac{1}{2}\pi(ka)^2 + 2\cdot\frac{1}{2}\int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2} [(ka)^2 - (2a\cos\theta)^2]d\theta = \frac{1}{2}k^2a^2\pi + a^2\int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2} (k^2 - 4\cos^2\theta)d\theta$$

$$= \frac{1}{2}k^2a^2\pi + a^2\int_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2} ((k^2 - 2) - 2\cos 2\theta)d\theta = \frac{1}{2}k^2a^2\pi + a^2\left[(k^2 - 2)\theta - \sin 2\theta\right]_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2}$$

$$= \frac{1}{2}k^2a^2\pi + a^2\left[k^2\theta - 2\theta - 2\sin\theta\cos\theta\right]_{\cos^{-1}\left(\frac{k}{2}\right)}^{\pi/2}$$

$$= \frac{1}{2}k^2a^2\pi + a^2\left[\frac{k^2\pi}{2} - \pi - k^2\cos^{-1}\left(\frac{k}{2}\right) + 2\cos^{-1}\left(\frac{k}{2}\right) + \frac{k\sqrt{4 - k^2}}{2}\right]$$

$$= a^2\left[(k^2 - 1)\pi + (2 - k^2)\cos^{-1}\left(\frac{k}{2}\right) + \frac{k\sqrt{4 - k^2}}{2}\right]$$

34. $|PT| = ka - \phi a$; ϕ goes from 0 to k.

$$A = \frac{1}{2}\int_0^k (ka - \phi a)^2 \, d\phi = \frac{1}{2}\left[-\frac{1}{3a}(ka - \phi a)^3\right]_0^k$$

$$= \frac{1}{6}a^2k^3$$

The grazing area is

$$\frac{1}{2}\pi(ka)^2 + 2A = a^2\left(\frac{\pi k^2}{2} + \frac{k^3}{3}\right).$$

35. The untethered goat has a grazing area of πa^2. From Problem 34, the tethered goat has a grazing area of $a^2\left(\frac{\pi k^2}{2} + \frac{k^3}{3}\right)$.

$$\pi a^2 = a^2\left(\frac{\pi k^2}{2} + \frac{k^3}{3}\right)$$

$$\pi = \frac{\pi k^2}{2} + \frac{k^3}{3}$$

$$2k^3 + 3\pi k^2 - 6\pi = 0$$

Using a numerical method or graphing calculator, $k \approx 1.26$. The length of the rope is approximately $1.26a$.

36. $f(\theta) = 2 + \cos\theta,\ f'(\theta) = -\sin\theta$

$$L = 2\int_0^\pi \sqrt{[2+\cos\theta]^2 + [-\sin\theta]^2}\, d\theta$$

$$= 2\int_0^\pi \sqrt{5 + 4\cos\theta}\, d\theta \approx 13.36$$

$f(\theta) = 2 + 4\cos\theta,\ f'(\theta) = -4\sin\theta$

$$L = 2\int_0^\pi \sqrt{\left[2+4\cos\theta\right]^2 + \left[-4\sin\theta\right]^2}\, d\theta$$

$$= 4\int_0^\pi \sqrt{5 + 4\cos\theta}\, d\theta \approx 26.73$$

37. $A = 3\cdot\frac{1}{2}\int_0^{\pi/3}(4\sin 3\theta)^2\, d\theta = 24\int_0^{\pi/3}\sin^2 3\theta\, d\theta$

$$= 12\int_0^{\pi/3}(1 - \cos 6\theta)\, d\theta$$

$$= 12\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = 4\pi$$

$f(\theta) = 4\sin 3\theta,\ f'(\theta) = 12\cos 3\theta$

$$L = 3\int_0^{\pi/3}\sqrt{(4\sin 3\theta)^2 + (12\cos 3\theta)^2}\, d\theta$$

$$= 3\int_0^{\pi/3}\sqrt{16\sin^2 3\theta + 144\cos^2 3\theta}\, d\theta$$

$$= 12\int_0^{\pi/3}\sqrt{1 + 8\cos^2 3\theta}\, d\theta \approx 26.73$$

38. $A = 4\cdot\frac{1}{2}\int_0^{\pi/4} 8\cos 2\theta\, d\theta = 2[4\sin 2\theta]_0^{\pi/4} = 8$

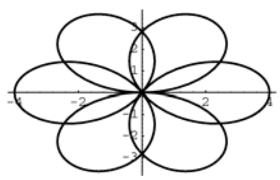
$f(\theta) = \sqrt{8\cos 2\theta},\ f'(\theta) = \frac{-8\sin 2\theta}{\sqrt{8\cos 2\theta}}$

$$L = 4\int_0^{\pi/4}\sqrt{8\cos 2\theta + \frac{8\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 4\int_0^{\pi/4}\sqrt{\frac{8}{\cos 2\theta}}\, d\theta$$

$$= 8\sqrt{2}\int_0^{\pi/4}\frac{1}{\sqrt{\cos 2\theta}}\, d\theta \approx 14.83$$

39. $r = 4\sin\left(\frac{3\theta}{2}\right), 0 \le \theta \le 4\pi$



$f(\theta) = 4\sin\left(\frac{3\theta}{2}\right),\ f'(\theta) = 6\cos\left(\frac{3\theta}{2}\right)$

$$L = \int_0^{4\pi}\sqrt{\left[4\sin\left(\frac{3\theta}{2}\right)\right]^2 + \left[6\cos\left(\frac{3\theta}{2}\right)\right]^2}\, d\theta$$

$$= \int_0^{4\pi}\sqrt{16\sin^2\left(\frac{3\theta}{2}\right) + 36\cos^2\left(\frac{3\theta}{2}\right)}\, d\theta$$

$$= \int_0^{4\pi}\sqrt{16 + 20\cos^2\left(\frac{3\theta}{2}\right)}\, d\theta \approx 63.46$$

10.8 Chapter Review

Concepts Test

1. False: If $a = 0$, the graph is a line.

2. True: The defining condition of a parabola is $|PF| = |PL|$. Since the axis of a parabola is perpendicular to the directrix and the distance from the vertex to the directrix is equal to the distance to the focus, the vertex is midway between the focus and the directrix.

3. False: The defining condition of an ellipse is $|PF| = e|PL|$ where $0 < e < 1$. Hence the distance from the vertex to a directrix is a greater than the distance to a focus.

4. True: See Problem 33 in Section 10.1.

5. True: The asymptotes for both hyperbolas are $y = \pm\frac{b}{a}x$.

6. True: $C = \int_0^{2\pi}\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt;$

$2\pi b = \int_0^{2\pi}\sqrt{b^2\sin^2 t + b^2\cos^2 t}\,dt$

$< C < \int_0^{2\pi}\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = 2\pi a$

7. True: As e approaches 0, the ellipse becomes more circular.

8. False: The equation can be rewritten as $\frac{x^2}{4} + \frac{y^2}{6} = 1$ which is a vertical ellipse with foci on the y-axis.

9. False: The equation $x^2 - y^2 = 0$ represents the two lines $y = \pm x$.

10. True: $(y^2 - 4x + 1)^2 = 0$ implies $y^2 - 4x + 1 = 0$ which is an equation for a parabola.

11. True: If $k > 0$, the equation is a horizontal hyperbola; if $k < 0$, the equation is a vertical hyperbola.

12. False: If $k < 0$, there is no graph.

13. False: If $b > a$, the distance is $2\sqrt{b^2 - a^2}$.

14. True: If $y = 0$, $\frac{x^2}{9} = -2$ which is not possible.

15. True: Since light from one focus reflects to the other focus, light emanating from a point between a focus and the nearest vertex will reflect beyond the other focus.

16. True: $a = \frac{8}{2} = 4, c = \frac{2}{2} = 1,$ $b = \sqrt{16-1} = \sqrt{15}$. The length of the minor diameter is $2b = \sqrt{60}$.

17. True: The equation is equivalent to $\left(x + \frac{C}{2}\right)^2 + \left(y + \frac{D}{2}\right)^2 = -F + \frac{C^2}{4} + \frac{D^2}{4}$. Thus, the graph is a circle if $-F + \frac{C^2}{4} + \frac{D^2}{4} > 0$, a point if $-F + \frac{C^2}{4} + \frac{D^2}{4} = 0$, or the empty set if $-F^2 + \frac{C^2}{4} + \frac{D^2}{4} < 0$.

18. False: The equation is equivalent to $2\left(x + \frac{C}{4}\right)^2 + \left(y + \frac{D}{2}\right)^2 = -F + \frac{C^2}{8} + \frac{D^2}{4}$. Thus, the graph can be a point if $-F + \frac{C^2}{8} + \frac{D^2}{4} = 0$.

19. False: The limiting forms of two parallel lines and the empty set cannot be formed in such a manner.

20. True: By definition, these curves are conic sections, which can be expressed by an equation of the form $Ax^2 + Cy^2 + Dx + Ex + F = 0$.

21. False: For example, $xy = 1$ is a hyperbola with coordinates only in the first and third quadrants.

22. False: For example, the graph of $x^2 + 3xy + y^2 = 1$ is a hyperbola that passes through the four points.

23. False: For example, $x = 0$, $y = t$, and $x = 0$, $y = -t$ both represent the line $x = 0$.

24. True: Eliminating the parameter gives $x = 2y$.

25. False: For example, the graph of $x = t^2, y = t$ does not represent y as a function of x. $y = \pm\sqrt{x}$, but $h(x) = \pm\sqrt{x}$ is not a function.

26. True: When $t = 1$, $x = 0$, and $y = 0$.

27. False: For example, if $x = t^3, y = t^3$ then $y = x$ so $\frac{d^2y}{dx^2} = 0$, but $\frac{g''(t)}{f''(t)} = 1$.

28. True: For example, the graph of the four-leaved rose has two tangent lines at the origin.

29. True: The graph of $r = 4\cos\theta$ is a circle of radius 2 centered at (2, 0). The graph $r = 4\cos\left(\theta - \frac{\pi}{3}\right)$ is the graph of

$r = 4\cos\theta$ rotated $\frac{\pi}{3}$

counter-clockwise about the pole.

30. True: (r, θ) can be expressed as $(r, \theta + 2\pi n)$ for any integer n.

31. False: For example, if $f(\theta) = \cos\theta$ and $g(\theta) = \sin\theta$, solving the two equations simultaneously does not give the pole (which is $\left(0, \frac{\pi}{2}\right)$ for $f(\theta)$ and (0, 0) for $g(\theta)$).

32. True: Since f is odd $f(-\theta) = -f(\theta)$. Thus, if we replace (r, θ) by $(-r, -\theta)$, the equation $-r = f(-\theta)$ is $-r = -f(\theta)$ or $r = f(\theta)$. Therefore, the graph is symmetric about the y-axis.

33. True: Since f is even $f(-\theta) = f(\theta)$. Thus, if we replace (r, θ) by $(r, -\theta)$, the equation $r = f(-\theta)$ is $r = f(\theta)$. Therefore, the graph is symmetric about the x-axis.

34. True: The graph has 3 leaves and the area is exactly one quarter of the circle $r = 4$. (See Problem 15 of Section 12.8.)

Sample Test Problems

1. **a.** $x^2 - 4y^2 = 0;\ y = \pm\frac{x}{2}$

(5) Two intersecting lines

b. $x^2 - 4y^2 = 0.01;\ \frac{x^2}{0.01} - \frac{y^2}{0.0025} = 1$

(9) A hyperbola

c. $x^2 - 4 = 0;\ x = \pm 2$

(4) Two parallel lines

d. $x^2 - 4x + 4 = 0;\ x = 2$

(3) A single line

e. $x^2 + 4y^2 = 0;\ (0, 0)$

(2) A single point

f. $x^2 + 4y^2 = x;\ x^2 - x + \frac{1}{4} + 4y^2 = \frac{1}{4};$

$$\frac{\left(x - \frac{1}{2}\right)^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{16}} = 1$$

(8) An ellipse

g. $x^2 + 4y^2 = -x;\ x^2 + x + \frac{1}{4} + 4y^2 = \frac{1}{4};$

$$\frac{\left(x + \frac{1}{2}\right)^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{16}} = 1$$

(8) An ellipse

h. $x^2 + 4y^2 = -1$

(1) No graph

i. $(x^2 + 4y - 1)^2 = 0;\ x^2 + 4y - 1 = 0$

(7) A parabola

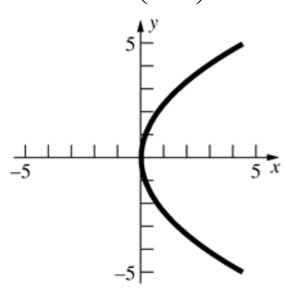
j. $3x^2 + 4y^2 = -x^2 + 1;\ x^2 + y^2 = \frac{1}{4}$

(6) A circle

2. $y^2 - 6x = 0;\ y^2 = 6x;\ y^2 = 4\left(\frac{3}{2}\right)x$

Horizontal parabola; opens to the right; $p = \frac{3}{2}$

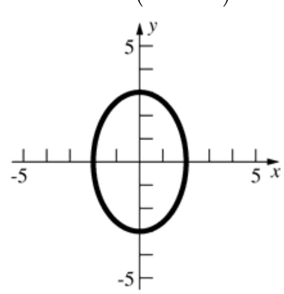
Focus is at $\left(\frac{3}{2}, 0\right)$ and vertex is at (0, 0).



3. $9x^2 + 4y^2 - 36 = 0;\ \frac{x^2}{4} + \frac{y^2}{9} = 1$

Vertical ellipse; $a = 3$, $b = 2$, $c = \sqrt{5}$

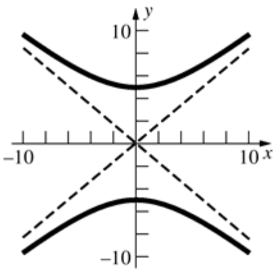
Foci are at $\left(0, \pm\sqrt{5}\right)$ and vertices are at (0, ±3).



4. $25x^2 - 36y^2 + 900 = 0; \dfrac{y^2}{25} - \dfrac{x^2}{36} = 1$

Vertical hyperbola; $a = 5$, $b = 6$, $c = \sqrt{61}$

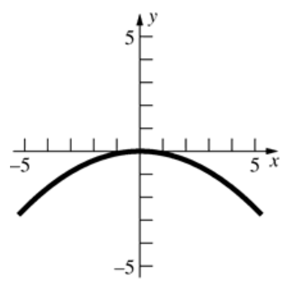
Foci are at $\left(0, \pm\sqrt{61}\right)$ and vertices are at (0, ±5).



5. $x^2 + 9y = 0; x^2 = -9y; x^2 = -4\left(\dfrac{9}{4}\right)y$

Vertical parabola; opens downward; $p = \dfrac{9}{4}$

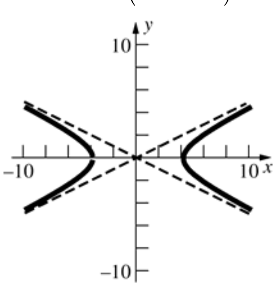
Focus at $\left(0, -\dfrac{9}{4}\right)$ and vertex at (0, 0).



6. $x^2 - 4y^2 - 16 = 0; \dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

Horizontal hyperbola; $a = 4$, $b = 2$, $c = 2\sqrt{5}$

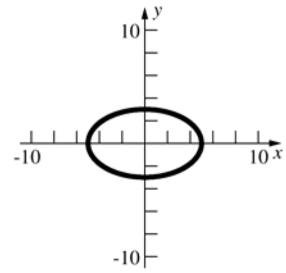
Foci are at $\left(\pm 2\sqrt{5}, 0\right)$ and vertices are at (±4, 0).



7. $9x^2 + 25y^2 - 225 = 0; \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

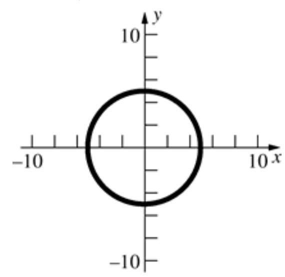
Horizontal ellipse, $a = 5$, $b = 3$, $c = 4$

Foci are at (±4, 0) and vertices are at (±5, 0).



8. $9x^2 + 9y^2 - 225 = 0; x^2 + y^2 = 25$

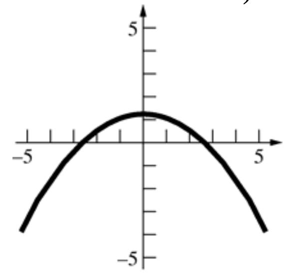
Circle; $r = 5$



9. $r = \dfrac{5}{2 + 2\sin\theta} = \dfrac{\left(\frac{5}{2}\right)(1)}{1 + (1)\cos\left(\theta - \frac{\pi}{2}\right)}$

$e = 1$; parabola

Focus is at (0, 0) and vertex is at $\left(0, \dfrac{5}{4}\right)$ (in Cartesian coordinates).



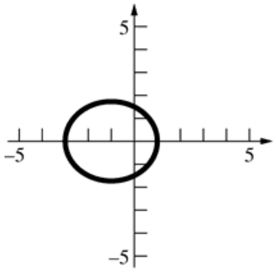
10. $r(2 + \cos\theta) = 3; r = \dfrac{3\left(\frac{1}{2}\right)}{1 + \frac{1}{2}\cos\theta}$

$e = \dfrac{1}{2}$, ellipse

At $\theta = 0, r = 1$. At $\theta = \pi, r = 3$. .

$a = \dfrac{1+3}{2} = 2, c = ea = 1$

Center is at (–1, 0) (in Cartesian coordinates).
Foci are (0, 0) and (–2, 0) and vertices are at (1, 0) and (–3, 0) (all in Cartesian coordinates).



11. Horizontal ellipse; center at (0, 0), $a = 4$,

$e = \frac{c}{a} = \frac{1}{2}$, $c = 2$, $b = \sqrt{16-4} = 2\sqrt{3}$

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

12. Vertical parabola; opens downward; $p = 3$

$$x^2 = -12y$$

13. Horizontal parabola;

$y^2 = ax$, $(3)^2 = a(-1)$, $a = -9$

$$y^2 = -9x$$

14. Vertical hyperbola; $a = 3$, $c = ae = 5$,

$b = \sqrt{25-9} = 4$, center at (0, 0)

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

15. Horizontal hyperbola, $a = 2$,

$x = \pm 2y, \frac{a}{b} = 2, b = 1$

$$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

16. Vertical parabola; opens downward; $p = 1$

$$(x-3)^2 = -4(y-3)$$

17. Horizontal ellipse; $2a = 10$, $a = 5$, $c = 4 - 1 = 3$,

$b = \sqrt{25-9} = 4$

$$\frac{(x-1)^2}{25} + \frac{(y-2)^2}{16} = 1$$

18. Vertical hyperbola; $2a = 6$, $a = 3$, $c = ae = 10$,

$b = \sqrt{100-9} = \sqrt{91}$, center at (2, 3)

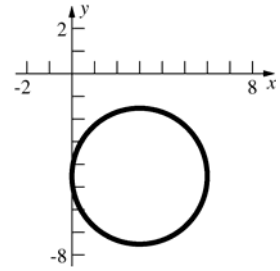
$$\frac{(y-3)^2}{9} - \frac{(x-2)^2}{91} = 1$$

19. $4x^2 + 4y^2 - 24x + 36y + 81 = 0$

$$4(x^2 - 6x + 9) + 4\left(y^2 + 9y + \frac{81}{4}\right) = -81 + 36 + 81$$

$$4(x-3)^2 + 4\left(y + \frac{9}{2}\right)^2 = 36$$

$(x-3)^2 + \left(y + \frac{9}{2}\right)^2 = 9$; circle

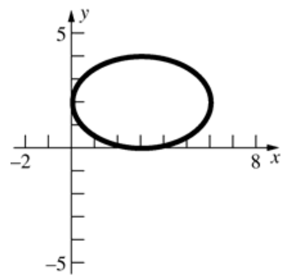


20. $4x^2 + 9y^2 - 24x - 36y + 36 = 0$

$$4(x^2 - 6x + 9) + 9(y^2 - 4y + 4) = -36 + 36 + 36$$

$$4(x-3)^2 + 9(y-2)^2 = 36$$

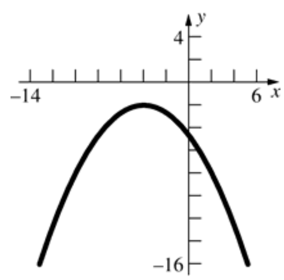
$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$; ellipse



21. $x^2 + 8x + 6y + 28 = 0$

$$(x^2 + 8x + 16) = -6y - 28 + 16$$

$(x+4)^2 = -6(y+2)$; parabola

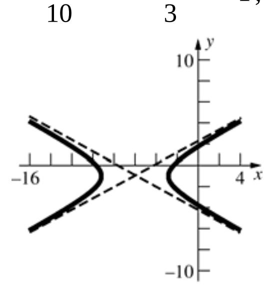


22. $3x^2 - 10y^2 + 36x - 20y + 68 = 0$

$$3(x^2 + 12x + 36) - 10(y^2 + 2y + 1) = -68 + 108 - 10$$

$$3(x+6)^2 - 10(y+1)^2 = 30$$

$\frac{(x+6)^2}{10} - \frac{(y+1)^2}{3} = 1$; hyperbola



23. $x = \frac{\sqrt{2}}{2}(u - v)$

$y = \frac{\sqrt{2}}{2}(u + v)$

$\frac{1}{2}(u-v)^2 + \frac{3}{2}(u-v)(u+v) + \frac{1}{2}(u+v)^2 = 10$

$\frac{5}{2}u^2 - \frac{1}{2}v^2 = 10$

$r = \frac{5}{2}, s = -\frac{1}{2}$

$\frac{u^2}{4} - \frac{v^2}{20} = 1$; hyperbola

$a = 2,\ b = 2\sqrt{5}, c = \sqrt{4+20} = 2\sqrt{6}$

The distance between foci is $4\sqrt{6}$.

24. $7x^2 + 8xy + y^2 = 9$

$\cot 2\theta = \frac{3}{4}$

$\cos 2\theta = \frac{3}{5}$

$\cos\theta = \sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}$

$\sin\theta = \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$

$\theta = \sin^{-1}\frac{1}{\sqrt{5}} \approx 0.4636$

$x = \frac{1}{\sqrt{5}}(2u - v)$

$y = \frac{1}{\sqrt{5}}(u + 2v)$

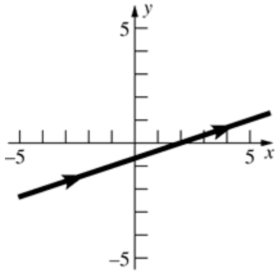
$\frac{7}{5}(2u-v)^2 + \frac{8}{5}(2u-v)(u+2v) + \frac{1}{5}(u+2v)^2 = 9$

$9u^2 - v^2 = 9$

$u^2 - \frac{v^2}{9} = 1$; hyperbola

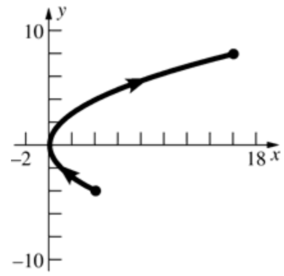
25. $t = \frac{1}{6}(x - 2)$

$y = \frac{1}{3}(x - 2)$



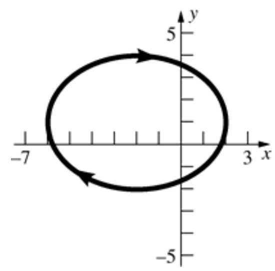
26. $t = \frac{y}{4}$

$x = \frac{y^2}{4}$ or $y^2 = 4x$



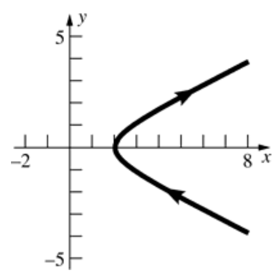
27. $\sin t = \frac{x+2}{4}, \cos t = \frac{y-1}{3}$

$\frac{(x+2)^2}{16} + \frac{(y-1)^2}{9} = 1$



28. $\frac{x^2}{4} = \sec^2 t, y^2 = \tan^2 t$

$\frac{x^2}{4} - y^2 = 1$



29. $\frac{dx}{dt} = 6t^2 - 4, \frac{dy}{dt} = 1 + \frac{1}{t+1} = \frac{t+2}{t+1}$

$\frac{dy}{dx} = \frac{\frac{t+2}{t+1}}{6t^2 - 4} = \frac{t+2}{(t+1)(6t^2-4)}$

At $t = 0$, $x = 7$, $y = 0$, and $\frac{dy}{dx} = -\frac{1}{2}$.

Tangent line: $y = -\frac{1}{2}(x-7)$ or $x + 2y - 7 = 0$

Normal line: $y = 2(x - 7)$ or $2x - y - 14 = 0$.

30. $\frac{dx}{dt} = -3e^{-t}, \frac{dy}{dt} = \frac{1}{2}e^{t}$

$$\frac{dy}{dx} = \frac{\frac{1}{2}e^{t}}{-3e^{-t}} = -\frac{1}{6}e^{2t}$$

At $t = 0$, $x = 3$, $y = \frac{1}{2}$, and $\frac{dy}{dx} = -\frac{1}{6}$.

Tangent line: $y - \frac{1}{2} = -\frac{1}{6}(x-3)$ or $x + 6y - 6 = 0$

Normal line: $y - \frac{1}{2} = 6(x-3)$ or $12x - 2y - 35 = 0$.

31. One approach is to use the arc length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^9 \sqrt{\frac{9t}{4} + \frac{9t}{4}}\, dt =$$

$$\frac{3\sqrt{2}}{2}\int_0^9 \sqrt{t}\, dt = \sqrt{2}\left[t^{3/2}\right]_0^9 = 27\sqrt{2}$$

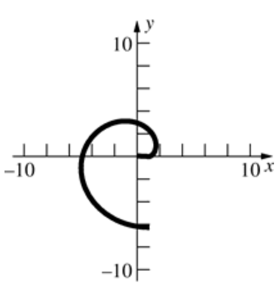
Another way is to note that when $t = 0$, $(x, y) = (1, 2)$, when $t = 9$, $(x, y) = (28, 29)$, and $y = x + 1$, which is a straight line. Thus the curve length is simply the distance between the points $(1, 2)$ and $(28, 29)$ or $\sqrt{(28-1)^2 + (29-2)^2} = 27\sqrt{2}$

32. $\frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t$

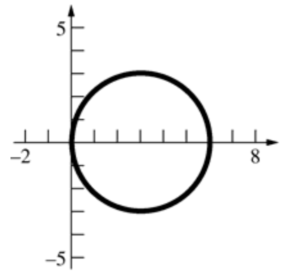
$$\frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t$$

$$L = \int_0^{2\pi} \sqrt{(t\cos t)^2 + (t\sin t)^2}\, dt = \int_0^{2\pi} t\, dt$$

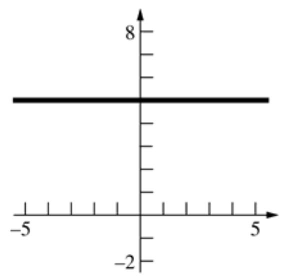
$$= \left[\frac{1}{2}t^2\right]_0^{2\pi} = 2\pi^2$$



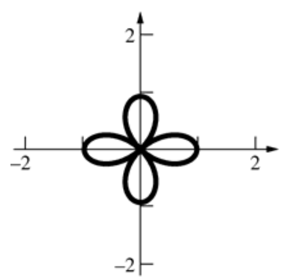
33 $r = 6\cos\theta$



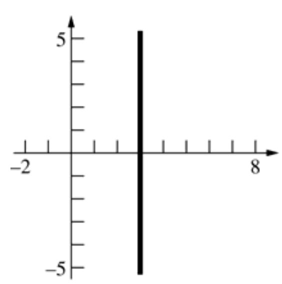
34. $r = \frac{5}{\sin\theta}$



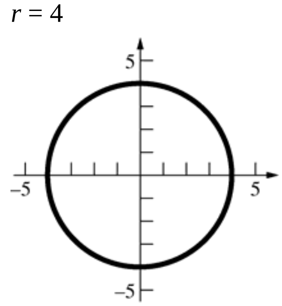
35. $r = \cos 2\theta$



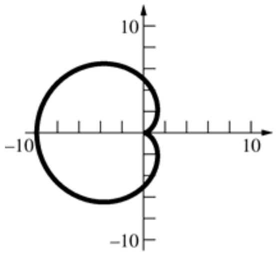
36. $r = \frac{3}{\cos\theta}$



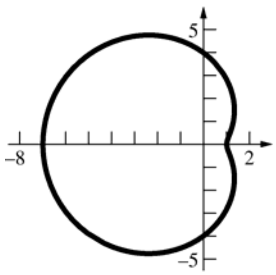
37. $r = 4$



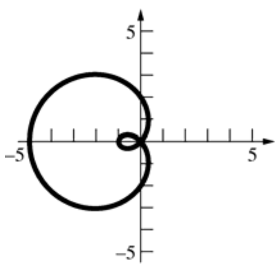
38. $r = 5 - 5\cos\theta$



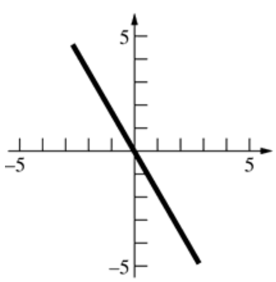
39. $r = 4 - 3\cos\theta$



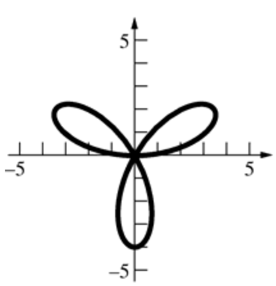
40. $r = 2 - 3\cos\theta$



41. $\theta = \frac{2}{3}\pi$

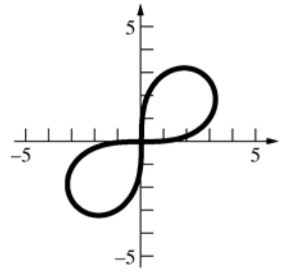


42. $r = 4\sin 3\theta$

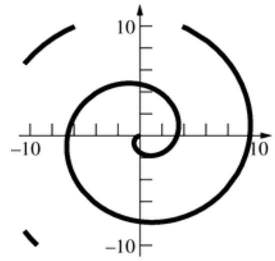


43. $r^2 = 16\sin 2\theta$

$r = \pm 4\sqrt{\sin 2\theta}$



44. $r = -\theta, \theta \ge 0$

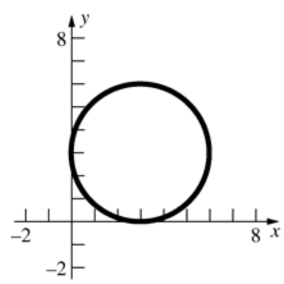


45. $r^2 - 6r(\cos\theta + \sin\theta) + 9 = 0$

$x^2 + y^2 - 6x - 6y + 9 = 0$

$(x^2 - 6x + 9) + (y^2 - 6y + 9) = -9 + 9 + 9$

$(x-3)^2 + (y-3)^2 = 9$

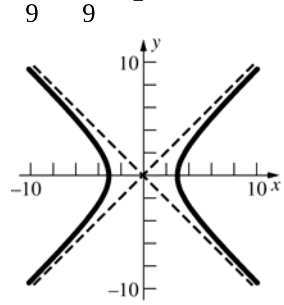


46. $r^2 \cos 2\theta = 9$

$r^2\cos^2\theta - r^2\sin^2\theta = 9$

$x^2 - y^2 = 9$

$\frac{x^2}{9} - \frac{y^2}{9} = 1$



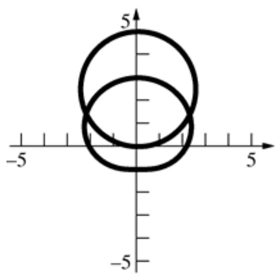
47. $f(\theta) = 3 + 3\cos\theta,\ f'(\theta) = -3\sin\theta$

$$m = \frac{(3+3\cos\theta)\cos\theta + (-3\sin\theta)\sin\theta}{-(3+3\cos\theta)\sin\theta + (-3\sin\theta)\cos\theta}$$

$$= \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta} = \frac{\cos\theta + \cos 2\theta}{-\sin\theta - \sin 2\theta}$$

At $\theta = \frac{\pi}{6}, m = \frac{\cos\frac{\pi}{6} + \cos\frac{\pi}{3}}{-\sin\frac{\pi}{6} - \sin\frac{\pi}{3}} = -1$.

48. $r = 5\sin\theta, r = 2 + \sin\theta$



$5\sin\theta = 2 + \sin\theta$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\left(\frac{5}{2}, \frac{\pi}{6}\right), \left(\frac{5}{2}, \frac{5\pi}{6}\right)$

49. $A = 2 \cdot \frac{1}{2}\int_0^{\pi}(5 - 5\cos\theta)^2\,d\theta$

$$= 25\int_0^{\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta$$

$$= 25\int_0^{\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= 25\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi} = \frac{75\pi}{2}$$

50. $A = 2 \cdot \frac{1}{2}\int_{\pi/6}^{\pi/2}\left[(5\sin\theta)^2 - (2+\sin\theta)^2\right]d\theta$

$$= \int_{\pi/6}^{\pi/2}(24\sin^2\theta - 4\sin\theta - 4)d\theta$$

$$= \int_{\pi/6}^{\pi/2}(8 - 12\cos 2\theta - 4\sin\theta)d\theta$$

$$= \left[8\theta - 6\sin 2\theta + 4\cos\theta\right]_{\pi/6}^{\pi/2} = \frac{8}{3}\pi + \sqrt{3}$$

51. $\frac{x^2}{400} + \frac{y^2}{100} = 1;\ \frac{x}{200} + \frac{yy'}{50} = 0$

$y' = -\frac{x}{4y};\ y' = -\frac{2}{3}$ at (16, 6)

Tangent line: $y - 6 = -\frac{2}{3}(x - 16)$

When x = 14, $y = -\frac{2}{3}(14 - 16) + 6 = \frac{22}{3}$.

$k = \frac{22}{3}$

52 **a.** III **b.** IV

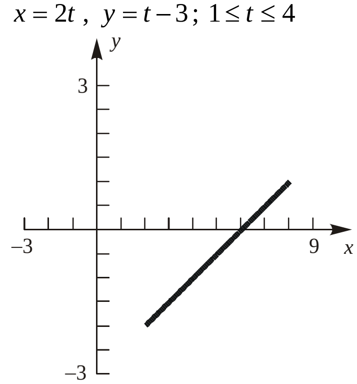
c. I **d.** II

53. **a.** I **b.** IV

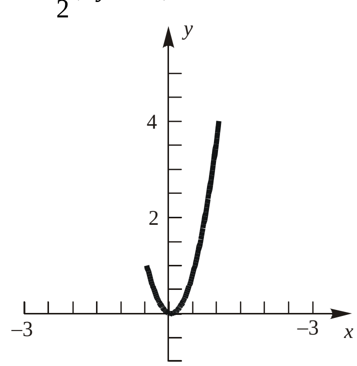
c. III **d.** II

Review and Preview Problems

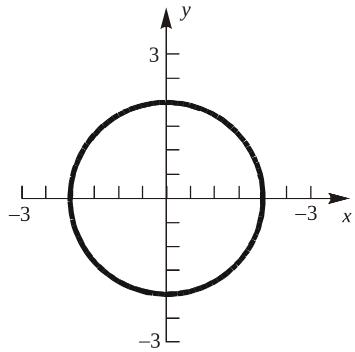
1. $x = 2t,\ y = t - 3;\ 1 \le t \le 4$



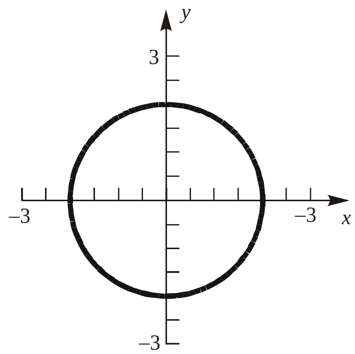
2. $x = \frac{t}{2},\ y = t^2;\ -1 \le t \le 2$



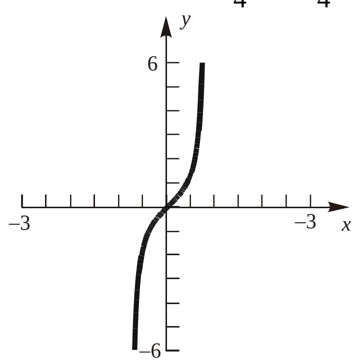
3. $x = 2\cos t$, $y = 2\sin t$; $0 \le t \le 2\pi$



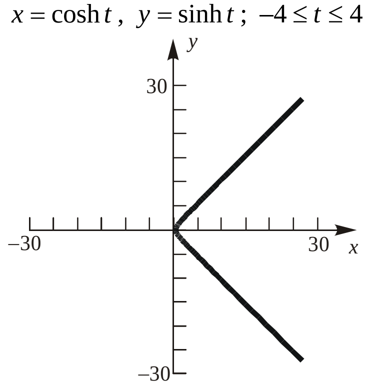
4. $x = 2\sin t$, $y = -2\cos t$; $0 \le t \le 2\pi$



5. $x = t$, $y = \tan 2t$; $-\frac{\pi}{4} < t < \frac{\pi}{4}$



6. $x = \cosh t$, $y = \sinh t$; $-4 \le t \le 4$



7. $x = h \cdot \cos\theta$

$y = h \cdot \sin\theta$

8. $x = h \cdot \cos\theta$

$y = h \cdot \sin\theta$

For problems 9-12, $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

9. $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = \frac{9}{2}\sqrt{t}$

$L = \int_0^4 \sqrt{1 + \frac{81}{4}t}\ dt = \int_0^4 \frac{1}{2}\sqrt{4 + 81t}\ dt =$

$u = 4 + 81t$
$du = 81dt$

$\frac{1}{162}\int_4^{328} \sqrt{u}\ du = \frac{2}{3} \cdot \frac{1}{162}\left[u^{3/2}\right]_4^{328} =$

$\frac{1}{243}\left[(328)^{3/2} - 8\right] \approx 24.4129$

10. $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2$

$L = \int_1^5 \sqrt{1+4}\ dt = \left[\sqrt{5}\,t\right]_1^5 = 4\sqrt{5} \approx 8.94$

11. $\frac{dx}{dt} = -2a\sin 2t,\ \frac{dy}{dt} = 2a\cos 2t$

$L = \int_0^{\pi/2} \sqrt{4a^2\sin^2 2t + 4a^2\cos^2 2t}\ dt =$

$\int_0^{\pi/2} 2|a|\sqrt{1}\ dt = 2|a|[\,t]_0^{\pi/2} = \pi|a|$

12. $\frac{dx}{dt} = \text{sech}^2 t,\ \frac{dy}{dt} = -\text{sech}\,t\tanh t$

$L = \int_0^4 \sqrt{\text{sech}^4 t + \text{sech}^2 t\tanh^2 t}\ \ dt =$

$\int_0^4 (\text{sech}\,t)\sqrt{\text{sech}^2 t + \tanh^2 t}\ \ dt =$

$\int_0^4 (\text{sech}\ t)\sqrt{\frac{1}{\cosh^2 t} + \frac{\sinh^2 t}{\cosh^2 t}}\ \ dt =$

$\int_0^4 (\text{sech}\ t)\sqrt{\frac{\cosh^2 t}{\cosh^2 t}}\ \ dt = \int_0^4 \text{sech}\ t\ \ dt =$

$\left[2\tan^{-1} e^t\right]_0^4 = 2\tan^{-1} e^4 - 2\tan^{-1} e^0 \approx 1.534$

13. Let $(x, 2x+1)$ represent any point on the line $y = 2x+1$; then the square of its distance from $(0,3)$ is $d(x) = x^2 + \left[(2x+1) - 3\right]^2 = 5x^2 - 8x + 4$ Now $d'(x) = 10x - 8$ so that $d'(0.8) = 0$; further, $d''(x) = 10 > 0$ so that the absolute minimum of the (square of the) distance occurs at the point $(0.8, 2.6)$ where the distance is

$\sqrt{d(0.8)} = \sqrt{5(0.8)^2 - 8(0.8) + 4} = \sqrt{0.8} \approx 0.894$

14. Results will vary. The only limitation on $\{a_1, a_2, b_1, b_2\}$ is that the t value that makes $x = 1$ must also make $y = -1$ *and* the t value that makes $x = 3$ must also make $y = 3$. Let $t = 0$ yield $(1, -1)$ and let $t = 1$ yield $(3, 3)$; then

$1 = a_1(0) + b_1 \quad -1 = a_2(0) + b_2$

$3 = a_1(1) + b_1 \quad 3 = a_2(1) + b_2$

From this we get $b_1 = 1,\ b_2 = -1,\ a_1 = 2,\ a_2 = 4$; thus one parametric representation is $x = 2t + 1,\ \ y = 4t - 1$. Using other t values to yield (1, -1) and (3, 3) will give other representations.

15. $s(t) = t^2 - 6t + 8$

a. $v(t) = s'(t) = 2t - 6$

$a(t) = v'(t) = 2$

b. The object is moving forward (in positive x-direction) when $v(t) > 0$ or $t > 3$.

16. $a(t) = 2$

a. $v(t) = \int a(t)\,dt = 2t + v(0)$; since the object is initially at rest, $v(0) = 0$ so $v(t) = 2t$.

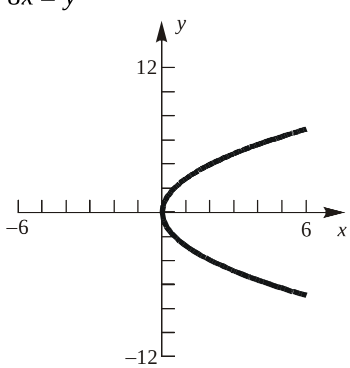
$s(t) = \int v(t)\,dt = t^2 + s(0)$; since $s(0) = 20$,

$s(t) = t^2 + 20$

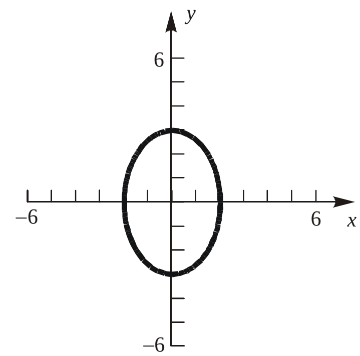
b. $s(t) = 100 \Rightarrow t^2 + 20 = 100 \Rightarrow t = \sqrt{80} \approx 8.944$

The object will reach position 100 after about 8.944 time units.

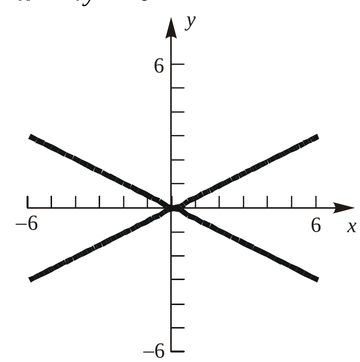
17. $8x = y^2$



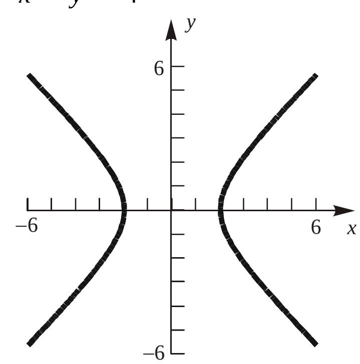
18. $\frac{x^2}{4} + \frac{y^2}{9} = 1$



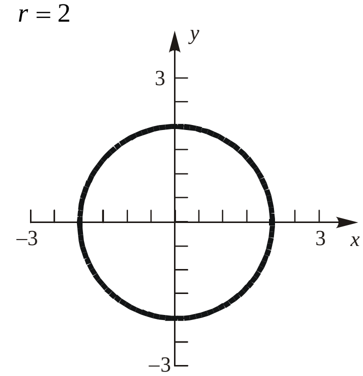
19. $x^2 - 4y^2 = 0$



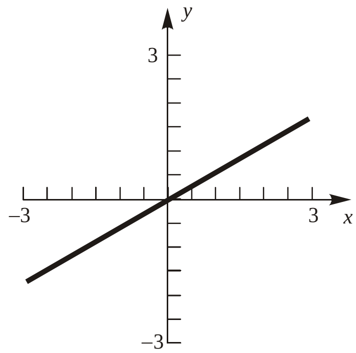
20. $x^2 - y^2 = 4$



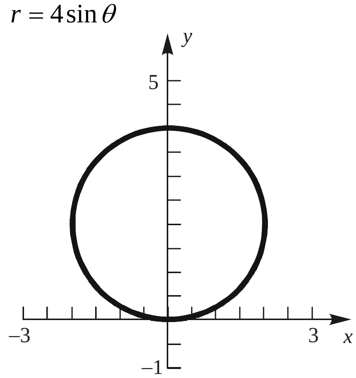
21. $r = 2$



22. $\theta = \frac{\pi}{6}$



23. $r = 4\sin\theta$



24. $r = \dfrac{1}{1+\frac{1}{2}\cos\theta}$