CHAPTER 11 Geometry in Space and Vectors

11.1 Concepts Review

1. coordinates

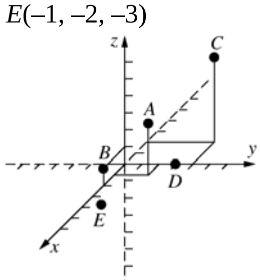
2. $\sqrt{(x+1)^2+(y-3)^2+(z-5)^2}$

3. (–1, 3, 5); 4

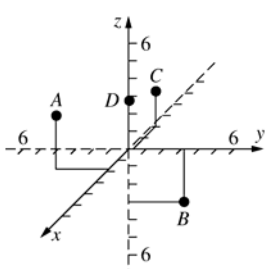
4. plane; 4; –6; 3

Problem Set 11.1

1. $A(1, 2, 3)$, $B(2, 0, 1)$, $C(-2, 4, 5)$, $D(0, 3, 0)$, $E(-1, -2, -3)$



2. $A\left(\sqrt{3}, -3, 3\right)$, $B(0, \pi, -3)$, $C\left(-2, \frac{1}{3}, 2\right)$, $D(0, 0, e)$



3. $x = 0$ in the yz-plane. $x = 0$ and $y = 0$ on the z-axis.

4. $y = 0$ in the xz-plane. $x = 0$ and $z = 0$ on the y-axis.

5. **a.** $\sqrt{(6-1)^2+(-1-2)^2+(0-3)^2} = \sqrt{43}$

   b. $\sqrt{(-2-2)^2+(-2+2)^2+(0+3)^2} = 5$

   c. $\sqrt{(e+\pi)^2+(\pi+4)^2+\left(0-\sqrt{3}\right)^2} \approx 9.399$

6. $P(4, 5, 3)$, $Q(1, 7, 4)$, $R(2, 4, 6)$

   $|PQ| = \sqrt{(4-1)^2+(5-7)^2+(3-4)^2} = \sqrt{14}$

   $|PR| = \sqrt{(4-2)^2+(5-4)^2+(3-6)^2} = \sqrt{14}$

   $|QR| = \sqrt{(1-2)^2+(7-4)^2+(4-6)^2} = \sqrt{14}$

   Since the distances are equal, the triangle formed by joining P, Q, and R is equilateral.

7. $P(2, 1, 6)$, $Q(4, 7, 9)$, $R(8, 5, -6)$

   $|PQ| = \sqrt{(2-4)^2+(1-7)^2+(6-9)^2} = 7$

   $|PR| = \sqrt{(2-8)^2+(1-5)^2+(6+6)^2} = 14$

   $|QR| = \sqrt{(4-8)^2+(7-5)^2+(9+6)^2} = \sqrt{245}$

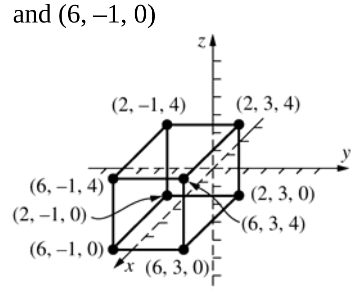
   $|PQ|^2+|PR|^2 = 49+196 = 245 = |QR|^2$, so the triangle formed by joining P, Q, and R is a right triangle, since it satisfies the Pythagorean Theorem.

8. **a.** The distance to the xy-plane is 1 since the point is 1 unit below the plane.

   b. The distance is $\sqrt{(2-0)^2+(3-3)^2+(-1-0)^2} = \sqrt{5}$ since the distance from a point to a line is the length of the shortest segment joining the point and the line. Using the point (0, 3, 0) on the y-axis clearly minimizes the length.

   c. $\sqrt{(2-0)^2+(3-0)^2+(-1-0)^2} = \sqrt{14}$

9. Since the faces are parallel to the coordinate planes, the sides of the box are in the planes $x = 2$, $y = 3$, $z = 4$, $x = 6$, $y = -1$, and $z = 0$ and the vertices are at the points where 3 of these planes intersect. Thus, the vertices are (2, 3, 4), (2, 3, 0), (2, –1, 4), (2, –1, 0), (6, 3, 4), (6, 3, 0), (6, –1, 4), and (6, –1, 0)



10. It is parallel to the y-axis; $x = 2$ and $z = 3$. (If it were parallel to the x-axis, the y-coordinate could not change, similarly for the z-axis.)

11. **a.** $(x-1)^2+(y-2)^2+(z-3)^2 = 25$

    b. $(x+2)^2+(y+3)^2+(z+6)^2 = 5$

    c. $(x-\pi)^2+(y-e)^2+\left(z-\sqrt{2}\right)^2 = \pi$

12. Since the sphere is tangent to the *xy*-plane, the point (2, 4, 0) is on the surface of the sphere. Hence, the radius of the sphere is 5 so the equation is $(x-2)^2+(y-4)^2+(z-5)^2=25.$

13. $(x^2-12x+36)+(y^2+14y+49)+(z^2-8z+16)=-1+36+49+16$

$(x-6)^2+(y+7)^2+(z-4)^2=100$

Center: (6, –7, 4); radius 10

14. $(x^2+2x+1)+(y^2-6y+9)+(z^2-10z+25)=-34+1+9+25$

$(x+1)^2+(y-3)^2+(z-5)^2=1$

Center: (–1, 3, 5); radius 1

15. $x^2+y^2+z^2-x+2y+4z=\frac{13}{4}$

$\left(x^2-x+\frac{1}{4}\right)+(y^2+2y+1)+(z^2+4z+4)=\frac{13}{4}+\frac{1}{4}+1+4$

$\left(x-\frac{1}{2}\right)^2+(y+1)^2+(z+2)^2=\frac{17}{2}$

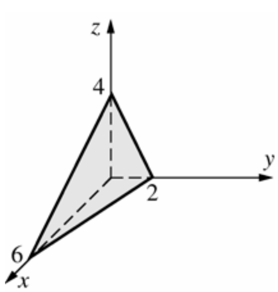
Center: $\left(\frac{1}{2},-1,-2\right)$; radius $\sqrt{\frac{17}{2}}\approx 2.92$

16. $(x^2+8x+16)+(y^2-4y+4)+(z^2-22z+121)=-77+16+4+121$

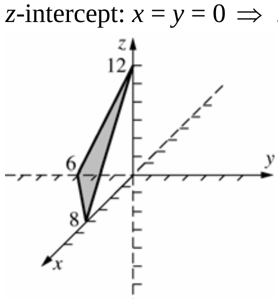
$(x+4)^2+(y-2)^2+(z-11)^2=64$

Center: (–4, 2, 11); radius 8

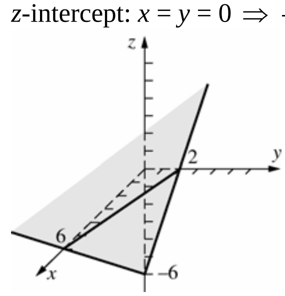
17. *x*-intercept: $y=z=0 \Rightarrow 2x=12, x=6$
y-intercept: $x=z=0 \Rightarrow 6y=12, y=2$
z-intercept: $x=y=0 \Rightarrow 3z=12, z=4$



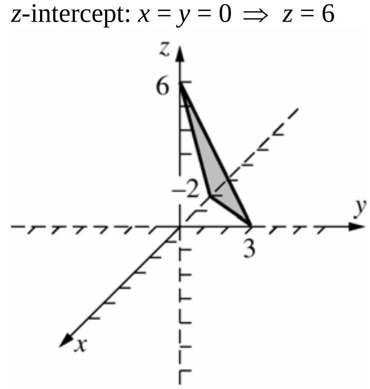
18. *x*-intercept: $y=z=0 \Rightarrow 3x=24, x=8$
y-intercept: $y=z=0 \Rightarrow -4y=24, y=-6$
z-intercept: $x=y=0 \Rightarrow 2z=24, z=12$



19. *x*-intercept: $y=z=0 \Rightarrow x=6$
y-intercept: $x=z=0 \Rightarrow 3y=6, y=2$
z-intercept: $x=y=0 \Rightarrow -z=6, z=-6$

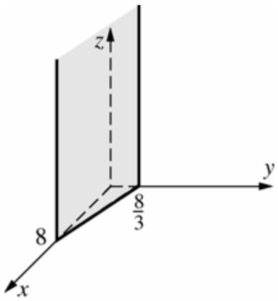


20. *x*-intercept: $y=z=0 \Rightarrow -3x=6, x=-2$
y-intercept: $x=z=0 \Rightarrow 2y=6, y=3$
z-intercept: $x=y=0 \Rightarrow z=6$

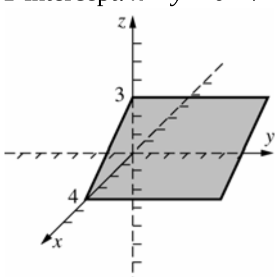


21. x and y cannot both be zero, so the plane is parallel to the z-axis.
x-intercept: $y = z = 0 \Rightarrow x = 8$

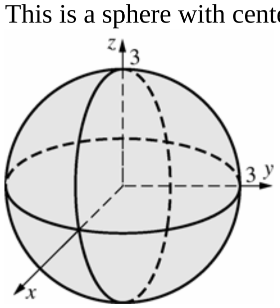
y-intercept: $x = z = 0 \Rightarrow 3y = 8,\ y = \frac{8}{3}$



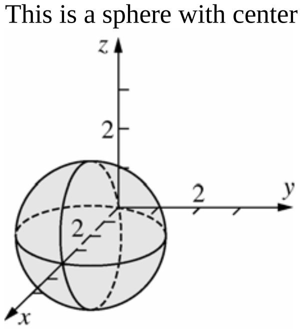
22. x and z cannot both be zero, so the plane is parallel to the y-axis.
x-intercept: $y = z = 0 \Rightarrow 3x = 12,\ x = 4$
z-intercept: $x = y = 0 \Rightarrow 4z = 12,\ z = 3$



23. This is a sphere with center (0, 0, 0) and radius 3.



24. This is a sphere with center (2, 0, 0) and radius 2.



For problems 25-36, $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

25. $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 2$

$L = \int_0^2 \sqrt{1^2 + 1^2 + 2^2}\, dt = \int_0^2 \sqrt{6}\, dt =$

$\left[\sqrt{6}t\right]_0^2 = 2\sqrt{6} \approx 4.899$

26. $\frac{dx}{dt} = \frac{1}{4}, \frac{dy}{dt} = \frac{1}{3}, \frac{dz}{dt} = \frac{1}{2}$

$L = \int_1^3 \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{2}\right)^2}\, dt = \int_1^3 \sqrt{\frac{61}{144}}\, dt =$

$\left[\sqrt{\frac{61}{144}}\, t\right]_1^3 = 2\sqrt{\frac{61}{144}} \approx 1.302$

27. $\frac{dx}{dt} = \frac{3}{2}\sqrt{t}, \frac{dy}{dt} = 3, \frac{dz}{dt} = 4$

$L = \int_1^4 \sqrt{\left(\frac{9}{4}t\right) + 9 + 16}\, dt = \int_1^4 \frac{1}{2}\underbrace{\sqrt{9t+100}}_{\substack{u = 9t+100 \\ du = 9\,dt}}\, dt =$

$\frac{1}{18}\int_{109}^{136} \sqrt{u}\, du = \frac{1}{27}\left[u^{3/2}\right]_{109}^{136} \approx 16.59$

28. $\frac{dx}{dt} = \frac{3}{2}\sqrt{t}, \frac{dy}{dt} = \frac{3}{2}\sqrt{t}, \frac{dz}{dt} = 1$

$L = \int_2^4 \sqrt{\left(\frac{9}{4}t\right) + \left(\frac{9}{4}t\right) + 1}\, dt = \int_2^4 \frac{1}{2}\underbrace{\sqrt{18t+4}}_{\substack{u = 18t+4 \\ du = 18\,dt}}\, dt =$

$\frac{1}{36}\int_{40}^{76} \sqrt{u}\, du = \frac{1}{54}\left[u^{3/2}\right]_{40}^{76} \approx 7.585$

29. $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2\sqrt{t}, \frac{dz}{dt} = 1$

$L = \int_0^8 \sqrt{4t^2 + 4t + 1}\, dt = \int_0^8 \sqrt{(2t+1)^2}\, dt =$

$\int_0^8 (2t+1)\, dt = \left[t^2 + t\right]_0^8 = 72$

30. $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2\sqrt{3t}, \frac{dz}{dt} = 3$

$L = \int_1^4 \sqrt{4t^2 + 12t + 9}\, dt = \int_1^4 \sqrt{(2t+3)^2}\, dt =$

$\int_1^4 (2t+3)\, dt = \left[t^2 + 3t\right]_1^4 = 28 - 4 = 24$

31. $\frac{dx}{dt} = -2\sin t, \frac{dy}{dt} = 2\cos t, \frac{dz}{dt} = 3$

$L = \int_{-\pi}^{\pi} \sqrt{4\sin^2 t + 4\cos^2 t + 9}\, dt = \int_{-\pi}^{\pi} \sqrt{13}\, dt =$

$\left[\sqrt{13}\, t\right]_{-\pi}^{\pi} = 2\pi\sqrt{13} \approx 22.654$

32. $\frac{dx}{dt} = -2\sin t,\ \frac{dy}{dt} = 2\cos t, \frac{dz}{dt} = \frac{1}{20}$

$L = \int_0^{8\pi} \sqrt{4\sin^2 t + 4\cos^2 t + \frac{1}{400}}\ dt =$

$\int_0^{8\pi} \frac{1}{40}\sqrt{1601}\, dt = \left[\frac{1}{40}\sqrt{1601}\, t\right]_0^{8\pi} =$

$\frac{\pi}{5}\sqrt{1601} \approx 25.14$

33. $\frac{dx}{dt} = \frac{1}{2\sqrt{t}},\ \frac{dy}{dt} = 1, \frac{dz}{dt} = 1$

$L = \int_1^6 \sqrt{\left(\frac{1}{4t}\right) + 1 + 1}\, dt = \int_1^6 \sqrt{2 + \left(\frac{1}{4t}\right)}\ dt$

By the Parabolic Rule (n = 10):

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	1	1.5000	1	1.5000
1	1.5	1.4720	4	5.8878
2	2	1.4577	2	2.9155
3	2.5	1.4491	4	5.7966
4	3	1.4434	2	2.8868
5	3.5	1.4392	4	5.7570
6	4	1.4361	2	2.8723
7	4.5	1.4337	4	5.7349
8	5	1.4318	2	2.8636
9	5.5	1.4302	4	5.7208
10	6	1.4289	1	1.4289
		approximation		**7.2273**

34. $\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2t, \frac{dz}{dt} = 3t^2$

$L = \int_1^2 \sqrt{1+4t^2+9t^4}\ dt$

By the Parabolic Rule (n = 10):

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	1	3.7417	1	3.7417
1	1.1	4.3608	4	17.4433
2	1.2	5.0421	2	10.0841
3	1.3	5.7849	4	23.1395
4	1.4	6.5890	2	13.1779
5	1.5	7.4540	4	29.8161
6	1.6	8.3799	2	16.7598
7	1.7	9.3664	4	37.4655
8	1.8	10.4134	2	20.8268
9	1.9	11.5208	4	46.0832
10	2	12.6886	1	12.6886
		approximation		**7.7075**

35. $\frac{dx}{dt} = -2\sin t,\ \frac{dy}{dt} = \cos t, \frac{dz}{dt} = 1$

$L = \int_0^{6\pi} \sqrt{4\sin^2 t + \cos^2 t + 1}\ dt =$

$\int_0^{6\pi} \sqrt{3\sin^2 t + 2}\, dt$

By the Parabolic Rule (n = 10):

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	0	1.4142	1	1.4142
1	1.88	2.1711	4	8.6843
2	3.77	1.7425	2	3.4851
3	5.65	1.7425	4	6.9702
4	7.54	2.1711	2	4.3421
5	9.42	1.4142	4	5.6569
6	11.3	2.1711	2	4.3421
7	13.2	1.7425	4	6.9702
8	15.1	1.7425	2	3.4851
9	17	2.1711	4	8.6843
10	18.8	1.4142	1	1.4142
		approximation		**34.8394**

36. $\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = -\sin t, \frac{dz}{dt} = \cos t$

$L = \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t + \cos^2 t}\ dt =$

$\int_0^{2\pi} \sqrt{\cos^2 t + 1}\, dt$

By the Parabolic Rule (n = 10):

i	x_i	$f(x_i)$	c_i	$c_i \cdot f(x_i)$
0	0	1.4142	1	1.4142
1	0.63	1.2863	4	5.1451
2	1.26	1.0467	2	2.0933
3	1.88	1.0467	4	4.1866
4	2.51	1.2863	2	2.5726
5	3.14	1.4142	4	5.6569
6	3.77	1.2863	2	2.5726
7	4.4	1.0467	4	4.1866
8	5.03	1.0467	2	2.0933
9	5.65	1.2863	4	5.1451
10	6.28	1.4142	1	1.4142
		approximation		**7.6405**

37. The center of the sphere is the midpoint of the diameter, so it is $\left(\frac{-2+4}{2}, \frac{3-1}{2}, \frac{6+5}{2}\right) = \left(1, 1, \frac{11}{2}\right)$. The radius is $\frac{1}{2}\sqrt{(-2-4)^2 + (3+1)^2 + (6-5)^2} = \frac{\sqrt{53}}{2}$. The equation is $(x-1)^2 + (y-1)^2 + \left(z - \frac{11}{2}\right)^2 = \frac{53}{4}$.

38. Since the spheres are tangent and have equal radii, the radius of each sphere is $\frac{1}{2}$ of the distance between the centers.

$r = \frac{1}{2}\sqrt{(-3-5)^2 + (1+3)^2 + (2-6)^2} = 2\sqrt{6}.$ The spheres are $(x+3)^2 + (y-1)^2 + (z-2)^2 = 24$ and $(x-5)^2 + (y+3)^2 + (z-6)^2 = 24.$

39. The center must be 6 units from each coordinate plane. Since it is in the first octant, the center is (6, 6, 6). The equation is $(x-6)^2 + (y-6)^2 + (z-6)^2 = 36.$

40. $x + y = 12$ is parallel to the z-axis. The distance from (1, 1, 4) to the plane $x + y = 12$ is the same as the distance in the xy-plane of (1, 1, 0) to the line $x + y - 12 = 0$. That distance is $\frac{|1+1-12|}{(1+1)^{1/2}} = 5\sqrt{2}.$ The equation of the sphere is $(x-1)^2 + (y-1)^2 + (z-4)^2 = 50.$

41. **a.** Plane parallel to and two units above the xy-plane

b. Plane perpendicular to the xy-plane whose trace in the xy-plane is the line $x = y$.

c. Union of the yz-plane ($x = 0$) and the xz-plane ($y = 0$)

d. Union of the three coordinate planes

e. Cylinder of radius 2, parallel to the z-axis

f. Top half of the sphere with center (0, 0, 0) and radius 3

42. The points of the intersection satisfy both $(x-1)^2 + (y+2)^2 + (z+1)^2 = 10$ and $z = 2$, so $(x-1)^2 + (y+2)^2 + (2+1)^2 = 10$. This simplifies to $(x-1)^2 + (y+2)^2 = 1,$ the equation of a circle of radius 1. The center is (1, –2, 2).

43. If $P(x, y, z)$ denotes the moving point, $\sqrt{(x-1)^2 + (y-2)^2 + (z+3)^2} = 2\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2},$ which simplifies to $(x-1)^2 + (y-2)^2 + (z-5)^2 = 16,$ is a sphere with radius 4 and center (1, 2, 5).

44. If $P(x, y, z)$ denotes the moving point, $\sqrt{(x-1)^2 + (y-2)^2 + (z+3)^2} = \sqrt{(x-2)^2 + (y-3)^2 + (z-2)^2},$ which simplifies to $x + y + 5z = 3/2$, a plane.

45. Note that the volume of a segment of height h in a hemisphere of radius r is $\pi h^2\left[r - \left(\frac{h}{3}\right)\right].$

The resulting solid is the union of two segments, one for each sphere. Since the two spheres have the same radius, each segment will have the same value for h. h is the radius minus half the distance between the centers of the two spheres.

$$h = 2 - \frac{1}{2}\sqrt{(2-1)^2 + (4-2)^2 + (3-1)^2} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$V = 2\left[\pi\left(\frac{1}{2}\right)^2\left(2 - \frac{1}{6}\right)\right] = \frac{11\pi}{12}$$

46. As in Problem 45, the resulting solid is the union of two segments. Since the radii are not the same, the segments will have different heights. Let h_1 be the height of the segment from the first sphere and let h_2 be the height from the second sphere. $r_1 = 2$ is the radius of the first sphere and $r_2 = 3$ is the radius of the second sphere .

Solving for the equation of the plane containing the intersection of the spheres $(x-1)^2 + (y-2)^2 + (z-1)^2 - 4 = 0$ and $(x-2)^2 + (y-4)^2 + (z-3)^2 - 9 = 0,$ we get $x + 2y + 2z - 9 = 0$.

The distance from (1, 2, 1) to the plane is $\frac{2}{3},$ and the distance from (2, 4, 3) to the plane is $\frac{7}{3}.$

$$h_1 = 2 - \frac{2}{3} = \frac{4}{3};\ h_2 = 3 - \frac{7}{3} = \frac{2}{3}$$

$$V = \pi\left(\frac{4}{3}\right)^2\left(2 - \frac{4}{9}\right) + \pi\left(\frac{2}{3}\right)^2\left(3 - \frac{2}{9}\right) = 4\pi$$

47. Plots will vary. We first note that the sign of c will influence the vertical direction an object moves (along the helix) with increasing time; if c is negative the object will spiral downward, whereas if c is positive it will spiral upward. The smaller $|c|$ is the "tighter" the spiral will be; that is the space between successive "coils" of the helix decreases as $|c|$ decreases.

48. Plots will vary. We first note that the sign of a will influence the rotational direction that an object moves (along the helix) with increasing time; if a is negative the object will rotate in a clockwise direction, whereas if a is positive rotation will be counterclockwise. The smaller $|a|$ is the narrower the spiral will be; that is the circles traced out will be of smaller radius as $|a|$ decreases

11.2 Concepts Review

1. magnitude; direction

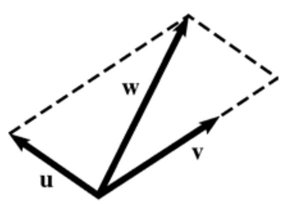
2. they have the same magnitude and direction.

3. the tail of **u**; the head of **v**

4. 3

Problem Set 11.2

1.



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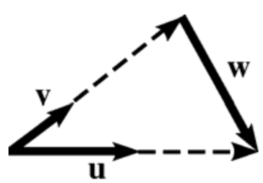

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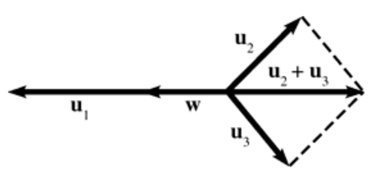


4. **0**

5. $\mathbf{w} = \frac{1}{2}(\mathbf{u}+\mathbf{v}) = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$

6. $\mathbf{n} = \frac{1}{2}(\mathbf{v}-\mathbf{u}) = \frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}$

$\mathbf{m} = \mathbf{v} - \mathbf{n} = \mathbf{v} - \frac{1}{2}(\mathbf{v}-\mathbf{u}) = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{u}$

7. $|\mathbf{w}| = |\mathbf{u}|\cos 60° + |\mathbf{v}|\cos 60° = \frac{1}{2} + \frac{1}{2} = 1$

8. $|\mathbf{w}| = |\mathbf{u}|\cos 45° + |\mathbf{v}|\cos 45° = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

9. $\mathbf{u}+\mathbf{v} = \langle -1+3, 0+4\rangle = \langle 2,4\rangle$

$\mathbf{u}-\mathbf{v} = \langle -1-3, 0-4\rangle = \langle -4,-4\rangle$

$\|\mathbf{u}\| = \sqrt{(-1)^2+(0)^2} = \sqrt{1} = 1$

$\|\mathbf{v}\| = \sqrt{(3)^2+(4)^2} = \sqrt{25} = 5$

10. $\mathbf{u}+\mathbf{v} = \langle 0+(-3), 0+4\rangle = \langle -3,4\rangle$

$\mathbf{u}-\mathbf{v} = \langle 0-(-3), 0-4\rangle = \langle 3,-4\rangle$

$\|\mathbf{u}\| = \sqrt{(0)^2+(0)^2} = \sqrt{0} = 0$

$\|\mathbf{v}\| = \sqrt{(-3)^2+(4)^2} = \sqrt{25} = 5$

11. $\mathbf{u}+\mathbf{v} = \langle 12+(-2), 12+2\rangle = \langle 10,14\rangle$

$\mathbf{u}-\mathbf{v} = \langle 12-(-2), 12-2\rangle = \langle 14,10\rangle$

$\|\mathbf{u}\| = \sqrt{(12)^2+(12)^2} = \sqrt{288} = 12\sqrt{2}$

$\|\mathbf{v}\| = \sqrt{(-2)^2+(2)^2} = \sqrt{8} = 2\sqrt{2}$

12. $\mathbf{u}+\mathbf{v} = \langle (-0.2)+(-2.1), 0.8+1.3\rangle = \langle -2.3, 2.1\rangle$

$\mathbf{u}-\mathbf{v} = \langle (-0.2)-(-2.1), 0.8-1.3\rangle = \langle 1.9,-0.5\rangle$

$\|\mathbf{u}\| = \sqrt{(-0.2)^2+(0.8)^2} = \sqrt{0.68} \approx 0.825$

$\|\mathbf{v}\| = \sqrt{(-2.1)^2+(1.3)^2} = \sqrt{6.10} \approx 2.47$

13. $\mathbf{u}+\mathbf{v} = \langle -1+3, 0+4, 0+0\rangle = \langle 2,4,0\rangle$

$\mathbf{u}-\mathbf{v} = \langle -1-3, 0-4, 0-0\rangle = \langle -4,-4,0\rangle$

$\|\mathbf{u}\| = \sqrt{(-1)^2+(0)^2+(0)^2} = \sqrt{1} = 1$

$\|\mathbf{v}\| = \sqrt{(3)^2+(4)^2+(0)^2} = \sqrt{25} = 5$

14. $\mathbf{u}+\mathbf{v}=\langle 0+(-3), 0+3, 0+1\rangle=\langle -3,3,1\rangle$

$\mathbf{u}-\mathbf{v}=\langle 0-(-3), 0-3, 0-1\rangle=\langle 3,-3,-1\rangle$

$\|\mathbf{u}\|=\sqrt{(0)^2+(0)^2+(0)^2}=\sqrt{0}=0$

$\|\mathbf{v}\|=\sqrt{(-3)^2+(3)^2+(1)^2}=\sqrt{19}\approx 4.359$

15. $\mathbf{u}+\mathbf{v}=\langle 1+(-5), 0+0, 1+0\rangle=\langle -4,0,1\rangle$

$\mathbf{u}-\mathbf{v}=\langle 1-(-5), 0-0, 1-0\rangle=\langle 6,0,1\rangle$

$\|\mathbf{u}\|=\sqrt{(1)^2+(0)^2+(1)^2}=\sqrt{2}\approx 1.414$

$\|\mathbf{v}\|=\sqrt{(-5)^2+(0)^2+(0)^2}=\sqrt{25}=5$

16. $\mathbf{u}+\mathbf{v}=\langle 0.3+2.2, 0.3+1.3, 0.5+(-0.9)\rangle=$

$\langle 2.5, 1.6, -0.4\rangle$

$\mathbf{u}-\mathbf{v}=\langle 0.3-2.2, 0.3-1.3, 0.5-(-0.9)\rangle=$

$\langle -1.9, -1.0, 1.4\rangle$

$\|\mathbf{u}\|=\sqrt{(0.3)^2+(0.3)^2+(0.5)^2}=\sqrt{0.43}\approx 0.656$

$\|\mathbf{v}\|=\sqrt{(2.2)^2+(1.3)^2+(-0.9)^2}=\sqrt{7.34}\approx 2.709$

17. Let θ be the angle of **w** measured clockwise from south.

$|\mathbf{w}|\cos\theta=|\mathbf{u}|\cos 30°+|\mathbf{v}|\cos 45°=25\sqrt{3}+25\sqrt{2}$

$=25\left(\sqrt{3}+\sqrt{2}\right)$

$|\mathbf{w}|\sin\theta=|\mathbf{v}|\sin 45°-|\mathbf{u}|\sin 30°=25\sqrt{2}-25$

$=25\left(\sqrt{2}-1\right)$

$|\mathbf{w}|^2=|\mathbf{w}|^2\cos^2\theta+|\mathbf{w}|^2\sin^2\theta$

$=625\left(\sqrt{3}+\sqrt{2}\right)^2+625\left(\sqrt{2}-1\right)^2$

$=625\left(8-2\sqrt{2}+2\sqrt{6}\right)$

$|\mathbf{w}|=\sqrt{625\left(8-2\sqrt{2}+2\sqrt{6}\right)}=25\sqrt{8-2\sqrt{2}+2\sqrt{6}}$

≈ 79.34

$\tan\theta=\dfrac{|\mathbf{w}|\sin\theta}{|\mathbf{w}|\cos\theta}=\dfrac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}$

$\theta=\tan^{-1}\left(\dfrac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}\right)=7.5°$

w has magnitude 79.34 lb in the direction S 7.5° W.

18. Let **v** be the resulting force. Let θ be the angle of **v** measured clockwise from south.

$|\mathbf{v}|\cos\theta=60\cos 30°+80\cos 60°=30\sqrt{3}+40$

$=10\left(3\sqrt{3}+4\right)$

$|v|\sin\theta=80\sin 60°-60\sin 30°=40\sqrt{3}-30$

$=10\left(4\sqrt{3}-3\right)$

$|v|^2=|v|^2\cos^2\theta+|v|^2\sin^2\theta$

$=100\left(3\sqrt{3}+4\right)^2+100\left(4\sqrt{3}-3\right)^2$

= 100(100) = 10,000

$|\mathbf{v}|=\sqrt{10,000}=100$

$\tan\theta=\dfrac{|\mathbf{v}|\sin\theta}{|\mathbf{v}|\cos\theta}=\dfrac{4\sqrt{3}-3}{3\sqrt{3}+4}$

$\theta=\tan^{-1}\left(\dfrac{4\sqrt{3}-3}{3\sqrt{3}+4}\right)\approx 23.13°$

The resultant force has magnitude 100 lb in the direction S 23.13° W.

19. The force of 300 N parallel to the plane has magnitude 300 sin 30° = 150 N. Thus, a force of 150 N parallel to the plane will just keep the weight from sliding.

20. Let a be the magnitude of the rope that makes an angle of 27.34°. Let b be the magnitude of the rope that makes an angle of 39.22°.

1. a sin 27.34° = b sin 39.22°

2. a cos 27.34° + b cos 39.22° = 258.5

Solve 1 for b and substitute in 2.

$a\cos 27.34°+a\dfrac{\sin 27.34°}{\sin 39.22°}\cos 39.22°=258.5$

$a=\dfrac{258.5}{\cos 27.34°+\sin 27.34°\cot 39.22°}\approx 178.15$

$b=\dfrac{a\sin 27.34°}{\sin 39.22°}\approx 129.40$

The magnitudes of the forces exerted by the ropes making angles of 27.34° and 39.22° are 178.15 lb and 129.40 lb, respectively.

21. Let θ be the angle the plane makes from north, measured clockwise.

425 sin θ = 45 sin 20°

$\sin\theta=\dfrac{9}{85}\sin 20°$

$\theta=\sin^{-1}\left(\dfrac{9}{85}\sin 20°\right)\approx 2.08°$

Let x be the speed of airplane with respect to the ground.

x = 45 cos 20° + 425 cos $\theta\approx$ 467

The plane flies in the direction N 2.08° E, flying 467 mi/h with respect to the ground.

22. Let **v** be his velocity relative to the surface. Let θ be the angle that his velocity relative to the surface makes with south, measured clockwise.

$|v|\cos\theta = 20, |v|\sin\theta = 3$

$|v|^2 = |v|^2\cos^2\theta + |v|^2\sin^2\theta = 400 + 9 = 409$

$|\mathbf{v}| = \sqrt{409} \approx 20.22$

$\tan\theta = \frac{|\mathbf{v}|\sin\theta}{|\mathbf{v}|\cos\theta} = \frac{3}{20}$

$\theta = \tan^{-1}\frac{3}{20} \approx 8.53°$

His velocity has magnitude 20.22 mi/h in the direction S 8.53° W.

23. Let x be the air speed.

$x\cos 60° = 40$

$x = \frac{40}{\cos 60°} = 80$

The air speed of the plane is 80 mi/hr

24. Let x be the air speed. Let θ be the angle that the plane makes with north measured counter-clockwise.

$x\cos\theta = 837 + 63\cos 11.5°$

$x\sin\theta = 63\sin 11.5°$

$x^2 = x^2\cos^2\theta + x^2\sin^2\theta$

$= (837 + 63\cos 11.5°)^2 + (63\sin 11.5°)^2$

$= 704{,}538 + 105{,}462\cos 11.5°$

$x = \sqrt{704{,}538 + 105{,}462\cos 11.5°} \approx 898.82$

$\tan\theta = \frac{x\sin\theta}{x\cos\theta} = \frac{63\sin 11.5°}{837 + 63\cos 11.5°}$

$\theta = \tan^{-1}\left(\frac{63\sin 11.5°}{837 + 63\cos 11.5°}\right) \approx 0.80°$

The plane should fly in the direction N 0.80° W at an air speed of 898.82 mi/h.

25. Let $\mathbf{u} = \langle u_1, u_2\rangle, \mathbf{v} = \langle v_1, v_2\rangle$, and $\mathbf{w} = \langle w_1, w_2\rangle$

a. $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle =$
$\langle v_1 + u_1, v_2 + u_2\rangle = \mathbf{v} + \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle =$
$\langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle =$
$\langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle =$
$\langle u_1, u_2\rangle\langle v_1 + w_1, v_2 + w_2\rangle = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

c. $\mathbf{u} + \mathbf{0} = \langle u_1 + 0, u_2 + 0\rangle = \langle u_1, u_2\rangle = \mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$ by part a.

d. $\mathbf{u} + \mathbf{(\text{-}u)} = \langle u_1, u_2\rangle + \langle -u_1, -u_2\rangle =$
$\langle u_1 + (-u_1), u_2 + (-u_2)\rangle = \langle 0, 0\rangle = \mathbf{0}$

e. $a(b\mathbf{u}) = a(\langle bu_1, bu_2\rangle) = \langle a(bu_1), a(bu_2)\rangle =$
$\langle (ab)u_1, (ab)u_2\rangle = (ab)\mathbf{u}$

f. $a(\mathbf{u} + \mathbf{v}) = a\langle u_1 + v_1, u_2 + v_2\rangle =$
$\langle a(u_1 + v_1), a(u_2 + v_2)\rangle =$
$\langle au_1 + av_1, au_2 + av_2\rangle =$
$\langle au_1, au_2\rangle + \langle av_1, av_2\rangle = a\mathbf{u} + a\mathbf{v}$

g. $(a + b)\mathbf{u} = \langle (a + b)u_1, (a + b)u_2\rangle =$
$\langle au_1 + bu_1, au_2 + bu_2\rangle =$
$\langle au_1, au_2\rangle + \langle bu_1, bu_2\rangle = a\mathbf{u} + b\mathbf{u}$

h. $1\mathbf{u} = \langle 1\cdot u_1, 1\cdot u_2\rangle = \langle u_1, u_2\rangle = \mathbf{u}$

i. $\|a\mathbf{u}\| = \|\langle au_1, au_2\rangle\| = \sqrt{(au_1)^2 + (au_2)^2} =$
$\sqrt{a^2u_1^2 + a^2u_2^2} = \sqrt{a^2(u_1^2 + u_2^2)} =$
$\sqrt{a^2}\sqrt{u_1^2 + u_2^2} = |a|\cdot\|\mathbf{u}\|$

26. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle, \mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$

a. $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle =$
$\langle v_1 + u_1, v_2 + u_2, v_3 + u_3\rangle = \mathbf{v} + \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$
$+ \langle w_1, w_2, w_3\rangle =$
$\langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3\rangle =$
$\langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3)\rangle =$
$\langle u_1, u_2, u_3\rangle\langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle =$
$\mathbf{u} + (\mathbf{v} + \mathbf{w})$

c. $\mathbf{u} + \mathbf{0} = \langle u_1 + 0, u_2 + 0, u_3 + 0\rangle = \langle u_1, u_2, u_3\rangle = \mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$ by part a.

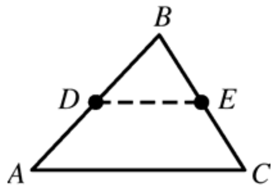
d. $\mathbf{u} + \mathbf{(\text{-}u)} = \langle u_1, u_2, u_3\rangle + \langle -u_1, -u_2, -u_3\rangle =$
$\langle u_1 + (-u_1), u_2 + (-u_2), u_3 + (-u_3)\rangle =$
$\langle 0, 0, 0\rangle = \mathbf{0}$

e. $a(b\mathbf{u}) = a(\langle bu_1, bu_2, bu_3\rangle) =$
$\langle a(bu_1), a(bu_2), a(bu_3)\rangle =$
$\langle (ab)u_1, (ab)u_2, (ab)u_3\rangle = (ab)\mathbf{u}$

g. $(a + b)\mathbf{u} = \langle (a + b)u_1, (a + b)u_2, (a + b)u_3\rangle =$
$\langle au_1 + bu_1, au_2 + bu_2, au_3 + bu_3\rangle =$
$\langle au_1, au_2, au_3\rangle + \langle bu_1, bu_2, bu_3\rangle = a\mathbf{u} + b\mathbf{u}$

h. $1\mathbf{u} = \langle 1\cdot u_1, 1\cdot u_2, 1\cdot u_3\rangle = \langle u_1, u_2, u_3\rangle = \mathbf{u}$

27. Given triangle *ABC*, let *D* be the midpoint of *AB* and *E* be the midpoint of *BC*. $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{AC}$, $\mathbf{z} = \overrightarrow{DE}$

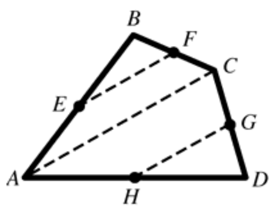


$\mathbf{u} + \mathbf{v} = \mathbf{w}$

$\mathbf{z} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} = \frac{1}{2}(\mathbf{u} + \mathbf{v}) = \frac{1}{2}\mathbf{w}$

Thus, *DE* is parallel to *AC*.

28. Given quadrilateral *ABCD*, let *E* be the midpoint of *AB*, *F* the midpoint of *BC*, *G* the midpoint of *CD*, and *H* the midpoint of *AD*.
ABC and *ACD* are triangles. From Problem 17, *EF* and *HG* are parallel to *AC*. Thus, *EF* is parallel to *HG*. By similar reasoning using triangles *ABD* and *BCD*, *EH* is parallel to *FG*. Therefore, *EFGH* is parallelogram.



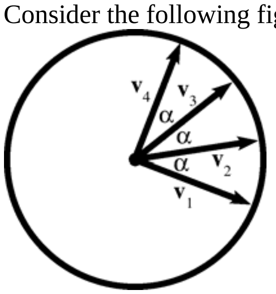
29. Let P_i be the tail of $\mathbf{v}_i$. Then

$\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n$

$= \overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \cdots + \overrightarrow{P_nP_1}$

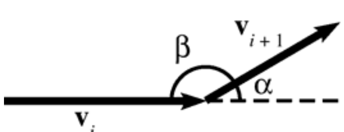
$= \overrightarrow{P_1P_1} = \mathbf{0}.$

30. Consider the following figure of the circle.



$\alpha = \frac{2\pi}{n}$

The vectors have the same length. Consider the following figure for adding vectors $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$.



Then $\beta = \pi - \frac{2\pi}{n}$. Note that the interior angle of a regular *n*-gon is $\pi - \frac{2\pi}{n}$. Thus the vectors (placed head to tail from $\mathbf{v}_1$ to $\mathbf{v}_n$) form a regular *n*-gon. From Problem 19, the sum of the vectors is **0**.

31. The components of the forces along the lines containing *AP*, *BP*, and *CP* are in equilibrium; that is,

$W = W\cos\alpha + W\cos\beta$
$W = W\cos\beta + W\cos\gamma$
$W = W\cos\alpha + W\cos\gamma$

Thus, $\cos\alpha + \cos\beta = 1$, $\cos\beta + \cos\gamma = 1$, and $\cos\alpha + \cos\gamma = 1$. Solving this system of equations results in $\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{2}$.

Hence $\alpha = \beta = \gamma = 60°$.
Therefore, $\alpha + \beta = \alpha + \gamma = \beta + \gamma = 120°$.

32. Let A', B', C' be the points where the weights are attached. The center of gravity is located $\frac{|AA'| + |BB'| + |CC'|}{3}$ units below the plane of the triangle. Then, using the hint, the system is in equilibrium when $|AA'| + |BB'| + |CC'|$ is maximum. Hence, it is in equilibrium when $|AP| + |BP| + |CP|$ is minimum, because the total length of the string is $|AP| + |AA'| + |BP| + |BB'| + |CP| + |CC'|$.

33. The components of the forces along the lines containing *AP*, *BP*, and *CP* are in equilibrium; that is,

$5w\cos\alpha + 4w\cos\beta = 3w$
$3w\cos\beta + 5w\cos\gamma = 4w$
$3w\cos\alpha + 4w\cos\gamma = 5w$

Thus,

$5\cos\alpha + 4\cos\beta = 3$
$3\cos\beta + 5\cos\gamma = 4$
$3\cos\alpha + 4\cos\gamma = 5.$

Solving this system of equations results in $\cos\alpha = \frac{3}{5}, \cos\beta = 0, \cos\gamma = \frac{4}{5},$ from which it follows that $\sin\alpha = \frac{4}{5}, \sin\beta = 1, \sin\gamma = \frac{3}{5}.$

Therefore, $\cos(\alpha+\beta) = -\frac{4}{5}, \cos(\alpha+\gamma) = 0,$

$\cos(\beta+\gamma) = -\frac{3}{5},$ so

$\alpha + \beta = \cos^{-1}\left(-\frac{4}{5}\right) \approx 143.13°,\ \ \alpha + \gamma = 90°,$

$\beta + \gamma = \cos^{-1}\left(-\frac{3}{5}\right) \approx 126.87°.$

This problem can be modeled with three strings going through A, four strings through B, and five strings through C, with equal weights attached to the twelve strings. Then the quantity to be minimized is $3|AP|+4|BP|+5|CP|$.

34. Written response.

35. By symmetry, the tension on each wire will be the same; denote it by $\|\mathbf{T}\|$ where $\mathbf{T}$ can be the tension vector along any of the wires. The chandelier exerts a force of 100 lbs. vertically downward. Each wire exerts a vertical tension of $\|\mathbf{T}\|\sin 45^\circ$ upward. Since a state of equilibrium exists,

$4\cdot\|\mathbf{T}\|\sin 45^\circ = 100$ or $\|\mathbf{T}\| = \dfrac{100}{2\sqrt{2}} \approx 35.36$

The tension in each wire is approximately 35.36 lbs.

36. By symmetry, the tension on each wire will be the same; denote it by $\|\mathbf{T}\|$ where $\mathbf{T}$ can be the tension vector along any of the wires. The chandelier exerts a force of 100 lbs. vertically downward. Each wire exerts a vertical tension of $\|\mathbf{T}\|\sin 45^\circ$ upward. Since a state of equilibrium exists, $3\cdot\|\mathbf{T}\|\sin 45^\circ = 100$ or $\|\mathbf{T}\| = \dfrac{200}{3\sqrt{2}} \approx 47.14$

The tension in each wire is approximately 47.14 lbs.

11.3 Concepts Review

1. $u_1v_1 + u_2v_2 + u_3v_3$; $\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$

2. 0

3. $\mathbf{F}\bullet\mathbf{D}$

4. $\langle A, B, C\rangle$

Problem Set 11.3

1. a. $2\mathbf{a} - 4\mathbf{b} = (-4\mathbf{i} + 6\mathbf{j}) + (-8\mathbf{i} + 12\mathbf{j})$
$= -12\mathbf{i} + 18\mathbf{j}$

b. $\mathbf{a}\cdot\mathbf{b} = (-2)(2) + (3)(-3) = -13$

c. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (-2\mathbf{i}+3\mathbf{j})\cdot(2\mathbf{i}-8\mathbf{j})$
$= (-2)(2) + (3)(-8) = -28$

d. $(-2\mathbf{a}+3\mathbf{b})\cdot 5\mathbf{c} = 5[(10\mathbf{i}-15\mathbf{j})\cdot(-5\mathbf{j})]$
$= 5[(10)(0) + (-15)(-5)] = 375$

e. $\|\mathbf{a}\|\mathbf{c}\cdot\mathbf{a} = \sqrt{4+9}[(0)(-2)+(-5)(3)] = -15\sqrt{13}$

f. $\mathbf{b}\cdot\mathbf{b} - \|\mathbf{b}\| = (2)(2)+(-3)(-3)-\sqrt{4+9}$
$= 13-\sqrt{13}$

2. a. $-4\mathbf{a} + 3\mathbf{b} = \langle -12, 4\rangle + \langle 3, -3\rangle = \langle -9, 1\rangle$

b. $\mathbf{b}\cdot\mathbf{c} = (1)(0)+(-1)(5) = -5$

c. $(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \langle 4,-2\rangle\cdot\langle 0,5\rangle$
$= (4)(0)+(-2)(5) = -10$

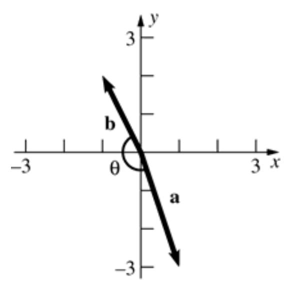
d. $2\mathbf{c}\cdot(3\mathbf{a}+4\mathbf{b}) = 2\langle 0,\ 5\rangle\cdot(\langle 9,\ -3\rangle + \langle 4,\ -4\rangle)$
$= 2\langle 0,\ 5\rangle\cdot\langle 13,\ -7\rangle = 2[(0)(13)+(5)(-7)]$
$= -70$

e. $\|\mathbf{b}\|\mathbf{b}\cdot\mathbf{a} = \sqrt{1+1}[(1)(3)+(-1)(-1)] = 4\sqrt{2}$

f. $\|\mathbf{c}\|^2 - \mathbf{c}\cdot\mathbf{c} = \left(\sqrt{0+25}\right)^2 - [(0)(0)+(5)(5)]$
$= 0$

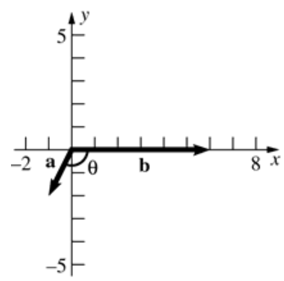
3. a. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(1)(-1)+(-3)(2)}{\left(\sqrt{10}\right)\left(\sqrt{5}\right)} = -\dfrac{7}{\sqrt{50}}$

$= -\dfrac{7}{5\sqrt{2}} \approx -0.9899$

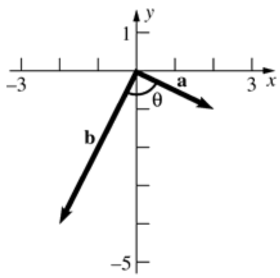


b. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(-1)(6)+(-2)(0)}{\left(\sqrt{5}\right)(6)} = -\dfrac{6}{6\sqrt{5}}$

$= -\dfrac{1}{\sqrt{5}} \approx -0.4472$

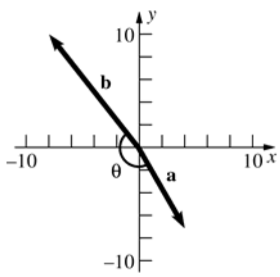


c. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(2)(-2)+(-1)(-4)}{\left(\sqrt{5}\right)\left(2\sqrt{5}\right)} = \dfrac{0}{10} = 0$



d. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(4)(-8)+(-7)(10)}{\left(\sqrt{65}\right)\left(2\sqrt{41}\right)}$

$= \dfrac{-102}{2\sqrt{2665}} = -\dfrac{51}{\sqrt{2665}} \approx -0.9879$

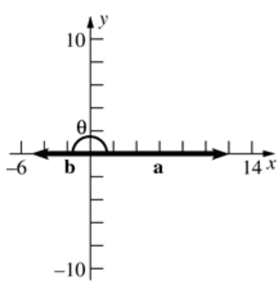


4. a.

$\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(12)(-5)+(0)(0)}{(12)(5)}$

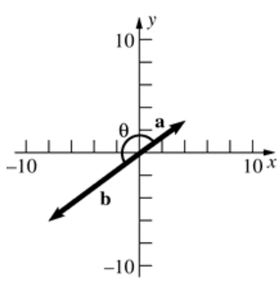
$= \dfrac{-60}{60} = -1$

$\theta = \cos^{-1}-1 = 180^\circ$



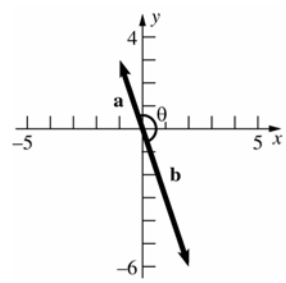
b. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(4)(-8)+(3)(-6)}{(5)(10)} = \dfrac{-50}{50} = -1$

$\theta = \cos^{-1}-1 = 180^\circ$



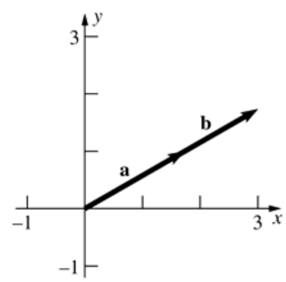
c. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(-1)(2)+(3)(-6)}{\left(\sqrt{10}\right)\left(2\sqrt{10}\right)} = \dfrac{-20}{20} = -1$

$\theta = \cos^{-1}-1 = 180^\circ$



d. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{\left(\sqrt{3}\right)(3)+(1)\left(\sqrt{3}\right)}{(2)\left(2\sqrt{3}\right)} = \dfrac{4\sqrt{3}}{4\sqrt{3}} = 1$

$\theta = \cos^{-1}1 = 0^\circ$



5. a. $\mathbf{a}\bullet\mathbf{b} = (1)(0)+(2)(1)+(-1)(1) = 1$

b. $(\mathbf{a}+\mathbf{c})\bullet\mathbf{b} = (3\mathbf{j}+\mathbf{k})\bullet(\mathbf{j}+\mathbf{k})$
$= (0)(0)+(3)(1)+(1)(1) = 4$

c. $\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{\sqrt{1^2+2^2+(-1)^2}}(\mathbf{i}+2\mathbf{j}-\mathbf{k})$
$= \dfrac{\sqrt{6}}{6}\mathbf{i} + \dfrac{\sqrt{6}}{3}\mathbf{j} - \dfrac{\sqrt{6}}{6}\mathbf{k}$

d. $(\mathbf{b}-\mathbf{c})\bullet\mathbf{a} = (\mathbf{i}-\mathbf{k})\bullet(\mathbf{i}+2\mathbf{j}-\mathbf{k})$
$= (1)(1)+(0)(2)+(-1)(-1) = 2$

e. $\dfrac{\mathbf{a}\bullet\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \dfrac{(1)(0)+(2)(1)+(-1)(1)}{\sqrt{1^2+2^2+(-1)^2}\sqrt{0^2+1^2+1^2}}$
$= \dfrac{1}{\sqrt{6}\sqrt{2}} = \dfrac{\sqrt{3}}{6}$

f. By Theorem A (5), $\mathbf{b}\bullet\mathbf{b} - \|\mathbf{b}\|^2 = 0$

6. a. $\mathbf{a}\bullet\mathbf{c} = (\sqrt{2})(-2)+(\sqrt{2})(2)+(0)(1) = 0$

b. $(\mathbf{a}-\mathbf{c})\bullet\mathbf{b} = \left\langle\sqrt{2}-(-2), \sqrt{2}-2, 0-1\right\rangle\bullet\langle 1,-1,1\rangle =$
$(2+\sqrt{2})(1)+(\sqrt{2}-2)(-1)+(-1)(1) = 3$

c. $\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + 0^2}} \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle =$

$\dfrac{1}{2}\left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle = \left\langle \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}, 0 \right\rangle$

d. $(\mathbf{b} - \mathbf{c}) \bullet \mathbf{a} = \langle 3, -3, 0 \rangle \bullet \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle =$

$(3)(\sqrt{2}) + (-3)(\sqrt{2}) + (0)(0) = 0$

e. $\dfrac{\mathbf{b} \bullet \mathbf{c}}{\|\mathbf{b}\|\|\mathbf{c}\|} = \dfrac{(1)(-2) + (-1)(2) + (1)(1)}{\sqrt{1^2 + (-1)^2 + 1^2}\sqrt{(-2)^2 + 2^2 + 1^2}} =$

$\dfrac{-3}{\sqrt{3}\sqrt{9}} = -\dfrac{\sqrt{3}}{3}$

f. By Theorem A (5), $\mathbf{a} \bullet \mathbf{a} - \|\mathbf{a}\|^2 = 0$

7. The basic formula is $\cos\theta = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$

$\theta_{\mathbf{a},\mathbf{b}} = \cos^{-1}\left(\dfrac{\sqrt{2} - \sqrt{2} + 0}{\sqrt{4} \cdot \sqrt{3}} \right) =$

$\cos^{-1} 0 = 90^\circ$

$\theta_{\mathbf{a},\mathbf{c}} = \cos^{-1}\left(\dfrac{-2\sqrt{2} + 2\sqrt{2} + 0}{\sqrt{4} \cdot \sqrt{9}} \right) =$

$\cos^{-1} 0 = 90^\circ$

$\theta_{\mathbf{b},\mathbf{c}} = \cos^{-1}\left(\dfrac{-2 - 2 + 1}{\sqrt{3} \cdot \sqrt{9}} \right) =$

$\cos^{-1}\left(-\dfrac{\sqrt{3}}{3} \right) \approx 125.26^\circ$

8. The basic formula is $\cos\theta_{\mathbf{u},\mathbf{v}} = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$

$\theta_{\mathbf{a},\mathbf{b}} = \cos^{-1}\left(\dfrac{\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + 0}{\sqrt{1} \cdot \sqrt{2}} \right) =$

$\cos^{-1} 0 = 90^\circ$

$\theta_{\mathbf{a},\mathbf{c}} = \cos^{-1}\left(\dfrac{\frac{-2\sqrt{3}}{3} + \frac{-2\sqrt{3}}{3} + \frac{\sqrt{3}}{3}}{\sqrt{1} \cdot \sqrt{9}} \right) =$

$\cos^{-1} - \dfrac{\sqrt{3}}{3} \approx 125.26^\circ$

$\theta_{\mathbf{b},\mathbf{c}} = \cos^{-1}\left(\dfrac{-2 + 2 + 0}{\sqrt{2} \cdot \sqrt{9}} \right) =$

$\cos^{-1} 0 = 90^\circ$

9. The basic formulae are

$\cos\alpha_{\mathbf{u}} = \dfrac{u_1}{\|\mathbf{u}\|} \quad \cos\beta_{\mathbf{u}} = \dfrac{u_2}{\|\mathbf{u}\|} \quad \cos\gamma_{\mathbf{u}} = \dfrac{u_3}{\|\mathbf{u}\|}$

a. $\mathbf{a} = \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle \qquad \|\mathbf{a}\| = 2$

$\cos\alpha_{\mathbf{a}} = \dfrac{a_1}{\|\mathbf{a}\|} = \dfrac{\sqrt{2}}{2}, \quad \alpha_{\mathbf{a}} = 45^\circ$

$\cos\beta_{\mathbf{a}} = \dfrac{a_2}{\|\mathbf{a}\|} = \dfrac{\sqrt{2}}{2}, \quad \beta_{\mathbf{a}} = 45^\circ$

$\cos\gamma_{\mathbf{a}} = \dfrac{a_3}{\|\mathbf{a}\|} = \dfrac{0}{2} = 0, \quad \gamma_{\mathbf{a}} = 90^\circ$

b. $\mathbf{b} = \langle 1, -1, 1 \rangle \qquad \|\mathbf{b}\| = \sqrt{3}$

$\cos\alpha_{\mathbf{b}} = \dfrac{b_1}{\|\mathbf{b}\|} = \dfrac{1}{\sqrt{3}} \approx 0.577, \quad \alpha_{\mathbf{b}} \approx 54.74^\circ$

$\cos\beta_{\mathbf{b}} = \dfrac{b_2}{\|\mathbf{b}\|} = \dfrac{-1}{\sqrt{3}} \approx -0.577, \quad \beta_{\mathbf{b}} \approx 125.26^\circ$

$\cos\gamma_{\mathbf{b}} = \dfrac{b_3}{\|\mathbf{b}\|} = \dfrac{1}{\sqrt{3}} \approx 0.577, \quad \gamma_{\mathbf{b}} \approx 54.74^\circ$

c. $\mathbf{c} = \langle -2, 2, 1 \rangle \qquad \|\mathbf{c}\| = 3$

$\cos\alpha_{\mathbf{c}} = \dfrac{c_1}{\|\mathbf{c}\|} = -\dfrac{2}{3}, \quad \alpha_{\mathbf{c}} \approx 131.81^\circ$

$\cos\beta_{\mathbf{c}} = \dfrac{c_2}{\|\mathbf{c}\|} = \dfrac{2}{3}, \quad \beta_{\mathbf{c}} \approx 48.19^\circ$

$\cos\gamma_{\mathbf{c}} = \dfrac{c_3}{\|\mathbf{c}\|} = \dfrac{1}{3}, \quad \gamma_{\mathbf{c}} \approx 70.53^\circ$

10. The basic formulae are

$\cos\alpha_{\mathbf{u}} = \dfrac{u_1}{\|\mathbf{u}\|} \quad \cos\beta_{\mathbf{u}} = \dfrac{u_2}{\|\mathbf{u}\|} \quad \cos\gamma_{\mathbf{u}} = \dfrac{u_3}{\|\mathbf{u}\|}$

a. $\mathbf{a} = \left\langle \dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3} \right\rangle \qquad \|\mathbf{a}\| = 1$

$\cos\alpha_{\mathbf{a}} = \dfrac{a_1}{\|\mathbf{a}\|} = \dfrac{\sqrt{3}}{3} \approx 0.577, \quad \alpha_{\mathbf{a}} \approx 54.74^\circ$

$\cos\beta_{\mathbf{a}} = \dfrac{a_2}{\|\mathbf{a}\|} = \dfrac{\sqrt{3}}{3} \approx 0.577, \quad \beta_{\mathbf{a}} \approx 54.74^\circ$

$\cos\gamma_{\mathbf{a}} = \dfrac{a_3}{\|\mathbf{a}\|} = \dfrac{\sqrt{3}}{3} \approx 0.577, \quad \gamma_{\mathbf{a}} \approx 54.74^\circ$

b. $\mathbf{b} = \langle 1, -1, 0 \rangle \qquad \|\mathbf{b}\| = \sqrt{2}$

$\cos\alpha_{\mathbf{b}} = \frac{b_1}{\|\mathbf{b}\|} = \frac{\sqrt{2}}{2}, \quad \alpha_{\mathbf{b}} = 45^\circ$

$\cos\beta_{\mathbf{b}} = \frac{b_2}{\|\mathbf{b}\|} = -\frac{\sqrt{2}}{2}, \quad \beta_{\mathbf{b}} \approx 135^\circ$

$\cos\gamma_{\mathbf{b}} = \frac{b_3}{\|\mathbf{b}\|} = \frac{0}{\sqrt{2}} = 0, \quad \gamma_{\mathbf{b}} = 90^\circ$

c. $\mathbf{c} = \langle -2, -2, 1 \rangle \qquad \|\mathbf{c}\| = 3$

$\cos\alpha_{\mathbf{c}} = \frac{c_1}{\|\mathbf{c}\|} = -\frac{2}{3}, \quad \alpha_{\mathbf{c}} \approx 131.81^\circ$

$\cos\beta_{\mathbf{c}} = \frac{c_2}{\|\mathbf{c}\|} = -\frac{2}{3}, \quad \beta_{\mathbf{c}} \approx 131.81^\circ$

$\cos\gamma_{\mathbf{c}} = \frac{c_3}{\|\mathbf{c}\|} = \frac{1}{3}, \quad \gamma_{\mathbf{c}} \approx 70.53^\circ$

11. $\langle 6, 3 \rangle \bullet \langle -1, 2 \rangle = (6)(-1) + (3)(2) = 0$
Therefore the vectors are orthogonal

12. $\mathbf{a} \bullet \mathbf{b} = (1)(1) + (1)(-1) + (1)(0) = 0$
$\mathbf{a} \bullet \mathbf{c} = (1)(-1) + (1)(-1) + (1)(2) = 0$
$\mathbf{b} \bullet \mathbf{c} = (1)(-1) + (-1)(-1) + (0)(2) = 0$
Therefore the vectors are mutually orthogonal.

13. $\mathbf{a} \bullet \mathbf{b} = (1)(1) + (-1)(1) + (0)(0) = 0$
$\mathbf{a} \bullet \mathbf{c} = (1)(0) + (-1)(0) + (0)(2) = 0$
$\mathbf{b} \bullet \mathbf{c} = (1)(0) + (1)(0) + (0)(2) = 0$
Therefore the vectors are mutually orthogonal.

14. $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v}$
$= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$
Thus, $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$ or $\|\mathbf{u}\| = \|\mathbf{v}\|$

15. If $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is perpendicular to $-4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + \mathbf{j}$, then $-4x + 5y + z = 0$ and $4x + y = 0$ since the dot product of perpendicular vectors is 0. Solving these equations yields $y = -4x$ and $z = 24x$. Hence, for any x, $x\mathbf{i} - 4x\mathbf{j} + 24x\mathbf{k}$ is perpendicular to the given vectors.

$\|x\mathbf{i} - 4x\mathbf{j} + 24x\mathbf{k}\| = \sqrt{x^2 + 16x^2 + 576x^2}$
$= |x|\sqrt{593}$

This length is 10 when $x = \pm\frac{10}{\sqrt{593}}$. The vectors are $\frac{10}{\sqrt{593}}\mathbf{i} - \frac{40}{\sqrt{593}}\mathbf{j} + \frac{240}{\sqrt{593}}\mathbf{k}$ and $-\frac{10}{\sqrt{593}}\mathbf{i} + \frac{40}{\sqrt{593}}\mathbf{j} - \frac{240}{\sqrt{593}}\mathbf{k}$.

16. If $\langle x, y, z \rangle$ is perpendicular to both $\langle 1, \text{-}2, \text{-}3 \rangle$ and $\langle \text{-}3, 2, 0 \rangle$, then $x - 2y - 3z = 0$ and $-3x + 2y = 0$. Solving these equations for y and z in terms of x yields $y = \frac{3}{2}x, z = -\frac{2}{3}x.$ Thus, all the vectors have the form $\left\langle x, \frac{3}{2}x, -\frac{2}{3}x \right\rangle$ where x is a real number.

17. A vector equivalent to $\overrightarrow{BA}$ is
$\mathbf{u} = \langle 1+4, 2-5, 3-6 \rangle = \langle 5, -3, -3 \rangle$.
A vector equivalent to $\overrightarrow{BC}$ is
$\mathbf{v} = \langle 1 + 4, 0 - 5, 1 - 6 \rangle = \langle 5, -5, -5 \rangle$.

$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{5 \cdot 5 + (-3)(-5) + (-3)(-5)}{\sqrt{25+9+9}\sqrt{25+25+25}}$

$= \frac{55}{\sqrt{43}\sqrt{75}} = \frac{11}{\sqrt{129}}$, so $\theta = \cos^{-1}\frac{11}{\sqrt{129}} \approx 14.4^\circ.$

18. A vector equivalent to $\overrightarrow{BA}$ is
$\mathbf{u} = \langle 6 - 3, 3 - 1, 3 + 1 \rangle = \langle 3, 2, 4 \rangle$.
A vector equivalent to $\overrightarrow{BC}$ is
$\mathbf{v} = \langle -1 - 3, 10 - 1, -2.5 + 1 \rangle = \langle -4, 9, -1.5 \rangle$.
$\mathbf{u} \cdot \mathbf{v} = 3(-4) + 2 \cdot 9 + 4 \cdot (-1.5) = 0$ so $\mathbf{u}$ is perpendicular to $\mathbf{v}$. Thus the angle at B is a right angle.

19. $\langle c, 6 \rangle \bullet \langle c, -4 \rangle = 0 \Rightarrow c^2 - 24 = 0 \Rightarrow$
$c^2 = 24 \Rightarrow c = \pm 2\sqrt{6}$

20. $(2c\mathbf{i} - 8\mathbf{j}) \bullet (3\mathbf{i} + c\mathbf{j}) = 0 \Rightarrow 6c - 8c = 0 \Rightarrow$
$-2c = 0 \Rightarrow c = 0$

21. $(c\mathbf{i} + \mathbf{j} + \mathbf{k}) \bullet (0\mathbf{i} + 2\mathbf{j} + d\mathbf{k}) = 0 \Rightarrow 0c + 2 + d = 0 \Rightarrow$
c is any number, $d = -2$

22. $\langle a, 0, 1 \rangle \bullet \langle 0, 2, b \rangle = 0 \Rightarrow b = 0$
$\langle a, 0, 1 \rangle \bullet \langle 1, c, 1 \rangle = 0 \Rightarrow a + 1 = 0$
$\langle 0, 2, b \rangle \bullet \langle 1, c, 1 \rangle = 0 \Rightarrow 2c + b = 0$
Thus: $a = -1$ and $c = b = 0$

For problems 23-34, the formula to use is

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{b} \bullet \mathbf{a}}{\|\mathbf{a}\|^2} \right) \mathbf{a}$$

23. $\mathbf{u} = \langle 1, 2 \rangle$, $\mathbf{v} = \langle 2, -1 \rangle$, $\mathbf{w} = \langle 1, 5 \rangle$

$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{(1)(2) + (2)(-1)}{2^2 + (-1)^2}\langle 2, -1 \rangle = \mathbf{0}$

24. $\mathbf{u}=\langle 1,2\rangle$, $\mathbf{v}=\langle 2,-1\rangle$, $\mathbf{w}=\langle 1,5\rangle$

$$\text{proj}_{\mathbf{u}}\mathbf{v}=\frac{(2)(1)+(-1)(2)}{1^2+2^2}\langle 1,2\rangle=\mathbf{0}$$

25. $\mathbf{u}=\langle 1,2\rangle$, $\mathbf{v}=\langle 2,-1\rangle$, $\mathbf{w}=\langle 1,5\rangle$

$$\text{proj}_{\mathbf{u}}\mathbf{w}=\frac{(1)(1)+(5)(2)}{1^2+2^2}\langle 1,2\rangle=\left\langle \frac{11}{5},\frac{22}{5}\right\rangle$$

26. $\mathbf{u}=\langle 1,2\rangle$, $\mathbf{v}=\langle 2,-1\rangle$, $\mathbf{w}=\langle 1,5\rangle$, $\mathbf{w}-\mathbf{v}=\langle -1,6\rangle$

$$\text{proj}_{\mathbf{u}}(\mathbf{w}-\mathbf{v})=\frac{(-1)(1)+(6)(2)}{1^2+2^2}\langle 1,2\rangle=\left\langle \frac{11}{5},\frac{22}{5}\right\rangle$$

27. $\mathbf{u}=\langle 1,2\rangle$, $\mathbf{v}=\langle 2,-1\rangle$, $\mathbf{w}=\langle 1,5\rangle$

$$\text{proj}_{\mathbf{j}}\mathbf{u}=\frac{(1)(0)+(2)(1)}{0^2+(1)^2}\langle 0,1\rangle=\langle 0,2\rangle$$

28. $\mathbf{u}=\langle 1,2\rangle$, $\mathbf{v}=\langle 2,-1\rangle$, $\mathbf{w}=\langle 1,5\rangle$

$$\text{proj}_{\mathbf{i}}\mathbf{u}=\frac{(1)(1)+(2)(0)}{1^2+(0)^2}\langle 1,0\rangle=\langle 1,0\rangle$$

29. $\mathbf{u}=\langle 3,2,1\rangle$, $\mathbf{v}=\langle 2,0,-1\rangle$, $\mathbf{w}=\langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{v}}\mathbf{u}=\frac{(3)(2)+(2)(0)+(1)(-1)}{2^2+0^2+(-1)^2}\langle 2,0,-1\rangle=$$
$$\langle 2,0,-1\rangle$$

30. $\mathbf{u}=\langle 3,2,1\rangle$, $\mathbf{v}=\langle 2,0,-1\rangle$, $\mathbf{w}=\langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{u}}\mathbf{v}=\frac{(2)(3)+(0)(2)+(-1)(1)}{3^2+2^2+1^2}\langle 3,2,1\rangle=$$
$$\left\langle \frac{15}{14},\frac{10}{14},\frac{5}{14}\right\rangle$$

31. $\mathbf{u}=\langle 3,2,1\rangle$, $\mathbf{v}=\langle 2,0,-1\rangle$, $\mathbf{w}=\langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{u}}\mathbf{w}=\frac{(1)(3)+(5)(2)+(-3)(1)}{3^2+2^2+1^2}\langle 3,2,1\rangle=$$
$$\left\langle \frac{15}{7},\frac{10}{7},\frac{5}{7}\right\rangle$$

32. $\mathbf{u}=\langle 3,2,1\rangle$, $\mathbf{v}=\langle 2,0,-1\rangle$, $\mathbf{w}=\langle 1,5,-3\rangle$

$\mathbf{w}+\mathbf{v}=\langle 3,5,-4\rangle$

$$\text{proj}_{\mathbf{u}}(\mathbf{w}+\mathbf{v})=\frac{(3)(3)+(5)(2)+(-4)(1)}{3^2+2^2+1^2}\langle 3,2,1\rangle=$$
$$\left\langle \frac{45}{14},\frac{30}{14},\frac{15}{14}\right\rangle$$

33. $\mathbf{u}=\langle 3,2,1\rangle$, $\mathbf{v}=\langle 2,0,-1\rangle$, $\mathbf{w}=\langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{k}}\mathbf{u}=\frac{(3)(0)+(2)(0)+(1)(1)}{0^2+0^2+(1)^2}\langle 0,0,1\rangle=\langle 0,0,1\rangle$$

34. $\mathbf{u}=\langle 3,2,1\rangle$, $\mathbf{v}=\langle 2,0,-1\rangle$, $\mathbf{w}=\langle 1,5,-3\rangle$

$$\text{proj}_{\mathbf{i}}\mathbf{u}=\frac{(3)(1)+(2)(0)+(1)(0)}{1^2+0^2+0^2}\langle 1,0,0\rangle=\langle 3,0,0\rangle$$

35. **a.** $$\text{proj}_{\mathbf{u}}\mathbf{u}=\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\right)\mathbf{u}=\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}=\mathbf{u}$$

b. $$\text{proj}_{-\mathbf{u}}\mathbf{u}=\left(\frac{\mathbf{u}\cdot(-\mathbf{u})}{\|-\mathbf{u}\|^2}\right)(-\mathbf{u})=\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}=\mathbf{u}$$

36. **a.** $$\text{proj}_{\mathbf{u}}(-\mathbf{u})=\left(\frac{(-\mathbf{u})\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\right)(-\mathbf{u})$$
$$=\left(\frac{-\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)(-\mathbf{u})=\mathbf{u}$$

b. $$\text{proj}_{-\mathbf{u}}(-\mathbf{u})=\left(\frac{(-\mathbf{u})\cdot(-\mathbf{u})}{\|-\mathbf{u}\|^2}\right)(-\mathbf{u})$$
$$=\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)(-\mathbf{u})=-\mathbf{u}$$

37. $\mathbf{u}\cdot\mathbf{v}=(-1)(-1)+5\cdot 1+3(-1)=3$

$\|\mathbf{v}\|=\sqrt{1+1+1}=\sqrt{3}$

$$\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}=\frac{3}{\sqrt{3}}=\sqrt{3}$$

38. $\mathbf{u}\cdot\mathbf{v}=5\left(-\sqrt{5}\right)+5(\sqrt{5})+2(1)=2$

$\|\mathbf{v}\|=\sqrt{5+5+1}=\sqrt{11}$

$$\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}=\frac{2}{\sqrt{11}}$$

39. $\|\mathbf{u}\|=(4+9+z^2)^{1/2}=5$ and $z>0$, so $z=2\sqrt{3}\approx 3.4641$.

40. $\cos^2(46°)+\cos^2(108°)+\cos^2\gamma=1$

$\Rightarrow \cos\gamma\approx \pm 0.6496$

$\Rightarrow \gamma\approx 49.49°$ or $\gamma\approx 130.51°$

41. There are infinitely many such pairs. Note that $\langle -4,2,5\rangle\cdot\langle 1,2,0\rangle=-4+4+0=0$, so $\mathbf{u}=\langle 1,2,0\rangle$ is perpendicular to $\langle -4,2,5\rangle$. For any c, $\langle -2,1,c\rangle\cdot\langle 1,2,0\rangle=-2+2+0=0$ so $\mathbf{v}=\langle -2,1,c\rangle$ is a candidate.

$\langle -4,2,5\rangle\cdot\langle -2,1,c\rangle=8+2+5c$

$8+2+5c=0\Rightarrow c=-2$, so one pair is $\mathbf{u}=\langle 1,2,0\rangle$, $\mathbf{v}=\langle -2,1,-2\rangle$.

42. The midpoint is
$$\left(\frac{3+5}{2}, \frac{2-7}{2}, \frac{-1+2}{2}\right) = \left(4, -\frac{5}{2}, \frac{1}{2}\right),$$ so the vector is $\left\langle 4, -\frac{5}{2}, \frac{1}{2}\right\rangle$.

43. The following do not make sense.

a. $\mathbf{v}\cdot\mathbf{w}$ is not a vector.

b. $\mathbf{u}\cdot\mathbf{w}$ is not a vector.

44. The following do not make sense.

c. $\|\mathbf{u}\|$ is not a vector.

d. $\mathbf{u}+\mathbf{v}$ is not a scalar.

45.
$$a\mathbf{u}+b\mathbf{u} = a\langle u_1, u_2\rangle + b\langle u_1, u_2\rangle$$
$$= \langle au_1, au_2\rangle + \langle bu_1, bu_2\rangle = \langle au_1+bu_1, au_2+bu_2\rangle$$
$$\langle (a+b)u_1, (a+b)u_2\rangle = (a+b)\langle u_1, u_2\rangle$$
$$= (a+b)\mathbf{u}$$

46. $\mathbf{u}\bullet\mathbf{v} = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = \mathbf{v}\bullet\mathbf{u}$

47.
$$c(\mathbf{u}\cdot\mathbf{v}) = c\left(\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle\right)$$
$$= c(u_1v_1 + u_2v_2) = c(u_1v_1) + c(u_2v_2)$$
$$= (cu_1)v_1 + (cu_2)v_2 = \langle cu_1, cu_2\rangle \cdot \langle v_1, v_2\rangle$$
$$= \left(c\langle u_1, u_2\rangle\right)\cdot\langle v_1, v_2\rangle = (c\mathbf{u})\cdot\mathbf{v}$$

48.
$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \langle u_1, u_2\rangle \cdot \left(\langle v_1, v_2\rangle + \langle w_1, w_2\rangle\right)$$
$$= \langle u_1, u_2\rangle \cdot \langle v_1 + w_1, v_2 + w_2\rangle$$
$$= u_1(v_1+w_1) + u_2(v_2+w_2)$$
$$= (u_1v_1 + u_2v_2) + (u_1w_1 + u_2w_2) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$$

49. $\mathbf{0}\bullet\mathbf{u} = 0u_1 + 0u_2 = 0$

50. $\mathbf{u}\bullet\mathbf{u} = u_1^{\ 2} + u_2^{\ 2} = \left(\sqrt{u_1^{\ 2}+u_2^{\ 2}}\right)^2 = \|\mathbf{u}\|^2$

51. $\mathbf{r} = k\mathbf{a} + m\mathbf{b} \Rightarrow 7 = k(3) + m(-3)$ and
$-8 = k(-2) + m(4)$
$3k - 3m = 7$
$-2k + 4m = -8$
Solve the system of equations to get
$k = \frac{2}{3}, m = -\frac{5}{3}.$

52. $\mathbf{r} = k\mathbf{a} + m\mathbf{b} \Rightarrow 6 = k(-4) + m(2)$ and
$-7 = k(3) + m(-1)$
$-4k + 2m = 6$
$3k - m = -7$
Solve the system of equations to get
$k = -4$, $m = -5$.

53. a and b cannot both be zero. If $a = 0$, then the line $ax + by = c$ is horizontal and $\mathbf{n} = b\mathbf{j}$ is vertical, so $\mathbf{n}$ is perpendicular to the line. Use a similar argument if $b = 0$. If $a \neq 0$ and $b \neq 0$, then $P_1\left(\frac{c}{a}, 0\right)$ and $P_2\left(0, \frac{c}{b}\right)$ are points on the line.
$$\mathbf{n}\cdot\overrightarrow{P_1P_2} = (a\mathbf{i} + b\mathbf{j})\cdot\left(-\frac{c}{a}\mathbf{i} + \frac{c}{b}\mathbf{j}\right) = -c + c = 0$$

54.
$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$
$$+(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = [\mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}]$$
$$+[\mathbf{u}\cdot\mathbf{u} - 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}] = 2(\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v})$$
$$= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

55.
$$\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$
$$-(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = [\mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}]$$
$$-[\mathbf{u}\cdot\mathbf{u} - 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}] = 4(\mathbf{u}\cdot\mathbf{v})$$
so $\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$

56. Place the cube so that its corners are at the points (0, 0, 0), (1, 0, 0), (0,1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0,1, 1), and (1, 1, 1). The main diagonals are (0, 0, 0) to (1, 1, 1), (1, 0, 0) to (0, 1, 1), (0, 1, 0) to (1, 0, 1), and (0, 0, 1) to (1, 1, 0). The corresponding vectors are $\langle 1, 1, 1\rangle$, $\langle -1,1,1\rangle$, $\langle 1,-1,1\rangle$, and $\langle 1,1,-1\rangle$.
Because of symmetry, we need only address one situation; let's choose the diagonal from $(0,0,0)$ to $(1,1,1)$ and the face that lies in the xy – plane. A vector in the direction of the diagonal is $\mathbf{d} = \langle 1,1,1\rangle$ and a vector normal to the chosen face is $\mathbf{n} = \langle 0,0,1\rangle$. The angle between the diagonal and the face is the *complement* of the angle between $\mathbf{d}$ and $\mathbf{n}$; that is
$$90^\circ - \theta = 90^\circ - \cos^{-1}\left(\frac{\mathbf{d}\bullet\mathbf{n}}{\|\mathbf{d}\|\|\mathbf{n}\|}\right) =$$
$$90^\circ - \cos^{-1}\left(\frac{1}{\sqrt{3}\sqrt{1}}\right) = 90^\circ - \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)$$
$$\approx 90^\circ - 54.74^\circ = 35.26^\circ$$

57. Place the box so that its corners are at the points (0, 0, 0), (4, 0, 0), (0, 6, 0), (4, 6, 0), (0, 0, 10), (4, 0, 10), (0, 6, 10), and (4, 6, 10). The main diagonals are (0, 0, 0) to (4, 6, 10), (4, 0, 0) to (0, 6, 10), (0, 6, 0) to (4, 0, 10), and (0, 0, 10) to (4, 6, 0). The corresponding vectors are $\langle 4, 6, 10\rangle$, $\langle -4, 6, 10\rangle$, $\langle 4, -6, 10\rangle$, and $\langle 4, 6, -10\rangle$.

All of these vectors have length $\sqrt{16+36+100} = \sqrt{152}$. Thus, the smallest angle θ between any pair, **u** and **v** of the diagonals is found from the largest value of $\mathbf{u}\cdot\mathbf{v}$, since

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{152}.$$

There are six ways of pairing the four vectors. The largest value of $\mathbf{u}\cdot\mathbf{v}$ is 120 which occurs with

u = $\langle 4, 6, 10\rangle$ and **v** = $\langle -4, 6, 10\rangle$. Thus,

$$\cos\theta = \frac{120}{152} = \frac{15}{19} \text{ so } \theta = \cos^{-1}\frac{15}{19} \approx 37.86°.$$

58. Place the box so that its corners are at the points (0, 0, 0), (1, 0, 0), (0,1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0,1, 1), and (1, 1, 1). The main diagonals are (0, 0, 0) to (1, 1, 1), (1, 0, 0) to (0, 1, 1), (0, 1, 0) to (1, 0, 1), and (0, 0, 1) to (1, 1, 0). The corresponding vectors are $\langle 1, 1, 1\rangle$, $\langle -1, 1, 1\rangle$, $\langle 1, -1, 1\rangle$, and $\langle 1, 1, -1\rangle$.

All of these vectors have length $\sqrt{1+1+1} = \sqrt{3}$. Thus, the angle θ between any pair, **u** and **v** of the diagonals is found from

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{3}.$$

There are six ways of pairing the four vectors, but due to symmetry, there are only two cases we need to consider. In these cases, $\mathbf{u}\cdot\mathbf{v} = 1$ or $\mathbf{u}\cdot\mathbf{v} = -1$. Thus we get that $\cos\theta = \frac{1}{3}$ or $\cos\theta = -\frac{1}{3}$. Solving for θ gives $\theta \approx 70.53^\circ$ or $\theta \approx 109.47^\circ$.

59. Work = $\mathbf{F}\cdot\mathbf{D} = (3\mathbf{i}+10\mathbf{j})\cdot(10\mathbf{j})$
= 0 + 100 = 100 joules

60. **F** = 100 sin 70°**i** – 100 cos 70°**j**
D = 30**i**
Work = $\mathbf{F}\cdot\mathbf{D}$
$= (100\sin 70°)(30) + (-100\cos 70°)(0)$
= 3000 sin 70° ≈ 2819 joules

61. **D** = 5**i** + 8**j**
Work = **F** · **D** = (6)(5) + (8)(8) = 94 ft-lb

62. **D** = 12**j**
Work = **F**· **D** = (–5)(0) + (8)(12) = 96 joules

63. **D** = (4 – 0)**i** + (4 – 0)**j** + (0– 8)**k** = 4 +4**j** – 8 **k**
Thus, $W = \mathbf{F}\cdot\mathbf{D} = 0(4) + 0(4) - 4(-8) = 32$ joules.

64. **D** = (9 – 2)**i** + (4 – 1)**j** + (6 – 3)**k** = 7**i** + 3**j** + 3**k**
Thus, $W = \mathbf{F}\cdot\mathbf{D} = 3(7) - 6(3) + 7(3) = 24$ ft-lbs.

65. $2(x-1) - 4(y-2) + 3(z+3) = 0$
$2x - 4y + 3z = -15$

66. $3(x+2) - 2(y+3) - 1(z-4) = 0$
$3x - 2y - z = -4$

67. $(x-1) + 4(y-2) + 4(z-1) = 0$
$x + 4y + 4z = 13$

68. $z + 3 = 0$
$z = -3$

69. The planes are 2x – 4y + 3z = –15 and 3x – 2y – z = –4. The normals to the planes are **u** = $\langle 2, -4, 3\rangle$ and **v** = $\langle 3, -2, -1\rangle$. If θ is the angle between the planes,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{6+8-3}{\sqrt{29}\sqrt{14}} = \frac{11}{\sqrt{406}}, \text{ so}$$

$$\theta = \cos^{-1}\frac{11}{\sqrt{406}} \approx 56.91°.$$

70. An equation of the plane has the form 2x + 4y – z = D. And this equation must satisfy 2(–1) + 4(2) – (–3) = D, so D = 9. Thus an equation of the plane is 2x + 4y – z = 9.

71. **a.** Planes parallel to the xy-plane may be expressed as $z = D$, so $z = 2$ is an equation of the plane.

b. An equation of the plane is 2(x + 4) – 3(y + 1) – 4(z – 2) = 0 or 2x – 3y – 4z = –13.

72. **a.** Planes parallel to the xy-plane may be expressed as $z = D$, so $z = 0$ is an equation of the plane.

b. An equation of the plane is $x + y + z = D$; since the origin is in the plane, $0 + 0 + 0 = D$. Thus an equation is $x + y + z = 0$.

73. $$\text{Distance} = \frac{|(1) + 3(-1) + (2) - 7|}{\sqrt{1+9+1}} = \frac{7}{\sqrt{11}} \approx 2.1106$$

74. The distance is 0 since the point is in the plane. $\left(|(-3)(2)+2(6)+(3)-9|=0\right)$

75. (0, 0, 9) is on $-3x + 2y + z = 9$. The distance from (0, 0, 9) to $6x - 4y - 2z = 19$ is $\frac{|6(0)-4(0)-2(9)-19|}{\sqrt{36+16+4}} = \frac{37}{\sqrt{56}} \approx 4.9443$ is the distance between the planes.

76. (1, 0, 0) is on $5x - 3y - 2z = 5$. The distance from (1, 0, 0) to $-5x + 3y + 2z = 7$ is $\frac{|-5(1)+3(0)+2(0)-7|}{\sqrt{25+9+4}} = \frac{12}{\sqrt{38}} \approx 1.9467.$

77. The equation of the sphere in standard form is $(x+1)^2+(y+3)^2+(z-4)^2=26,$ so its center is (–1, –3, 4) and radius is $\sqrt{26}$. The distance from the sphere to the plane is the distance from the center to the plane minus the radius of the sphere or

$$\frac{|3(-1)+4(-3)+1(4)-15|}{\sqrt{9+16+1}}-\sqrt{26}=\sqrt{26}-\sqrt{26}=0,$$

so the sphere is tangent to the plane.

78. The line segment between the points is perpendicular to the plane and its midpoint, (2, 1, 1), is in the plane. Then $\langle 6-(-2), 1-1, -2-4\rangle = \langle 8, 0, -6\rangle$ is perpendicular to the plane. The equation of the plane is

$8(x - 2) + 0(y - 1) - 6(z - 1) = 0$ or $8x - 6z = 10$.

79. $|\mathbf{u}\cdot\mathbf{v}| = |\cos\theta|\|\mathbf{u}\|\|\mathbf{v}\| \le \|\mathbf{u}\|\|\mathbf{v}\|$ since $|\cos\theta| \le 1$.

80. $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$

$\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2$

$\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\|+\|\mathbf{v}\|)^2$

Therefore, $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\|+\|\mathbf{v}\|$.

81. The 3 wires must offset the weight of the object, thus

$(3\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}) + (-8\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}) + (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$
$= 0\mathbf{i} + 0\mathbf{j} + 30\mathbf{k}$

Thus, $3 - 8 + a = 0$, so $a = 5$;
$4 - 2 + b = 0$, so $b = -2$;
$15 + 10 + c = 30$, so $c = 5$.

82. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ represent the sides of the polygon connected tail to head in succession around the polygon.

Then $\mathbf{v}_1+\mathbf{v}_2+\ldots+\mathbf{v}_n=\mathbf{0}$ since the polygon is closed, so

$\mathbf{F}\cdot\mathbf{v}_1+\mathbf{F}\cdot\mathbf{v}_2+\ldots+\mathbf{F}\cdot\mathbf{v}_n=\mathbf{F}\cdot(\mathbf{v}_1+\mathbf{v}_2+\ldots+\mathbf{v}_n)$
$= \mathbf{F}\cdot\mathbf{0} = 0$.

83. Let $\mathbf{x} = \langle x, y, z\rangle$, so

$(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b}) =$

$\langle x-a_1, y-a_2, z-a_3\rangle\cdot\langle x-b_1, y-b_2, z-b_3\rangle$

$= x^2-(a_1+b_1)x+a_1b_1+y^2-(a_2+b_2)y+a_2b_2$

$+z^2-(a_3+b_3)z+a_3b_3$

Setting this equal to 0 and completing the squares yields

$$\left(x-\frac{a_1+b_1}{2}\right)^2+\left(y-\frac{a_2+b_2}{2}\right)^2+\left(z-\frac{a_3+b_3}{2}\right)^2.$$

$$=\frac{1}{4}\left[(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2\right]$$

A sphere with center $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$ and radius

$$\frac{1}{4}\left[(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2\right]=\frac{1}{4}|\mathbf{a}-\mathbf{b}|^2$$

84. Let $P(x_0, y_0, z_0)$ be any point on $Ax + By + Cz = D$, so $Ax_0+By_0+Cz_0=D$. The distance between the planes is the distance from $P(x_0, y_0, z_0)$ to $Ax + By + Cz = E$, which is

$$\frac{|Ax_0+By_0+Cz_0-E|}{\sqrt{A^2+B^2+C^2}}=\frac{|D-E|}{\sqrt{A^2+B^2+C^2}}.$$

85. If **a**, **b**, and **c** are the position vectors of the vertices labeled *A*, *B*, and *C*, respectively, then the side *BC* is represented by the vector **c** – **b**. The position vector of the midpoint of *BC* is $\mathbf{b}+\frac{1}{2}(\mathbf{c}-\mathbf{b})=\frac{1}{2}(\mathbf{b}+\mathbf{c}).$ The segment from *A* to the midpoint of *BC* is $\frac{1}{2}(\mathbf{b}+\mathbf{c})-\mathbf{a}.$ Thus, the position vector of *P* is

$$\mathbf{a}+\frac{2}{3}\left[\frac{1}{2}(\mathbf{b}+\mathbf{c})-\mathbf{a}\right]=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$$

If the vertices are (2, 6, 5), (4, –1, 2), and (6, 1, 2), the corresponding position vectors are $\langle 2, 6, 5\rangle$, $\langle 4, -1, 2\rangle$, and $\langle 6, 1, 2\rangle$. The position vector of *P* is

$\frac{1}{3}\langle 2+4+6, 6-1+1, 5+2+2\rangle = \frac{1}{3}\langle 12, 6, 9\rangle =$
$\langle 4, 2, 3\rangle$. Thus *P* is (4, 2, 3).

86. Let A, B, C, and D be the vertices of the tetrahedron with corresponding position vectors **a**, **b**, **c**, and **d**. The vector representing the segment from A to the centroid of the opposite face, triangle BCD, is $\frac{1}{3}(\mathbf{b}+\mathbf{c}+\mathbf{d})-\mathbf{a}$ by Problem 85. Similarly, the segments from B, C, and D to the opposite faces are $\frac{1}{3}(\mathbf{a}+\mathbf{c}+\mathbf{d})-\mathbf{b}, \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{d})-\mathbf{c},$ and $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})-\mathbf{d},$ respectively. If these segments meet in one point which has a nice formulation as some fraction of the way from a vertex to the centroid of the opposite face, then there is some number k, such that

$$\mathbf{a}+k\left[\frac{1}{3}(\mathbf{b}+\mathbf{c}+\mathbf{d})-\mathbf{a}\right]=\mathbf{b}+k\left[\frac{1}{3}(\mathbf{a}+\mathbf{c}+\mathbf{d})-\mathbf{b}\right]$$
$$=\mathbf{c}+k\left[\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{d})-\mathbf{c}\right]=\mathbf{d}+k\left[\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})-\mathbf{d}\right].$$

Thus, $\mathbf{a}(1-k)=\frac{k}{3}\mathbf{a},$ so $k=\frac{3}{4}.$ Hence the segments joining the vertices to the centroids of the opposite faces meet in a common point which is $\frac{3}{4}$ of the way from a vertex to the corresponding centroid. With $k=\frac{3}{4}$, all of the above formulas simplify to $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}),$ the position vector of the point.

87. After reflecting from the xy-plane, the ray has direction $a\mathbf{i} + b\mathbf{j} - c\mathbf{k}$. After reflecting from the xz-plane, the ray now has direction $a\mathbf{i} - b\mathbf{j} - c\mathbf{k}$. After reflecting from the yz-plane, the ray now has direction $-a\mathbf{i} - b\mathbf{j} - c\mathbf{k}$, the opposite of its original direction. ($a<0$, $b<0$, $c<0$)

11.4 Concepts Review

1. $$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 1\\ 3 & 1 & -1\end{vmatrix}=\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & 1\\ 3 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & 2\\ 3 & 1\end{vmatrix}\mathbf{k}$$
$= (-2 - 1)\mathbf{i} - (1 - 3)\mathbf{j} + (-1 - 6)\mathbf{k}$
$= -3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k} = \langle -3, 2, -7\rangle$

2. $\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$

3. $-(\mathbf{v} \times \mathbf{u})$

4. parallel

Problem Set 11.4

1. a. $$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -3 & 2 & -2\\ -1 & 2 & -4\end{vmatrix}$$
$$=\begin{vmatrix}2 & -2\\ 2 & -4\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3 & -2\\ -1 & -4\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3 & 2\\ -1 & 2\end{vmatrix}\mathbf{k}$$
$=(-8 + 4)\mathbf{i} - (12 - 2)\mathbf{j} + (-6 + 2)\mathbf{k}$
$= -4\mathbf{i} - 10\mathbf{j} - 4\mathbf{k}$

b. $\mathbf{b} + \mathbf{c} = 6\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$, so
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -3 & 2 & -2\\ 6 & 5 & -8\end{vmatrix}$$
$$=\begin{vmatrix}2 & -2\\ 5 & -8\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3 & -2\\ 6 & -8\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3 & 2\\ 6 & 5\end{vmatrix}\mathbf{k}$$
$= (-16 + 10)\mathbf{i} - (24 + 12)\mathbf{j} + (-15 - 12)\mathbf{k}$
$= -6\mathbf{i} - 36\mathbf{j} - 27\mathbf{k}$

c. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = -3(6)+2(5)-2(-8) = 8$

d. $$\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & -4\\ 7 & 3 & -4\end{vmatrix}$$
$$=\begin{vmatrix}2 & -4\\ 3 & -4\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & -4\\ 7 & -4\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & 2\\ 7 & 3\end{vmatrix}\mathbf{k}$$
$= (-8 + 12)\mathbf{i} - (4 + 28)\mathbf{j} + (-3 - 14)\mathbf{k}$
$= 4\mathbf{i} - 32\mathbf{j} - 17\mathbf{k}$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -3 & 2 & -2\\ 4 & -32 & -17\end{vmatrix}$$
$$=\begin{vmatrix}2 & -2\\ -32 & -17\end{vmatrix}\mathbf{i}-\begin{vmatrix}-3 & -2\\ 4 & -17\end{vmatrix}\mathbf{j}+\begin{vmatrix}-3 & 2\\ 4 & -32\end{vmatrix}\mathbf{k}$$
$= (-34 - 64)\mathbf{i} - (51 + 8)\mathbf{j} + (96 - 8)\mathbf{k}$
$= -98\mathbf{i} - 59\mathbf{j} + 88\mathbf{k}$

2. a. $$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 3 & 1\\ -2 & -1 & 0\end{vmatrix}$$
$$=\begin{vmatrix}3 & 1\\ -1 & 0\end{vmatrix}\mathbf{i}-\begin{vmatrix}3 & 1\\ -2 & 0\end{vmatrix}\mathbf{j}+\begin{vmatrix}3 & 3\\ -2 & -1\end{vmatrix}\mathbf{k}$$
$= (0 + 1)\mathbf{i} - (0 + 2)\mathbf{j} + (-3 + 6)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
$= \langle 1, -2, 3\rangle$

b. $\mathbf{b} + \mathbf{c} = \langle -2 - 2, -1 - 3, 0 - 1\rangle$

$= \langle -4, -4, -1\rangle$

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ -4 & -4 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ -4 & -4 \end{vmatrix}\mathbf{k}$$

$= (-3 + 4)\mathbf{i} - (-3 + 4)\mathbf{j} + (-12 + 12)\mathbf{k}$

$= \mathbf{i} - \mathbf{j} = \langle 1, -1, 0\rangle$

c. $$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 0 \\ -2 & -3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 0 \\ -2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & -1 \\ -2 & -3 \end{vmatrix}\mathbf{k}$$

$= (1 - 0)\mathbf{i} - (2 - 0)\mathbf{j} + (6 - 2)\mathbf{k} = \langle 1, -2, 4\rangle$

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 3(1) + 3(-2) + 1(4) = 1$

d. $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$= (12 + 2)\mathbf{i} - (12 - 1)\mathbf{j} + (-6 - 3)\mathbf{k}$

$= \langle 14, -11, -9\rangle$

3. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 2 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$= (-8 - 6)\mathbf{i} - (-4 + 6)\mathbf{j} + (2 + 4)\mathbf{k} = -14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ is perpendicular to both **a** and **b**. All perpendicular vectors will have the form $c(-14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ where c is a real number.

4. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 5 & -2 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -2 \\ -2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & -2 \\ 3 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 5 \\ 3 & -2 \end{vmatrix}\mathbf{k}$$

$= (20 - 4)\mathbf{i} - (-8 + 6)\mathbf{j} + (4 - 15)\mathbf{k}$

$= 16\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}$

All vectors perpendicular to both **a** and **b** will have the form $c(16\mathbf{i} + 2\mathbf{j} - 11\mathbf{k})$ where c is a real number.

5. $\mathbf{u} = \langle 3-1, -1-3, 2-5\rangle = \langle 2, -4, -3\rangle$ and $\mathbf{v} = \langle 4-1, 0-3, 1-5\rangle = \langle 3, -3, -4\rangle$ are in the plane.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -3 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & -3 \\ -3 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -4 \\ 3 & -3 \end{vmatrix}\mathbf{k}$$

$= (16 - 9)\mathbf{i} - (-8 + 9)\mathbf{j} + (-6 + 12)\mathbf{k}$

$= \langle 7, -1, 6\rangle$ is perpendicular to the plane.

$$\pm\frac{1}{\sqrt{49+1+36}}\langle 7, -1, 6\rangle = \pm\left\langle \frac{7}{\sqrt{86}}, -\frac{1}{\sqrt{86}}, \frac{6}{\sqrt{86}}\right\rangle$$

are the unit vectors perpendicular to the plane.

6. $\mathbf{u} = \langle 5+1, 1-3, 2-0\rangle = \langle 6, -2, 2\rangle$ and $\mathbf{v} = \langle 4+1, -3-3, -1-0\rangle = \langle 5, -6, -1\rangle$ are in the plane.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & 2 \\ 5 & -6 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 2 \\ -6 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 6 & 2 \\ 5 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 6 & -2 \\ 5 & -6 \end{vmatrix}\mathbf{k}$$

$= (2 + 12)\mathbf{i} - (-6 - 10)\mathbf{j} + (-36 + 10)\mathbf{k}$

$= \langle 14, 16, -26\rangle$

is perpendicular to the plane.

$$\pm\frac{1}{\sqrt{196+256+676}}\langle 14, 16, -26\rangle$$

$$= \pm\left\langle \frac{7}{\sqrt{282}}, \frac{8}{\sqrt{282}}, -\frac{13}{\sqrt{282}}\right\rangle$$

are the unit vectors perpendicular to the plane.

7. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -3 \\ 4 & 2 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -3 \\ 4 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 1 \\ 4 & 2 \end{vmatrix}\mathbf{k}$$

$= (-4 + 6)\mathbf{i} - (4 + 12)\mathbf{j} + (-2 - 4)\mathbf{k}$

$= 2\mathbf{i} - 16\mathbf{j} - 6\mathbf{k}$

Area of parallelogram = $\|\mathbf{a}\times\mathbf{b}\|$

$= \sqrt{4 + 256 + 36} = 2\sqrt{74}$

8. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 1 & -4 \end{vmatrix}$

$= \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{k}$

$= (-8 + 1)\mathbf{i} - (-8 - 1)\mathbf{j} + (2 + 2)\mathbf{k}$

$= -7\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$

Area of parallelogram $= \|\mathbf{a}\times\mathbf{b}\| = \sqrt{49+81+16}$

$= \sqrt{146}$

9. $\mathbf{a} = \langle 2-3,\ 4-2,\ 6-1\rangle = \langle -1,\ 2,\ 5\rangle$ and

$\mathbf{b} = \langle -1-3, 2-2, 5-1\rangle = \langle -4, 0, 4\rangle$ are adjacent sides of the triangle. The area of the triangle is half the area of the parallelogram with **a** and **b** as adjacent sides.

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 5 \\ -4 & 0 & 4 \end{vmatrix}$

$= \begin{vmatrix} 2 & 5 \\ 0 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 5 \\ -4 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ -4 & 0 \end{vmatrix}\mathbf{k}$

$= (8 - 0)\mathbf{i} - (-4 + 20)\mathbf{j} + (0 + 8)\mathbf{k} = \langle 8, -16, 8\rangle$

Area of triangle $= \frac{1}{2}\sqrt{64+256+64} = \frac{1}{2}\left(8\sqrt{6}\right)$

$= 4\sqrt{6}$

10. $\mathbf{a} = \langle 3-1, 1-2, 5-3\rangle = \langle 2, -1, 2\rangle$ and

$\mathbf{b} = \langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle$ are adjacent sides of the triangle.

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 3 & 3 & 3 \end{vmatrix}$

$= \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}\mathbf{k}$

$= (-3 - 6)\mathbf{i} - (6 - 6)\mathbf{j} + (6 + 3)\mathbf{k} = \langle -9, 0, 9\rangle$

Area of triangle $= \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\sqrt{81+81} = \frac{9\sqrt{2}}{2}$

11. $\mathbf{u} = \langle 0-1, 3-3, 0-2\rangle = \langle -1, 0, -2\rangle$ and

$\mathbf{v} = \langle 2-1, 4-3, 3-2\rangle = \langle 1, 1, 1\rangle$ are in the plane.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{k}$

$= (0 + 2)\mathbf{i} - (-1 + 2)\mathbf{j} + (-1 - 0)\mathbf{k} = \langle 2, -1, -1\rangle$

The plane through (1, 3, 2) with normal $\langle 2, -1, -1\rangle$ has equation

$2(x - 1) - 1(y - 3) - 1(z - 2) = 0$ or

$2x - y - z = -3$.

12. $\mathbf{u} = \langle 0-1, 0-1, 1-2\rangle = \langle -1, -1, -1\rangle$ and

$\mathbf{v} = \langle -2-1, -3-1, 0-2\rangle = \langle -3, -4, -2\rangle$ are in the plane.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \\ -3 & -4 & -2 \end{vmatrix}$

$= \begin{vmatrix} -1 & -1 \\ -4 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -1 \\ -3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & -1 \\ -3 & -4 \end{vmatrix}\mathbf{k}$

$= (2 - 4)\mathbf{i} - (2 - 3)\mathbf{j} + (4 - 3)\mathbf{k} = \langle -2, 1, 1\rangle$

The plane through (0, 0, 1) with normal $\langle -2, 1, 1\rangle$ has equation $-2(x - 0) + 1(y - 0) + 1(z - 1) = 0$ or

$-2x + y + z = 1$.

13. $\mathbf{u} = \langle 0-7, 3-0, 0-0\rangle = \langle -7, 3, 0\rangle$ and

$\mathbf{v} = \langle 0-7, 0-0, 5-0\rangle = \langle -7, 0, 5\rangle$ are in the plane.

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & 3 & 0 \\ -7 & 0 & 5 \end{vmatrix} =$

$\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -7 & 0 \\ -7 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -7 & 3 \\ -7 & 0 \end{vmatrix}\mathbf{k}$

$= (15 - 0)\mathbf{i} - (-35 - 0)\mathbf{j} + (0 + 21)\mathbf{k} = \langle 15, 35, 21\rangle$

The plane through (7, 0, 0) with normal $\langle 15, 35, 21\rangle$ has equation $15(x - 7) + 35(y - 0) + 21(z - 0) = 0$ or $15x + 35y + 21z = 105$.

14. $\mathbf{u} = \langle 0-a, b-0, 0-0\rangle = \langle -a, b, 0\rangle$ and $\mathbf{v} = \langle 0-a, 0-0, c-0\rangle = \langle -a, 0, c\rangle$ are in the plane.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} =$$

$$\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix}\mathbf{i} - \begin{vmatrix} -a & 0 \\ -a & c \end{vmatrix}\mathbf{j} + \begin{vmatrix} -a & b \\ -a & 0 \end{vmatrix}\mathbf{k}$$

$= bc\mathbf{i} - (-ac)\mathbf{j} + ab\mathbf{k} = \langle bc, ac, ab\rangle$

The plane through $(a, 0, 0)$ with normal $\langle bc, ac, ab\rangle$ has equation $bc(x - a) + ac(y - 0) + ab(z - 1) = 0$ or $bcx + ac\,y + abz = abc$.

15. An equation of the plane is $1(x-2) - 1(y - 5) + 2(z - 1) = 0$ or $x - y + 2z = -1$.

16. An equation of the plane is $1(x-0) + 1(y - 0) + 1(z - 2) = 0$ or $x + y + z = 2$.

17. The plane's normals will be perpendicular to the normals of the other two planes. Then a normal is $\langle 1, -3, 2\rangle \times \langle 2, -2, -1\rangle = \langle 7, 5, 4\rangle$. An equation of the plane is $7(x + 1) + 5(y + 2) + 4(z - 3) = 0$ or $7x + 5y + 4z = -5$.

18. $\mathbf{u} = \langle 1, 1, 1\rangle$ is normal to the plane $x + y + z = 2$ and $\mathbf{v} = \langle 1, -1, -1\rangle$ is normal to the plane $x - y - z = 4$. A normal vector to the required plane must be perpendicular to both the other normals; thus one possibility is

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = \langle 0, 2, -2\rangle$$ The plane has an equation of the form: $2y - 2z = D$. Since the point $(2, -1, 4)$ is in the plane, $2(-1) - 2(4) = D$; thus $D = -10$ and an equation for the plane is $2y - 2z = -10$ or $y - z = -5$.

19. $(4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}) = 13\mathbf{i} - 26\mathbf{j} - 26\mathbf{k}$
$= 13(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$
is normal to the plane. An equation of the plane is $1(x - 2) - 2(y + 3) - 2(z - 2) = 0$ or $x - 2y - 2z = 4$.

20. $\mathbf{k} = \langle 0, 0, 1\rangle$ is normal to the xy-plane and $\mathbf{v} = \langle 3, -2, 1\rangle$ is normal to the plane $3x - 2y + z = 4$. A normal vector to the required plane must be perpendicular to both the other normals; thus one possibility is:

$$\mathbf{k}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 3 & -2 & 1 \end{vmatrix} = \langle 2, 3, 0\rangle$$ The plane has an equation of the form: $2x + 3y = D$. Since the point $(0, 0, 0)$ is in the plane, $2(0) + 3(0) = D$; thus $D = 0$ and an equation for the plane is $2x + 3y = 0$.

21. Each vector normal to the plane we seek is parallel to the line of intersection of the given planes. Also, the cross product of vectors normal to the given planes will be parallel to both planes, hence parallel to the line of intersection. Thus, $\langle 4, -3, 2\rangle \times \langle 3, 2, -1\rangle = \langle -1, 10, 17\rangle$ is normal to the plane we seek. An equation of the plane is $-1(x - 6) + 10(y - 2) + 17(z + 1) = 0$ or $-x + 10y + 17z = -3$.

22. $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$, hence $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is perpendicular to $\mathbf{a} \times \mathbf{b}$ hence it is parallel to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

23. $\text{Volume} = |\langle 2, 3, 4\rangle \cdot (\langle 0, 4, -1\rangle \times \langle 5, 1, 3\rangle)| = |\langle 2, 3, 4\rangle \cdot \langle 13, -5, -20\rangle| = |-69| = 69$

24. $\text{Volume} = |(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot [(-\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})| = |(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \cdot (12\mathbf{i} + 8\mathbf{j} - 4\mathbf{k})| = |-4| = 4$

25. **a.** $\text{Volume} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\langle 3, 2, 1\rangle \cdot \langle -3, -1, 2\rangle| = |-9| = 9$

b. $\text{Area} = \|\mathbf{u} \times \mathbf{v}\| = |\langle 3, -5, 1\rangle| = \sqrt{9 + 25 + 1} = \sqrt{35}$

c. Let θ be the angle. Then θ is the complement of the smaller angle between **u** and **v** × **w**.

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=\frac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{u}\|\|\mathbf{v}\times\mathbf{w}\|}=\frac{9}{\sqrt{9+4+1}\sqrt{9+1+4}}=\frac{9}{14},\ \theta\approx 40.01°$$

26. From Theorem C, $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$, which leads to $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$. Again from Theorem C, $|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|=|(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}|=|-(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b}|$, which leads to $|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|=|\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})|$. Therefore, we have that $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=|\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})|=|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$.

27. Choice (c) does not make sense because $(\mathbf{a}\cdot\mathbf{b})$ is a scalar and can't be crossed with a vector. Choice (d) does not make sense because $(\mathbf{a}\times\mathbf{b})$ is a vector and can't be added to a constant.

28. **a** × **b** and **c** × **d** will both be perpendicular to the common plane. Hence **a** × **b** and **c** × **d** are parallel so (**a** × **b**) × (**c** × **d**) = **0**.

29. Let **b** and **c** determine the (triangular) base of the tetrahedron. Then the area of the base is $\frac{1}{2}\|\mathbf{b}\times\mathbf{c}\|$ which is half of the area of the parallelogram determined by **b** and **c**. Thus,

$$\frac{1}{3}(\text{area of base})(\text{height})=\frac{1}{3}\left[\frac{1}{2}(\text{area of corresponding parallelogram})(\text{height})\right]$$

$$=\frac{1}{6}(\text{area of corresponding parallelpiped})=\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

30. $\mathbf{a}=\langle 4+1,-1-2,2-3\rangle=\langle 5,-3,-1\rangle$,

$\mathbf{b}=\langle 5+1,6-2,3-3\rangle=\langle 6,4,0\rangle$,

$\mathbf{c}=\langle 1+1,1-2,-2-3\rangle=\langle 2,-1,-5\rangle$

$$\text{Volume}=\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|=\frac{1}{6}|\langle 5,-3,-1\rangle\cdot\langle -20,30,-14\rangle|=\frac{1}{6}|-176|=\frac{88}{3}$$

31. Let $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ then

$\mathbf{u}\times\mathbf{v}=\langle u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1\rangle$

$$\|\mathbf{u}\times\mathbf{v}\|^2=u_2^2v_3^2-2u_2u_3v_2v_3+u_3^2v_2^2+u_3^2v_1^2-2u_1u_3v_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_1u_2v_1v_2+u_2^2v_1^2$$

$$=u_1^2(v_1^2+v_2^2+v_3^2)-u_1^2v_1^2+u_2^2(v_1^2+v_2^2+v_3^2)-u_2^2v_2^2+u_3^2(v_1^2+v_2^2+v_3^2)$$
$$-u_3^2v_3^2-2u_2u_3v_2v_3-2u_1u_3v_1v_3-2u_1u_2v_1v_2$$

$$=(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2+2u_2u_3v_2v_3+2u_1u_3v_1v_3+2u_1u_2v_1v_2)$$

$$=(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2=\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2$$

32. $\mathbf{u}=\langle u_1,u_2,u_3\rangle,\ \mathbf{v}=\langle v_1,v_2,v_3\rangle,\ \mathbf{w}=\langle w_1,w_2,w_3\rangle$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=\langle (u_2v_3-u_3v_2)+(u_2w_3-u_3w_2),\ (u_3v_1-u_1v_3)+(u_3w_1-u_1w_3),\ (u_1v_2-u_2v_1)+(u_1w_2-u_2w_1)\rangle$$

= (**u** × **v**) + (**u** × **w**)

33. (**v** + **w**) × **u** = –[**u** × (**v** + **w**)]
= –[(**u** × **v**) + (**u** × **w**)] = –(**u** × **v**) – (**u** × **w**)
= (**v** × **u**) + (**w** × **u**)

34. **u** × **v** = **0** ⇒ **u** and **v** are parallel. $\mathbf{u}\cdot\mathbf{v}=0\Rightarrow$ **u** and **v** are perpendicular. Thus, either **u** or **v** is **0**.

35. $\overrightarrow{PQ} = \langle -a, b, 0\rangle$, $\overrightarrow{PR} = \langle -a, 0, c\rangle$,

The area of the triangle is half the area of the parallelogram with $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as adjacent sides, so area

$(\Delta PQR) = \frac{1}{2}\left\|\langle -a, b, 0\rangle \times \langle -a, 0, c\rangle\right\|$

$= \frac{1}{2}\left\|\langle bc, ac, ab\rangle\right\| = \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$.

36. The area of the triangle is

$\frac{1}{2}\left\|\langle x_2 - x_1, y_2 - y_1, 0\rangle \times \langle x_3 - x_1, y_3 - y_1, 0\rangle\right\| =$

$\frac{1}{2}\left\|\langle 0, 0, (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\rangle\right\|$

$= \frac{1}{2}\left|(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right|$

which is half of the absolute value of the determinant given. (Expand the determinant along the third column to see the equality.)

37. From Problem 35, $D^2 = \frac{1}{4}(b^2c^2 + a^2c^2 + a^2b^2)$

$= \left(\frac{1}{2}bc\right)^2 + \left(\frac{1}{2}ac\right)^2 + \left(\frac{1}{2}ab\right)^2 = A^2 + B^2 + C^2.$

38. Note that the area of the face determined by **a** and **b** is $\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$.

Label the tetrahedron so that $\mathbf{m} = \frac{1}{2}(\mathbf{a}\times\mathbf{b})$,

$\mathbf{n} = \frac{1}{2}(\mathbf{b}\times\mathbf{c})$, and $\mathbf{p} = \frac{1}{2}(\mathbf{c}\times\mathbf{a})$ point outward.

The fourth face is determined by **a** – **c** and **b** – **c**, so

$\mathbf{q} = \frac{1}{2}[(\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{c})]$

$= \frac{1}{2}[(\mathbf{b}\times\mathbf{a}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a}) + (\mathbf{c}\times\mathbf{c})]$

$= \frac{1}{2}[-(\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a})].$

$\mathbf{m}+\mathbf{n}+\mathbf{p}+\mathbf{q} = \frac{1}{2}[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a})$

$-(\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a})] = \mathbf{0}$

39. The area of the triangle is $A = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$. Thus,

$A^2 = \frac{1}{4}\|\mathbf{a}\times\mathbf{b}\|^2 = \frac{1}{4}\left(\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2\right)$

$= \frac{1}{4}\left[\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \frac{1}{4}\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right)^2\right]$

$= \frac{1}{16}\left[4a^2b^2 - (a^2 + b^2 - c^2)^2\right]$

$= \frac{1}{16}(2a^2b^2 - a^4 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4).$

Note that $s - a = \frac{1}{2}(b + c - a)$,

$s - b = \frac{1}{2}(a + c - b)$, and $s - c = \frac{1}{2}(a + b - c)$.

$s(s-a)(s-b)(s-c)$

$= \frac{1}{16}(a+b+c)(b+c-a)(a+c-b)(a+b-c)$

$= \frac{1}{16}(2a^2b^2 - a^4 + 2a^2c^2 - b^4 + 2b^2c^2 - c^4)$

which is the same as was obtained above.

40. $\mathbf{u}\times\mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\times(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$

$= (u_1v_1)(\mathbf{i}\times\mathbf{i}) + (u_1v_2)(\mathbf{i}\times\mathbf{j}) + (u_1v_3)(\mathbf{i}\times\mathbf{k}) +$

$(u_2v_1)(\mathbf{j}\times\mathbf{i}) + (u_2v_2)(\mathbf{j}\times\mathbf{j}) + (u_2v_3)(\mathbf{j}\times\mathbf{k}) +$

$(u_3v_1)(\mathbf{k}\times\mathbf{i}) + (u_3v_2)(\mathbf{k}\times\mathbf{j}) + (u_3v_3)(\mathbf{k}\times\mathbf{k})$

$= (u_1v_1)(0) + (u_1v_2)(\mathbf{k}) + (u_1v_3)(-\mathbf{j}) +$

$(u_2v_1)(-\mathbf{k}) + (u_2v_2)(0) + (u_2v_3)(\mathbf{i}) +$

$(u_3v_1)(\mathbf{j}) + (u_3v_2)(-\mathbf{i}) + (u_3v_3)(0)$

$= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

11.5 Concepts Review

1. a vector-valued function of a real variable

2. f and g are continuous at c; $f'(t)\mathbf{i}+g'(t)\mathbf{j}$

3. position

4. $\mathbf{r}'(t); \mathbf{r}''(t)$; tangent; concave

Problem Set 11.5

1. $\lim_{t\to1}[2t\mathbf{i}-t^2\mathbf{j}]=\lim_{t\to1}(2t)\mathbf{i}-\lim_{t\to1}(t^2)\mathbf{j}=2\mathbf{i}-\mathbf{j}$

2. $\lim_{t\to3}[2(t-3)^2\mathbf{i}-7t^3\mathbf{j}]$
$=\lim_{t\to3}[2(t-3)^2]\mathbf{i}-\lim_{t\to3}(7t^3)\mathbf{j}=-189\mathbf{j}$

3. $\lim_{t\to1}\left[\frac{t-1}{t^2-1}\mathbf{i}-\frac{t^2+2t-3}{t-1}\mathbf{j}\right]$
$=\lim_{t\to1}\left(\frac{t-1}{(t-1)(t+1)}\right)\mathbf{i}-\lim_{t\to1}\left(\frac{(t-1)(t+3)}{t-1}\right)\mathbf{j}$
$=\lim_{t\to1}\left(\frac{1}{t+1}\right)\mathbf{i}-\lim_{t\to1}(t+3)\mathbf{j}=\frac{1}{2}\mathbf{i}-4\mathbf{j}$

4. $\lim_{t\to-2}\left[\frac{2t^2-10t-28}{t+2}\mathbf{i}-\frac{7t^3}{t-3}\mathbf{j}\right]$
$=\lim_{t\to-2}\left(\frac{2t^2-10t-28}{t+2}\right)\mathbf{i}-\lim_{t\to-2}\left(\frac{7t^3}{t-3}\right)\mathbf{j}$
$=\lim_{t\to-2}(2t-14)\mathbf{i}-\frac{56}{5}\mathbf{j}=-18\mathbf{i}-\frac{56}{5}\mathbf{j}$

5. $\lim_{t\to0}\left[\frac{\sin t\cos t}{t}\mathbf{i}-\frac{7t^3}{e^t}\mathbf{j}+\frac{t}{t+1}\mathbf{k}\right]$
$=\lim_{t\to0}\left(\frac{\sin t\cos t}{t}\right)\mathbf{i}-\lim_{t\to0}\left(\frac{7t^3}{e^t}\right)\mathbf{j}$
$+\lim_{t\to0}\left(\frac{t}{t+1}\right)\mathbf{k}=\mathbf{i}$

6. $\lim_{t\to\infty}\left[\frac{t\sin t}{t^2}\mathbf{i}-\frac{7t^3}{t^3-3t}\mathbf{j}-\frac{\sin t}{t}\mathbf{k}\right]$
$=\lim_{t\to\infty}\left(\frac{t\sin t}{t^2}\right)\mathbf{i}-\lim_{t\to\infty}\left(\frac{7t^3}{t^3-3t}\right)\mathbf{j}-\lim_{t\to\infty}\left(\frac{\sin t}{t}\right)\mathbf{k}$
$=\lim_{t\to\infty}\left(\frac{\sin t}{t}\right)\mathbf{i}-7\mathbf{j}-\lim_{t\to\infty}\left(\frac{\sin t}{t}\right)\mathbf{k}=-7\mathbf{j}$

7. $\lim_{t\to0^+}\left\langle \ln(t^3), t^2\ln t, t\right\rangle$ does not exist because $\lim_{t\to0^+}\ln(t^3)=-\infty.$

8. $\lim_{t\to0^-}\left\langle e^{-1/t^2}, \frac{t}{|t|}, |t|\right\rangle$
$=\left\langle \lim_{t\to0^-}e^{-1/t^2}, \lim_{t\to0^-}\frac{t}{|t|}, \lim_{t\to0^-}|t|\right\rangle=\langle 0,-1,0\rangle$

9. **a.** The domain of $f(t)=\frac{2}{t-4}$ is $(-\infty,4)\cup(4,\infty)$. The domain of $g(t)=\sqrt{3-t}$ is $(-\infty, 3]$. The domain of $h(t)=\ln|4-t|$ is $(-\infty,4)\cup(4,\infty)$. Thus, the domain of $\mathbf{r}$ is $(-\infty, 3]$ or $\{t\in\mathbb{R}: t\le 3\}$.

b. The domain of $f(t)=[\![t^2]\!]$ is $(-\infty,\infty)$. The domain of $g(t)=\sqrt{20-t}$ is $(-\infty, 20]$. The domain of $h(t)=3$ is $(-\infty,\infty)$. Thus, the domain of $\mathbf{r}$ is $(-\infty, 20]$ or $\{t\in\mathbb{R}: t\le 20\}$.

c. The domain of $f(t)=\cos t$ is $(-\infty,\infty)$. The domain of $g(t)=\sin t$ is also $(-\infty,\infty)$. The domain of $h(t)=\sqrt{9-t^2}$ is $[-3,3]$. Thus, the domain of $\mathbf{r}$ is $[-3,3]$, or $\{t\in\mathbb{R}: -3\le t\le 3\}$.

10. **a.** The domain of $f(t)=\ln(t-1)$ is $(1,\infty)$. The domain of $g(t)=\sqrt{20-t}$ is $(-\infty, 20]$. Thus, the domain of $\mathbf{r}$ is $(1, 20]$ or $\{t\in\mathbb{R}: 1<t\le 20\}$.

b. The domain of $f(t)=\ln(t^{-1})$ is $(0,\infty)$). The domain of $g(t)=\tan^{-1}t$ is $(-\infty,\infty)$. The domain of $h(t)=t$ is $(-\infty,\infty)$. Thus, the domain of $\mathbf{r}$ is $(0,\infty)$ or $\{t\in\mathbb{R}: t>0\}$.

c. The domain of $g(t)=1/\sqrt{1-t^2}$ is $(-1,1)$. The domain of $h(t)=1/\sqrt{9-t^2}$ is $(-3,3)$. (The function f is $f(x)=0$ which has domain $(-\infty,\infty)$.) Thus, domain of $\mathbf{r}$ is $(-1,1)$.

11. a. $f(t)=\dfrac{2}{t-4}$ is continuous on $(-\infty,4)\cup(4,\infty)$. $g(t)=\sqrt{3-t}$ is continuous on $(-\infty, 3]$. $h(t)=\ln|4-t|$ is continuous on $(-\infty,4)$ and on $(4,\infty)$. Thus, **r** is continuous on $(-\infty, 3]$ or $\{t\in\mathbb{R}: t\le 3\}$.

b. $f(t)=[\![t^2]\!]$ is continuous on $\left(-\sqrt{n+1},-\sqrt{n}\right)\cup\left(\sqrt{n},\sqrt{n+1}\right)$ where n is a non-negative integer. $g(t)=\sqrt{20-t}$ is continuous on $(-\infty,20)$ or $\{t\in\mathbb{R}: t<20\}$. $h(t)=3$ is continuous on $(-\infty,\infty)$. Thus, **r** is continuous on $\left(-\sqrt{n+1},-\sqrt{n}\right)\cup\left(\sqrt{k},\sqrt{k+1}\right)$ where n and k are non-negative integers and $k<400$ or $\{t\in\mathbb{R}: t<20, t^2 \text{ not an integer}\}$.

c. $f(t)=\cos t$ and $f(t)=\sin t$ are continuous on $(-\infty,\infty)$. $h(t)=\sqrt{9-t^2}$ is continuous on $[-3,3]$. Thus, **r** is continuous on $[-3,3]$.

12. a. $f(t)=\ln(t-1)$ is continuous on $(1,\infty)$. $g(t)=\sqrt{20-t}$ is continuous on $(-\infty, 20)$. Thus, **r** is continuous on (1, 20) or $\{t\in\mathbb{R}: 1<t<20\}$.

b. $f(t)=\ln(t^{-1})$ is continuous on $(0,\infty)$. $g(t)=\tan^{-1}t$ is continuous on $(-\infty,\infty)$. $h(t)=t$ is continuous on $(-\infty,\infty)$. Thus, **r** is continuous on $(0,\infty)$ or $\{t\in\mathbb{R}: t>0\}$.

c. $g(t)=1/\sqrt{1-t^2}$ is continuous on $(-1,1)$. $h(t)=1/\sqrt{9-t^2}$ is continuous on $(-3,3)$. (The function f is $f(x)=0$ which is continuous on $(-\infty,\infty)$.) Thus, **r** is continuous on $(-1,1)$.

13. a. $D_t\mathbf{r}(t)=9(3t+4)^2\mathbf{i}+2te^{t^2}\mathbf{j}+0\mathbf{k}$

$D_t^2\mathbf{r}(t)=54(3t+4)\mathbf{i}+2(2t^2+1)e^{t^2}\mathbf{j}$

b. $D_t\mathbf{r}(t)=\sin 2t\mathbf{i}-3\sin 3t\mathbf{j}+2t\mathbf{k}$

$D_t^2\mathbf{r}(t)=2\cos 2t\mathbf{i}-9\cos 3t\mathbf{j}+2\mathbf{k}$

14. a. $\mathbf{r}'(t)=\left(e^t-2te^{-t^2}\right)\mathbf{i}+(\ln 2)2^t\mathbf{j}+\mathbf{k}$

$\mathbf{r}''(t)=\left(e^t+4t^2e^{-t^2}-2e^{-t^2}\right)\mathbf{i}+(\ln 2)^2 2^t\mathbf{j}$

b. $\mathbf{r}'(t)=2\sec^2 2t\mathbf{i}+\dfrac{1}{1+t^2}\mathbf{j}$

$\mathbf{r}''(t)=8\tan 2t\sec^2 2t\mathbf{i}-\dfrac{2t}{(1+t^2)^2}\mathbf{j}$

15. $\mathbf{r}'(t)=-e^{-t}\mathbf{i}-\dfrac{2}{t}\mathbf{j}$; $\mathbf{r}''(t)=e^{-t}\mathbf{i}+\dfrac{2}{t^2}\mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}''(t)=e^{-2t}-\dfrac{2}{t^2}\ln\left(t^2\right)$

$$D_t[\mathbf{r}(t)\cdot\mathbf{r}''(t)]=-2e^{-2t}-\left[\frac{2}{t^2}\cdot\frac{1}{t^2}\cdot 2t-\frac{4}{t^3}\ln\left(t^2\right)\right]$$

$$=-2e^{-2t}-\frac{4}{t^3}+\frac{4\ln\left(t^2\right)}{t^3}$$

16. $\mathbf{r}'(t)=3\cos 3t\mathbf{i}+3\sin 3t\mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$

$D_t[\mathbf{r}(t)\cdot\mathbf{r}'(t)]=0$

17. $h(t)\mathbf{r}(t)=e^{-3t}\sqrt{t-1}\mathbf{i}+e^{-3t}\ln(2t^2)\mathbf{j}$

$$D_t[h(t)\mathbf{r}(t)]=-\frac{e^{-3t}}{2}\left(\frac{6t-7}{\sqrt{t-1}}\right)\mathbf{i}$$

$$+e^{-3t}\left(\frac{2}{t}-3\ln(2t^2)\right)\mathbf{j}$$

18. $h(t)\mathbf{r}(t)=\ln(3t-2)\sin 2t\mathbf{i}+\ln(3t-2)\cosh t\mathbf{j}$

$$D_t[h(t)\mathbf{r}(t)]=\left[2\ln(3t-2)\cos 2t+\frac{3\sin 2t}{3t-2}\right]\mathbf{i}$$

$$+\left[\ln(3t-2)\sinh t+\frac{3\cosh t}{3t-2}\right]\mathbf{j}$$

19. $\mathbf{v}(t) = \mathbf{r}'(t) = 4\mathbf{i} + 10t\mathbf{j} + 2\mathbf{k}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 10\mathbf{j}$

$\mathbf{v}(1) = 4\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}$; $\mathbf{a}(1) = 10\mathbf{j}$;

$s(1) = \sqrt{16+100+4} = 2\sqrt{30} \approx 10.954$

20. $\mathbf{v}(t) = \mathbf{i} + 2(t-1)\mathbf{j} + 3(t-3)^2\mathbf{k}$

$\mathbf{a}(t) = 2\mathbf{j} + 6(t-3)\mathbf{k}$

$\mathbf{v}(0) = \mathbf{i} - 2\mathbf{j} + 27\mathbf{k}$; $\mathbf{a}(0) = 2\mathbf{j} - 18\mathbf{k}$;

$s(0) = \sqrt{1+4+729} = \sqrt{734} \approx 27.092$

21. $\mathbf{v}(t) = -\frac{1}{t^2}\mathbf{i} - \frac{2t}{(t^2-1)^2}\mathbf{j} + 5t^4\mathbf{k}$

$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{i} + \frac{2+6t^2}{(t^2-1)^3}\mathbf{j} + 20t^3\mathbf{k}$

$\mathbf{v}(2) = -\frac{1}{4}\mathbf{i} - \frac{4}{9}\mathbf{j} + 80\mathbf{k}$;

$\mathbf{a}(2) = \frac{1}{4}\mathbf{i} + \frac{26}{27}\mathbf{j} + 160\mathbf{k}$;

$s(2) = \sqrt{\frac{1}{16} + \frac{16}{81} + 6400} = \frac{\sqrt{8,294,737}}{36}$

≈ 80.002

22. $\mathbf{v}(t) = 6t^5\mathbf{i} + 72t(6t^2-5)^5\mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 30t^4\mathbf{i} + 72(66t^2-5)(6t^2-5)^4\mathbf{j}$

$\mathbf{v}(1) = 6\mathbf{i} + 72\mathbf{j} + \mathbf{k}$; $\mathbf{a}(1) = 30\mathbf{i} + 4392\mathbf{j}$;

$s(1) = \sqrt{36+5184+1} = \sqrt{5221} \approx 72.256$

23. $\mathbf{v}(t) = t^2\mathbf{j} + \frac{2}{3}t^{-1/3}\mathbf{k}$; $\mathbf{a}(t) = 2t\mathbf{j} - \frac{2}{9}t^{-4/3}\mathbf{k}$

$\mathbf{v}(2) = 4\mathbf{j} + \frac{2^{2/3}}{3}\mathbf{k}$; $\mathbf{a}(2) = 4\mathbf{j} - \frac{2^{-1/3}}{9}\mathbf{k}$

$s(2) = \sqrt{16 + \frac{2^{4/3}}{9}} \approx 4.035$

$\mathbf{v}(2) = 4\mathbf{j} + \frac{2^{2/3}}{3}\mathbf{k}$; $\mathbf{a}(2) = 4\mathbf{j} - \frac{1}{9\sqrt[3]{2}}\mathbf{k}$;

$s(2) = \sqrt{16 + \frac{2^{4/3}}{9}} \approx 4.035$

24. $\mathbf{v}(t) = t^2\mathbf{i} + 5(t-1)^3\mathbf{j} + \sin\pi t\mathbf{k}$

$\mathbf{a}(t) = 2t\mathbf{i} + 15(t-1)^2\mathbf{j} + \pi\cos\pi t\mathbf{k}$

$\mathbf{v}(2) = 4\mathbf{i} + 5\mathbf{j}$; $\mathbf{a}(2) = 4\mathbf{i} + 15\mathbf{j} + \pi\mathbf{k}$;

$s(2) = \sqrt{16+25} = \sqrt{41} \approx 6.403$

25. $\mathbf{v}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

$\mathbf{v}(\pi) = -\mathbf{j} + \mathbf{k}$; $\mathbf{a}(\pi) = \mathbf{i}$; $s(\pi) = \sqrt{1+1} = \sqrt{2} \approx 1.414$

26. $\mathbf{v}(t) = 2\cos 2t\,\mathbf{i} - 3\sin 3t\,\mathbf{j} - 4\sin 4t\,\mathbf{k}$

$\mathbf{a}(t) = -4\sin 2t\,\mathbf{i} - 9\cos 3t\,\mathbf{j} - 16\cos 4t\,\mathbf{k}$

$\mathbf{v}\left(\frac{\pi}{2}\right) = -2\mathbf{i} + 3\mathbf{j}$; $\mathbf{a}\left(\frac{\pi}{2}\right) = -16\mathbf{k}$;

$s\left(\frac{\pi}{2}\right) = \sqrt{4+9} = \sqrt{13} \approx 3.606$

27. $\mathbf{v}(t) = \sec^2 t\mathbf{i} + 3e^t\mathbf{j} - 4\sin 4t\mathbf{k}$

$\mathbf{a}(t) = 2\sec^2 t\tan t\mathbf{i} + 3e^t\mathbf{j} - 16\cos 4t\mathbf{k}$

$\mathbf{v}\left(\frac{\pi}{4}\right) = 2\mathbf{i} + 3e^{\pi/4}\mathbf{j}$; $\mathbf{a}\left(\frac{\pi}{4}\right) = 4\mathbf{i} + 3e^{\pi/4}\mathbf{j} + 16\mathbf{k}$;

$s\left(\frac{\pi}{4}\right) = \sqrt{4 + 9e^{\pi/2}} \approx 6.877$

28. $\mathbf{v}(t) = -e^t\mathbf{i} - \sin\pi t\mathbf{j} + \frac{2}{3}t^{-1/3}\mathbf{k}$

$\mathbf{a}(t) = -e^t\mathbf{i} - \pi\cos\pi t\mathbf{j} - \frac{2}{9}t^{-4/3}\mathbf{k}$

$\mathbf{v}(2) = -e^2\mathbf{i} + \frac{2^{2/3}}{3}\mathbf{k}$; $\mathbf{a}(2) = -e^2\mathbf{i} - \pi\mathbf{j} - \frac{1}{9\sqrt[3]{2}}\mathbf{k}$;

$s(2) = \sqrt{e^4 + \frac{2^{4/3}}{9}} \approx 7.408$

29. $\mathbf{v}(t) = (\pi t\cos\pi t + \sin\pi t)\mathbf{i}$

$+ (\cos\pi t - \pi t\sin\pi t)\mathbf{j} - e^{-t}\mathbf{k}$

$\mathbf{a}(t) = (2\pi\cos\pi t - \pi^2 t\sin\pi t)\mathbf{i}$

$+ (-2\pi\sin\pi t - \pi^2 t\cos\pi t)\mathbf{j} + e^{-t}\mathbf{k}$

$\mathbf{v}(2) = 2\pi\mathbf{i} + \mathbf{j} - e^{-2}\mathbf{k}$; $\mathbf{a}(2) = 2\pi\mathbf{i} - 2\pi^2\mathbf{j} + e^{-2}\mathbf{k}$;

$s(2) = \sqrt{4\pi^2 + 1 + e^{-4}} \approx 6.364$

30. $\mathbf{v}(t) = \frac{1}{t}\mathbf{i} + \frac{2}{t}\mathbf{j} + \frac{3}{t}\mathbf{k}$

$\mathbf{a}(t) = -\frac{1}{t^2}\mathbf{i} - \frac{2}{t^2}\mathbf{j} - \frac{3}{t^2}\mathbf{k}$

$\mathbf{v}(2) = \frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k}$; $\mathbf{a}(2) = -\frac{1}{4}\mathbf{i} - \frac{1}{2}\mathbf{j} - \frac{3}{4}\mathbf{k}$;

$s(2) = \sqrt{\frac{1}{4} + 1 + \frac{9}{4}} = \frac{\sqrt{14}}{2} \approx 1.871$

31. If $\|\mathbf{v}\| = C$, then $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = C$ Differentiate implicitly to get $D_t(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\mathbf{v}' = 0.$ Thus, $\mathbf{v}\cdot\mathbf{v}' = \mathbf{v}\cdot\mathbf{a} = 0,$ so **a** is perpendicular to **v**.

32. If $\|\mathbf{r}(t)\| = C,$ similar to Problem 31, $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0.$ Conversely, if $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0,$ then $2\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0.$ But since $2\mathbf{r}(t)\cdot\mathbf{r}'(t) = D_t[\mathbf{r}(t)\cdot\mathbf{r}(t)],$ this means that $\mathbf{r}(t)\cdot\mathbf{r}(t) = \|\mathbf{r}(t)\|^2$ is a constant, so $\|\mathbf{r}(t)\|$ is constant.

33. $s = \int_0^2 \sqrt{1^2 + \cos^2 t + (-\sin t)^2}\,dt$

$= \int_0^2 \sqrt{1+\cos^2 t + \sin^2 t}\,dt = \int_0^2 \sqrt{1+1}\,dt$

$= \sqrt{2}\int_0^2 dt = \sqrt{2}(2-0) = 2\sqrt{2}$

34. $s = \int_0^2 \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2}\,dt$

$= \int_0^2 \sqrt{t^2+3}\,dt$

$= \left[\frac{t}{2}\sqrt{t^2+3} + \frac{3}{2}\ln\left|t+\sqrt{t^2+3}\right|\right]_0^2$

$= \sqrt{7} + \frac{3}{2}\ln\left(2+\sqrt{7}\right) - \frac{3}{2}\ln\sqrt{3} \approx 4.126$

Use Formula 44 with $u = t$ and $a = \sqrt{3}$ for $\int\sqrt{t^2+3}\,dt.$

35. $s = \int_3^6 \sqrt{24t^2 + 4t^4 + 36}\,dt$

$= \int_3^6 2\sqrt{t^4 + 6t^2 + 9}\,dt$

$= \int_3^6 2(t^2+3)\,dt = 2\left[\frac{t^3}{3} + 3t\right]_3^6$

$= 2[72 + 18 - (9 + 9)] = 144$

36. $s = \int_0^1 \sqrt{4t^2 + 36t^4 + 324t^4}\,dt$

$= \int_0^1 2t\sqrt{1+90t^2}\,dt = \left[\frac{1}{135}(1+90t^2)^{3/2}\right]_0^1$

$= \frac{1}{135}(91^{3/2} - 1) \approx 6.423$

37. $s = \int_0^1 \sqrt{9t^4 + 36t^4 + 324t^4}\,dt$

$= \int_0^1 3\sqrt{41}t^2\,dt = \left[\sqrt{41}t^3\right]_0^1 = \sqrt{41} \approx 6.403$

38. $s = \int_0^1 \sqrt{343t^{12} + 98t^{12} + 1764t^{12}}\,dt$

$= \int_0^1 21\sqrt{5}t^6\,dt = \left[3\sqrt{5}t^7\right]_0^1 = 3\sqrt{5} \approx 6.708$

39. $\mathbf{f}'(u) = -\sin u\mathbf{i} + 3e^{3u}\mathbf{j};\ g'(t) = 6t$

$\mathbf{F}'(t) = \mathbf{f}'(g(t))g'(t)$

$= -6t\sin(3t^2 - 4)\mathbf{i} + 18te^{9t^2-12}\mathbf{j}$

40. $\mathbf{f}'(u) = 2u\mathbf{i} + \sin 2u\mathbf{j};\ g'(t) = \sec^2 t$

$\mathbf{F}'(t) = \mathbf{f}'(g(t))g'(t)$

$= 2\tan t\sec^2 t\mathbf{i} + \sec^2 t\sin(2\tan t)\mathbf{j}$

41. $\int_0^1 (e^t\mathbf{i} + e^{-t}\mathbf{j})dt = \left[e^t\mathbf{i} - e^{-t}\mathbf{j}\right]_0^1$

$= (e-1)\mathbf{i} + (1-e^{-1})\mathbf{j}$

42. $\int_{-1}^1 [(1+t)^{3/2}\mathbf{i} + (1-t)^{3/2}\mathbf{j}]dt$

$= \left[\frac{2}{5}(1+t)^{5/2}\mathbf{i} - \frac{2}{5}(1-t)^{5/2}\mathbf{j}\right]_{-1}^1$

$= \frac{8\sqrt{2}}{5}\mathbf{i} + \frac{8\sqrt{2}}{5}\mathbf{j}$

43. $\mathbf{r}(t) = 5\cos 6t\mathbf{i} + 5\sin 6t\mathbf{j}$

$\mathbf{v}(t) = -30\sin 6t\mathbf{i} + 30\cos 6t\mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{900\sin^2 6t + 900\cos^2 6t} = 30$

$\mathbf{a}(t) = -180\cos 6t\mathbf{i} - 180\sin 6t\mathbf{j}$

44. **a.** $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + (2t-3)\mathbf{k}$

$2t - 3 < 0$ for $t < \frac{3}{2},$ so the particle moves downward for $0 \le t < \frac{3}{2}.$

b. $|\mathbf{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t + (2t-3)^2}$

$= \sqrt{4t^2 - 12t + 10}$

$4t^2 - 12t + 10 = 0$ has no real-number solutions, so the particle never has speed 0, i.e., it never stops moving.

c. $t^2 - 3t + 2 = 12$ when $t^2 - 3t - 10 = (t+2)(t-5) = 0,$ $t = -2, 5.$ Since $t \ge 0,$ the particle is 12 meters above the ground when $t = 5.$

d. $\mathbf{v}(5) = \cos 5\mathbf{i} - \sin 5\mathbf{j} + 7\mathbf{k}$

45. a. The motion of the planet with respect to the sun can be given by

$x = R_p \cos t,\quad y = R_p \sin t.$

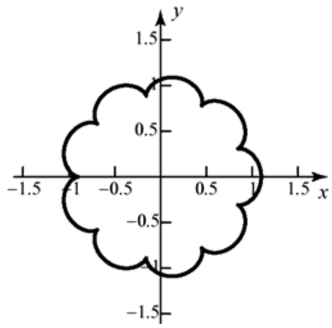
Assume that when $t = 0$, both the planet and the moon are on the x-axis. Since the moon orbits the planet 10 times for every time the planet orbits the sun, the motion of the moon with respect to the planet can be given by

$x = R_m \cos 10t,\quad y = R_m \sin 10t.$

Combining these equations, the motion of the moon with respect to the sun is given by $x = R_p \cos t + R_m \cos 10t,$

$y = R_p \sin t + R_m \sin 10t.$

b.



c. $x'(t) = -R_p \sin t - 10R_m \sin 10t$

$y'(t) = R_p \cos t + 10R_m \cos 10t$

The moon is motionless with respect to the sun when $x'(t)$ and $y'(t)$ are both 0.

Solve $x'(t) = 0$ for sin t and $y'(t) = 0$ for cos t to get $\sin t = -\frac{10R_m}{R_p}\sin 10t,$

$\cos t = -\frac{10R_m}{R_p}\cos 10t$.

Since $\sin^2 t + \cos^2 t = 1$

$1 = \frac{100R_m^2}{R_p^2}\sin^2 10t + \frac{100R_m^2}{R_p^2}\cos^2 10t$

$= \frac{100R_m^2}{R_p^2}$. Thus, $R_p^2 = 100R_m^2$ or

$R_p = 10R_m$. Substitute this into

$x'(t) = 0$ and $y'(t) = 0$ to get

$-R_p(\sin t + \sin 10t) = 0$ and

$R_p(\cos t + \cos 10t) = 0.$

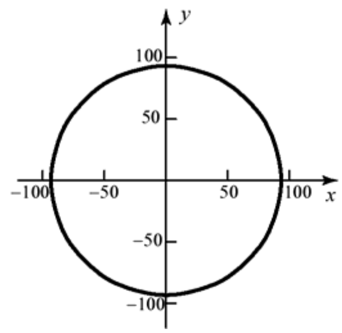
If $0 \le t \le \frac{\pi}{2},$ then to have sin t + sin $10t$ = 0 and cos t + cos $10t$ = 0 it must be that

$10t = \pi + t$ or $t = \frac{\pi}{9}.$

Thus, when the radius of the planet's orbit around the sun is ten times the radius of the moon's orbit around the planet and $t = \frac{\pi}{9},$ the moon is motionless with respect to the sun.

46. a. Years

b.



The moon's orbit is almost indistinguishable from a circle.

c. The sun orbits the earth once each year while the moon orbits the earth roughly 13 times each year.

d. $\|\mathbf{r}(0)\| = \|93.24\mathbf{i}\| = 93.24$ million mi, the sum of the orbital radii.

e. 93 – 0.24 = 92.76 million mi

f. No; since the moon orbits the earth 13 times for each time the earth orbits the sun, the moon could not be stationary with respect to the sun unless the radius of its orbit around the earth were $\frac{1}{13}$ th the radius of the earth's orbit around the sun.

g. $\mathbf{v}(t) = [-186\pi \sin(2\pi t) - 6.24\pi \sin(26\pi t)]\mathbf{i} + [186\pi \cos(2\pi t) + 6.24\pi \cos(26\pi t)]\mathbf{j}$

$\mathbf{a}(t) = [-372\pi^2\cos(2\pi t) - 162.24\pi^2\cos(26\pi t)]\mathbf{i}$
$+ [-372\pi^2\sin(2\pi t) - 162.24\pi^2\sin(26\pi t)]\mathbf{j}$

$\mathbf{v}\left(\frac{1}{2}\right) = 0\mathbf{i} + (-186\pi - 6.24\pi)\mathbf{j} = -192.24\pi\mathbf{j}$

$s\left(\frac{1}{2}\right) = 192.24\pi$ million mi/yr

$\mathbf{a}\left(\frac{1}{2}\right) = (372\pi^2 + 162.24\pi^2)\mathbf{i} + 0\mathbf{j} = 534.24\pi^2\mathbf{i}$

47. a. Winding upward around the right circular cylinder $x = \sin t$, $y = \cos t$ as t increases.

b. Same as part a, but winding faster/slower by a factor of $3t^2$.

c. With standard orientation of the axes, the motion is winding to the right around the right circular cylinder $x = \sin t$, $z = \cos t$.

d. Spiraling upward, with increasing radius, along the spiral $x = t \sin t$, $y = t \cos t$.

48. For this problem, keep in mind that $\mathbf{r}$, θ, $\mathbf{u}_1$, and $\mathbf{u}_2$ are all functions of t and that prime indicates differentiation with respect to t.

a. $\mathbf{u}_1 = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and $\mathbf{u}_2 = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$. Applying the Chain Rule to $\mathbf{u}_1$ gives

$$\mathbf{u}_1' = (-\sin\theta)\theta'\mathbf{i} + (\cos\theta)\theta'\mathbf{j}$$
$$= \theta'(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = \theta'\mathbf{u}_2$$

Similarly, applying the Chain Rule to $\mathbf{u}_2$ gives

$$\mathbf{u}_2' = (-\cos\theta)\theta'\mathbf{i} + (-\sin\theta)\theta'\mathbf{j}$$
$$= -\theta'(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = -\theta'\mathbf{u}_1$$

b. $$\mathbf{v}(t) = \mathbf{r}'(t) = D_t(r\,\mathbf{u}_1) = r\,\mathbf{u}_1' + r'\mathbf{u}_1$$
$$= r'\mathbf{u}_1 + r\theta'\mathbf{u}_2$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = D_t(r'\mathbf{u}_1 + r\theta'\mathbf{u}_2)$$
$$= r'\mathbf{u}_1' + r''\mathbf{u}_1 + r\theta'\mathbf{u}_2' + r\theta''\mathbf{u}_2 + \theta' r'\mathbf{u}_2$$
$$= \left(r'' - r(\theta')^2\right)\mathbf{u}_1 + (2r'\theta' + r\theta'')\mathbf{u}_2$$

c. The only force acting on the planet is the gravitational attraction of the sun, which is a force directed along the line from the sun to the planet. Thus, by Newton's Second Law,

$$m\mathbf{a} = \mathbf{F} = -c\mathbf{u}_1 + 0\mathbf{u}_2$$

From Newton's Law of Gravitation

$$\|\mathbf{F}\| = \frac{GMm}{r^2}$$

so from part (b)

$$m\mathbf{a} = -\frac{GMm}{r^2}\mathbf{u}_1$$

$$\left(r'' - r(\theta')^2\right)\mathbf{u}_1 + (2r'\theta' + r\theta'')\mathbf{u}_2 = \mathbf{a} = -\frac{GM}{r^2}\mathbf{u}_1$$

Equating the coefficients of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ gives

$$r'' - r(\theta')^2 = -\frac{GM}{r^2}$$
$$2r'\theta' r\theta'' = 0$$

d. $\mathbf{r}\times\mathbf{r}'$ is a constant vector by Example 8. Call it $\mathbf{D} = \mathbf{r}\times\mathbf{r}'$. Thus,

$$\mathbf{D} = \mathbf{r}\times\mathbf{r}' = \mathbf{r}\times(r_1\,\mathbf{u}_1 + r\theta'\mathbf{u}_2)$$
$$= r_1\,\mathbf{r}\times\mathbf{u}_1 + r\theta'\mathbf{r}\times\mathbf{u}_2$$
$$= \mathbf{0} + (r\theta')(r\,\mathbf{u}_1)\times\mathbf{u}_2$$
$$= r^2\theta'(\mathbf{u}_1\times\mathbf{u}_2)$$
$$= r^2\theta'\mathbf{k}$$

e. The speed at $t = 0$ is the distance from the sun times the angular velocity, that is, $v_0 = r_0\,\theta'(0)$. Thus, $\theta'(0) = v_0 / r_0$. Substituting these into the expression from part (d) gives

$$(r(0))^2\,\theta'(0)\mathbf{k} = \mathbf{D}$$

$$(r(0))^2 \frac{v_0}{r_0}\mathbf{k} = \mathbf{D}$$

$$\mathbf{D} = r_0 v_0\,\mathbf{k}$$

Since **D** is a constant vector (i.e., constant for all *t*), we conclude that

$$r^2\theta'\mathbf{k} = \mathbf{D} = r_0 v_0\,\mathbf{k}$$

$$r^2\theta' = r_0 v_0$$

for all *t*.

f. Let $q = r'$. From (c)

$$q = r' = \frac{dr}{dt}$$

$$r'' = \frac{d}{dt}\frac{dr}{dt} = \frac{dq}{dt} = \frac{dq}{dr}\frac{dr}{dt} = q\frac{dq}{dr}$$

$$r'' - r(\theta')^2 = -\frac{GM}{r^2}$$

$$q\frac{dq}{dr} - r\left(\frac{r_0 v_0}{r^2}\right)^2 = -\frac{GM}{r^2}$$

$$q\frac{dq}{dr} = r\left(\frac{r_0 v_0}{r^2}\right)^2 - \frac{GM}{r^2}$$

$$q\frac{dq}{dr} = \frac{r_0^2 v_0^2}{r^3} - \frac{GM}{r^2}$$

g. Integrating the result from (f) gives

$$\int q\frac{dq}{dr}\,dr = \int \left(\frac{r_0^2 v_0^2}{r^3} - \frac{GM}{r^2}\right)dr$$

$$\frac{1}{2}q^2 = \frac{r_0^2 v_0^2}{(-2)r^2} + \frac{GM}{r} + C$$

$$q^2 = -\frac{r_0^2 v_0^2}{r^2} + \frac{2GM}{r} + C_1$$

When $t = 0$, $r'(0) = q(0) = 0$ since the rate of change of distance from the origin is 0 at the perihelion. Also, when $t = 0$, $r(0) = r_0$.

Thus

$$0 = -\frac{r_0^2 v_0^2}{r_0^2} + \frac{2GM}{r_0} + C_1$$

$$C_1 = -\frac{2GM}{r_0} + v_0^2$$

Thus,

$$q^2 = -\frac{r_0^2 v_0^2}{r^2} + \frac{2GM}{r} + C_1 = -\frac{r_0^2 v_0^2}{r^2} + \frac{2GM}{r} - \frac{2GM}{r_0} + v_0^2 = 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) + v_0^2\left(1 - \frac{r_0^2}{r^2}\right)$$

h. Let $p = 1/r$. Then $r = 1/p$ and $r' = -\frac{1}{p^2}p'$. Thus,

$$q^2 = (r')^2 = \left(-\frac{p'}{p^2}\right)^2 = \frac{(p')^2}{p^4}$$

Dividing both sides of the equation from (g) by $\left(r^2\theta'\right)^2$ and using the result from (e) that $r^2\theta' = r_0v_0$ gives

$$\frac{q^2}{\left(r^2\theta'\right)^2} = \frac{2GM}{\left(r^2\theta'\right)^2}\left(\frac{1}{r}-\frac{1}{r_0}\right)+\frac{v_0^2}{\left(r^2\theta'\right)^2}\left(1-\frac{r_0^2}{r^2}\right)$$

$$\frac{q^2}{\left(r^2\theta'\right)^2} = \frac{2GM}{(r_0v_0)^2}\left(\frac{1}{r}-\frac{1}{r_0}\right)+\frac{v_0^2}{(r_0v_0)^2}\left(1-\frac{r_0^2}{r^2}\right)$$

$$\frac{q^2}{\left(r^2\theta'\right)^2} = \frac{2GM}{r_0^2v_0^2}\left(\frac{1}{r}-\frac{1}{r_0}\right)+\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)$$

Now substitute the result from above to get

$$\frac{\frac{1}{p^4}(p')^2}{\frac{1}{p^4}(\theta')^2} = \frac{2GM}{v_0^2r_0^2}(p-p_0)+\left(p_0^2-p^2\right)$$

$$\frac{v_0^2r_0^2}{(\theta')^2}\left(\frac{dp}{dt}\right)^2 = 2GM\,(p-p_0)+v_0^2r_0^2\left(p_0^2-p^2\right)$$

$$\frac{v_0^2r_0^2}{(\theta')^2}\left(\frac{dp}{dt}\right)^2 = 2GM\,(p-p_0)+v_0^2\left(1-\frac{p^2}{p_0^2}\right)$$

i. Continuing with the equation from (h), and using the Chain Rule gives

$$v_0^2r_0^2\left(\frac{dp/dt}{d\theta/dt}\right)^2 = 2GM\,(p-p_0)+v_0^2\left(1-\frac{p^2}{p_0^2}\right)$$

$$\left(\frac{dp}{d\theta}\right)^2 = \frac{2GMp_0^2}{v_0^2}(p-p_0)+\left(p_0^2-p^2\right)$$

$$\left(\frac{dp}{d\theta}\right)^2 = p_0^2-\frac{2GMp_0^2}{v_0^2}p_0+\left(\frac{GMp_0^2}{v_0^2}\right)^2-\left(p^2-\frac{2GMp_0^2}{v_0^2}p+\left(\frac{GMp_0^2}{v_0^2}\right)^2\right)$$

$$\left(\frac{dp}{d\theta}\right)^2 = \left(p_0-\frac{GMp_0^2}{v_0^2}\right)^2-\left(p-\frac{GMp_0^2}{v_0^2}\right)^2$$

j. Taking the square root of both sides from part (i) gives

$$\frac{dp}{d\theta} = \pm\sqrt{\left(p_0-\frac{GMp_0^2}{v_0^2}\right)^2-\left(p-\frac{GMp_0^2}{v_0^2}\right)^2}$$

From (e) we have $r^2\theta' = r_0v_0$, so $\theta' = r_0v_0/r^2 > 0$. Recall that the planet is at its perihelion at time $t = 0$, so this is as close as it gets to the sun. Thus, for t near 0, the distance from the sun r must increase with t.

Thus, $\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{r'}{\theta'} > 0$ and from the beginning of (h) $r' = -\frac{1}{p^2}p'$. Thus, $\frac{dp}{d\theta} = \frac{dp/dt}{d\theta/dt} = \frac{-p^2r'}{d\theta/dt} < 0$

The minus sign is the correct sign to take.

k. Separating variables in this differential equation gives

$$\frac{-dp}{\sqrt{\left(p_0 - \frac{GMp_0^2}{v_0^2}\right)^2 - \left(p - \frac{GMp_0^2}{v_0^2}\right)^2}} = d\theta$$

$$\int 1/\sqrt{\left(p_0 - GMp_0^2/v_0^2\right)^2 - \left(p - GMp_0^2/v_0^2\right)^2}\, dp = \int d\theta$$

$$\cos^{-1}\left(\frac{p - GMp_0^2/v_0^2}{p_0 - GMp_0^2/v_0^2}\right) = \theta + C_2$$

The initial condition for this differential equation is $p = p_0$ when $t = 0$. Thus,

$$\cos^{-1}\left(\frac{p_0 - GMp_0^2/v_0^2}{p_0 - GMp_0^2/v_0^2}\right) = \theta(0) + C_2$$

$$\cos^{-1}(1) = 0 + C_2$$

$$0 = C_2$$

The solution is therefore $\theta = \cos^{-1}\left(\frac{p - GMp_0^2/v_0^2}{p_0 - GMp_0^2/v_0^2}\right)$

l. Finally (!)

$$\cos\theta = \frac{p - GMp_0^2/v_0^2}{p_0 - GMp_0^2/v_0^2}$$

Solving this for p gives $p = \left(p_0 - GMp_0^2/v_0^2\right)(\cos\theta) + GMp_0^2/v_0^2$

Recall that $p = 1/r$, so that

$$\frac{1}{r} = \left(p_0 - GMp_0^2/v_0^2\right)(\cos\theta) + GMp_0^2/v_0^2$$

$$r = \frac{1}{GMp_0^2/v_0^2 + \left(p_0 - GMp_0^2/v_0^2\right)(\cos\theta)} = \frac{r_0}{\frac{GM}{r_0 v_0^2} + \left(1 - \frac{GM}{r_0 v_0^2}\right)\cos\theta}$$

$$= \frac{r_0 \frac{r_0 v_0^2}{GM}}{1 + \left(\frac{r_0 v_0^2}{GM} - 1\right)\cos\theta} = \frac{r_0(1+e)}{1 + e\cos\theta}$$

where $e = \frac{r_0 v_0^2}{GM} - 1$ is the eccentricity. This is the polar equation of an ellipse.

11.6 Concepts Review

1. $1 + 4t$; $-3 - 2t$; $2 - t$

2. $\frac{x-1}{4} = \frac{y+3}{-2} = \frac{z-2}{-1}$

3. $2t\mathbf{i} - 3\mathbf{j} + 3t^2\mathbf{k}$

4. $\langle 2, -3, 3\rangle$; $\frac{x-1}{2} = \frac{y+3}{-3} = \frac{z-1}{3}$

Problem Set 11.6

1. A parallel vector is
$\mathbf{v} = \langle 4-1, 5+2, 6-3\rangle = \langle 3, 7, 3\rangle$.
$x = 1 + 3t, y = -2 + 7t, z = 3 + 3t$

2. A parallel vector is
$\mathbf{v} = \langle 7-2, -2+1, 3+5\rangle = \langle 5, -1, 8\rangle$
$x = 2 + 5t, y = -1 - t, z = -5 + 8t$

3. A parallel vector is
$\mathbf{v} = \langle 6-4, 2-2, -1-3\rangle = \langle 2, 0, -4\rangle$ or
$\langle 1, 0, -2\rangle$.
$x = 4 + t, y = 2, z = 3 - 2t$

4. A parallel vector is
$\mathbf{v} = \langle 5-5, 4+3, 2+3\rangle = \langle 0, 7, 5\rangle$
$x = 5, y = -3 + 7t, z = -3 + 5t$

5. $x = 4 + 3t, y = 5 + 2t, z = 6 + t$
$\frac{x-4}{3} = \frac{y-5}{2} = \frac{z-6}{1}$

6. $x = -1 - 2t, y = 3, z = -6 + 5t$
Since the second direction number is 0, the line does not have symmetric equations.

7. Another parallel vector is $\langle 1, 10, 100\rangle$.
$x = 1 + t, y = 1 + 10t, z = 1 + 100t$
$\frac{x-1}{1} = \frac{y-1}{10} = \frac{z-1}{100}$

8. $x = -2 + 7t, y = 2 - 6t, z = -2 + 3t$
$\frac{x+2}{7} = \frac{y-2}{-6} = \frac{z+2}{3}$

9. Set $z = 0$. Solving $4x + 3y = 1$ and $10x + 6y = 10$ yields $x = 4, y = -5$. Thus $P_1(4, -5, 0)$ is on the line. Set $y = 0$. Solving $4x - 7z = 1$ and $10x - 5z = 10$ yields $x = \frac{13}{10}, z = \frac{3}{5}$. Thus $P_2\left(\frac{13}{10}, 0, \frac{3}{5}\right)$ is on the line.
$\overrightarrow{P_1P_2} = \left\langle \frac{13}{10} - 4, 0 - (-5), \frac{3}{5} - 0\right\rangle = \left\langle -\frac{27}{10}, 5, \frac{3}{5}\right\rangle$ is a direction vector. Thus,
$\langle 27, -50, -6\rangle = -10\overrightarrow{P_1P_2}$ is also a direction vector. The symmetric equations are thus
$\frac{x-4}{27} = \frac{y+5}{-50} = \frac{z}{-6}$

10. With $x = 0$, $y - z = 2$ and $-2y + z = 3$ yield $(0, -5, -7)$.
With $y = 0$, $x - z = 2$ and $3x + z = 3$ yield $\left(\frac{5}{4}, 0, -\frac{3}{4}\right)$.
A vector parallel to the line is
$\left\langle \frac{5}{4}, 5, -\frac{3}{4} + 7\right\rangle = \left\langle \frac{5}{4}, 5, \frac{25}{4}\right\rangle$ or $\langle 1, 4, 5\rangle$.
$\frac{x}{1} = \frac{y+5}{4} = \frac{z+7}{5}$

11. $\mathbf{u} = \langle 1, 4, -2\rangle$ and $\mathbf{v} = \langle 2, -1, -2\rangle$ are both perpendicular to the line, so $\mathbf{u} \times \mathbf{v} = \langle -10, -2, -9\rangle$, and hence $\langle 10, 2, 9\rangle$ is parallel to the line.
With $y = 0$, $x - 2z = 13$ and $2x - 2z = 5$ yield $\left(-8, 0, -\frac{21}{2}\right)$. The symmetric equations are
$\frac{x+8}{10} = \frac{y}{2} = \frac{z + \frac{21}{2}}{9}$

12. $\mathbf{u} = \langle 1, -3, 1\rangle$ and $\mathbf{v} = \langle 6, -5, 4\rangle$ are both perpendicular to the line, so $\mathbf{u} \times \mathbf{v} = \langle -7, 2, 13\rangle$ is parallel to the line.
With $x = 0$, $-3y + z = -1$ and $-5y + 4z = 9$ yield $\left(0, \frac{13}{7}, \frac{32}{7}\right)$.
$\frac{x}{-7} = \frac{y - \frac{13}{7}}{2} = \frac{z - \frac{32}{7}}{13}$

13. $\langle 1, -5, 2\rangle$ is a vector in the direction of the line.
$\frac{x-4}{1} = \frac{y}{-5} = \frac{z-6}{2}$

14. $\langle 2,1,-3\rangle \times \langle 5,4,-1\rangle = \langle 11,-13,3\rangle$ is in the direction of the line.

$$\frac{x+5}{11}=\frac{y-7}{-13}=\frac{z+2}{3}$$

15. The point of intersection on the z-axis is (0, 0, 4). A vector in the direction of the line is $\langle 5-0,-3-0,4-4\rangle = \langle 5,-3,0\rangle$. Parametric equations are $x = 5t$, $y = -3t$, $z = 4$.

16. $\langle 3,1,-2\rangle \times \langle 2,3,-1\rangle = \langle 5,-1,7\rangle$ is in the direction of the line since the line is perpendicular to $\langle 3,1,-2\rangle$ and $\langle 2,3,-1\rangle$.

$$\frac{x-2}{5}=\frac{y+4}{-1}=\frac{z-5}{7}$$

17. Using $t = 0$ and $t = 1$, two points on the first line are (–2, 1, 2) and (0, 5, 1). Using $t = 0$, a point on the second line is (2, 3, 1). Thus, two nonparallel vectors in the plane are

$\langle 0+2,5-1,1-2\rangle = \langle 2,4,1\rangle$ and

$\langle 2+2,3-1,1-2\rangle = \langle 4,2,-1\rangle$

Hence, $\langle 2,4,-1\rangle \times \langle 4,2,-1\rangle = \langle -2,-2,-12\rangle$ is a normal to the plane, and so is $\langle 1, 1, 6\rangle$. An equation of the plane is

$1(x + 2) + 1(y - 1) + 6(z - 2) = 0$ or

$x + y + 6z = 11$.

18. Solve $\frac{x-1}{-4}=\frac{y-2}{3}$ and $\frac{x-2}{-1}=\frac{y-1}{1}$ simultaneously to get $x = 1$, $y = 2$. From the first line $\frac{1-1}{-4}=\frac{z-4}{-2}$, so $z = 4$ and (1, 2, 4) is on the first line. This point also satisfies the equations of the second line, so the lines intersect.

$\langle -4,3,-2\rangle$ and $\langle -1,1,6\rangle$ are parallel to the plane determined by the lines, so

$\langle -4,3,-2\rangle \times \langle -1,1,6\rangle = \langle 20,26,-1\rangle$ is a normal to the plane. An equation of the plane is

$20(x - 1) + 26(y - 2) - 1(z - 4) = 0$ or

$20x + 26y - z = 68$.

19. Using $t = 0$, another point in the plane is (1, –1, 4) and $\langle 2, 3, 1\rangle$ is parallel to the plane. Another parallel vector is

$\langle 1-1,-1+1,5-4\rangle = \langle 0, 0, 1\rangle$. Thus,

$\langle 2, 3, 1\rangle \times \langle 0, 0, 1\rangle = \langle 3,-2,0\rangle$ is a normal to the plane. An equation of the plane is

$3(x - 1) - 2(y + 1) + 0(z - 5) = 0$ or $3x - 2y = 5$.

20. Using $t = 0$, one point of the plane is (0, 1, 0).

$\langle 2,-1,1\rangle \times \langle 0, 1, 1\rangle = \langle -2,-2,2\rangle = -2\langle 1,1,-1\rangle$ is perpendicular to the normals of both planes, hence parallel to their line of intersection.

$\langle 3, 1, 2\rangle$ is parallel to the line in the plane we seek, thus $\langle 3, 1, 2\rangle \times \langle 1,1,-1\rangle = \langle -3,5,2\rangle$ is a normal to the plane. An equation of the plane is

$-3(x - 0) + 5(y - 1) + 2(z - 0) = 0$ or

$-3x + 5y + 2z = 5$.

21. **a.** With $t = 0$ in the first line, $x = 2 - 0 = 2$, $y = 3 + 4\cdot 0 = 3$, $z = 2\cdot 0 = 0$, so (2, 3, 0) is on the first line.

b. $\langle -1,4,2\rangle$ is parallel to the first line, while $\langle 1, 0, 2\rangle$ is parallel to the second line, so

$\langle -1,4,2\rangle \times \langle 1, 0, 2\rangle = \langle 8,4,-4\rangle = 4\langle 2,1,-1\rangle$

is normal to both. Thus, π has equation

$2(x - 2) + 1(y - 3) - 1(z - 0) = 0$ or

$2x + y - z = 7$, and contains the first line.

c. With $t = 0$ in the second line,

$x = -1 + 0 = -1$, $y = 2$, $z = -1 + 2\cdot 0 = -1$, so $Q(-1, 2, -1)$ is on the second line.

d. From Example 10 of Section 11.3, the distance from Q to π is

$$\frac{|2(-1)+(2)-(-1)-7|}{\sqrt{4+1+1}}=\frac{6}{\sqrt{6}}=\sqrt{6}\approx 2.449.$$

22. With $t = 0$, (1, –3, –1) is on the first line.

$\langle 2,4,-1\rangle \times \langle -2,3,2\rangle = \langle 11,-2,14\rangle$ is perpendicular to both lines, so

$11(x - 1) - 2(y + 3) + 14(z + 1) = 0$ or

$11x - 2y + 14z = 3$ is parallel to both lines and contains the first line.

With $t = 0$, (4, 1, 0) is on the second line. The distance from (4, 1, 0) to $11x - 2y + 14z = 3$ is

$$\frac{|11(4)-2(1)+14(0)-3|}{\sqrt{121+4+196}}=\frac{39}{\sqrt{321}}\approx 2.1768.$$

23. $\mathbf{r}\left(\frac{\pi}{3}\right)=\mathbf{i}+3\sqrt{3}\mathbf{j}+\frac{\pi}{3}\mathbf{k}$, so $\left(1, 3\sqrt{3}, \frac{\pi}{3}\right)$ is on the tangent line.

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 6\cos t\mathbf{j} + \mathbf{k}$, so

$\mathbf{r}'\left(\frac{\pi}{3}\right)=-\sqrt{3}\mathbf{i}+3\mathbf{j}+\mathbf{k}$ is parallel to the tangent line at $t=\frac{\pi}{3}$. The symmetric equations of the line are $\frac{x-1}{-\sqrt{3}}=\frac{y-3\sqrt{3}}{3}=\frac{z-\frac{\pi}{3}}{1}$.

24. The curve is given by $\mathbf{r}(t) = 2t^2\mathbf{i} + 4t\mathbf{j} + t^3\mathbf{k}$. $\mathbf{r}(1) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, so (2, 4, 1) is on the tangent line.
$\mathbf{r}'(t) = 4t\mathbf{i} + 4\mathbf{j} + 3t^2\mathbf{k}$, so $\mathbf{r}'(1) = 4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ is parallel to the tangent line. The parametric equations of the line are $x = 2 + 4t$, $y = 4 + 4t$, $z = 1 + 3t$.

25. The curve is given by $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j} + t^5\mathbf{k}$. $\mathbf{r}(-1) = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, so (–3, 2, –1) is on the plane.
$\mathbf{r}'(t) = 3\mathbf{i} + 4t\mathbf{j} + 5t^4\mathbf{k}$, so $\mathbf{r}'(-1) = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ is in the direction of the curve at $t = -1$, hence normal to the plane. An equation of the plane is $3(x + 3) - 4(y - 2) + 5(z + 1) = 0$ or $3x - 4y + 5z = -22$.

26. $\mathbf{r}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\mathbf{i} + \frac{3\pi}{2}\mathbf{j}$, so $\left(\frac{\pi}{2}, \frac{3\pi}{2}, 0\right)$ is on the plane.
$\mathbf{r}'(t) = (t\cos t + \sin t)\mathbf{i} + 3\mathbf{j} + (2\cos t - 2t\sin t)\mathbf{k}$ so

$\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{i} + 3\mathbf{j} - \pi\mathbf{k}$ is in the direction of the curve at $\frac{\pi}{2}$, hence normal to the plane. An equation of the plane is

$$1\left(x - \frac{\pi}{2}\right) + 3\left(y - \frac{3\pi}{2}\right) - \pi(z - 0) = 0 \text{ or}$$

$x + 3y - \pi z = 5\pi$.

27. **a.** $[x(t)]^2 + [y(t)]^2 + [z(t)]^2$

$$= (2t)^2 + \left(\sqrt{7t}\right)^2 + \left(\sqrt{9 - 7t - 4t^2}\right)^2$$

$$= 4t^2 + 7t + 9 - 7t - 4t^2 = 9$$

Thus, the curve lies on the sphere $x^2 + y^2 + z^2 = 9$ whose center is at the origin.

b. $\mathbf{r}(t) = 2t\,\mathbf{i} + \sqrt{7t}\,\mathbf{j} + \sqrt{9 - 7t - 4t^2}\,\mathbf{k}$

$$\mathbf{r}(1/4) = \frac{1}{2}\mathbf{i} + \sqrt{7/4}\,\mathbf{j} + \sqrt{9 - 7/4 - 1/4}\,\mathbf{k}$$

$$= \frac{1}{2}\mathbf{i} + \frac{\sqrt{7}}{2}\mathbf{j} + \sqrt{7}\,\mathbf{k}$$

$$\mathbf{r}'(t) = 2\mathbf{i} + \frac{\sqrt{7}}{2\sqrt{t}}\mathbf{j} + \frac{-7 - 8t}{2\sqrt{9 - 7t - 4t^2}}\mathbf{k}$$

$$\mathbf{r}'(1/4) = 2\mathbf{i} + \sqrt{7}\,\mathbf{j} - \frac{9}{2\sqrt{7}}\mathbf{k}$$

The tangent line is therefore

$$x = \frac{1}{2} + 2t$$

$$y = \frac{\sqrt{7}}{2} + \sqrt{7}\,t$$

$$z = \sqrt{7} - \frac{9}{2\sqrt{7}}t$$

This line intersects the xz-plane when $y = 0$, which occurs when $0 = \frac{\sqrt{7}}{2} + \sqrt{7}t$, that is, when $t = -\frac{1}{2}$. For this value of t, $x = -\frac{1}{2}$, $y = 0$, and $z = \frac{9}{4\sqrt{7}} + \sqrt{7} = \frac{37}{4\sqrt{7}}$. The point of intersection is therefore $\left(-\frac{1}{2}, 0, \frac{37}{4\sqrt{7}}\right)$.

28. **a.** $[x(t)]^2 + [y(t)]^2 + [z(t)]^2$

$$= (\sin t\cos t)^2 + (\sin^2 t)^2 + \cos^2 t$$

$$= \sin^2 t\cos^2 t + \sin^4 t + \cos^2 t$$

$$= \sin^2 t(\cos^2 t + \sin^2 t) + \cos^2 t$$

$$= \sin^2 t + \cos^2 t = 1$$

Thus, the curve lies on the sphere

$$x^2 + y^2 + z^2 = 1$$

whose center is at the origin.

b. $\mathbf{r}\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$

$= \frac{\sqrt{3}}{4}\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$, so $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ is on the tangent line.

$\mathbf{r}'(t) = (\cos^2 t - \sin^2 t)\mathbf{i} + 2\cos t\sin t\mathbf{j} - \sin t\mathbf{k}$

so $\mathbf{r}'\left(\frac{\pi}{6}\right) = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$ is parallel to the line. The line has equations

$$x = \frac{\sqrt{3}}{4} + t,\ y = \frac{1}{4} + \sqrt{3}t,\ z = \frac{\sqrt{3}}{2} - t.$$

The line intersects the xy-plane when $z = 0$, so $t = \frac{\sqrt{3}}{2}$, hence

$$x = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4},\ y = \frac{1}{4} + \frac{3}{2} = \frac{7}{4}.$$

The point is $\left(\frac{3\sqrt{3}}{4}, \frac{7}{4}, 0\right)$.

29. **a.** $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\mathbf{j} + (1-t^2)\mathbf{k}$. Notice that $x(t) + y(t) + z(t) = 2t + t^2 + 1 - t^2 = 2t + 1$. Since $x = 2t$, we have $x + y + z = x + 1.$, so this curve lies on the plane with equation $y + z = 1$.

b. Since $\mathbf{r}(2) = 4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, $\mathbf{r}'(t) = 2\,\mathbf{i} + 2t\,\mathbf{j} - 2t\,\mathbf{k}$ and $\mathbf{r}'(2) = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, the equation of the tangent line is $x = 4 + 2t$, $y = 4 + 4t$, $z = -3 - 4t$. The line intersects the *xy*-plane when $z = 0$, that is, when $t = -\frac{3}{4}$, which gives $x = \frac{5}{2}$, $y = 1$, and $z = 0$. The point of intersection is therefore $\left(\frac{5}{2}, 1, 0\right)$.

30. In Figure 7, *d* is the magnitude of the scalar projection of $\overrightarrow{PQ}$ on **n**. $\mathbf{pr_n}\,\overrightarrow{PQ} = \frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}$, so

$$\left|\mathbf{pr_n}\,\overrightarrow{PQ}\right| = \left|\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}\right| = \frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$$

The point (0, 0, 1) is on the plane $4x - 4y + 2z = 2$. With $P(0, 0, 1)$, $Q(4, -2, 3)$, and $\mathbf{n} = \langle 2, -2, 1\rangle = \frac{1}{2}\langle 4, -4, 2\rangle$,

$\overrightarrow{PQ} = \langle 4-0, -2-0, 3-1\rangle = \langle 4, -2, 2\rangle$ and

$$d = \frac{|\langle 4, -2, 2\rangle\cdot\langle 2, -2, 1\rangle|}{\sqrt{4+4+1}} = \frac{14}{3}.$$

From Example 10 of Section 11.3,

$$d = \frac{|4(4) - 4(-2) + 2(3) - 2|}{\sqrt{16+16+4}} = \frac{28}{6} = \frac{14}{3}.$$

31. Let $\overrightarrow{PR}$ be the scalar projection of $\overrightarrow{PQ}$ on **n**.

Then $\left\|\overrightarrow{PQ}\right\|^2 = \left\|\overrightarrow{PR}\right\|^2 + d^2$ so

$$d^2 = \left\|\overrightarrow{PQ}\right\|^2 - \left\|\overrightarrow{PR}\right\|^2 = \left\|\overrightarrow{PQ}\right\|^2 - \left(\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|^2}\right)^2$$

$$= \frac{\left\|\overrightarrow{PQ}\right\|^2\|\mathbf{n}\|^2 - \left(\overrightarrow{PQ}\cdot\mathbf{n}\right)^2}{\|\mathbf{n}\|^2} = \frac{\left\|\overrightarrow{PQ}\times\mathbf{n}\right\|^2}{\|\mathbf{n}\|^2}$$

by Lagrange's Identity. Thus,

$$d = \frac{\left\|\overrightarrow{PQ}\times\mathbf{n}\right\|}{\|\mathbf{n}\|}.$$

a. $P(3, -2, 1)$ is on the line, so $\overrightarrow{PQ} = \langle 1-3, 0+2, -4-1\rangle = \langle -2, 2, -5\rangle$ while $\mathbf{n} = \langle 2, -2, 1\rangle$, so

$$d = \frac{\|\langle -2, 2, -5\rangle\times\langle 2, -2, 1\rangle\|}{\sqrt{4+4+1}} = \frac{\|\langle -8, -8, 0\rangle\|}{3} = \frac{8\sqrt{2}}{3} \approx 3.771$$

b. $P(1, -1, 0)$ is on the line, so $\overrightarrow{PQ} = \langle 2-1, -1+1, 3-0\rangle = \langle 1, 0, 3\rangle$ while $\mathbf{n} = \langle 2, 3, -6\rangle$.

$$d = \frac{\|\langle 1, 0, 3\rangle\times\langle 2, 3, -6\rangle\|}{\sqrt{4+9+36}} = \frac{\|\langle -9, 12, 3\rangle\|}{7} = \frac{3\sqrt{26}}{7} \approx 2.185$$

32. *d* is the distance between the parallel planes containing the lines. Since **n** is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, it is normal to the planes. Thus, *d* is the magnitude of the scalar projection of $\overrightarrow{PQ}$ on **n**, which is $\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$.

a. $P(3, -2, 1)$ is on the first line, $Q(-4, -5, 0)$ is on the second line, $\mathbf{n}_1 = \langle 1, 1, 2\rangle$, and $\mathbf{n}_2 = \langle 3, 4, 5\rangle$.

$\overrightarrow{PQ} = \langle -4-3, -5+2, 0-1\rangle = \langle -7, -3, -1\rangle$

$\mathbf{n} = \langle 1, 1, 2\rangle\times\langle 3, 4, 5\rangle = \langle -3, 1, 1\rangle$

$$d = \frac{|\langle -7, -3, -1\rangle\cdot\langle -3, 1, 1\rangle|}{\sqrt{9+1+1}} = \frac{17}{\sqrt{11}} \approx 5.126$$

b. $P(1, –2, 0)$ is on the first line, $Q(0, 1, 0)$ is on the second line, $\mathbf{n}_1 = \langle 2, 3, -4\rangle$, and $\mathbf{n}_2 = \langle 3, 1, -5\rangle$.

$\overrightarrow{PQ} = \langle 0-1, 1+2, 0-0\rangle = \langle -1, 3, 0\rangle$

$\mathbf{n} = \langle 2, 3, -4\rangle \times \langle 3, 1, -5\rangle = \langle -11, -2, -7\rangle$

$$d = \frac{|\langle -1, 3, 0\rangle \cdot \langle -11, -2, -7\rangle|}{\sqrt{121+4+49}}$$

$$= \frac{5}{\sqrt{174}} \approx 0.379$$

11.7 Concepts Review

1. $\frac{d\mathbf{T}}{ds}$

2. $1/a$; 0

3. $\frac{d^2s}{dt^2}; \left(\frac{ds}{dt}\right)^2 \kappa$

4. 0

Problem Set 11.7

1. $\mathbf{r}(t) = \langle t, t^2\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t\rangle$

$\mathbf{v}(1) = \langle 1, 2\rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2\rangle$

$\mathbf{a}(1) = \langle 0, 2\rangle$

$$\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle$$

$$\mathbf{T}(1) = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

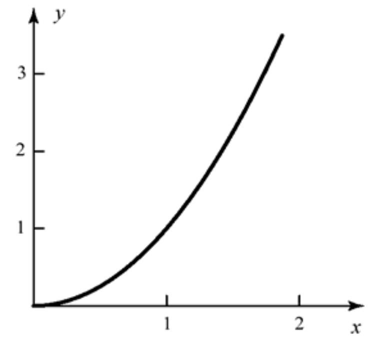
$$\mathbf{T}'(t) = \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle$$

$$\mathbf{T}'(1) = \left\langle -\frac{4}{5^{3/2}}, \frac{2}{5^{3/2}} \right\rangle$$

$$\|\mathbf{T}'(1)\| = \frac{2}{5}$$

$$\|\mathbf{v}(1)\| = \sqrt{5}$$

$$\kappa = \frac{\|\mathbf{T}'(1)\|}{\|\mathbf{v}(1)\|} = \frac{2}{5\sqrt{5}} = \frac{2}{5^{3/2}}$$



2. $\mathbf{r}(t) = \langle t^2, 1+2t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 2\rangle$

$\mathbf{v}(1) = \langle 2, 2\rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 0\rangle$

$\mathbf{a}(1) = \langle 2, 0\rangle$

$$\mathbf{T}(t) = \left\langle \frac{2t}{\sqrt{4+4t^2}}, \frac{2}{\sqrt{4+4t^2}} \right\rangle$$

$$= \left\langle \frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}} \right\rangle$$

$$\mathbf{T}(1) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

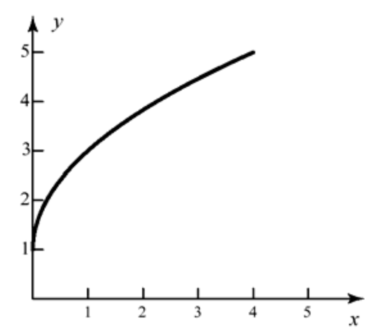
$$\mathbf{T}'(t) = \left\langle \frac{1}{(1+t^2)^{3/2}}, -\frac{t}{(1+t^2)^{3/2}} \right\rangle$$

$$\mathbf{T}'(1) = \left\langle \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}} \right\rangle$$

$$\|\mathbf{T}'(1)\| = \frac{1}{2}$$

$$\|\mathbf{v}(t)\| = 2\sqrt{2}$$

$$\kappa = \frac{\|\mathbf{T}'(1)\|}{\|\mathbf{v}(1)\|} = \frac{1/2}{2\sqrt{2}} = \frac{1}{4\sqrt{2}}$$



3. $\mathbf{r}(t) = \langle t, 2\cos t, 2\sin t \rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -2\sin t, 2\cos t \rangle$

$\mathbf{v}(\pi) = \langle 1, 0, -2 \rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, -2\cos t, -2\sin t \rangle$

$\mathbf{a}(\pi) = \langle 0, 2, 0 \rangle$

$$\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{5}}, -\frac{2\sin t}{\sqrt{5}}, \frac{2\cos t}{\sqrt{5}} \right\rangle$$

$$\mathbf{T}(\pi) = \left\langle \frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \right\rangle$$

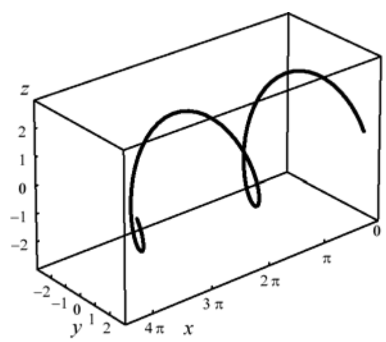
$$\mathbf{T}'(t) = \left\langle 0, -\frac{2\cos t}{\sqrt{5}}, -\frac{2\sin t}{\sqrt{5}} \right\rangle$$

$$\mathbf{T}'(\pi) = \left\langle 0, \frac{2}{\sqrt{5}}, 0 \right\rangle$$

$$\|\mathbf{T}'(\pi)\| = \frac{2}{\sqrt{5}}$$

$$\|\mathbf{v}(t)\| = \sqrt{5}$$

$$\kappa = \frac{\|\mathbf{T}'(\pi)\|}{\|\mathbf{v}(\pi)\|} = \frac{2/\sqrt{5}}{\sqrt{5}} = \frac{2}{5}$$



4. $\mathbf{r}(t) = \langle 5\cos t, 2t, 5\sin t \rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -5\sin t, 2, 5\cos t \rangle$

$\mathbf{v}(\pi) = \langle 0, 2, -5 \rangle$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -5\cos t, 0, -5\sin t \rangle$

$\mathbf{a}(\pi) = \langle 5, 0, 0 \rangle$

$$\mathbf{T}(t) = \left\langle -\frac{5\sin t}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{5\cos t}{\sqrt{29}} \right\rangle$$

$$\mathbf{T}(\pi) = \left\langle 0, \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}} \right\rangle$$

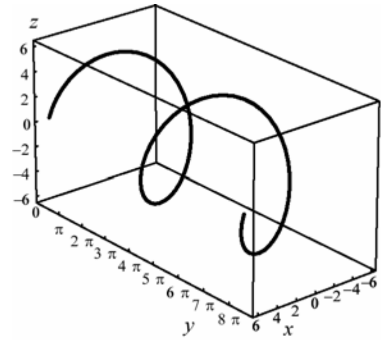
$$\mathbf{T}'(t) = \left\langle -\frac{5\cos t}{\sqrt{29}}, 0, -\frac{5\sin t}{\sqrt{29}} \right\rangle$$

$$\mathbf{T}'(\pi) = \left\langle \frac{5}{\sqrt{29}}, 0, 0 \right\rangle$$

$$\|\mathbf{T}'(\pi)\| = \frac{5}{\sqrt{29}}$$

$$\|\mathbf{v}(t)\| = \sqrt{29}$$

$$\kappa = \frac{\|\mathbf{T}'(\pi)\|}{\|\mathbf{v}(\pi)\|} = \frac{5/\sqrt{29}}{\sqrt{29}} = \frac{5}{29}$$



5. $\mathbf{r}(t) = \left\langle \frac{t^2}{8}, 5\cos t, 5\sin t \right\rangle$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{t}{4}, -5\sin t, 5\cos t \right\rangle$$

$$\mathbf{v}(\pi) = \left\langle \frac{\pi}{4}, 0, -5 \right\rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \left\langle \frac{1}{4}, -5\cos t, -5\sin t \right\rangle$$

$$\mathbf{a}(\pi) = \left\langle \frac{1}{4}, 5, 0 \right\rangle$$

$$\mathbf{T}(t) = \left\langle \frac{t}{\sqrt{400+t^2}}, -\frac{20\sin t}{\sqrt{400+t^2}}, \frac{20\cos t}{\sqrt{400+t^2}} \right\rangle$$

$$\mathbf{T}(\pi) = \left\langle \frac{\pi}{\sqrt{400+\pi^2}}, 0, -\frac{20}{\sqrt{400+\pi^2}} \right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{400}{\left(400+t^2\right)^{3/2}}, \frac{-20\left((400+t^2)\cos t - t\sin t\right)}{\left(400+t^2\right)^{3/2}}, \frac{-20\left((400+t^2)\sin t + t\cos t\right)}{\left(400+t^2\right)^{3/2}} \right\rangle$$

$$\mathbf{T}'(\pi) = \left\langle \frac{400}{\left(400+\pi^2\right)^{3/2}}, \frac{-20\left((400+\pi^2)\cos\pi - \pi\sin\pi\right)}{\left(400+\pi^2\right)^{3/2}}, \frac{-20\left((400+\pi^2)\sin\pi + \pi\cos\pi\right)}{\left(400+\pi^2\right)^{3/2}} \right\rangle$$

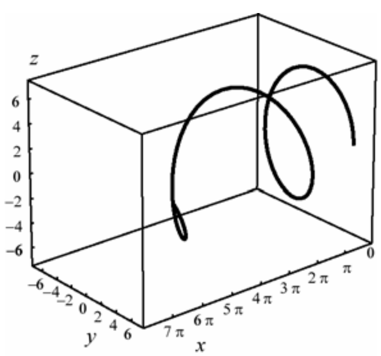
$$= \left\langle \frac{400}{\left(400+\pi^2\right)^{3/2}}, \frac{-20}{\left(400+\pi^2\right)^{1/2}}, \frac{20\pi}{\left(400+\pi^2\right)^{3/2}} \right\rangle$$

$$\left\|\mathbf{T}'(\pi)\right\| = \frac{400^2}{\left(400+\pi^2\right)^3} + \frac{400}{\left(400+\pi^2\right)} + \frac{400\pi^2}{\left(400+\pi^2\right)^3}$$

$$\approx 0.989091$$

$$\left\|\mathbf{v}(t)\right\| = \sqrt{\frac{\pi^2}{16}+25}$$

$$\kappa = \frac{\left\|\mathbf{T}'(\pi)\right\|}{\left\|\mathbf{v}(\pi)\right\|} \approx 0.195422$$



6. $\mathbf{r}(t) = \left\langle \frac{t^2}{4}, 2\cos t, 2\sin t \right\rangle$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{t}{2}, -2\sin t, 2\cos t \right\rangle$$

$$\mathbf{v}(\pi) = \left\langle \frac{\pi}{2}, 0, -2 \right\rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \left\langle \frac{1}{2}, -2\cos t, -2\sin t \right\rangle$$

$$\mathbf{a}(\pi) = \left\langle \frac{1}{2}, 2, 0 \right\rangle$$

$$\mathbf{T}(t) = \left\langle \frac{t}{\sqrt{16+t^2}}, \frac{-4\sin t}{\sqrt{16+t^2}}, \frac{4\cos t}{\sqrt{16+t^2}} \right\rangle$$

$$\mathbf{T}(\pi) = \left\langle \frac{\pi}{\sqrt{16+\pi^2}}, 0, \frac{-4}{\sqrt{16+\pi^2}} \right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{16}{\left(16+t^2\right)^{3/2}}, \frac{-4\left(16+t^2\right)\cos t + 4t\sin t}{\left(16+t^2\right)^{3/2}}, \frac{-4t\cos t + 4\left(16+t^2\right)\sin t}{\left(16+t^2\right)^{3/2}} \right\rangle$$

$$\mathbf{T}'(\pi) = \left\langle \frac{16}{\left(16+\pi^2\right)^{3/2}}, \frac{-4\left(16+\pi^2\right)\cos\pi}{\left(16+\pi^2\right)^{3/2}}, \frac{-4\pi\cos\pi}{\left(16+\pi^2\right)^{3/2}} \right\rangle$$

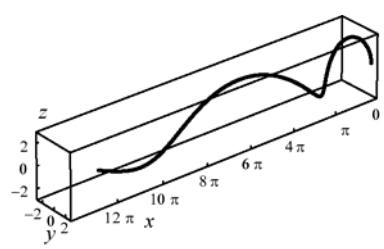
$$\left\|\mathbf{T}'(\pi)\right\| = \sqrt{\frac{16^2 + 4^2\left(16+\pi^2\right)^2 + 4^2\pi^2}{\left(16+\pi^2\right)^3}}$$

$$= \frac{4\sqrt{16+\pi^2+\left(16+\pi^2\right)^2}}{\left(16+\pi\right)^{3/2}}$$

$$\approx 0.801495$$

$$\left\|\mathbf{v}(t)\right\| = \sqrt{4+\frac{\pi^2}{4}}$$

$$\kappa = \frac{\left\|\mathbf{T}'(\pi)\right\|}{\left\|\mathbf{v}(\pi)\right\|} \approx 0.315164$$



7. $\mathbf{u}'(t) = 8t\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{u}'(t)\| = 4\sqrt{4t^2+1}$

$$\mathbf{T}(t) = \frac{\mathbf{u}'(t)}{\|\mathbf{u}'(t)\|} = \frac{2t}{\sqrt{4t^2+1}}\mathbf{i} + \frac{1}{\sqrt{4t^2+1}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = 8t$ $\quad y'(t) = 4$

$x''(t) = 8$ $\quad y''(t) = 0$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{32}{64(4t^2+1)^{3/2}}$$

$$= \frac{1}{2(4t^2+1)^{3/2}}$$

$$\kappa\left(\frac{1}{2}\right) = \frac{1}{2(2)^{3/2}} = \frac{1}{4\sqrt{2}}$$

8. $\mathbf{r}'(t) = t^2\mathbf{i} + t\mathbf{j}$

$\|\mathbf{r}'(t)\| = t\sqrt{t^2+1}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{t}{\sqrt{t^2+1}}\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\mathbf{j}$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = t^2$ $\quad y'(t) = t$

$x''(t) = 2t$ $\quad y''(t) = 1$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{t^2}{t^3(t^2+1)^{3/2}} = \frac{1}{t(t^2+1)^{3/2}}$$

$$\kappa(1) = \frac{1}{1(2)^{3/2}} = \frac{1}{2\sqrt{2}}$$

9. $\mathbf{z}'(t) = -3\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\|\mathbf{z}'(t)\| = \sqrt{9\sin^2 t + 16\cos^2 t}$

$$\mathbf{T}(t) = \frac{\mathbf{z}'(t)}{\|\mathbf{z}'(t)\|}$$

$$= -\frac{3\sin t}{\sqrt{9\sin^2 t + 16\cos^2 t}}\mathbf{i} + \frac{4\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}}\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$x'(t) = -3\sin t$ $\quad y'(t) = 4\cos t$

$x''(t) = -3\cos t$ $\quad y''(t) = -4\sin t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{12}{\left(9\sin^2 t + 16\cos^2 t\right)^{3/2}}$$

$$\kappa\left(\frac{\pi}{4}\right) = \frac{12}{\left(\frac{25}{2}\right)^{3/2}} = \frac{24\sqrt{2}}{125}$$

10. $\mathbf{r}'(t) = e^t\mathbf{i} + e^t\mathbf{j}$

$\|\mathbf{r}'(t)\| = e^t\sqrt{2}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}(\ln 2) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = e^t$ $\quad y'(t) = e^t$

$x''(t) = e^t$ $\quad y''(t) = e^t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = 0$$

$\kappa(\ln 2) = 0$

11. $x = 1 - t^2, \quad y = 1 - t^3$

$x'(t) = -2t, \quad y'(t) = -3t^2$

$x''(t) = -2, \quad y''(t) = -6t$

$\mathbf{r}(t) = \left(1 - t^2\right)\mathbf{i} + \left(1 - t^3\right)\mathbf{j}$

$\mathbf{r}'(t) = -2t\,\mathbf{i} - 3t^2\,\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-2t\,\mathbf{i} - 3t^2\,\mathbf{j}}{\sqrt{4t^2 + 9t^4}}$$

$$\mathbf{T}(1) = \frac{-2(1)\mathbf{i} - 3(1)^2\,\mathbf{j}}{\sqrt{4(1)^2 + 9(1)^4}} = -\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}$$

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\left((x')^2 + (y')^2\right)^{3/2}}$$

$$= \frac{\left|(-2t)(-6t) - (-2)\left(-3t^2\right)\right|}{\left((-2t)^2 + \left(-3t^2\right)^2\right)^{3/2}}$$

$$= \frac{\left|12t^2 - 6t^2\right|}{\left(4t^2 + 9t^4\right)^{3/2}} = \frac{6t^2}{\left(4t^2 + 9t^4\right)^{3/2}}$$

When $t = 1$, the curvature is

$$\kappa = \frac{6}{(4+9)^{3/2}} = \frac{6}{13^{3/2}} \approx 0.128008$$

12. $\mathbf{r}'(t) = \cosh t\,\mathbf{i} + \sinh t\,\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\cosh t}{\sqrt{\sinh^2 t + \cosh^2 t}}\mathbf{i} + \frac{\sinh t}{\sqrt{\sinh^2 t + \cosh^2 t}}\mathbf{j}$$

$$\mathbf{T}(\ln 3) = \frac{5}{\sqrt{41}}\mathbf{i} + \frac{4}{\sqrt{41}}\mathbf{j}$$

$x'(t) = \cosh t \qquad y'(t) = \sinh t$

$x''(t) = \sinh t \qquad y''(t) = \cosh t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{1}{(\sinh^2 t + \cosh^2 t)^{3/2}};$$

$$\kappa(\ln 3) = \frac{1}{\left(\frac{41}{9}\right)^{3/2}} = \frac{27}{41\sqrt{41}}$$

13. $\mathbf{r}'(t) = -(\cos t + \sin t)e^{-t}\mathbf{i} + (\cos t - \sin t)e^{t}\mathbf{j}$

$|\mathbf{r}'(t)| = \sqrt{2}e^{-t}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\frac{\cos t + \sin t}{\sqrt{2}}\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}}\mathbf{j}$$

$$\mathbf{T}(0) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$x'(t) = -(\cos t - \sin t)e^{-t} \quad y'(t) = (\cos t - \sin t)e^{-t}$

$x''(t) = (2\sin t)e^{-t} \qquad y''(t) = (-2\cos t)e^{-t}$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{2e^{-2t}}{2\sqrt{2}e^{-3t}} = \frac{e^t}{\sqrt{2}}$$

$$\kappa(0) = \frac{1}{\sqrt{2}}$$

14. $\mathbf{r}'(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{t^2 + 1}$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\cos t - t\sin t}{\sqrt{t^2+1}}\mathbf{i} + \frac{\sin t + t\cos t}{\sqrt{t^2+1}}\mathbf{j}$$

$$\mathbf{T}(1) = \frac{\cos 1 - \sin 1}{\sqrt{2}}\mathbf{i} + \frac{\sin 1 + \cos 1}{\sqrt{2}}\mathbf{j}$$

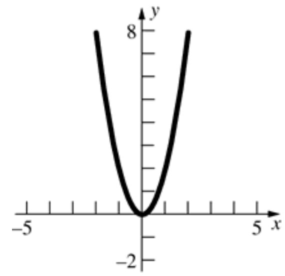
$x'(t) = \cos t - t\sin t \qquad y'(t) = \sin t + t\cos t$

$x''(t) = -2\sin t - t\cos t \quad y''(t) = 2\cos t - t\sin t$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{t^2 + 2}{(t^2+1)^{3/2}}$$

$$\kappa(1) = \frac{3}{2\sqrt{2}}$$

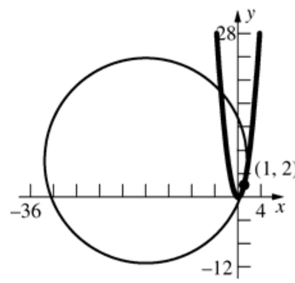
15.



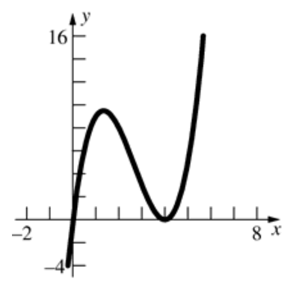
$y' = 4x,\ y'' = 4$

$$\kappa = \frac{4}{(1+16x^2)^{3/2}}$$

At (1, 2), $\kappa = \frac{4}{17\sqrt{17}}$ and $R = \frac{17\sqrt{17}}{4}$.



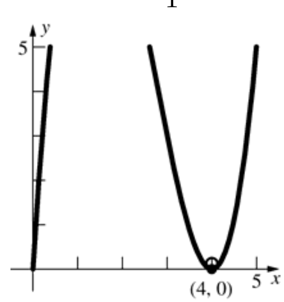
16.



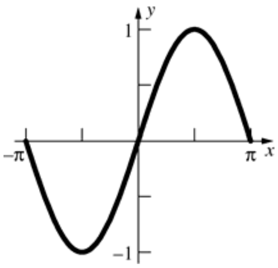
$y' = 3x^2 - 16x + 16,\ y'' = 6x - 16$

$$\kappa = \frac{|6x - 16|}{\left[1 + (3x^2 - 16x + 16)^2\right]^{3/2}}$$

At (4, 0), $\kappa = \frac{8}{1} = 8$ and $R = \frac{1}{8}$.



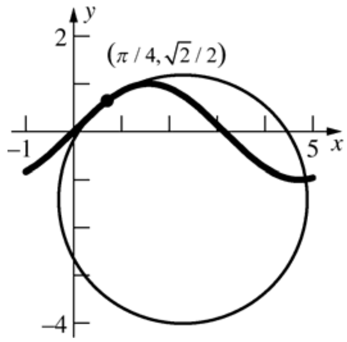
17.



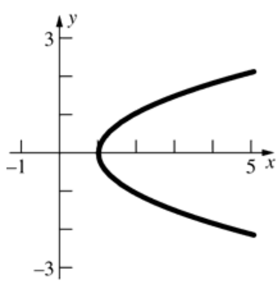
$y' = \cos x,\ y'' = -\sin x$

$\kappa = \dfrac{|\sin x|}{(1+\cos^2 x)^{3/2}}$

At $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right), \kappa = \dfrac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2}\right)^{3/2}} = \dfrac{2}{3\sqrt{3}}$ and $R = \dfrac{3\sqrt{3}}{2}$.



18.

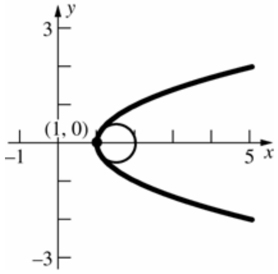


$2yy' = 1$

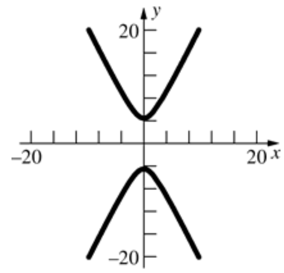
$y' = \dfrac{1}{2y},\ y'' = -\dfrac{y'}{2y^2} = -\dfrac{1}{4y^3}$

$\kappa = \dfrac{\left|\frac{1}{4y^3}\right|}{\left(1+\frac{1}{4y^2}\right)^{3/2}} = \dfrac{2}{(4y^2+1)^{3/2}}$

At (1, 0), $\kappa = \dfrac{2}{1} = 2$ and $R = \dfrac{1}{2}$.



19.

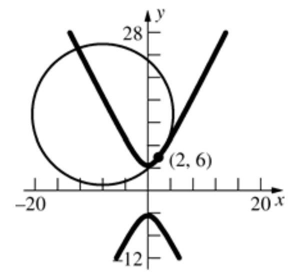


$2yy' - 8x = 0$

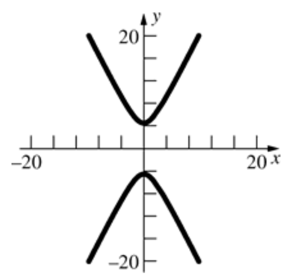
$y' = \dfrac{4x}{y},\ y'' = \dfrac{4(y-xy')}{y^2} = \dfrac{4(y^2-4x^2)}{y^3}$

$\kappa = \dfrac{\frac{4y^2-16x^2}{y^3}}{\left(1+\frac{16x^2}{y^2}\right)^{3/2}} = \dfrac{4(y^2-4x^2)}{(y^2+16x^2)^{3/2}}$

At (2, 6), $\kappa = \dfrac{80}{1000} = \dfrac{2}{25}$ and $R = \dfrac{25}{2}$.



20.

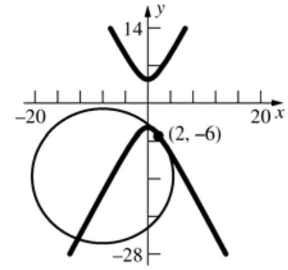


$2yy' - 8x = 0$

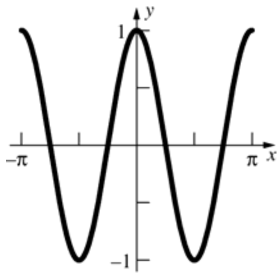
$y' = \dfrac{4x}{y},\ y'' = \dfrac{4(y-xy')}{y^2} = \dfrac{4(y^2-4x^2)}{y^3}$

$\kappa = \dfrac{\frac{4(y^2-4x^2)}{y^3}}{\left(1+\frac{16x^2}{y^2}\right)^{3/2}} = \dfrac{4(y^2-4x^2)}{(y^2+16x^2)^{3/2}}$

At (2, –6), $\kappa = \dfrac{80}{1000} = \dfrac{2}{25}$ and $R = \dfrac{25}{2}$.



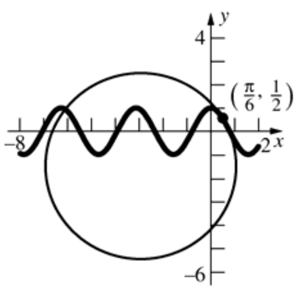
21.



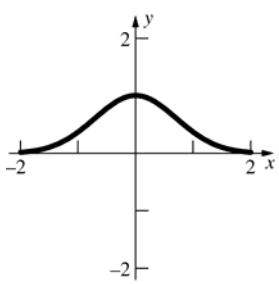
$y' = -2\sin 2x,\ y'' = -4\cos 2x$

$\kappa = \dfrac{|4\cos 2x|}{(1+4\sin^2 2x)^{3/2}}$

At $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$, $\kappa = \dfrac{2}{8} = \dfrac{1}{4}$ and $R = 4$.



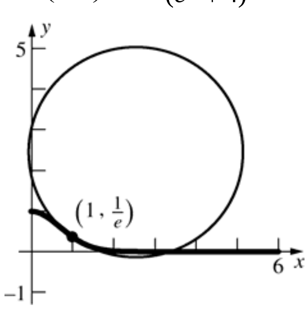
22



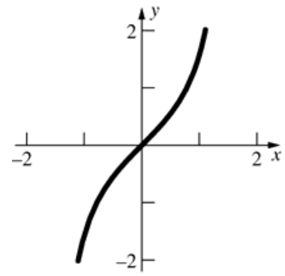
$y' = -2xe^{-x^2},\ y'' = (4x^2-2)e^{-x^2}$

$\kappa = \dfrac{\left|(4x^2-2)e^{-x^2}\right|}{(1+4x^2e^{-2x^2})^{3/2}} = \dfrac{e^{2x^2}\left|4x^2-2\right|}{(e^{2x^2}+4x^2)^{3/2}}$

At $\left(1, \dfrac{1}{e}\right)$, $\kappa = \dfrac{2e^2}{(e^2+4)^{3/2}}$ and $R = \dfrac{(e^2+4)^{3/2}}{2e^2}$.



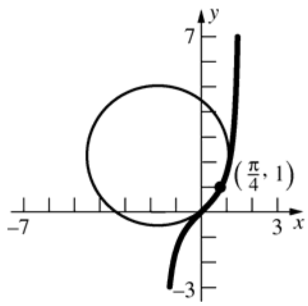
23



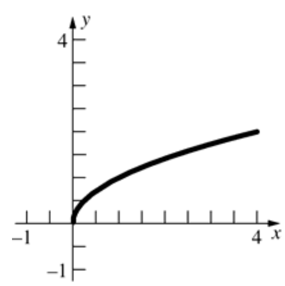
$y' = \sec^2 x,\ y'' = 2\sec^2 x\tan x$

$\kappa = \dfrac{\left|2\sec^2 x\tan x\right|}{(1+\sec^4 x)^{3/2}}$

At $\left(\dfrac{\pi}{4}, 1\right)$, $\kappa = \dfrac{4}{5\sqrt{5}}$ and $R = \dfrac{5\sqrt{5}}{4}$.



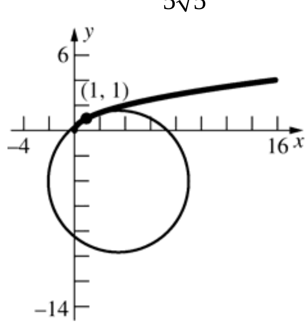
24



$y' = \dfrac{1}{2\sqrt{x}},\ y'' = -\dfrac{1}{4x^{3/2}}$

$\kappa = \dfrac{\left|\frac{1}{4x^{3/2}}\right|}{\left(1+\dfrac{1}{4x}\right)^{3/2}} = \dfrac{2}{(4x+1)^{3/2}}$

At (1, 1), $\kappa = \dfrac{2}{5\sqrt{5}}$ and $R = \dfrac{5\sqrt{5}}{2}$.



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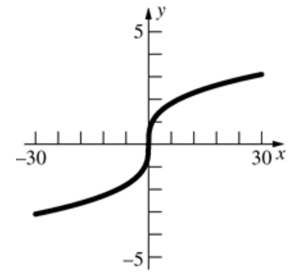
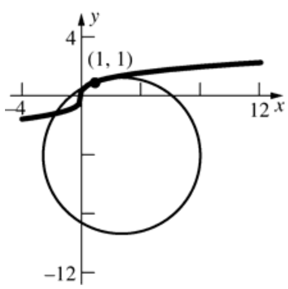


$$y' = \frac{1}{3x^{2/3}},\ y'' = -\frac{2}{9x^{5/3}}$$

$$\kappa = \frac{\left|\frac{2}{9x^{5/3}}\right|}{\left(1+\frac{1}{9x^{4/3}}\right)^{3/2}} = \frac{6x^{1/3}}{(9x^{4/3}+1)^{3/2}}$$

At (1, 1), $\kappa = \frac{6}{10\sqrt{10}} = \frac{3}{5\sqrt{10}}$ and $R = \frac{5\sqrt{10}}{3}.$



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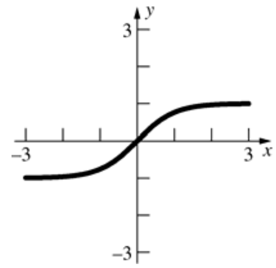
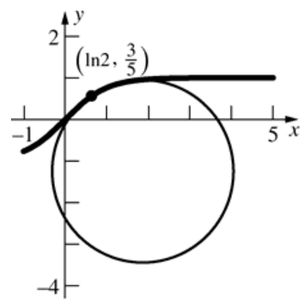


$$y' = \text{sech}^2 x,\ y'' = -2\text{sech}^2 x \tanh x$$

$$\kappa = \frac{\left|2\text{sech}^2 x \tanh x\right|}{(1+\text{sech}^4 x)^{3/2}}$$

At $\left(\ln 2, \frac{3}{5}\right), \kappa = \frac{\frac{96}{125}}{\left(\frac{881}{625}\right)^{3/2}} = \frac{12,000}{881\sqrt{881}}$ and

$R = \frac{881\sqrt{881}}{12,000}.$



27. $\mathbf{r}'(t) = t\mathbf{i}+\mathbf{j}+t^2\mathbf{k}$

$\mathbf{r}''(t) = \mathbf{i}+2t\mathbf{k}$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{2\mathbf{i}+\mathbf{j}+4\mathbf{k}}{\sqrt{4+1+16}} = \frac{2}{\sqrt{21}}\mathbf{i}+\frac{1}{\sqrt{21}}\mathbf{j}+\frac{4}{\sqrt{21}}\mathbf{k}$$

$$a_T(2) = \frac{\mathbf{r}'(2)\cdot\mathbf{r}''(2)}{\|\mathbf{r}'(2)\|} = \frac{(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+4\mathbf{k})}{\sqrt{21}} = \frac{18}{\sqrt{21}}$$

$$a_N(2) = \frac{\|\mathbf{r}'(2)\times\mathbf{r}''(2)\|}{\|\mathbf{r}'(2)\|} = \frac{\|(2\mathbf{i}+\mathbf{j}+4\mathbf{k})\times(\mathbf{i}+4\mathbf{k})\|}{\sqrt{21}} = \frac{1}{\sqrt{21}}\|4\mathbf{i}-4\mathbf{j}-\mathbf{k}\| = \frac{\sqrt{33}}{\sqrt{21}} = \sqrt{\frac{11}{7}}$$

$$\mathbf{N}(2) = \frac{\mathbf{r}''(2)-a_T(2)\mathbf{T}(2)}{a_N(2)} = \frac{(\mathbf{i}+4\mathbf{k})-\frac{18}{\sqrt{21}}\left(\frac{2}{\sqrt{21}}\mathbf{i}+\frac{1}{\sqrt{21}}\mathbf{j}+\frac{4}{\sqrt{21}}\mathbf{k}\right)}{\sqrt{\frac{11}{7}}} = \sqrt{\frac{7}{11}}\left(-\frac{15}{21}\mathbf{i}-\frac{18}{21}\mathbf{j}+\frac{12}{21}\mathbf{k}\right)$$

$$= \sqrt{\frac{7}{11}}\left(-\frac{5}{7}\mathbf{i}-\frac{6}{7}\mathbf{j}+\frac{4}{7}\mathbf{k}\right) = -\frac{5}{\sqrt{77}}\mathbf{i}-\frac{6}{\sqrt{77}}\mathbf{j}+\frac{4}{\sqrt{77}}\mathbf{k}$$

$$\kappa(2) = \frac{\|\mathbf{r}'(2)\times\mathbf{r}''(2)\|}{\|\mathbf{r}'(2)\|^3} = \frac{\sqrt{33}}{\left(\sqrt{21}\right)^3} = \sqrt{\frac{33}{9261}} = \sqrt{\frac{11}{3087}} = \frac{\sqrt{11}}{21\sqrt{7}}$$

$$\mathbf{B}(2) = \mathbf{T}(2)\times\mathbf{N}(2) = \left(\frac{2}{\sqrt{21}}\mathbf{i}+\frac{1}{\sqrt{21}}\mathbf{j}+\frac{4}{\sqrt{21}}\mathbf{k}\right)\times\left(-\frac{5}{\sqrt{77}}\mathbf{i}-\frac{6}{\sqrt{77}}\mathbf{j}+\frac{4}{\sqrt{77}}\mathbf{k}\right) = \frac{4}{\sqrt{33}}\mathbf{i}-\frac{4}{\sqrt{33}}\mathbf{j}-\frac{1}{\sqrt{33}}\mathbf{k}$$

28. $\mathbf{r}(t) = \langle \sin 3t, \cos 3t, t \rangle$

$\mathbf{r}'(t) = \langle 3\cos 3t, -3\sin 3t, 1 \rangle$

$\mathbf{r}''(t) = \langle -9\sin 3t, -9\cos 3t, 0 \rangle$

$$\mathbf{T}\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{9}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\right\|} = \frac{\left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1 \right\rangle}{\sqrt{\frac{9}{4}+\frac{27}{4}+1}} = \left\langle \frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$$

$$a_T\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{9}\right)\cdot\mathbf{r}''\left(\frac{\pi}{9}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\right\|} = \frac{\left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1 \right\rangle \cdot \left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0 \right\rangle}{\sqrt{10}} = 0$$

$$a_N\left(\frac{\pi}{9}\right) = \frac{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\times\mathbf{r}''\left(\frac{\pi}{9}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\right\|} = \frac{\left|\left\langle \frac{3}{2}, -\frac{3\sqrt{3}}{2}, 1 \right\rangle \times \left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0 \right\rangle\right|}{\sqrt{10}} = \frac{1}{\sqrt{10}}\left|\left\langle \frac{9}{2}, -\frac{9\sqrt{3}}{2}, -27 \right\rangle\right| = \frac{1}{\sqrt{10}}\left(9\sqrt{10}\right) = 9$$

$$\mathbf{N}\left(\frac{\pi}{9}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{9}\right) - a_T\left(\frac{\pi}{9}\right)\mathbf{T}\left(\frac{\pi}{9}\right)}{a_N} = \frac{1}{9}\left\langle -\frac{9\sqrt{3}}{2}, -\frac{9}{2}, 0 \right\rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right\rangle$$

$$\kappa\left(\frac{\pi}{9}\right) = \frac{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\times\mathbf{r}''\left(\frac{\pi}{9}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{9}\right)\right\|^3} = \frac{9\sqrt{10}}{\left(\sqrt{10}\right)^3} = \frac{9}{10}$$

$$\mathbf{B}\left(\frac{\pi}{9}\right) = \mathbf{T}\left(\frac{\pi}{9}\right)\times\mathbf{N}\left(\frac{\pi}{9}\right) = \left\langle \frac{3}{2\sqrt{10}}, -\frac{3\sqrt{3}}{2\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle \times \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right\rangle = \left\langle \frac{1}{2\sqrt{10}}, -\frac{\sqrt{3}}{2\sqrt{10}}, -\frac{3}{\sqrt{10}} \right\rangle$$

29. $\mathbf{r}(t) = \langle 7\sin 3t, 7\cos 3t, 14t \rangle$

$\mathbf{r}'(t) = \langle 21\cos 3t, -21\sin 3t, 14 \rangle$

$\mathbf{r}''(t) = \langle -63\sin 3t, -63\cos 3t, 0 \rangle$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{\langle -21, 0, 14 \rangle}{\sqrt{441+196}} = \frac{1}{7\sqrt{13}}\langle -21, 0, 14 \rangle = \left\langle -\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}} \right\rangle$$

$$a_T\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)\cdot\mathbf{r}''\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{\langle -21, 0, 14 \rangle \cdot \langle 0, -63, 0 \rangle}{7\sqrt{13}} = 0$$

$$a_N\left(\frac{\pi}{3}\right) = \frac{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{\left|\langle -21, 0, 14 \rangle \times \langle 0, -63, 0 \rangle\right|}{7\sqrt{13}} = \frac{\left|\langle 882, 0, -1323 \rangle\right|}{7\sqrt{13}} = \frac{441\sqrt{13}}{7\sqrt{13}} = 63$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{3}\right) - a_T\left(\frac{\pi}{3}\right)\mathbf{T}\left(\frac{\pi}{3}\right)}{a_N\left(\frac{\pi}{3}\right)} = \frac{1}{63}\langle 0, 63, 0 \rangle = \langle 0, 1, 0 \rangle$$

$$\kappa\left(\frac{\pi}{3}\right) = \frac{\left\|\mathbf{r}\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|^3} = \frac{441\sqrt{13}}{\left(7\sqrt{13}\right)^3} = \frac{9}{91}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right)\times\mathbf{N}\left(\frac{\pi}{3}\right) = \left\langle -\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}} \right\rangle \times \langle 0, 1, 0 \rangle = \left\langle -\frac{2}{\sqrt{13}}, 0, -\frac{3}{\sqrt{13}} \right\rangle$$

30. $\mathbf{r}'(t) = -3\cos^2 t\sin t\mathbf{i} + 3\sin^2 t\cos t\mathbf{k}$

$\mathbf{r}''(t) = (6\cos t\sin^2 t - 3\cos^3 t)\mathbf{i} + (6\cos^2 t\sin t - 3\sin^3 t)\mathbf{k}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{0}$ so the object is motionless at $t_1 = \frac{\pi}{2}$.

κ, **T**, **N**, and **B** do not exist.

31. $\mathbf{r}'(t) = \sinh\frac{t}{3}\mathbf{i} + \mathbf{j}$

$\mathbf{r}''(t) = \frac{1}{3}\cosh\frac{t}{3}\mathbf{i}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\sinh\frac{1}{3}\mathbf{i} + \mathbf{j}}{\sqrt{\sinh^2\frac{1}{3} + 1}} = \frac{\sinh\frac{1}{3}\mathbf{i} + \mathbf{j}}{\cosh\frac{1}{3}} = \tanh\frac{1}{3}\mathbf{i} + \operatorname{sech}\frac{1}{3}\mathbf{j}$$

$$a_T(1) = \frac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{\|\mathbf{r}'(1)\|} = \frac{\left(\sinh\frac{1}{3}\mathbf{i} + \mathbf{j}\right)\cdot\left(\frac{1}{3}\cosh\frac{1}{3}\mathbf{i}\right)}{\cosh\frac{1}{3}} = \frac{1}{3}\sinh\frac{1}{3}$$

$$a_N(1) = \frac{\|\mathbf{r}'(1)\times\mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|} = \frac{\left\|\left(\sinh\frac{1}{3}\mathbf{i} + \mathbf{j}\right)\times\left(\frac{1}{3}\cosh\frac{1}{3}\mathbf{i}\right)\right\|}{\cosh\frac{1}{3}} = \frac{\left\|-\frac{1}{3}\cos\frac{1}{3}\mathbf{k}\right\|}{\cosh\frac{1}{3}} = \frac{1}{3}$$

$$\mathbf{N}(1) = \frac{\mathbf{r}''(1) - a_T(1)\mathbf{T}(1)}{a_N(1)} = 3\left[\frac{1}{3}\cosh\frac{1}{3}\mathbf{i} - \frac{1}{3}\sinh\frac{1}{3}\left(\tanh\frac{1}{3}\mathbf{i} + \operatorname{sech}\frac{1}{3}\mathbf{j}\right)\right] = \left(\cosh\frac{1}{3} - \frac{\sinh^2\frac{1}{3}}{\cosh\frac{1}{3}}\right)\mathbf{i} - \tanh\frac{1}{3}\mathbf{j}$$

$$= \operatorname{sech}\frac{1}{3}\mathbf{i} - \tanh\frac{1}{3}\mathbf{j}$$

$$\kappa(1) = \frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{\frac{1}{3}\cosh\frac{1}{3}}{\cosh^3\frac{1}{3}} = \frac{1}{3}\operatorname{sech}^2\frac{1}{3}$$

$$\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \left(\tanh\frac{1}{3}\mathbf{i} + \operatorname{sech}\frac{1}{3}\mathbf{j}\right)\times\left(\operatorname{sech}\frac{1}{3}\mathbf{i} - \tanh\frac{1}{3}\mathbf{j}\right) = \left(-\operatorname{sech}^2\frac{1}{3} - \tanh^2\frac{1}{3}\right)\mathbf{k} = -\mathbf{k}$$

32. $\mathbf{r}'(t) = e^{7t}(7\cos 2t - 2\sin 2t)\mathbf{i} + e^{7t}(7\sin 2t + 2\cos 2t)\mathbf{j} + 7e^{7t}\mathbf{k}$

$\mathbf{r}''(t) = e^{7t}(45\cos 2t - 28\sin 2t)\mathbf{i} + e^{7t}(45\sin 2t + 28\cos 2t)\mathbf{j} + 49e^{7t}\mathbf{k}$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{e^{7\pi/3}\left(-\frac{7}{2} - \sqrt{3}\right)\mathbf{i} + e^{7\pi/3}\left(\frac{7\sqrt{3}}{2} - 1\right)\mathbf{j} + 7e^{7\pi/3}\mathbf{k}}{e^{7\pi/3}\sqrt{\left(-\frac{7}{2} - \sqrt{3}\right)^2 + \left(\frac{7\sqrt{3}}{2} - 1\right)^2 + 49}} = \left(-\frac{7 + 2\sqrt{3}}{2\sqrt{102}}\right)\mathbf{i} + \left(\frac{7\sqrt{3} - 2}{2\sqrt{102}}\right)\mathbf{j} + \frac{7}{\sqrt{102}}\mathbf{k}$$

$$a_T\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{3}\right)\cdot\mathbf{r}''\left(\frac{\pi}{3}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|} = \frac{714e^{14\pi/3}}{e^{7\pi/3}\sqrt{102}} = 7\sqrt{102}e^{7\pi/3}$$

$$a_N\left(\frac{\pi}{3}\right) = \frac{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right|}{\left|\mathbf{r}'\left(\frac{\pi}{3}\right)\right|} = \frac{\left\|e^{14\pi/3}\left(49 + 14\sqrt{3}\right)\mathbf{i} + e^{14\pi/3}\left(14 - 49\sqrt{3}\right)\mathbf{j} + 106e^{14\pi/3}\mathbf{k}\right\|}{e^{7\pi/3}\sqrt{102}} = 2\sqrt{53}e^{7\pi/3}$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{\mathbf{r}''\left(\frac{\pi}{3}\right) - a_T\left(\frac{\pi}{3}\right)\mathbf{T}\left(\frac{\pi}{3}\right)}{a_N} = \frac{2 - 7\sqrt{3}}{2\sqrt{53}}\mathbf{i} - \frac{7 + 2\sqrt{3}}{2\sqrt{53}}\mathbf{j}; \quad \kappa\left(\frac{\pi}{3}\right) = \frac{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\times\mathbf{r}''\left(\frac{\pi}{3}\right)\right\|}{\left\|\mathbf{r}'\left(\frac{\pi}{3}\right)\right\|^3} = \frac{2\sqrt{5406}e^{14\pi/3}}{\sqrt{102^3}e^{7\pi}} = \frac{\sqrt{53}}{51}e^{-7\pi/3};$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right)\times\mathbf{N}\left(\frac{\pi}{3}\right) = \frac{42 + 49\sqrt{3}}{6\sqrt{1802}}\mathbf{i} + \frac{14\sqrt{3} - 147}{6\sqrt{1802}}\mathbf{j} + \sqrt{\frac{53}{102}}\mathbf{k}$$

33. $\mathbf{r}'(t) = -2e^{-2t}\mathbf{i} + 2e^{2t}\mathbf{j} + 2\sqrt{2}\mathbf{k}$

$\mathbf{r}''(t) = 4e^{-2t}\mathbf{i} + 4e^{2t}\mathbf{j}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{-2\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k}}{\sqrt{4+4+8}} = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

$$a_T(0) = \frac{\mathbf{r}'(0)\cdot\mathbf{r}''(0)}{\|\mathbf{r}'(0)\|} = \frac{\left(-2\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k}\right)\cdot(4\mathbf{i} + 4\mathbf{j})}{4} = 0$$

$$a_N(0) = \frac{\|\mathbf{r}'(0)\times\mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|} = \frac{\left|-8\sqrt{2}\mathbf{i} + 8\sqrt{2}\mathbf{j} - 16\mathbf{k}\right|}{4} = \frac{16\sqrt{2}}{4} = 4\sqrt{2}$$

$$\mathbf{N}(0) = \frac{\mathbf{r}''(0) - a_T(0)\mathbf{T}(0)}{a_N(0)} = \frac{4\mathbf{i} + 4\mathbf{j}}{4\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\kappa(0) = \frac{\|\mathbf{r}'(0)\times\mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|^3} = \frac{16\sqrt{2}}{64} = \frac{\sqrt{2}}{4}$$

$$\mathbf{B}(0) = \mathbf{T}(0)\times\mathbf{N}(0) = \left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}\right)\times\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$$

34. $\mathbf{r}(t) = \left\langle \ln t,\ 3t,\ t^2 \right\rangle$

$$\mathbf{r}'(t) = \left\langle \frac{1}{t},\ 3,\ 2t \right\rangle$$

$$\mathbf{r}''(t) = \left\langle -\frac{1}{t^2},\ 0,\ 2 \right\rangle$$

$$T(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{\left\langle \frac{1}{2}, 3, 4\right\rangle}{\sqrt{\frac{1}{4}+9+16}} = \frac{\left\langle \frac{1}{2}, 3, 4\right\rangle}{\frac{\sqrt{101}}{2}} = \left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}} \right\rangle$$

$$a_T(2) = \frac{\mathbf{r}'(2)\cdot\mathbf{r}''(2)}{\|\mathbf{r}'(2)\|} = \frac{2}{\sqrt{101}}\left(\left\langle \frac{1}{2}, 3, 4\right\rangle \cdot \left\langle -\frac{1}{4}, 0, 2\right\rangle\right) = \frac{2}{\sqrt{101}}\left(\frac{63}{8}\right) = \frac{63}{4\sqrt{101}}$$

$$a_N(2) = \frac{\|\mathbf{r}'(2)\times\mathbf{r}''(2)\|}{\|\mathbf{r}'(2)\|} = \frac{2}{\sqrt{101}}\left|\left\langle 6, -2, \frac{3}{4}\right\rangle\right| = \frac{2}{\sqrt{101}}\left(\frac{\sqrt{649}}{4}\right) = \frac{\sqrt{649}}{2\sqrt{101}}$$

$$\mathbf{N}(2) = \frac{\mathbf{r}''(2) - a_T(2)\mathbf{T}(2)}{a_N(2)} = \frac{2\sqrt{101}}{\sqrt{649}}\left(\left\langle -\frac{1}{4}, 0, 2\right\rangle - \frac{63}{4\sqrt{101}}\left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}}\right\rangle\right)$$

$$= \frac{2\sqrt{101}}{\sqrt{649}}\left\langle -\frac{41}{101}, -\frac{189}{202}, \frac{76}{101}\right\rangle = \left\langle -\frac{82}{\sqrt{65,549}}, -\frac{189}{\sqrt{65,549}}, \frac{152}{\sqrt{65,549}}\right\rangle$$

$$\kappa(2) = \frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3} = \left(\frac{\sqrt{649}}{4}\right)\left(\frac{2}{\sqrt{101}}\right)^3 = \frac{2\sqrt{649}}{101\sqrt{101}}$$

$$\mathbf{B}(2) = \mathbf{T}(2)\times\mathbf{N}(2) = \left\langle \frac{1}{\sqrt{101}}, \frac{6}{\sqrt{101}}, \frac{8}{\sqrt{101}}\right\rangle \times \left\langle -\frac{82}{\sqrt{65,549}}, -\frac{189}{\sqrt{65,549}}, \frac{152}{\sqrt{65,549}}\right\rangle = \left\langle \frac{24}{\sqrt{649}}, -\frac{8}{\sqrt{649}}, \frac{3}{\sqrt{649}}\right\rangle$$

35. $y' = \frac{1}{x},\ y'' = -\frac{1}{x^2}$

$$\kappa = \frac{\left|\frac{1}{x^2}\right|}{\left(1+\frac{1}{x^2}\right)^{3/2}} = \frac{|x|}{(x^2+1)^{3/2}}$$

Since $0 < x < \infty$, $\kappa = \frac{x}{(x^2+1)^{3/2}}$.

$$\kappa' = \frac{(x^2+1)^{3/2} - 3x^2(x^2+1)^{1/2}}{(x^2+1)^3} = \frac{-2x^2+1}{(x^2+1)^{5/2}}$$

$\kappa' = 0$ when $x = \frac{1}{\sqrt{2}}$. Since $\kappa' > 0$ on $\left(0, \frac{1}{\sqrt{2}}\right)$ and $\kappa' < 0$ on $\left(\frac{1}{\sqrt{2}}, \infty\right)$, so κ is maximum when $x = \frac{1}{\sqrt{2}},\ y = \ln\frac{1}{\sqrt{2}} = -\frac{\ln 2}{2}$. The point of maximum curvature is $\left(\frac{1}{\sqrt{2}}, -\frac{\ln 2}{2}\right)$.

36 $y' = \cos x,\ y'' = -\sin x$

$$\kappa = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$$

$$\kappa' = \frac{\frac{|\sin x|}{\sin x}\cos x(1+\cos^2 x)^{3/2} + 3|\sin x|\cos x \sin x(1+\cos^2 x)^{1/2}}{(1+\cos^2 x)^3} = \frac{2|\sin x|\cot x(2+\cos^2 x)}{(1+\cos^2 x)^{5/2}}$$

$k' = 0$ when $x = -\frac{\pi}{2}, \frac{\pi}{2}$. κ' is not defined when $x = -\pi, 0$. Since $\kappa' > 0$ on $\left(-\pi, -\frac{\pi}{2}\right) \cup \left(0, \frac{\pi}{2}\right)$ and $\kappa' < 0$ on $\left(-\frac{\pi}{2}, 0\right) \cup \left(\frac{\pi}{2}, \pi\right)$, so κ has local maxima when $x = -\frac{\pi}{2},\ y = -1$ and $x = \frac{\pi}{2},\ y = 1$.

$$\kappa\left(-\frac{\pi}{2}\right) = \kappa\left(\frac{\pi}{2}\right) = 1$$

The points of maximum curvature are $(0,1)$ and $\left(\frac{\pi}{2}, 1\right)$.

37. $y' = \sinh x,\ y'' = \cosh x$

$$\kappa = \frac{\cosh x}{(1+\sinh^2 x)^{3/2}} = \text{sech}^2 x$$

$\kappa' = -2\,\text{sech}^2 x \tanh x$

$\kappa' = 0$ when $x = 0$. Since $\kappa' > 0$ on $(-\infty, 0)$ and $\kappa' < 0$ on $(0, \infty)$, so κ is maximum when $x = 0$, $y = 1$. The point of maximum curvature is $(0,1)$.

38. $y' = \cosh x,\ y'' = \sinh x$

$$\kappa = \frac{|\sinh x|}{(1+\cosh^2 x)^{3/2}}$$

$$\kappa' = \frac{\frac{|\sinh x|}{\sinh x}\cosh x(1+\cosh^2 x)^{3/2} - 3|\sinh x|\cosh x \sinh x(1+\cosh^2 x)^{1/2}}{(1+\cosh^2 x)^3} = \frac{2|\sinh x|\coth x(2-\cosh^2 x)}{(1+\cosh^2 x)^{5/2}}$$

κ' is not defined when $x = 0$ and $\kappa' = 0$ when $\cosh x = \sqrt{2}$ or $x = \pm\ln\left(\sqrt{2}+1\right)$. Since $\kappa' > 0$ on $\left(-\infty, -\ln\left(\sqrt{2}+1\right)\right) \cup \left(0, \ln\left(\sqrt{2}+1\right)\right)$ and $\kappa' < 0$ on $\left(-\ln\left(\sqrt{2}+1\right), 0\right) \cup \left(\ln\left(\sqrt{2}+1\right), \infty\right)$, κ has local maxima when $x = -\ln\left(\sqrt{2}+1\right)$, $y = -1$ and $x = \ln\left(\sqrt{2}+1\right)$, $y = 1$.

$$\kappa\left(-\ln\left(\sqrt{2}+1\right)\right) = \kappa\left(\ln\left(\sqrt{2}+1\right)\right) = \frac{1}{3\sqrt{3}}$$

The points of maximum curvature are $\left(-\ln\left(\sqrt{2}+1\right), -1\right)$ and $\left(\ln\left(\sqrt{2}+1\right), 1\right)$.

39. $y' = e^x,\ y'' = e^x$

$$\kappa = \frac{e^x}{(1+e^{2x})^{3/2}}$$

$$\kappa' = \frac{e^x(1+e^{2x})^{3/2} - 3e^{3x}(1+e^{2x})^{1/2}}{(1+e^{2x})^3}$$

$$= \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{5/2}}$$

$\kappa' = 0$ when $x = -\frac{1}{2}\ln 2$. Since $\kappa' > 0$ on $\left(-\infty, -\frac{1}{2}\ln 2\right)$ and $\kappa' < 0$ on $\left(-\frac{1}{2}\ln 2, \infty\right)$, so κ is maximum when $x = -\frac{1}{2}\ln 2,\ y = \frac{1}{\sqrt{2}}$. The point of maximum curvature is $\left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$.

40. $y' = -\tan x,\ \ y'' = -\sec^2 x$

$$\kappa = \frac{|\sec^2 x|}{(1+\tan^2 x)^{3/2}} = |\cos x|$$

Since $-\frac{\pi}{2} < x < \frac{\pi}{2}, \kappa = \cos x$.

$\kappa' = -\sin x$

$\kappa' = 0$ when $x = 0$. Since $\kappa' > 0$ on $\left(-\frac{\pi}{2}, 0\right)$ and $\kappa' < 0$ on $\left(0, \frac{\pi}{2}\right)$, κ is maximum when $x = 0$, $y = 0$. The point of maximum curvature is (0, 0).

41. $\mathbf{r}'(t) = 3\mathbf{i} + 6t\mathbf{j}$

$\mathbf{r}''(t) = 6\mathbf{j}$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = 3\sqrt{1+4t^2}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{12t}{\sqrt{1+4t^2}}$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = 36 - \frac{144t^2}{1+4t^2} = \frac{36}{1+4t^2}$$

$$a_N = \frac{6}{\sqrt{1+4t^2}}$$

At $t_1 = \frac{1}{3}, a_T = \frac{12}{\sqrt{13}}$ and $a_N = \frac{18}{\sqrt{13}}$.

42. $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{i}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{4t^2+1}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{4t}{\sqrt{4t^2+1}}$$

$$a_N^2 = \|\mathbf{r}''(t)\|^2 - a_T^2 = 4 - \frac{16t^2}{4t^2+1} = \frac{4}{4t^2+1}$$

$$a_N = \frac{2}{\sqrt{4t^2+1}}$$

At $t_1 = 1, a_T = \frac{4}{\sqrt{5}}$ and $a_N = \frac{2}{\sqrt{5}}$.

43. $\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{j}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = 2\sqrt{1+t^2}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{2t}{\sqrt{1+t^2}}$$

$$a_N^2 = \|\mathbf{r}''(t)\|^2 - a_T^2 = 4 - \frac{4t^2}{1+t^2} = \frac{4}{1+t^2}$$

$$a_N = \frac{2}{\sqrt{1+t^2}}$$

At $t_1 = -1, a_T = -\sqrt{2}$ and $a_N = \sqrt{2}$.

44. $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$

$\mathbf{r}''(t) = -a\cos t\mathbf{i} - a\sin t\mathbf{j}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = a$$

$$a_T = \frac{d^2s}{dt^2} = 0$$

$$a_N^2 = |\mathbf{r}''(t)|^2 - a_T^2 = a^2$$

$$a_N = a$$

At $t_1 = \frac{\pi}{6}, a_T = 0$ and $a_N = a$.

45. $\mathbf{r}'(t) = a\sinh t\mathbf{i} + a\cosh t\mathbf{j}$

$\mathbf{r}''(t) = a\cosh t\mathbf{i} + a\sinh t\mathbf{j}$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = a\sqrt{\sinh^2 t + \cosh^2 t}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{2a\cosh t\sinh t}{\sqrt{\sinh^2 t + \cosh^2 t}}$$

$$a_N^2 = \|\mathbf{r}''(t)\|^2\, a_T^2$$

$$= a^2(\cosh^2 t + \sinh^2 t) - \frac{4a^2\cosh^2 t\sinh^2 t}{\sinh^2 t + \cosh^2 t}$$

$$= \frac{a^2(\cosh^2 t - \sinh t^2)^2}{\sinh^2 t + \cosh^2 t}$$

$$a_N = \frac{a(\cosh^2 t - \sinh^2 t)}{\sqrt{\sinh^2 t + \cosh^2 t}}$$

At $t_1 = \ln 3, \cosh t = \frac{5}{3}, \sinh t = \frac{4}{3}$, and

$\sqrt{\sinh^2 t + \cosh^2 t} = \frac{\sqrt{41}}{3}$, so $a_T = \frac{40a}{3\sqrt{41}}$ and

$a_N = \frac{3a}{\sqrt{41}}$.

46. $x'(t) = 3$ $\quad y'(t) = -6$

$x''(t) = 0$ $\quad y''(t) = 0$

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = 3\sqrt{5}$$

$$a_T = \frac{d^2s}{dt^2} = 0$$

$$a_N^2 = [x''(t)^2 + y''(t)^2] - a_T^2 = 0;\ a_N = 0$$

At $t_1 = 0, a_T = 0$ and $a_N = 0$.

47. $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

$\mathbf{r}''(t) = 2\mathbf{k}$

$$a_T(t) = \frac{(\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k})\cdot(2\mathbf{k})}{\sqrt{1+9+4t^2}} = \frac{4t}{\sqrt{10+4t^2}}$$

$$a_T(1) = \frac{4}{\sqrt{14}}$$

$$a_N(t) = \frac{|(\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k})\times(2\mathbf{k})|}{\sqrt{10+4t^2}} = \frac{|6\mathbf{i} - 2\mathbf{j}|}{\sqrt{10+4t^2}}$$

$$= \frac{\sqrt{36+4}}{\sqrt{10+4t^2}} = 2\sqrt{\frac{10}{10+4t^2}} = 2\sqrt{\frac{5}{5+2t^2}}$$

$$a_N(1) = 2\sqrt{\frac{5}{7}}$$

48. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$

$\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$

$\mathbf{r}''(t) = \langle 0, 2, 6t\rangle$

$$a_T(t) = \frac{\langle 1, 2t, 3t^2\rangle \cdot \langle 0, 2, 6t\rangle}{\sqrt{1+4t^2+9t^4}} = \frac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}$$

$$a_T(2) = \frac{152}{\sqrt{161}}$$

$$a_N(t) = \frac{\left|\langle 1, 2t, 3t^2\rangle \times \langle 0, 2, 6t\rangle\right|}{\sqrt{1+4t^2+9t^4}} = \frac{\left|\langle 6t^2, -6t, 2\rangle\right|}{\sqrt{1+4t^2+9t^4}}$$

$$= \sqrt{\frac{36t^4+36t^2+4}{1+4t^2+9t^4}} = 2\sqrt{\frac{9t^4+9t^2+1}{1+4t^2+9t^4}}$$

$$a_N(2) = 2\sqrt{\frac{181}{161}}$$

49. $\mathbf{r}(t) = \left\langle e^{-t}, 2t, e^t \right\rangle$; $\mathbf{r}'(t) = \left\langle -e^{-t}, 2, e^t \right\rangle$

$\mathbf{r}''(t) = \left\langle e^{-t}, 0, e^t \right\rangle$; $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -e^{-2t} + e^{2t}$

$\|\mathbf{r}'(t)\| = \sqrt{e^{-2t} + 4 + e^{2t}}$

$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \left|\left\langle 2e^t, 2, -2e^{-t} \right\rangle\right|$

$= \sqrt{4e^{2t} + 4 + 4e^{-2t}}$

$= 2\sqrt{e^{2t} + 1 + e^{-2t}}$

$a_T(t) = \dfrac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + 4 + e^{-2t}}}$

$a_T(0) = 0$

$a_N(t) = 2\sqrt{\dfrac{e^{2t} + 1 + e^{-2t}}{e^{2t} + 4 + e^{-2t}}}$

$a_N(0) = 2\sqrt{\dfrac{3}{6}} = \sqrt{2}$

50. $\mathbf{r}'(t) = 2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = 2\mathbf{i} - 2\mathbf{j}$

$a_T(t) = \dfrac{[2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}] \cdot (2\mathbf{i} - 2\mathbf{j})}{\sqrt{4(t-2)^2 + 4t^2 + 1}}$

$= \dfrac{4(t-2) + 4t}{\sqrt{8t^2 - 16t + 17}} = \dfrac{8t - 8}{\sqrt{8t^2 - 16t + 17}}$

$a_T(2) = \dfrac{8}{\sqrt{17}}$

$a_N(t) = \dfrac{\|[2(t-2)\mathbf{i} - 2t\mathbf{j} + \mathbf{k}] \times (2\mathbf{i} - 2\mathbf{j})\|}{\sqrt{8t^2 + -16t + 17}}$

$= \dfrac{\|2\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}\|}{\sqrt{8t^2 - 16t + 17}} = \dfrac{6\sqrt{2}}{\sqrt{8t^2 - 16t + 17}}$

$a_N(2) = \dfrac{6\sqrt{2}}{\sqrt{17}}$

51. $\mathbf{r}'(t) = (1 - t^2)\mathbf{i} - (1 + t^2)\mathbf{j} + \mathbf{k}$

$\mathbf{r}''(t) = -2t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -2t(1 - t^2) + 2t(1 + t^2) = 4t^3$

$\|\mathbf{r}'(t)\| = \sqrt{(1 - t^2)^2 + (1 + t^2)^2 + 1} = \sqrt{2t^4 + 3}$

$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|2t\mathbf{i} - 2t\mathbf{j} - 4t\mathbf{k}\|$

$= \sqrt{4t^2 + 4t^2 + 16t^2}$

$= 2\sqrt{6}\,|t|$

$a_T(t) = \dfrac{4t^3}{\sqrt{2t^4 + 3}}$; $\quad a_T(3) = 36\sqrt{\dfrac{3}{55}}$

$a_N(t) = \dfrac{2\sqrt{6}\,|t|}{\sqrt{2t^4 + 3}}$; $\quad a_N(3) = 6\sqrt{\dfrac{2}{55}}$

52. $\mathbf{r}'(t) = \mathbf{i} + t^2\mathbf{j} - \dfrac{1}{t^2}\mathbf{k}, t > 0$

$\mathbf{r}''(t) = 2t\mathbf{j} + \dfrac{2}{t^3}\mathbf{k}$

$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 2t^3 - \dfrac{2}{t^5} = \dfrac{2}{t^5}(t^8 - 1)$

$\|\mathbf{r}'(t)\| = \sqrt{1 + t^4 + \dfrac{1}{t^4}} = \dfrac{1}{t^2}\sqrt{t^4 + t^8 + 1}$

$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \left\|\dfrac{4}{t}\mathbf{i} - \dfrac{2}{t^3}\mathbf{j} + 2t\mathbf{k}\right\|$

$= \sqrt{\dfrac{4}{t^6} + \dfrac{16}{t^2} + 4t^2} = \dfrac{2}{t^3}\sqrt{1 + 4t^4 + t^8}$

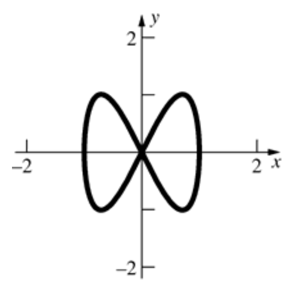
$a_T(t) = \dfrac{\frac{2}{t^5}(t^8 - 1)}{\frac{1}{t^2}\sqrt{t^4 + t^8 + 1}} = \dfrac{2(t^8 - 1)}{t^3\sqrt{t^8 + t^4 + 1}}$

$a_T(1) = 0$

$a_N(t) = \dfrac{\frac{2}{t^3}\sqrt{1 + 4t^4 + t^8}}{\frac{1}{t^2}\sqrt{t^4 + t^8 + 1}} = \dfrac{2}{t}\sqrt{\dfrac{t^8 + 4t^4 + 1}{t^8 + t^4 + 1}}$

$a_N(1) = 2\sqrt{\dfrac{6}{3}} = 2\sqrt{2}$

53.



$\mathbf{v}(t) = \langle \cos t, 2\cos 2t \rangle$, $\mathbf{a}(t) = \langle -\sin t, -4\sin 2t \rangle$

$\mathbf{a}(t) = 0$ if and only if $-\sin t = 0$ and $-4\sin 2t = 0$, which occurs if and only if $t = 0, \pi, 2\pi$, so it occurs only at the origin.

$\mathbf{a}(t)$ points to the origin if and only if $\mathbf{a}(t) = -k\mathbf{r}(t)$ for some k and $\mathbf{r}(t)$ is not $\mathbf{0}$. This occurs if and only if $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$, so it occurs only at (1, 0) and (–1, 0).

54. $\mathbf{v}(t) = \langle -\sin t + t\cos t + \sin t, \cos t + t\sin t - \cos t\rangle$
$= t\cos t\mathbf{i} + t\sin t\mathbf{j}$
$\mathbf{a}(t) = \langle -t\sin t + \cos t, t\cos t + \sin t\rangle$

a. $\frac{ds}{dt} = \|\mathbf{v}(t)\| = \left|t(\cos^2 t + \sin^2 t)^{1/2}\right| = t$
(since $t \geq 0$)

b. $a_T = \frac{d^2s}{dt^2} = \left(\frac{d}{dt}\right)(t) = 1$
$a_N^2 = |a|^2 - a_T^2$
$= [t^2(\sin^2 t + \cos^2 t) + (\cos^2 t + \sin^2 t)] - 1 = t^2$
Therefore, $a_N = t$.

55. $s''(t) = a_T = 0 \Rightarrow$ speed $= s'(t) = c$ (a constant)
$\kappa\left(\frac{ds}{dt}\right)^2 = a_N = 0 \Rightarrow \kappa = 0$ or $\frac{ds}{dt} = 0 \Rightarrow \kappa = 0$

56. $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$;
$\mathbf{v}(t) = -a\omega\sin\omega t\,\mathbf{i} + b\omega\cos\omega t\,\mathbf{j}$
$\mathbf{a}(t) = \left\langle -a\omega^2\cos\omega t, -b\omega^2\sin\omega t\right\rangle = -\omega^2\mathbf{r}(t)$

$\mathbf{T} = \frac{\mathbf{v}}{(\mathbf{v}\cdot\mathbf{v})^{1/2}}$;

$\frac{d\mathbf{T}}{dt} = \frac{(\mathbf{v}\cdot\mathbf{v})\mathbf{a} - (\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{(\mathbf{v}\cdot\mathbf{v})^{3/2}}$

$= \frac{-ab\omega}{(a^2\sin^2\omega t + b^2\cos\omega t)^{3/2}}(b\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j})$

$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{ab\omega(b^2\cos^2\omega t + a^2\sin^2\omega t)^{1/2}}{(a^2\sin^2\omega t + b^2\cos\omega t)^{3/2}}$

$= \frac{ab\omega}{a^2\sin^2\omega t + b^2\cos^2\omega t}$

Then

$\frac{\frac{d\mathbf{T}}{dt}}{\left\|\frac{d\mathbf{T}}{dt}\right\|} = \frac{-1}{(a^2\sin^2\omega t + b^2\cos^2\omega t)^{1/2}}$

$\times(b\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j})$

Note that this was done assuming $ab > 0$; if $ab < 0$, drop the negative sign in the numerator.

57. $\mathbf{v}(5)$ is tangent to the helix at the point where the particle is 12 meters above the ground. Its path is described by $\mathbf{v}(5) = \cos 5\mathbf{i} - \sin 5\mathbf{j} + 7\mathbf{k}$.

58. $a_N = 0$ wherever $\kappa = 0$ or $\frac{ds}{dt} = 0$. κ, the curvature, is 0 at the inflection points, which occur at multiples of $\frac{\pi}{2}$. However, $\frac{ds}{dt} \neq 0$ on this curve. Therefore, $a_N = 0$ at multiples of $\frac{\pi}{2}$.

59. It is given that at (–12, 16), $s'(t) = 10$ ft/s and $s''(t) = 5$ ft/s^2. From Example 2, $\kappa = \frac{1}{20}$.

Therefore, $a_T = 5$ and $a_N = \left(\frac{1}{20}\right)(10)^2 = 5$, so $\mathbf{a} = 5\mathbf{T} + 5\mathbf{N}$.
Let $\mathbf{r}(t) = \langle 20\cos t, 20\sin t\rangle$ describe the circle.
$\mathbf{r}(t) = \langle -12, 16\rangle \Rightarrow \cos t = -\frac{3}{5}$ and $\sin t = \frac{4}{5}$.
$\mathbf{v}(t) = \langle -20\sin t, 20\cos t\rangle$, so $|\mathbf{v}(t)| = 20$.

Then $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle -\sin t, \cos t\rangle$.

Thus, at (–12, 16), $\mathbf{T} = \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$ and

$\mathbf{N} = \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$ since $\mathbf{N}$ is a unit vector perpendicular to $\mathbf{T}$ and pointing to the concave side of the curve.
Therefore,
$\mathbf{a} = 5\left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle + 5\left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle = -\mathbf{i} - 7\mathbf{j}.$

60. $s'(t) = 4$ and $s''(t) = 0$.
$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$
Therefore,
$\mathbf{a} = (0)\mathbf{T} + (4)^2\frac{2}{(1 + 4x^2)^{3/2}}\mathbf{N} = \frac{32}{(1 + 4x^2)^{3/2}}\mathbf{N}.$

61. Let $\mu mg = \frac{mv_R^2}{R}$. Then $v_R = \sqrt{\mu gR}$. At the values given,
$v_R = \sqrt{(0.4)(32)(400)} = \sqrt{5120} \approx 71.55$ ft/s
(about 49.79 mi/h).

62. **a.** $\frac{R\|\mathbf{F}\|\sin\theta}{v_R^2} = \frac{\|\mathbf{F}\|\cos\theta}{g}$ (from the given equations, equating m in each.)
Therefore, $v_R = \sqrt{Rg\tan\theta}$.

b. For the values given,
$v_R = \sqrt{(400)(32)(\tan 10°)} \approx 47.51$ ft/s.

63. $\tan\phi = y'$
$(1 + y'^2)^{3/2} = (1 + \tan^2\phi)^{3/2} = \sec^3\phi$
$\kappa = \frac{|y''|}{\left[1 + y'^2\right]^{3/2}} = \frac{|y''|}{\left|\sec^3\phi\right|} = \left|y''\cos^3\phi\right|$

64. $\mathbf{N} = \dfrac{\frac{d\mathbf{T}}{ds}}{\left\|\frac{d\mathbf{T}}{ds}\right\|} = \dfrac{\langle -\sin\phi, \cos\phi\rangle\left(\frac{d\phi}{ds}\right)}{(\sin^2\phi + \cos^2\phi)^{1/2}\left|\frac{d\phi}{ds}\right|}$

$= \dfrac{\frac{d\phi}{ds}}{\left|\frac{d\phi}{ds}\right|}\langle -\sin\phi, \cos\phi\rangle$

If $\dfrac{d\phi}{ds} > 0, \mathbf{N} = \langle -\sin\phi, \cos\phi\rangle$ and if $\dfrac{d\phi}{ds} < 0, \mathbf{N} = \langle \sin\phi, -\cos\phi\rangle$, so **N** points to the concave side of the curve in either case.

65. $\mathbf{B} = \mathbf{T}\times\mathbf{N}$. Left-multiply by $\mathbf{T}$ and use Theorem 11.4C

$$\begin{aligned}\mathbf{T}\times\mathbf{B} &= \mathbf{T}\times(\mathbf{T}\times\mathbf{N})\\ &= (\mathbf{T}\cdot\mathbf{N})\mathbf{T} - (\mathbf{T}\cdot\mathbf{T})\mathbf{N}\\ &= 0\mathbf{T} - 1\mathbf{N}\\ &= -\mathbf{N}\end{aligned}$$

Thus, $\mathbf{N} = -\mathbf{T}\times\mathbf{B} = \mathbf{B}\times\mathbf{T}$.

To derive a result for **T** in terms of **N** and **B**, begin with $\mathbf{N} = \mathbf{B}\times\mathbf{T}$ and left multiply by **B**:

$$\begin{aligned}\mathbf{B}\times\mathbf{N} &= \mathbf{B}\times(\mathbf{B}\times\mathbf{T})\\ &= (\mathbf{B}\cdot\mathbf{T})\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})\mathbf{T}\\ &= 0\mathbf{B} - 1\mathbf{T}\\ &= -\mathbf{T}\end{aligned}$$

Thus, $\mathbf{T} = -\mathbf{B}\times\mathbf{N} = \mathbf{N}\times\mathbf{B}$.

66. Since $\lim_{x\to 0^-} y = \lim_{x\to 0^+} y = 0 = y(0)$, y is continuous.

$$y'(x) = \begin{cases} 0 & \text{if } x \le 0\\ 3x^2 & \text{if } x > 0\end{cases}$$

is continuous since $\lim_{x\to 0^-} y' = \lim_{x\to 0^+} y' = 0 = y'(0)$.

$$y''(x) = \begin{cases} 0 & \text{if } x \le 0\\ 6x & \text{if } x > 0\end{cases}$$

is continuous since $\lim_{x\to 0^-} y'' = \lim_{x\to 0^+} y'' = 0 = y''(0)$. Thus,

$\kappa = \dfrac{|y''|}{(1+y'^2)^{3/2}}$ is continuous also. If $x \ne 0$ then y' and κ are continuous as elementary functions.

67. Let

$P_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$.

$P_5(0) = 0 \Rightarrow a_0 = 0$

$P_5'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4$, so

$P_5'(0) = 0 \Rightarrow a_1 = 0$.

$P''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3$, so

$P_5''(0) = 0 \Rightarrow a_2 = 0$.

Thus, $P_5(x) = a_3x^3 + a_4x^4 + a_5x^5$,

$P_5'(x) = 3a_3x^2 + 4a_4x^3 + 5a_5x^4$, and

$P_5''(x) = 6a_3x + 12a_4x^2 + 20a_5x^3$.

$P_5(1) = 1, P_5'(1) = 0,$ and $P_5''(1) = 0 \Rightarrow$

$a_3 + a_4 + a_5 = 1$

$3a_3 + 4a_4 + 5a_5 = 0$

$6a_3 + 12a_4 + 20a_5 = 0$

The simultaneous solution to these equations is $a_3 = 10, a_4 = -15, a_5 = 6$, so

$P_5(x) = 10x^3 - 15x^4 + 6x^5$.

68. Let

$P_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$.

Then $P_5(x)$ must satisfy

$P_5(0) = 0$; $P_5(1) = 1$; $P_5'(0) = 0$; $P_5'(1) = 1$;

$P_5''(0) = 0$; $P_5''(1) = 0$

As in Problem 67, the three conditions at 0 imply $a_0 = a_1 = a_2 = 0$. The three conditions at 0 lead to the system of equations

$a_3 + a_4 + a_5 = 1$

$3a_3 + 4a_4 + 5a_5 = 1$

$6a_3 + 12a_4 + 20a_5 = 0$

The solution to this system is $a_3 = 6$, $a_4 = -8$, $a_5 = 3$. Thus, the required polynomial is $P_5(x) = 6x^3 - 8x^4 + 3x^5$.

69. Let the polar coordinate equation of the curve be $r = f(\theta)$. Then the curve is parameterized by $x = r\cos\theta$ and $y = r\sin\theta$.

$x' = -r\sin\theta + r'\cos\theta \qquad y' = r\cos\theta + r'\sin\theta$

$x'' = -r\cos\theta - 2r'\sin\theta + r''\cos\theta \qquad y'' = -r\sin\theta + 2r'\cos\theta + r''\sin\theta$

By Theorem A, the curvature is

$$\kappa = \frac{|x'y'' - y'x''|}{\left|x'^2 + y'^2\right|^{3/2}}$$

$$= \frac{|(-r\sin\theta + r'\cos\theta)(-r\sin\theta + 2r'\cos\theta + r''\sin\theta) - (r\cos\theta + r'\sin\theta)(-r\cos\theta - 2r'\sin\theta + r''\cos\theta)|}{[(-r\sin\theta + r'\cos\theta)^2 + (r\cos\theta + r'\sin\theta)^2]^{3/2}}$$

$$= \frac{\left|r^2 + 2r'^2 - rr''\right|}{(r^2 + r'^2)^{3/2}}.$$

70. $r' = -4\sin\theta,\ r'' = -4\cos\theta$

$$\kappa = \frac{\left|16\cos^2\theta + 32\sin^2\theta + 16\cos^2\theta\right|}{(16\cos^2\theta + 16\sin^2\theta)^{3/2}} = \frac{32}{64} = \frac{1}{2}$$

71. $r' = -\sin\theta,\ r'' = -\cos\theta$

At $\theta = 0$, $r = 2$, $r' = 0$, and $r'' = -1$.

$$\kappa = \frac{|4 + 0 + 2|}{(4 + 0)^{3/2}} = \frac{6}{8} = \frac{3}{4}$$

72. $r' = 1,\ r'' = 0$

At $\theta = 1$, $r = 1$, $r' = 1$, and $r'' = 0$.

$$\kappa = \frac{|1 + 2 - 0|}{(1 + 1)^{3/2}} = \frac{3}{2\sqrt{2}}$$

73. $r' = -4\sin\theta,\ r'' = -4\cos\theta$

At $\theta = \frac{\pi}{2}$, $r = 4$, $r' = -4$, and $r'' = 0$.

$$\kappa = \frac{|16 + 32 - 0|}{(16 + 16)^{3/2}} = \frac{48}{128\sqrt{2}} = \frac{3}{8\sqrt{2}}$$

74. $r' = 3e^{3\theta},\ r'' = 9e^{3\theta}$

At $\theta = 1$, $r = e^3$, $r' = 3e^3$, and $r'' = 9e^3$.

$$\kappa = \frac{\left|e^6 + 18e^6 - 9e^6\right|}{(e^6 + 9e^6)^{3/2}} = \frac{10e^6}{10\sqrt{10}e^9} = \frac{1}{\sqrt{10e^9}}$$

75. $r' = 4\cos\theta,\ r'' = -4\sin\theta$

At $\theta = \frac{\pi}{2}$, $r = 8$, $r' = 0$, $r'' = -4$.

$$\kappa = \frac{|64 + 0 + 32|}{(64 + 0)^{3/2}} = \frac{96}{512} = \frac{3}{16}$$

76. $r' = 6e^{6\theta},\ r'' = 36e^{6\theta}$

$$\kappa = \frac{\left|e^{12\theta} + 72e^{12\theta} - 36e^{12\theta}\right|}{(e^{12\theta} + 36e^{12\theta})^{3/2}} = \frac{37e^{12\theta}}{37\sqrt{37}e^{18\theta}}$$

$$= \frac{1}{\sqrt{37}}\left(\frac{1}{e^{6\theta}}\right) = \frac{1}{\sqrt{37}}\left(\frac{1}{r}\right)$$

77. $r = \sqrt{\cos 2\theta}$; $r' = -\dfrac{\sin 2\theta}{\sqrt{\cos 2\theta}},\ r'' = -\dfrac{\cos^2 2\theta + 1}{(\cos 2\theta)^{3/2}}$

$$\kappa = \frac{\left|\cos 2\theta + \frac{2\sin^2 2\theta}{\cos 2\theta} + \frac{\cos^2 2\theta + 1}{\cos 2\theta}\right|}{\left(\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}\right)^{3/2}} = \frac{\left|\frac{3}{\cos 2\theta}\right|}{\left(\frac{1}{\cos 2\theta}\right)^{3/2}}$$

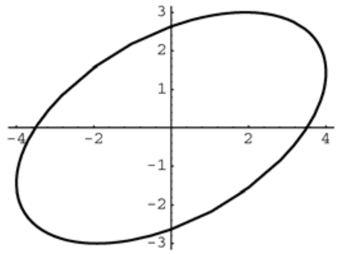
$$= 3\left(\sqrt{\cos 2\theta}\right) = 3r$$

78. $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where $x = f(t)$ and $y = g(t)$; $\mathbf{v}(t) = \mathbf{r}'(t)$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{x'}{\sqrt{x'^2 + y'^2}}\mathbf{i} + \frac{y'}{\sqrt{x'^2 + y'^2}}\mathbf{j} = \frac{x''(x'^2 + y'^2) - x'(x'x'' + y'y'')}{(x'^2 + y'^2)^{3/2}}\mathbf{i} + \frac{y''(x'^2 + y'^2) - y'(x'x'' + y'y'')}{(x'^2 + y'^2)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(t) = \frac{(x''y' - x'y'')y'}{(x'^2 + y'^2)^{3/2}}\mathbf{i} + \frac{(y''x' - y'x'')x'}{(x'^2 + y'^2)^{3/2}}\mathbf{j} = \frac{(x'y'' - y'x'')}{(x'^2 + y'^2)^{3/2}}(-y'\mathbf{i} + x'\mathbf{j})$$

$$\|\mathbf{T}'(t)\| = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}\sqrt{x'^2 + y'^2} = \frac{|x'y'' - y'x''|}{x'^2 + y'^2} \quad \Rightarrow \quad \kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$$

79. 

Maximum curvature ≈ 0.7606,
minimum curvature ≈ 0.1248

80. $\mathbf{B}\cdot\mathbf{B} = 1$

$\frac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 2\mathbf{B}\cdot\frac{d\mathbf{B}}{ds} = 0$

Thus, $\frac{d\mathbf{B}}{ds}$ is perpendicular to **B**.

81. $\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds}$

Since $\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\|\frac{d\mathbf{T}}{ds}\right\|}$, $\frac{d\mathbf{T}}{ds}\times\mathbf{N} = 0$, so $\frac{d\mathbf{B}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}$.

Thus $\mathbf{T}\cdot\frac{d\mathbf{B}}{ds} = \mathbf{T}\cdot\left(\mathbf{T}\times\frac{d\mathbf{N}}{ds}\right) = (\mathbf{T}\times\mathbf{T})\cdot\frac{d\mathbf{N}}{ds} = 0$, so

$\frac{d\mathbf{B}}{ds}$ is perpendicular to **T**.

82. **N** is perpendicular to **T**, and **B** = **T** × **N** is perpendicular to both **T** and **N**. Thus, since $\frac{d\mathbf{B}}{ds}$ is perpendicular to both **T** and **B**, it is parallel to **N**, and hence there is some number $\tau(s)$ such that

$\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}$.

83. Let $ax + by + cz + d = 0$ be the equation of the plane containing the curve. Since **T** and **N** lie in the plane $\mathbf{B} = \pm\frac{a\mathbf{i}+b\mathbf{j}+c\mathbf{k}}{\sqrt{a^2+b^2+c^2}}$. Thus, **B** is a constant vector and $\frac{d\mathbf{B}}{ds} = \mathbf{0}$, so $\tau(s) = 0$, since **N** will not necessarily be **0** everywhere.

84. $\mathbf{r}'(t) = a_0\mathbf{i} + b_0\mathbf{j} + c_0\mathbf{k}$

$\mathbf{r}''(t) = \mathbf{0}$

Thus, $\mathbf{r}'(t)\times\mathbf{r}''(t) = \mathbf{0}$ and since

$\kappa = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$, $\kappa = 0$.

To show that $\tau = 0$, note that the curve is confined to a plane. This means that the curve is two-dimensional and thus $\tau = 0$.

85. $\mathbf{r}(t) = 6\cos\pi t\mathbf{i} + 6\sin\pi t\mathbf{j}, + 2t\mathbf{k}$, $t > 0$

Let $(6\cos\pi t)^2 + (6\sin\pi t)^2 + (2t)^2 = 100$.

Then $36(\cos^2\pi t + \sin^2\pi t) + 4t^2 = 100$;

$4t^2 = 64$; $t = 4$.

$\mathbf{r}(4) = 6\mathbf{i} + 8\mathbf{k}$, so the fly will hit the sphere at the point (6, 0, 8).

$\mathbf{r}'(t) = -6\pi\sin\pi t\mathbf{i} + 6\pi\cos\pi t\mathbf{j} + 2\mathbf{k}$, so the fly will have traveled

$\int_0^4\sqrt{(-6\pi\sin\pi t)^2 + (6\pi\cos\pi t)^2 + (2)^2}\,dt$

$= \int_0^4\sqrt{36\pi^2 + 4}\,dt$

$= \sqrt{36\pi^2+4}(4-0) = 8\sqrt{9\pi^2+1} \approx 75.8214$

86. $\mathbf{r}(t) = \left\langle 10\cos t, 10\sin t, \left(\frac{34}{2\pi}\right)t\right\rangle$

Using the result of Example 1 with $a = 10$ and $c = \frac{34}{2\pi}$, the length of one complete turn is

$2\pi\sqrt{(10)^2 + \left(\frac{34}{2\pi}\right)^2}$ angstroms

$= 10^{-8}\sqrt{400\pi^2 + 34^2}$ cm. Therefore, the total length of the helix is

$(2.9)(10^8)(10^{-8})\sqrt{400\pi^2 + 34^2} \approx 207.1794$ cm.

11.8 Concepts Review

1. traces; cross sections

2. cylinders; z-axis

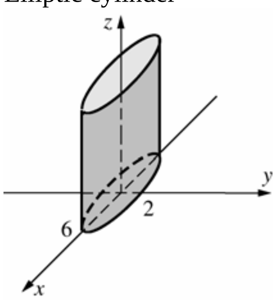
3. ellipsoid

4. elliptic paraboloid

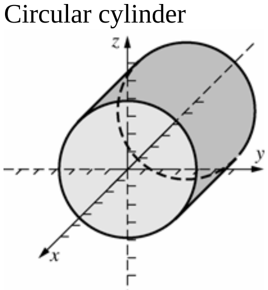
Problem Set 11.8

1. $\dfrac{x^2}{36}+\dfrac{y^2}{4}=1$

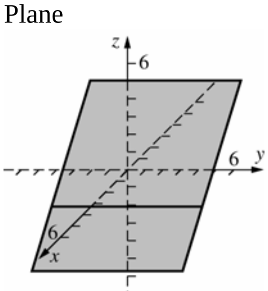
Elliptic cylinder



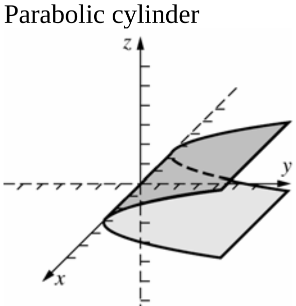
2. Circular cylinder



3. Plane

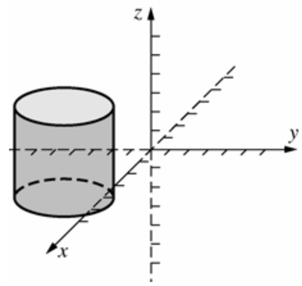


4. Parabolic cylinder

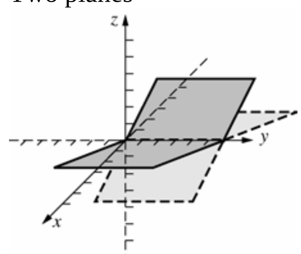


5. $(x-4)^2+(y+2)^2=7$

Circular cylinder

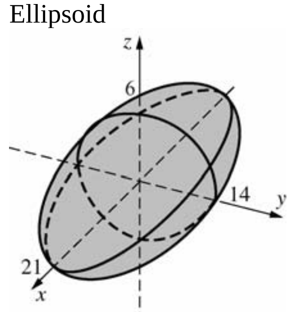


6. Two planes



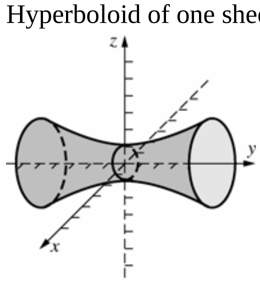
7. $\dfrac{x^2}{441}+\dfrac{y^2}{196}+\dfrac{z^2}{36}=1$

Ellipsoid



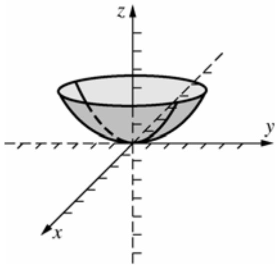
8. $\dfrac{x^2}{1}-\dfrac{y^2}{9}+\dfrac{z^2}{1}=1$

Hyperboloid of one sheet

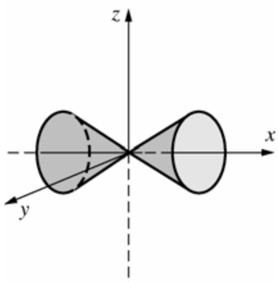


9. $z = \frac{x^2}{8} + \frac{y^2}{2}$

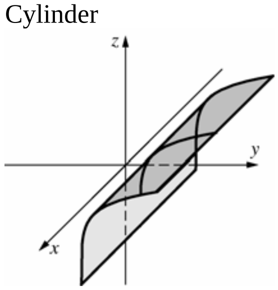
Elliptic paraboloid



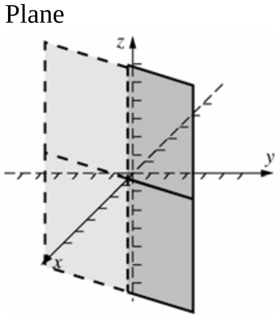
10. Circular cone



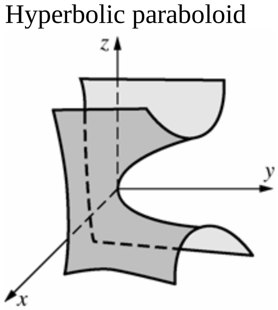
11. Cylinder



12. Plane

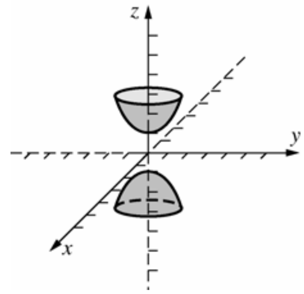


13. Hyperbolic paraboloid



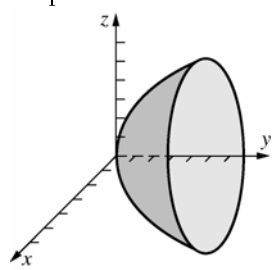
14. $-\frac{x^2}{4} - \frac{y^2}{4} + \frac{z^2}{1} = 1$

Hyperboloid of two sheets



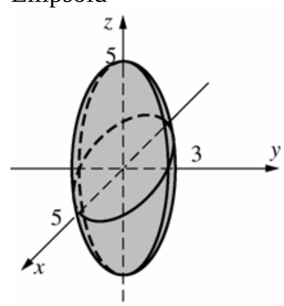
15. $y = \frac{x^2}{4} + \frac{z^2}{9}$

Elliptic Paraboloid

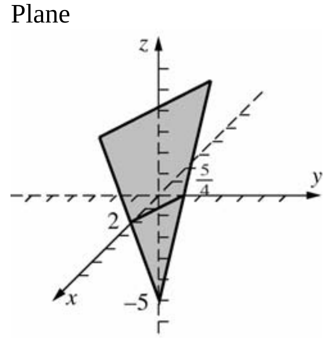


16. $\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{25} = 1$

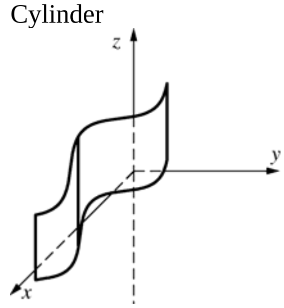
Ellipsoid



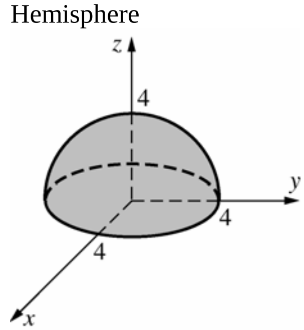
17. Plane



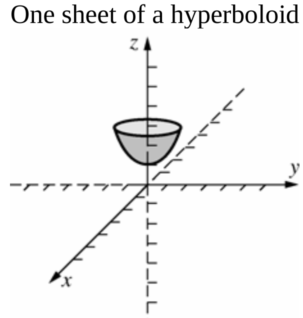
18. Cylinder



19. Hemisphere



20. One sheet of a hyperboloid of two sheets.



21. a. Replacing x by $-x$ results in an equivalent equation.

b. Replacing x by $-x$ and y by $-y$ results in an equivalent equation.

c. Replacing x by $-x$, y by $-y$, and z with $-z$, results in an equivalent equation.

22. a. Replacing y by $-y$ results in an equivalent equation.

b. Replacing x with $-x$ and z with $-z$ results in an equivalent equation.

c. Replacing y by $-y$ and z with $-z$ results in an equivalent equation.

23. All central ellipsoids are symmetric with respect to (a) the origin, (b) the *x*-axis, and the (c) *xy*-plane.

24. All central hyperboloids of one sheet are symmetric with respect to (a) the origin, (b) the *y*-axis, and (c) the *xy*-plane.

25. All central hyperboloids of two sheets are symmetric with respect to (a) the origin, (b) the *z*-axis, and (c) the *yz*-plane.

26. a. 1, 2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 18

b. 1, 2, 6, 7, 8, 9, 10, 14, 16, 19, 20

27. At $y = k$, the revolution generates a circle of radius $x = \sqrt{\frac{y}{2}} = \sqrt{\frac{k}{2}}$. Thus, the cross section in the plane $y = k$ is the circle $x^2 + z^2 = \frac{k}{2}$ or $2x^2 + 2z^2 = k$. The equation of the surface is $y = 2x^2 + 2z^2$.

28. At $z = k$, the revolution generates a circle of radius $y = \frac{z}{2} = \frac{k}{2}$. Thus, the cross section in the plane

$z = k$ is the circle $x^2 + y^2 = \frac{k^2}{4}$ or $4x^2 + 4y^2 = k^2$. The equation of the surface is $z^2 = 4x^2 + 4y^2$.

29. At $y = k$, the revolution generates a circle of radius $x = \sqrt{3 - \frac{3}{4}y^2} = \sqrt{3 - \frac{3}{4}k^2}$. Thus, the cross section in the plane $y = k$ is the circle $x^2 + z^2 = 3 - \frac{3}{4}k^2$ or $12 - 4x^2 - 4z^2 = 3k^2$. The equation of the surface is $4x^2 + 3y^2 + 4z^2 = 12$.

30. At $x = k$, the revolution generates a circle of radius $y = \sqrt{\frac{4}{3}x^2 - 4} = \sqrt{\frac{4}{3}k^2 - 4}$. Thus, the cross section in the plane $x = k$ is the circle $y^2 + z^2 = \frac{4}{3}k^2 - 4$ or $12 + 3y^2 + 3z^2 = 4k^2$. The equation of the surface is $4x^2 = 12 + 3y^2 + 3z^2$.

31. When $z = 4$ the equation is $4 = \frac{x^2}{4} + \frac{y^2}{9}$ or $1 = \frac{x^2}{16} + \frac{y^2}{36}$, so $a^2 = 36, b^2 = 16$, and $c^2 = a^2 - b^2 = 20$, hence $c = \pm 2\sqrt{5}$. The major axis of the ellipse is on the *y*-axis so the foci are at $\left(0, \pm 2\sqrt{5}, 4\right)$.

32. When $x = 4$, the equation is $z = \frac{16}{4} + \frac{y^2}{9}$ or

$y^2 = 9(z-4) = 4 \cdot \frac{9}{4}(z-4)$, hence $p = \frac{9}{4}$. The vertex is at (4, 0, 4) so the focus is $\left(4, 0, 4+\frac{9}{4}\right) = \left(4, 0, \frac{25}{4}\right)$.

33. When $z = h$, the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{h^2}{c^2} = 1$ or

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{c^2 - h^2}{c^2}$ which is equivalent to

$\frac{x^2}{\frac{a^2(c^2-h^2)}{c^2}} + \frac{y^2}{\frac{b^2(c^2-h^2)}{c^2}} = 1$, which is $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

with $A = \frac{a}{c}\sqrt{c^2 - h^2}$ and $B = \frac{b}{c}\sqrt{c^2 - h^2}$. Thus, the area is

$\pi\left(\frac{a}{c}\sqrt{c^2-h^2}\right)\left(\frac{b}{c}\sqrt{c^2-h^2}\right) = \frac{\pi ab(c^2 - h^2)}{c^2}$

34. The equation of the elliptical cross section is

$\frac{x^2}{a^2(h-z)} + \frac{y^2}{b^2(h-z)} = 1$, for each z in $[0, h)$.

Therefore, $\Delta V \approx \pi\left(a\sqrt{h-z}\right)\left(b\sqrt{h-z}\right)\Delta z$

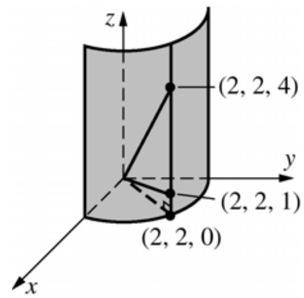
$= \pi ab(h-z)\Delta z$, using the area formula mentioned in Problem 33.

Therefore, $V = \int_0^h \pi ab(h-z)dz = \pi ab\left[hz - \frac{z^2}{2}\right]_0^h$

$= \pi ab\left[\left(h - \frac{h^2}{2}\right) - 0\right] = \frac{\pi abh^2}{2}$, which is the height times one half the area of the base ($z = 0$), $\pi\left(a\sqrt{h}\right)\left(b\sqrt{h}\right) = \pi abh.$

35. Equating the expressions for y, $4 - x^2 = x^2 + z^2$ or $1 = \frac{x^2}{2} + \frac{z^2}{4}$ which is the equation of an ellipse in the xz-plane with major diameter of $2\sqrt{4} = 4$ and minor diameter $2\sqrt{2}$.

36.



$y = x$ intersects the cylinder when $x = y = 2$. Thus, the vertices of the triangle are (0, 0, 0), (2, 2, 1), and (2, 2, 4). The area of the triangle with sides represented by $\langle 2, 2, 1\rangle$ and $\langle 2, 2, 4\rangle$ is

$\frac{1}{2}\|\langle 2, 2, 1\rangle \times \langle 2, 2, 4\rangle\| = \frac{1}{2}\|\langle 6, -6, 0\rangle\|$

$= \frac{1}{2}\left(6\sqrt{2}\right) = 3\sqrt{2}$.

37. $(t\cos t)^2 + (t\sin t)^2 - t^2$

$= t^2(\cos^2 t + \sin^2 t) - t^2 = t^2 - t^2 = 0,$

hence every point on the spiral is on the cone. For $\mathbf{r} = 3t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k}$, every point satisfies $x^2 + 9y^2 - 9z^2 = 0$ so the spiral lies on the elliptical cone.

38. It is clear that $x = y$ at each point on the curve. Thus, the curve lies in the plane $x = y$. Since $z = t^2 = y^2$, the curve is the intersection of the plane $x = y$ with the parabolic cylinder $z = y^2$. Let the line $y = x$ in the xy-plane be the u-axis, then the curve determined by $\mathbf{r}$ is in the uz-plane. The u-coordinate of a point on the curve, (y, y, z) is the signed distance of the point $(y, y, 0)$ from the origin, i.e., $u = \sqrt{2}y$. Thus, $u^2 = 2y^2 = 2z$ or $u^2 = 4\left(\frac{1}{2}\right)z$. This is a parabola in the uz-plane with vertex at $u = 0$, $z = 0$ and focus at $u = 0$, $z = \frac{1}{2}$. The focus is at $\left(0, 0, \frac{1}{2}\right)$.

11.9 Concepts Review

1. circular cylinder; sphere

2. plane; cone

3. $\rho^2 = r^2 + z^2$

4. $x^2 + y^2 + z^2 = 4z$, so

$x^2 + y^2 + z^2 - 4z + 4 = 4$ or

$x^2 + y^2 + (z-2)^2 = 4.$

Problem Set 11.9

1. Cylindrical to Spherical:

$\rho = \sqrt{r^2 + z^2}$

$\cos\phi = \dfrac{z}{\sqrt{r^2 + z^2}}$

$\theta = \theta$

Spherical to Cylindrical:

$r = \rho \sin\phi$

$z = \rho\cos\phi$

$\theta = \theta$

2. a. $(\rho, \theta, \phi) = \left(\sqrt{2}, \dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$

b. Note: $\left(-2, \dfrac{\pi}{4}, 2\right) = \left(2, \dfrac{5\pi}{4}, 2\right)$

$(\rho, \theta, \phi) = \left(2\sqrt{2}, \dfrac{5\pi}{4}, \dfrac{\pi}{4}\right)$

3. a. $x = 6\cos\left(\dfrac{\pi}{6}\right) = 3\sqrt{3}$

$y = 6\sin\left(\dfrac{\pi}{6}\right) = 3$

$z = -2$

b. $x = 4\cos\left(\dfrac{4\pi}{3}\right) = -2$

$y = 4\sin\left(\dfrac{4\pi}{3}\right) = -2\sqrt{3}$

$z = -8$

4. a. $x = 8\sin\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$

$y = 8\sin\left(\dfrac{\pi}{6}\right)\sin\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$

$z = 8\cos\left(\dfrac{\pi}{6}\right) = 4\sqrt{3}$

b. $x = 4\sin\left(\dfrac{3\pi}{4}\right)\cos\left(\dfrac{\pi}{3}\right) = \sqrt{2}$

$y = 4\sin\left(\dfrac{3\pi}{4}\right)\sin\left(\dfrac{\pi}{3}\right) = \sqrt{6}$

$z = 4\cos\left(\dfrac{3\pi}{4}\right) = -2\sqrt{2}$

5. a. $\rho = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{4 + 12 + 16} = 4\sqrt{2}$

$\tan\theta = \dfrac{y}{x} = \dfrac{-2\sqrt{3}}{2} = -\sqrt{3}$ and (x, y) is in the 4th quadrant so $\theta = \dfrac{5\pi}{3}$.

$\cos\phi = \dfrac{z}{\rho} = \dfrac{4}{4\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ so $\phi = \dfrac{\pi}{4}$.

Spherical: $\left(4\sqrt{2}, \dfrac{5\pi}{3}, \dfrac{\pi}{4}\right)$

b. $\rho = \sqrt{2 + 2 + 12} = 4$

$\tan\theta = \dfrac{\sqrt{2}}{-\sqrt{2}} = -1$ and (x, y) is in the 2nd quadrant so $\theta = \dfrac{3\pi}{4}$.

$\cos\phi = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2}$ so $\dfrac{\pi}{6}$.

Spherical: $\left(4, \dfrac{3\pi}{4}, \dfrac{\pi}{6}\right)$

6. a. $r = \sqrt{4+4} = 2\sqrt{2}$

$\tan\theta = \dfrac{2}{2} = 1$, $x > 0, y > 0$, so $\theta = \dfrac{\pi}{4}$. $z = 3$

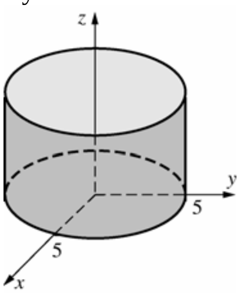
b. $r = \sqrt{48 + 16} = 8$

$\tan\theta = -\dfrac{4}{4\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$, $x > 0, y < 0$, so

$\theta = \dfrac{11\pi}{6}$. $z = 6$

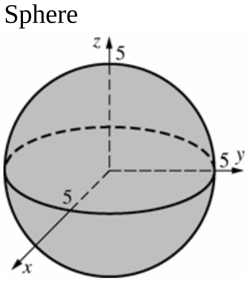
7. $r = 5$

Cylinder



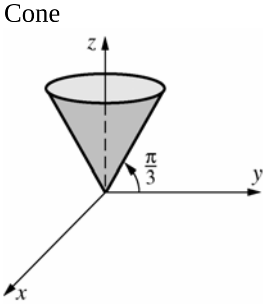
8. $\rho = 5$

Sphere



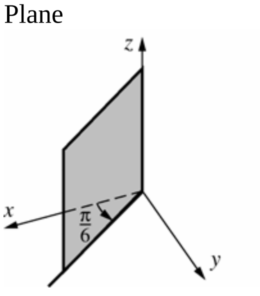
9. $\phi = \frac{\pi}{6}$

Cone



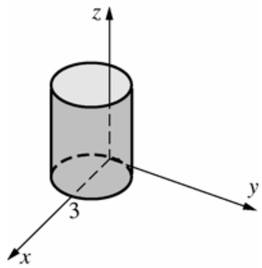
10. $\theta = \frac{\pi}{6}$

Plane



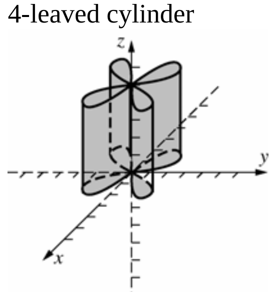
11. $r = 3 \cos \theta$

Circular cylinder



12. $r = 2 \sin 2\theta$

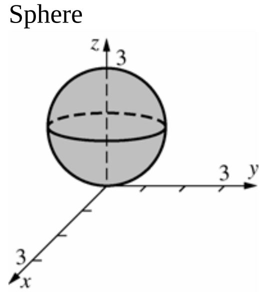
4-leaved cylinder



13. $\rho = 3 \cos \phi$

$$x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = \frac{9}{4}$$

Sphere

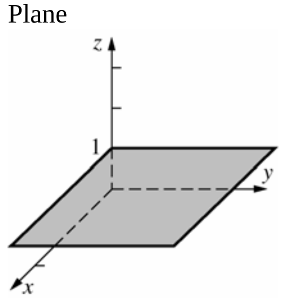


14. $\rho = \sec \phi$

$\rho \cos \phi = 1$

$z = 1$

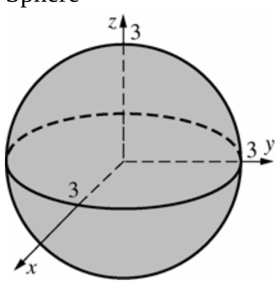
Plane



15. $r^2 + z^2 = 9$

$x^2 + y^2 + z^2 = 9$

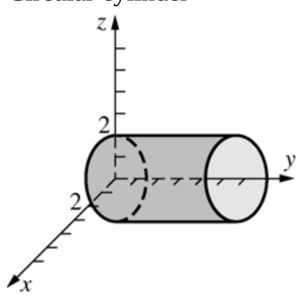
Sphere



16. $r^2 \cos^2 \theta + z^2 = 4$

$x^2 + z^2 = 4$

Circular cylinder



17. $x^2 + y^2 = 9; r^2 = 9;\ r = 3$

18. $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 25; r^2 \cos 2\theta = 25;$

$r^2 = 25\sec^2 \theta; r = 5\sec\theta$

19. $r^2 + 4z^2 = 10$

20. $(x^2 + y^2 + z^2) + 3z^2 = 10; \rho^2 + 3\rho^2 \cos^2 \phi = 10;$

$\rho^2 = \dfrac{10}{1 + 3\cos^2 \phi}$

21. $(x^2 + y^2 + z^2) - 3z^2 = 0; \rho^2 - 3\rho^2 \cos^2 \phi = 0;$

$\cos^2 \phi = \dfrac{1}{3}$ (pole is not lost); $\cos^2 \phi = \dfrac{1}{3}$ (or

$\sin^2 \phi = \dfrac{2}{3}$ or $\tan^2 \phi = 2$)

22. $\rho^2[\sin^2 \phi \cos^2 \theta - \sin^2 \phi \sin^2 \theta - \cos^2 \phi] = 1;$

$\rho^2[\sin^2 \phi \cos^2 \theta - \sin^2 \phi \sin^2 \theta - 1 + \sin^2 \phi] = 1;$

$\rho^2[\sin^2 \phi \cos^2 \theta - 1 + \sin^2 \phi(1 - \sin^2 \theta)] = 1;$

$\rho^2[\sin^2 \phi \cos^2 \theta - 1 + \sin^2 \phi \cos^2 \theta] = 1;$

$\rho^2[2\sin^2 \phi \cos^2 \theta - 1] = 1;\ \ \rho^2 = \dfrac{1}{2\sin^2 \phi \cos^2 \theta - 1}$

23. $(r^2 + z^2) + z^2 = 4; \rho^2 + \rho^2 \cos^2 \phi = 4;$

$\rho^2 = \dfrac{4}{1 + \cos^2 \phi}$

24. $\rho^2 = 2\rho \cos \phi; r^2 + z^2 = 2z; r^2 = 2z - z^2;$

$r = \sqrt{2z - z^2}$

25. $r \cos \theta + r \sin \theta = 4;\ r = \dfrac{4}{\sin\theta + \cos\theta}$

26. $\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta + \rho \cos \phi = 1;$

$\rho = \dfrac{1}{\sin\phi(\sin\theta + \cos\theta) + \cos\phi}$

27. $(x^2 + y^2 + z^2) - z^2 = 9;\ \ \rho^2 - \rho^2 \cos^2 \phi = 9;$

$\rho^2(1 - \cos^2 \phi) = 9; \rho^2 \sin^2 \phi = 9$

$\rho \sin \phi = 3$

28. $r^2 = 2r\sin\theta;\ x^2 + y^2 = 2y;\ x^2 + (y-1)^2 = 1$

29. $r^2 \cos 2\theta = z; r^2(\cos^2 \theta - \sin^2 \theta) = z;$

$(r\cos\theta)^2 - (r\sin\theta)^2 = z;\ \ x^2 - y^2 = z$

30. $\rho \sin \phi = 1$ (spherical); $r = 1$ (cylindrical);

$x^2 + y^2 = 1$ (Cartesian)

31. $z = 2x^2 + 2y^2 = 2(x^2 + y^2)$ (Cartesian); $z = 2r^2$ (cylindrical)

32. $2x^2 + 2y^2 - z^2 = 2$ (Cartesian); $2r^2 - z^2 = 2$ (cylindrical)

33. For St. Paul:
$\rho = 3960$,
$\theta = 360° - 93.1° = 266.9° \approx 4.6583$ rad
$\phi = 90° - 45° = 45° = \frac{\pi}{4}$ rad

$$x = 3960 \sin\frac{\pi}{4} \cos 4.6583 \approx -151.4$$

$$y = 3960 \sin\frac{\pi}{4} \sin 4.6583 \approx -2796.0$$

$$z = 3960 \cos\frac{\pi}{4} \approx 2800.1$$

For Oslo:
$\rho = 3960$, $\theta = 10.5° \approx 0.1833$ rad,
$\phi = 90° - 59.6° = 30.4° \approx 0.5306$ rad
$x = 3960 \sin 0.5306 \cos 0.1833 \approx 1970.4$
$y = 3960 \sin 0.5306 \sin 0.1833 \approx 365.3$
$z = 3960 \cos 0.5306 \approx 3415.5$
As in Example 7,

$$\cos\gamma \approx \frac{(-151.4)(1970.4) + (-2796.0)(365.3) + (2800.1)(3415.5)}{3960^2} \approx 0.5257$$

so $\gamma \approx 1.0173$ and the great-circle distance is $d \approx 3960(1.0173) \approx 4029$ mi

34. For New York:
$\rho = 3960$, $\theta = 360° - 74° = 286° \approx 4.9916$ rad
$\phi = 90° - 40.4° = 49.6° \approx 0.8657$ rad
$x = 3960 \sin 0.8657 \cos 4.9916 \approx 831.1$
$y = 3960 \sin 0.8657 \sin 4.9916 \approx -2898.9$
$z = 3960 \cos 0.8657 \approx 2566.5$
For Greenwich:
$\rho = 3960$, $\theta = 0$, $\phi = 90° - 51.3° = 38.7° \approx 0.6754$ rad
$x = 3960 \sin 0.6754 \cos 0 \approx 2475.8$
$y = 0$
$z = 3960 \cos 0.6754 \approx 3090.6$

$$\cos\gamma = \frac{(831.1)(2475.8) + (-2898.9)(0) + (2566.5)(3090.6)}{3960^2} \approx 0.6370$$

so $\gamma \approx 0.8802$ and the great-circle distance is $d \approx 3960(0.8802) \approx 3485$ mi

35. From Problem 33, the coordinates of St. Paul are $P(-151.4, -2796.0, 2800.1)$.
For Turin:
$\rho = 3960$, $\theta = 7.4° \approx 0.1292$ rad, $\phi = \frac{\pi}{4}$ rad

$$x = 3960 \sin\frac{\pi}{4} \cos 0.1292 \approx 2776.8$$

$$y = 3960 \sin\frac{\pi}{4} \sin 0.1292 \approx 360.8$$

$$z = 3960 \cos\frac{\pi}{4} \approx 2800.1$$

$$\cos\gamma \approx \frac{(-151.4)(2776.8) + (-2796.0)(360.8) + (2800.1)(2800.1)}{3960^2} \approx 0.4088$$

so $\gamma \approx 1.1497$ and the great-circle distance is $d \approx 3960(1.1497) \approx 4553$ mi

36. The circle inscribed on the earth at 45° parallel (ϕ = 45°) has radius $3960\cos\frac{\pi}{4}$. The longitudinal angle between St. Paul and Turin is 93.1° + 7.4° = 100.5° $\approx$ 1.7541 rad

Thus, the distance along the 45° parallel is $\left(3960\cos\frac{\pi}{4}\right)(1.7541)\approx 4912$ mi

37. Let St. Paul be at $P_1(-151.4, -2796.0, 2800.1)$ and Turin be at $P_2(2776.8, 360.8, 2800.1)$ and O be the center of the earth. Let β be the angle between the z-axis and the plane determined by O, P_1, and P_2. $\overrightarrow{OP_1}\times\overrightarrow{OP_2}$ is normal to the plane. The angle between the z-axis and $\overrightarrow{OP_1}\times\overrightarrow{OP_2}$ is complementary to β. Hence

$$\beta=\frac{\pi}{2}-\cos^{-1}\left(\frac{\left(\overrightarrow{OP_1}\times\overrightarrow{OP_2}\right)\cdot\mathbf{k}}{\left\|\overrightarrow{OP_1}\times\overrightarrow{OP_2}\right\|\|\mathbf{k}\|}\right)\approx\frac{\pi}{2}-\cos^{-1}\left(\frac{7.709\times10^6}{1.431\times10^7}\right)\approx 0.5689\,.$$

The distance between the North Pole and the St. Paul-Turin great-circle is 3960(0.5689) $\approx$ 2253 mi

38. $x_i=\rho_i\sin\phi_i\cos\theta_i,\ y_i=\rho_i\sin\phi_i\sin\theta_i,\ z_i=\rho_i\cos\phi_i$ for i = 1, 2.

$d^2=(\rho_2\sin\phi_2\cos\theta_2-\rho_1\sin\phi_1\cos\theta_1)^2+(\rho_2\sin\phi_2\sin\theta_2-\rho_1\sin\phi_1\sin\theta_1)^2$
$\quad+(\rho_2\cos\phi_2-\rho_1\cos\phi_1)^2$

$=\rho_2^2\sin^2\phi_2(\cos^2\theta_2+\sin^2\theta_2)+\rho_1^2\sin^2\phi_1(\cos^2\theta_1+\sin^2\theta_1)+\rho_2^2\cos^2\phi_2+\rho_1^2\cos^2\phi_1$
$\quad-2\rho_1\rho_2\sin\phi_1\sin\phi_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)-2\rho_1\rho_2\cos\phi_1\cos\phi_2$

$=\rho_2^2\sin^2\phi_2+\rho_1^2\sin^2\phi_1+\rho_2^2\cos^2\phi_2+\rho_1^2\cos^2\phi_1-2\rho_1\rho_2\sin\phi_1\sin\phi_2[\cos(\theta_1-\theta_2)]$
$\quad-2\rho_1\rho_2\cos\phi_1\cos\phi_2$

$=\rho_2^2(\sin^2\phi_2+\cos^2\phi_2)+\rho_1^2(\sin^2\phi_1+\cos^2\phi_1)+2\rho_1\rho_2[-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2-\cos\phi_1\cos\phi_2]$

$=\rho_1^2+\rho_2^2+2\rho_1\rho_2[-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2-\cos\phi_1\cos\phi_2]$

$=\rho_1^2-2\rho_1\rho_2+\rho_2^2+2\rho_1\rho_2[1-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2-\cos\phi_1\cos\phi_2]$

$=(\rho_1-\rho_2)^2+2\rho_1\rho_2[1-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2-\cos\phi_1\cos\phi_2]$

Hence, $d=\{(\rho_1-\rho_2)^2+2\rho_1\rho_2[1-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2-\cos\phi_1\cos\phi_2]\}^{1/2}$

39. Let P_1 be (a_1, θ_1, ϕ_1) and P_2 be (a_2, θ_2, ϕ_2). If γ is the angle between $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ then the great-circle distance between P_1 and P_2 is $a\gamma$. $|OP_1|=|OP_2|=a$ while the straight-line distance between P_1 and P_2 is (from Problem 38) $d^2=(a-a)^2+2a^2[1-\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2-\cos\phi_1\cos\phi_2]$

$=2a^2\{1-[\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2+\cos\phi_1\cos\phi_2]\}$.

Using the Law Of Cosines on the triangle OP_1P_2,

$d^2=a^2+a^2-2a^2\cos\gamma\ =2a^2(1-\cos\gamma)$.

Thus, γ is the central angle and $\cos\gamma=\cos(\theta_1-\theta_2)\sin\phi_1\sin\phi_2+\cos\phi_1\cos\phi_2$.

40. The longitude/latitude system (α, β) is related to a spherical coordinate system (ρ, θ, ϕ) by the following relations: $\rho = 3960$; the trigonometric function values of α and θ are identical but

$-\pi \le \alpha \le \pi$ rather than $0 \le \theta \le 2\pi$, and $\beta = \frac{\pi}{2} - \phi$ so $\sin\beta = \cos\phi$ and $\cos\beta = \sin\phi$.

From Problem 39, the great-circle distance between $(3960, \theta_1, \phi_1)$ and $(3960, \theta_2, \phi_2)$ is 3960γ where $0 \le \gamma \le \pi$ and $\cos\gamma = \cos(\theta_1 - \theta_2)\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 = \cos(\alpha_1 - \alpha_2)\cos\beta_1 \cos\beta_2 + \sin\beta_1 \sin\beta_2$.

41. a. New York (–74°, 40.4°); Greenwich (0°, 51.3°)
$\cos\gamma = \cos(-74° - 0°)\cos(40.4°)\cos(51.3°) + \sin(40.4°)\sin(51.3°) \approx 0.637$
Then $\gamma \approx 0.880$ rad, so $d \approx 3960(0.8801) \approx 3485$ mi.

b. St. Paul (–93.1°, 45°); Turin (7.4°, 45°)
$\cos\gamma = \cos(-93.1° - 7.4°)\cos(45°)\cos(45°) + \sin(45°)\sin(45°) \approx 0.4089$
Then $\gamma \approx 1.495$ rad, so $d \approx 3960(1.1495) \approx 4552$ mi.

c. South Pole (7.4°, –90°); Turin (7.4°, 45°)
Note that any value of α can be used for the poles.
$\cos\gamma = \cos 0° \cos(-90°)\cos(45°) + \sin(-90°)\sin(45°) = -\frac{1}{\sqrt{2}}$
thus $\gamma = 135° = \frac{3\pi}{4}$ rad, so $d = 3960\left(\frac{3\pi}{4}\right) \approx 9331$ mi.

d. New York (–74°, 40.4°); Cape Town (18.4°, –33.9°)
$\cos\gamma = \cos(-74° - 18.4°)\cos(40.4°)\cos(-33.9°) + \sin(40.4°)\sin(-33.9°) \approx -0.3880$
Then $\gamma \approx 1.9693$ rad, so $d \approx 3960(1.9693) \approx 7798$ mi.

e. For these points $\alpha_1 = 100°$ and $\alpha_2 = -80°$ while $\beta_1 = \beta_2 = 0$, hence $\cos\gamma = \cos 180°$ and $\gamma = \pi$ rad, so $d = 3960\pi \approx 12{,}441$ mi.

42. $\rho = 2a\sin\phi$ is independent of θ so the cross section in each half-plane, $\theta = k$, is a circle tangent to the origin and with radius 2*a*. Thus, the graph of $\rho = 2a\sin\phi$ is the surface of revolution generated by revolving about the *z*-axis a circle of radius 2*a* and tangent to the *z*-axis at the origin.

11.10 Chapter Review

Concepts Test

1. True: The coordinates are defined in terms of distances from the coordinate planes in such a way that they are unique.

2. False: The equation is $(x-2)^2 + y^2 + z^2 = -5$, so the solution set is the empty set.

3. True: See Section 11.2.

4. False: See previous problem. It represents a plane if *A* and *B* are not both zero.

5. False: The distance between (0, 0, 3) and (0, 0, –3) (a point from each plane) is 6, so the distance between the planes is less than or equal to 6 units.

6. False: It is normal to the plane.

7. True: Let $t = \frac{1}{2}$.

8. True: Direction cosines are $\frac{a}{\|\mathbf{u}\|}, \frac{b}{\|\mathbf{u}\|}, \frac{c}{\|\mathbf{u}\|}$

9. True: $(2\mathbf{i} - 3\mathbf{j}) \cdot (6\mathbf{i} + 4\mathbf{j}) = 0$ if and only if the vectors are perpendicular.

10. True: Since **u** and **v** are unit vectors, $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \mathbf{u} \cdot \mathbf{v}$.

11. False: The dot product for three vectors $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ is not defined.

12. True: $\|\mathbf{u}\cdot\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\cos\theta| \le \|\mathbf{u}\|\|\mathbf{v}\|$ since $|\cos\theta| \le 1$.

13. True: If $\|\mathbf{u}\cdot\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|$, then $|\cos\theta| = 1$ since $\|\mathbf{u}\cdot\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\cos\theta|$. Thus, **u** is a scalar multiple of **v**. If **u** is a scalar multiple of **v**, $\|\mathbf{u}\cdot\mathbf{v}\| = \|\mathbf{u}\cdot k\mathbf{u}\| = k\|\mathbf{u}\|^2$

$= \|\mathbf{u}\|\|k\mathbf{u}\| = \|\mathbf{u}\|\|\mathbf{v}\|.$

14. False: If **u** = –**i** and $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$, then $\mathbf{u}+\mathbf{v} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ and $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\| = 1.$

15. True: $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$ so $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$ or $|\mathbf{u}| = |\mathbf{v}|$.

16. True: $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$

$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}$

17. True: Theorem 11.5A

18. True: $D_t[\mathbf{F}(t)\cdot\mathbf{F}(t)]$

$= \mathbf{F}(t)\cdot\mathbf{F}'(t) + \mathbf{F}(t)\cdot\mathbf{F}'(t)$

$= 2\mathbf{F}(t)\cdot\mathbf{F}'(t)$

19. True: $\|\,\|\mathbf{u}\|\mathbf{u}\,\| = \|\mathbf{u}\|\|\mathbf{u}\| = \|\mathbf{u}\|^2$

20. False: The dot product of a scalar and a vector is not defined.

21. True: $\|\mathbf{u}\times\mathbf{v}\| = \|-\mathbf{v}\times\mathbf{u}\| = |-1|\|\mathbf{v}\times\mathbf{u}\|$

$= \|\mathbf{v}\times\mathbf{u}\|$

22. True: $(k\mathbf{v})\times\mathbf{v} = k(\mathbf{v}\times\mathbf{v}) = k(\mathbf{0}) = \mathbf{0}$

23. False: Obviously not true if **u** = **v**. (More generally, it is only true when **u** and **v** are also perpendicular.)

24. False: It multiplies **v** by *a*; it multiplies the length of **v** by $|a|$.

25. True: $\frac{\|\mathbf{u}\times\mathbf{v}\|}{(\mathbf{u}\cdot\mathbf{v})} = \frac{\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta}{\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta} = \tan\theta$

26. True: The vectors are both parallel and perpendicular, so one or both must be **0**.

27. True: $|(2\mathbf{i}\times 2\mathbf{j})\cdot(\mathbf{j}\times\mathbf{i})| = 4|(\mathbf{k})\cdot(-\mathbf{k})|$

$= 4(\mathbf{k}\cdot\mathbf{k}) = 4$

28. False: Let **u** = **v** = **i**, **w** = **j**.
Then (**u** × **v**) × **w** = **0** × **w** = **0**; but **u** × (**v** × **w**) = **i** × **k** = –**j**.

29. True: Since $\langle b_1, b_2, b_3\rangle$ is normal to the plane.

30. False: Each line can be represented by parametric equations, but lines with any zero direction number cannot be represented by symmetric equations.

31. True: $\kappa = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3} = 0 \Rightarrow \|\mathbf{r}'\times\mathbf{r}''\|$

$= \|\mathbf{r}'\|\|\mathbf{r}''\|\sin\theta = 0.$ Thus, either $\mathbf{r}'$ and $\mathbf{r}''$ are parallel or either $\mathbf{r}'$ or $\mathbf{r}''$ is **0**, which implies that the path is a straight line.

32. True: An ellipse bends the sharpest at points on the major axis.

33. False: κ depends only on the shape of the curve.

34. True: $x' = 3 \qquad y' = 2$

$x'' = 0 \qquad y'' = 0$

Thus, $\kappa = \frac{|x'y'' - y'x''|}{\left[x'^2 + y'^2\right]^{3/2}} = 0.$

35. False: $x' = -2\sin t \qquad y' = 2\cos t$

$x'' = -2\cos t \qquad y'' = -2\sin t$

Thus, $\kappa = \frac{|x'y'' - y'x''|}{[x'^2 + y'^2]^{3/2}} = \frac{4}{8} = \frac{1}{2}.$

36. True: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$;

$$\mathbf{T}'(t) = -\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3}\mathbf{r}'(t) + \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}''(t);$$

$\mathbf{T}(t)\cdot\mathbf{T}'(t)$

$$= \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t)\cdot\left[-\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3}\mathbf{r}'(t) + \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}''(t)\right]$$

$$= -\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2} + \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2} = 0$$

37. False: Consider uniform circular motion: $|dv/dt| = 0$ but $\|\mathbf{v}\| = a\omega$.

38. True: $\kappa = \frac{|y''|}{[1+y'^{\,2}]^{3/2}} = 0$

39. False: If $y'' = k$ then $y' = kx + C$ and $\kappa = \frac{|y''|}{[1+y'^2]^{3/2}} = \frac{k}{[1+(kx+C)^2]^{3/2}}$ is not constant.

40. False: For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$, then $\mathbf{u}\cdot\mathbf{v} = 0$.

41. False: For example, if $\mathbf{r}(t) = \cos t^2\mathbf{i} + \sin t^2\mathbf{j}$, then $\mathbf{r}'(t) = -2t\sin t^2\mathbf{i} + 2t\cos t^2\mathbf{j}$, so $\|\mathbf{r}(t)\| = 1$ but $\|\mathbf{r}'(t)\| = 2t$.

42. True: If $\mathbf{v}\cdot\mathbf{v} = \text{constant}$, differentiate both sides to get $\mathbf{v}\cdot\mathbf{v}' + \mathbf{v}\cdot\mathbf{v}' = 2\mathbf{v}\cdot\mathbf{v}' = 0$, so $\mathbf{v}\cdot\mathbf{v}' = 0$.

43. True If $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j} + ct\mathbf{k}$, then $\mathbf{r}'(t) = -a\omega\sin\omega t\,\mathbf{i} + a\omega\cos\omega t\,\mathbf{j} + c\mathbf{k}$ so

$$\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\| = \frac{-a\omega\sin\omega t\,\mathbf{i} + a\omega\cos\omega t\,\mathbf{j} + c\mathbf{k}}{\sqrt{a^2\omega^2 + c^2}}$$

$\mathbf{T}'(t) = \frac{-a\omega^2\cos\omega t\,\mathbf{i} - a\omega^2\sin\omega t\,\mathbf{j}}{\sqrt{a\omega^2 + c^2}}$ which points directly to the z-axis. Therefore $\mathbf{N}(t) = \mathbf{T}'(t)/\|\mathbf{T}'(t)\|$ points directly to the z-axis.

44. False: Suppose $\mathbf{v}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, then $\|\mathbf{v}(t)\| = 1$ but $\mathbf{a}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$ which is non-zero.

45. True: **T** depends only upon the shape of the curve, hence **N** and **B** also.

46. True: If **v** is perpendicular to **a**, then **T** is also perpendicular to **a**, so $\frac{d}{dt}\left(\frac{ds}{dt}\right) = a_T = \mathbf{T}\cdot\mathbf{a} = 0$. Thus speed $= \frac{ds}{dt}$ is a constant.

47. False: If $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}$ then **v** is perpendicular to **a**, but the path of motion is a circular helix, not a circle.

48. False: The circular helix (see Problem 27) has constant curvature.

49. True: The curves are identical, although the motion of an object moving along the curves would be different.

50. False: At any time $0 < t < 1$, $\mathbf{r}_1(t) \ne \mathbf{r}_2(t)$

51. True: The parameterization affects only the rate at which the curve is traced out.

52. True: If a curve lies in a plane, then **T** and **N** will lie in the plane, so **T** × **N** = **B** will be a unit vector normal to the plane.

53. False: For $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\,\mathbf{j}$, $\|\mathbf{r}(t)\| = 1$, but $\mathbf{r}'(t) \ne \mathbf{0}$.

54. True: The plane passes through the origin so its intersection with the sphere is a great circle. The radius of the circle is 1, so is curvature is $\frac{1}{1} = 1$.

55. False: The graph of $\rho = 0$ is the origin.

56. False: It is a parabolic cylinder.

57. False: The origin, $\rho = 0$, has infinitely many spherical coordinates, since any value of θ and ϕ can be used.

Sample Test Problems

1. The center of the sphere is the midpoint $\left(\frac{-2+4}{2}, \frac{3+1}{2}, \frac{3+5}{2}\right) = (1, 2, 4)$ of the diameter. The radius is $r = \sqrt{(1+2)^2 + (2-3)^2 + (4-3)^2}$ $= \sqrt{9+1+1} = \sqrt{11}$. The equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-4)^2 = 11$

2. $(x^2 - 6x + 9) + (y^2 + 2y + 1) + (z^2 - 8z + 16)$ $= 9 + 1 + 16;\ (x-3)^2 + (y+1)^2 + (z-4)^2 = 26$
Center: (3, –1, 4); radius: $\sqrt{26}$

3. **a.** $3\langle 2, -5\rangle - 2\langle 1, 1\rangle = \langle 6, -15\rangle - \langle 2, 2\rangle = \langle 4, -17\rangle$

b. $\langle 2, -5\rangle \cdot \langle 1, 1\rangle = 2 + (-5) = -3$

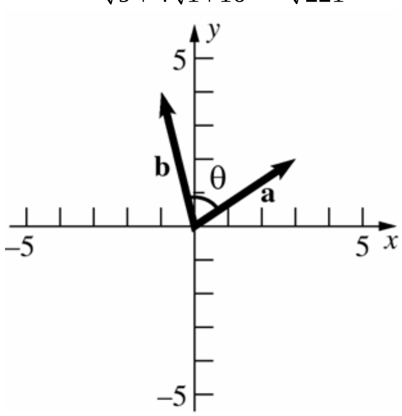
c. $\langle 2, -5\rangle \cdot \left(\langle 1, 1\rangle + \langle -6, 0\rangle\right) = \langle 2, -5\rangle \cdot \langle -5, 1\rangle$ $= -10 + (-5) = -15$

d. $\left(4\langle 2, -5\rangle + 5\langle 1, 1\rangle\right) \cdot 3\langle -6, 0\rangle$ $= \langle 13, -15\rangle \cdot \langle -18, 0\rangle = -234 + 0 = -234$

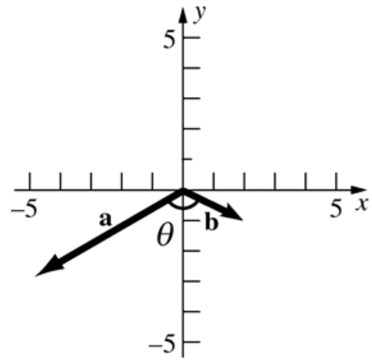
e. $\sqrt{36+0}\,\langle -6, 0\rangle \cdot \langle 1, -1\rangle = 6(-6+0) = -36$

f. $\langle -6, 0\rangle \cdot \langle -6, 0\rangle - \sqrt{36+0}$ $= (36+0) - 6 = 30$

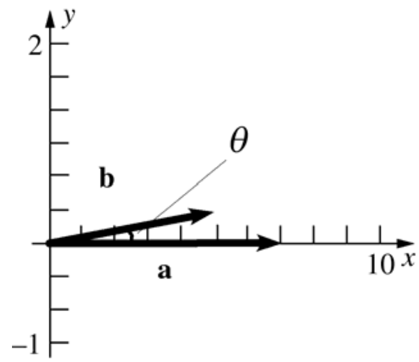
4. **a.** $\cos\theta = \frac{(3)(-1) + (2)(4)}{\sqrt{9+4}\sqrt{1+16}} = \frac{5}{\sqrt{221}} \approx 0.3363$



b. $\cos\theta = \frac{(-5)(2) + (-3)(-1)}{\sqrt{25+9}\sqrt{4+1}} = -\frac{7}{\sqrt{170}}$ ≈ -0.5369



c. $\cos\theta = \frac{(7)(5) + (0)(1)}{\sqrt{49+0}\sqrt{25+1}} = \frac{5}{\sqrt{26}} \approx 0.9806$



5. **a.** $\mathbf{a} + \mathbf{b} + \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

b. $\mathbf{b} \bullet \mathbf{c} = (0)(3) + (1)(-1) + (-2)(4) = -9$

c. $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k})$
$= -2 - 6 - 6 = -14$

d. $\mathbf{b} \cdot \mathbf{c}$ is a scalar, and $\mathbf{a}$ crossed with a scalar doesn't exist.

e. $\|\mathbf{a} - \mathbf{b}\| = \|-\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}\|$
$= \sqrt{1^2 + 0^2 + 4^2} = \sqrt{17}$

f. From part (c), $\mathbf{b} \times \mathbf{c} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ so $\|\mathbf{b} \times \mathbf{c}\| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7$.

6. a. $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{0+5-3}{(3\sqrt{3})(\sqrt{10})}$

≈ 0.121716

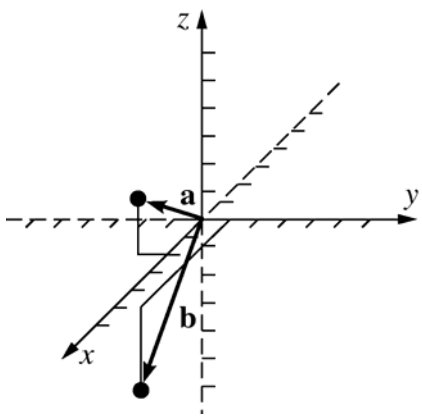
$\theta = \cos^{-1} 0.121716 \approx 83.009°$

b. $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{-1-0+6}{(\sqrt{5})(\sqrt{11})}$

≈ 0.67200

$\theta = \cos^{-1} 0.67200 \approx 47.608°$

7.



a. $\|\mathbf{a}\| = \sqrt{4+1+4} = 3;$

$\|\mathbf{b}\| = \sqrt{25+1+9} = \sqrt{35}$

b. $\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, direction cosines $\frac{2}{3}, -\frac{1}{3}$, and $\frac{2}{3}$.

$\frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{5}{\sqrt{35}}\mathbf{i} + \frac{1}{\sqrt{35}}\mathbf{j} - \frac{3}{\sqrt{35}}\mathbf{k}$, direction cosines $\frac{5}{\sqrt{35}}, \frac{1}{\sqrt{35}}, -\frac{3}{\sqrt{35}}$

c. $\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

d. $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{10-1-6}{3\sqrt{35}}$

$= \frac{3}{3\sqrt{35}} = \frac{1}{\sqrt{35}}$

$\theta = \cos^{-1}\frac{1}{\sqrt{35}} \approx 1.4010 \approx 80.27°$

8. a. $\langle -5,-5,5\rangle = -5\langle 1,1,-1\rangle$

b. $\langle 2,-1,1\rangle \times \langle 0,5,1\rangle = \langle 6,-2,10\rangle$

c. $\langle 2,-1,1\rangle \cdot \langle -7,1,-5\rangle = -20$

d. $\langle 2,-1,1\rangle \times \langle -7,1,-5\rangle = \langle 4,3,-5\rangle$

9. $c\langle 3,3,-1\rangle \times \langle -1,-2,4\rangle = c\langle 10,-11,-3\rangle$ for any c in $\mathbb{R}$.

10. Two vectors determined by the points are $\langle -1,7,-3\rangle$ and $\langle 3,-1,-3\rangle$. Then $\langle -1,7,-3\rangle \times \langle 3,-1,-3\rangle = -4\langle 6,3,5\rangle$ and $\langle 6,3,5\rangle$ are normal to the plane. $\frac{\pm\langle 6,3,5\rangle}{\sqrt{36+9+25}} = \frac{\pm\langle 6,3,5\rangle}{\sqrt{70}}$ are the unit vectors normal to the plane.

11. a. $y = 7$, since y must be a constant.

b. $x = -5$, since it is parallel to the yz-plane.

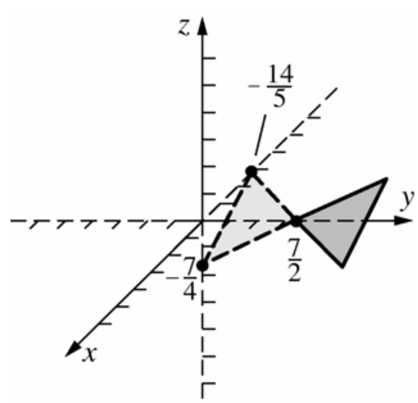
c. $z = -2$, since it is parallel to the xy-plane.

d. $3x - 4y + z = -45$, since it can be expressed as $3x - 4y + z = D$ and D must satisfy $3(-5) - 4(7) + (-2) = D$, so $D = -45$.

12. a. $\langle 4+1, 1-5, 1+7\rangle = \langle 5,-4,8\rangle$ is along the line, hence normal to the plane, which has equation $\langle x-2,\ y+4,\ z+5\rangle\cdot\langle 5,-4,8\rangle = 0$.

b. $5(x-2) - 4(y+4) + 8(z+5) = 0$ or $5x - 4y + 8z = -14$

c.



13. If the planes are perpendicular, their normals will also be perpendicular. Thus
$0 = \langle 1, 5, C\rangle \cdot \langle 4, -1, 1\rangle = 4 - 5 + C$,
so $C = 1$.

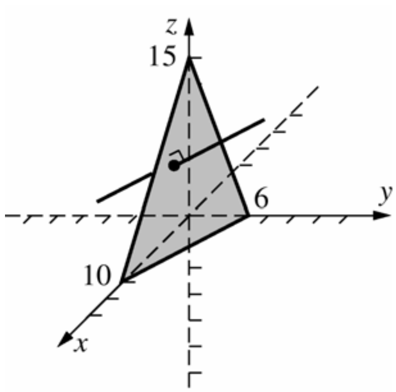
14. Two vectors in the same plane are $\langle 3,-2,-3\rangle$ and $\langle 3, 7, 0\rangle$. Their cross product, $3\langle 7,-3,9\rangle$, is normal to the plane. An equation of the plane is
$7(x-2) - 3(y-3) + 9(z+1) = 0$ or
$7x - 3y + 9z = -4$.

15. A vector in the direction of the line is $\langle 8,1,-8\rangle$.
Parametric equations are
$x = -2 + 8t,\ y = 1 + t,\ z = 5 - 8t$.

16. In the *yz*-plane, $x = 0$. Solve $-2y + 4z = 14$ and $2y - 5z = -30$, obtaining $y = 25$ and $z = 16$.
In the *xz*-plane, $y = 0$. Solve $x + 4z = 14$ and $-x - 5z = -30$, obtaining $x = -50$ and $z = 16$. Therefore, the points are
(0, 25, 16) and (–50, 0, 16).

17. (0, 25, 16) and (–50, 0, 16) are on the line, so $\langle 50, 25, 0\rangle = 25\langle 2, 1, 0\rangle$ is in the direction of the line. Parametric equations are $x = 0 + 2t$, $y = 25 + 1t$, $z = 16 + 0t$ or $x = 2t$, $y = 25 + t$, $z = 16$.

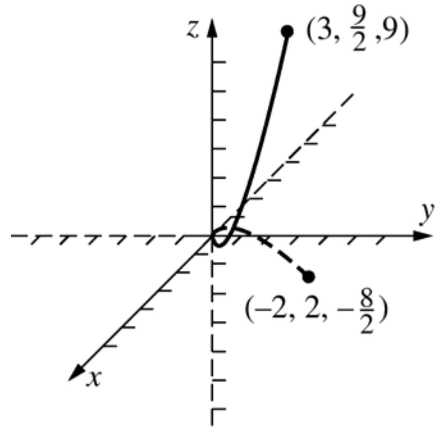
18. $\langle 3, 5, 2\rangle$ is normal to the plane, so is in the direction of the line. Symmetric equations of the line are $\frac{x-4}{3} = \frac{y-5}{5} = \frac{z-8}{2}$.



19. $\langle 5,-4,-3\rangle$ is a vector in the direction of the line, and $\langle 2,-2,1\rangle$ is a position vector to the line. Then a vector equation of the line is $\mathbf{r}(t)$
$= \langle 2,-2,1\rangle + t\langle 5,-4,-3\rangle$.

20.

t	x	y	z
–2	–2	2	–8/3
–1	–1	1/2	–1/3
0	0	0	0
1	1	1/2	1/3
2	2	2	8/3
3	3	9/2	9



21. $\mathbf{r}'(t) = \langle 1, t, t^2\rangle, \mathbf{r}'(2) = \langle 1, 2, 4\rangle$ and
$\mathbf{r}(2) = \left\langle 2, 2, \frac{8}{3}\right\rangle$. Symmetric equations for the tangent line are $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z-\frac{8}{3}}{4}$. Normal plane is $1(x-2) + 2(y-2) + 4\left(z - \frac{8}{3}\right) = 0$ or
$3x + 6y + 12z = 50$.

22. $\mathbf{r}(t) = \langle t \cos t, t \sin t, 2t \rangle$;

$\mathbf{r}'(t) = \langle -t\sin t + \cos t,\ t\cos t + \sin t,\ 2 \rangle$;

$\mathbf{r}''(t) = \langle -t\cos t - 2\sin t, -t\sin t + 2\cos t,\ 0 \rangle$

$\mathbf{r}'\left(\frac{\pi,}{2}\right) = \left\langle -\frac{\pi}{2}, 1, 2 \right\rangle; \mathbf{r}''\left(\frac{\pi}{2}\right) = \left\langle -2, -\frac{\pi}{2}, 0 \right\rangle$

$\left|\mathbf{r}'\left(\frac{\pi}{2}\right)\right| = \frac{\sqrt{\pi^2 + 20}}{2}$;

$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{\mathbf{r}'\left(\frac{\pi}{2}\right)}{\left\|\mathbf{r}'\left(\frac{\pi}{2}\right)\right\|} = \frac{\langle -\pi, 2, 4 \rangle}{\sqrt{\pi^2 + 20}}$

23. $\mathbf{r}'(t) = e^t \langle \cos t + \sin t, -\sin t + \cos t, 1 \rangle$

$\|\mathbf{r}'(t)\| = \sqrt{3}e^t$

Length is $\int_1^5 \sqrt{3}e^t dt = \left[\sqrt{3}e^t\right]_1^5$

$= \sqrt{3}(e^5 - e) \approx 252.3509.$

24. $-(\mathbf{F}_1 + \mathbf{F}_2) = -5\mathbf{i} - 9\mathbf{j}$

25. Let the wind vector be
$\mathbf{w} = \langle 100\cos 30°, 100\sin 30° \rangle$
$= \langle 50\sqrt{3}, 50 \rangle$.

Let $\mathbf{p} = \langle p_1, p_2 \rangle$ be the plane's air velocity vector.

We want $\mathbf{w} + \mathbf{p} = 450\,\mathbf{j} = \langle 0, 450 \rangle$.

$\langle 50\sqrt{3}, 50 \rangle + \langle p_1, p_2 \rangle = \langle 0, 450 \rangle$

$\Rightarrow 50\sqrt{3} + p_1 = 0, 50 + p_2 = 450$

$\Rightarrow p_1 = -50\sqrt{3}, p_2 = 400$

Therefore, $\mathbf{p} = \langle -50\sqrt{3}, 400 \rangle$. The angle β formed with the vertical satisfies

$\cos\beta = \frac{\mathbf{p}\cdot\mathbf{j}}{\|\mathbf{p}\|\|\mathbf{j}\|} = \frac{400}{\sqrt{167,500}}$; $\beta \approx 12.22°$. Thus, the heading is N12.22°W. The air speed is $\|\mathbf{p}\| = \sqrt{167,500} \approx 409.27$ mi/h.

26. $\mathbf{r}(t) = \langle e^{2t}, e^{-t} \rangle; \mathbf{r}'(t) = \langle 2e^{2t}, -e^{-t} \rangle$

a. $\lim_{t\to 0} \langle e^{2t}, e^{-t} \rangle = \left\langle \lim_{t\to 0} e^{2t}, \lim_{t\to 0} e^{-t} \right\rangle = \langle 1, 1 \rangle$

b. $\lim_{h\to 0} \frac{\mathbf{r}(0+h) - \mathbf{r}(0)}{h} = \mathbf{r}'(0) = \langle 2, -1 \rangle$

c. $\int_0^{\ln 2} \langle e^{2t}, e^{-t} \rangle dt$

$= \left[\left\langle \left(\frac{1}{2}\right)e^{2t}, -e^{-t} \right\rangle\right]_0^{\ln 2}$

$= \left\langle 2, -\frac{1}{2} \right\rangle - \left\langle \frac{1}{2}, -1 \right\rangle = \left\langle \frac{3}{2}, \frac{1}{2} \right\rangle$

d. $D_t[t\mathbf{r}(t)] = t\mathbf{r}'(t) + \mathbf{r}(t)$

$= t\langle 2e^{2t}, -e^{-t} \rangle + \langle e^{2t}, e^{-t} \rangle$

$= \langle e^{2t}(2t+1), e^{-t}(1-t) \rangle$

e. $D_t[\mathbf{r}(3t+10)] = [\mathbf{r}'(3t+10)](3)$

$= 3\langle 2e^{6t+20}, -e^{-3t-10} \rangle$

$= \langle 6e^{6t+20}, -3e^{-3t-10} \rangle$

f. $D_t[\mathbf{r}(t)\cdot\mathbf{r}'(t)] = D_t[2e^{4t} - e^{-2t}]$

$= 8e^{4t} + 2e^{-2t}$

27. a. $\mathbf{r}'(t) = \left\langle \frac{1}{t}, -6t \right\rangle; \mathbf{r}''(t) = \langle -t^{-2}, -6 \rangle$

b. $\mathbf{r}'(t) = \langle \cos t, -2\sin 2t \rangle$;

$\mathbf{r}''(t) = \langle -\sin t, -4\cos 2t \rangle$

c. $\mathbf{r}'(t) = \langle \sec^2 t, -4t^3 \rangle$;

$\mathbf{r}''(t) = \langle 2\sec^2 t \tan t, -12t^2 \rangle$

28. $\mathbf{v}(t) = \langle e^t, -e^{-t}, 2 \rangle$

$\mathbf{a}(t) = \langle e^t, e^{-t}, 0 \rangle$; $\mathbf{v}(\ln 2) = \left\langle 2, -\frac{1}{2}, 2 \right\rangle$

$\mathbf{a}(\ln 2) = \left\langle 2, \frac{1}{2}, 0 \right\rangle$

$\kappa(\ln 2) = \frac{|\mathbf{v}(\ln 2)\times\mathbf{a}(\ln 2)|}{|\mathbf{v}(\ln 2)|^3} = \frac{|\langle -1, 4, 2 \rangle|}{\left(\sqrt{\frac{33}{4}}\right)^3}$

$= \frac{8}{33^{3/2}}\sqrt{21} = 8\sqrt{\frac{21}{35937}} = \frac{8\sqrt{7}}{\sqrt{11979}}$

≈ 0.1934

29. $\mathbf{v}(t) = \left\langle 1, 2t, 3t^2 \right\rangle; \mathbf{a}(t) = \langle 0, 2, 6t \rangle$

$\mathbf{v}(1) = \langle 1, 2, 3 \rangle$

$\|\mathbf{v}(1)\| = \sqrt{14}$

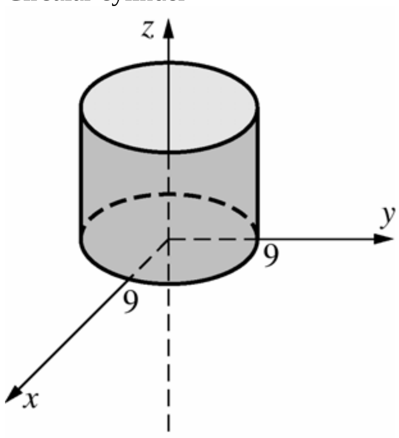
$\mathbf{a}(1) = \langle 0, 2, 6 \rangle$

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{0+4+18}{\sqrt{14}} = \frac{22}{\sqrt{14}} \approx 5.880;$$

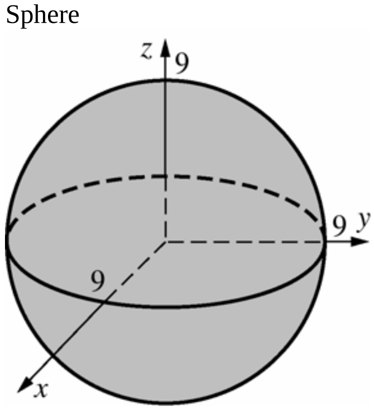
$$a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{|\langle 6, -6, 2 \rangle|}{\sqrt{14}}$$

$$= \frac{2\sqrt{19}}{\sqrt{14}} \approx 2.330$$

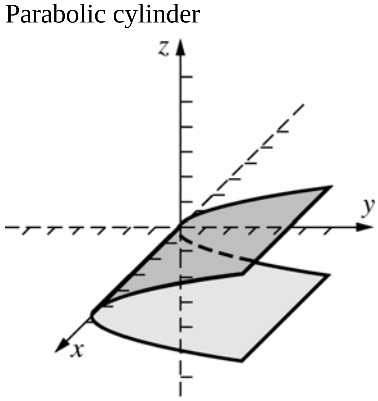
30. Circular cylinder



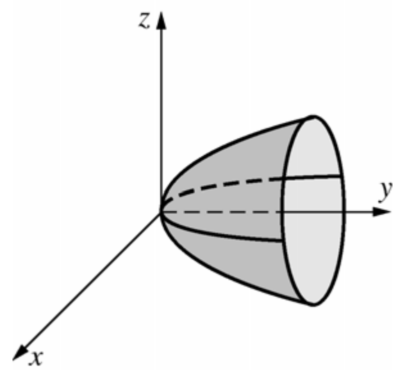
31. Sphere



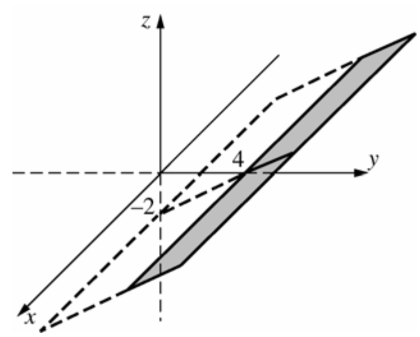
32. Parabolic cylinder



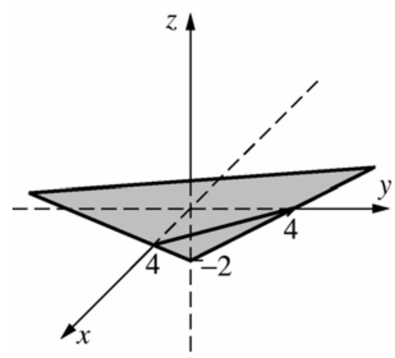
33. Circular paraboloid



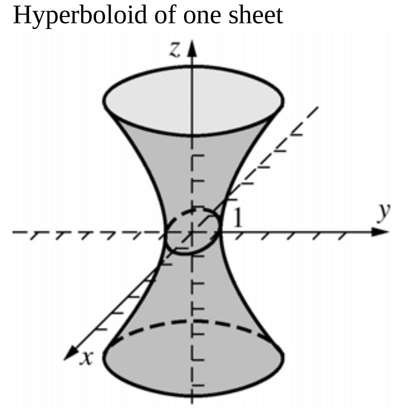
34. Plane



35. Plane

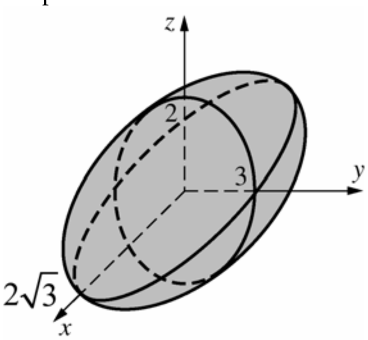


36. Hyperboloid of one sheet



37. $\frac{x^2}{12}+\frac{y^2}{9}+\frac{z^2}{4}=1$

Ellipsoid



38. The graph of $3x^2+4y^2+9z^2=-36$ is the empty set.

39. **a.** $r^2=9;\ r=3$

b. $(x^2+y^2)+3y^2=16$

$r^2+3r^2\sin^2\theta=16, r^2=\frac{16}{1+3\sin^2\theta}$

c. $r^2=9z$

d. $r^2+4z^2=10$

40. **a.** $x^2+y^2+z^2=9$

b. $x^2+z^2=4$

c. $r^2(\cos^2\theta-\sin^2\theta)+z^2=1;$

$x^2-y^2+z^2=1$

41. **a.** $\rho^2=4; \rho=2$

b. $x^2+y^2+z^2-2z^2=0;$

$\rho^2-2\rho^2\cos^2\phi=0;\quad \rho^2(1-2\cos^2\phi)=0;$

$1-2\cos^2\phi=0;\ \ \cos^2\phi=\frac{1}{2};\ \ \phi=\frac{\pi}{4}$ or

$\phi=\frac{3\pi}{4}.$

Any of the following (as well as others) would be acceptable:

$\left(\phi-\frac{\pi}{4}\right)\left(\phi-\frac{3\pi}{4}\right)=0$

$\cos^2\phi=\frac{1}{2}$

$\sec^2\phi=2$

$\tan^2\phi=1$

c. $2x^2-(x^2+y^2+z^2)=1;$

$2\rho^2\sin^2\phi\cos^2\theta-\rho^2=1;$

$\rho^2=\frac{1}{2\sin^2\phi\cos^2\theta-1}$

d. $x^2+y^2=z;$

$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta=\rho\cos\phi$

$\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)=\rho\cos\phi;$

$\rho\sin^2\phi=\cos\phi;\ \ \rho=\cot\phi\csc\phi$

(Note that when we divided through by ρ in part c and d we did not lose the pole since it is also a solution of the resulting equations.)

42. Cartesian coordinates are $\left(2\sqrt{2},\ 2\sqrt{2},\ 4\sqrt{3}\right)$ and $\left(\sqrt{2},\ \sqrt{6},\ -2\sqrt{2}\right)$. Distance

$\left[2+\left(2\sqrt{2}-\sqrt{6}\right)^2+\left(4\sqrt{3}+2\sqrt{2}\right)^2\right]^{1/2}\approx 9.8659.$

43. (2, 0, 0) is a point of the first plane. The distance between the planes is

$\frac{\left|2(2)-3(0)+\sqrt{3}(0)-9\right|}{\sqrt{4+9+3}}=\frac{5}{\sqrt{16}}=1.25$

44. $\langle 2,-4,1\rangle$ and $\langle 3,2,-5\rangle$ are normal to the respective planes. The acute angle between the two planes is the same as the acute angle θ between the normal vectors.

$\cos\theta=\frac{|6-8-5|}{\sqrt{21}\sqrt{38}}=\frac{7}{\sqrt{798}},$

so $\theta\approx$ 1.3204 rad $\approx$ 75.65°

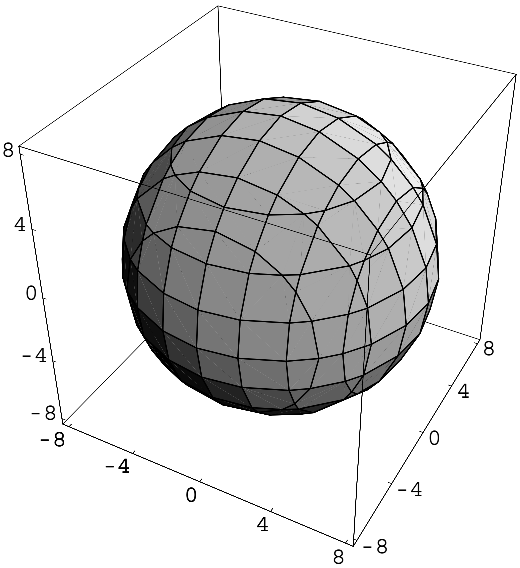
45. If speed $=\frac{ds}{dt}=c,$ a constant, then

$\mathbf{a}=\frac{d^2s}{dt^2}\mathbf{T}+\left(\frac{ds}{dt}\right)^2\kappa\mathbf{N}=c^2\kappa\mathbf{N}$ since $\frac{d^2s}{dt^2}=0.$

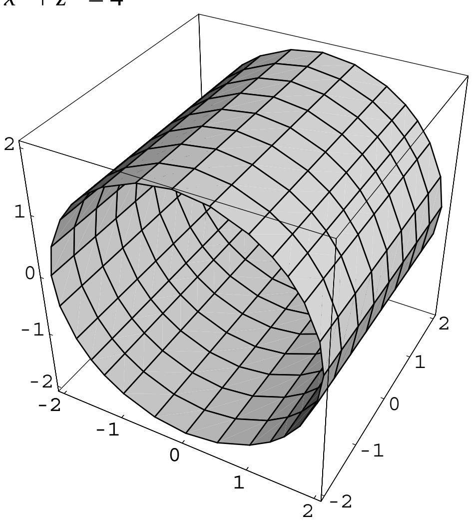
T is in the direction of **v**, while **N** is perpendicular to **T** and hence to **v** also. Thus, $\mathbf{a}=c^2\kappa\mathbf{N}$ is perpendicular to **v**.

Review and Preview

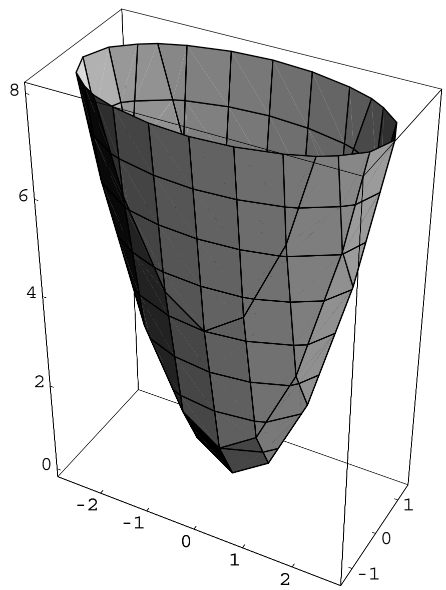
1. $x^2 + y^2 + z^2 = 64$



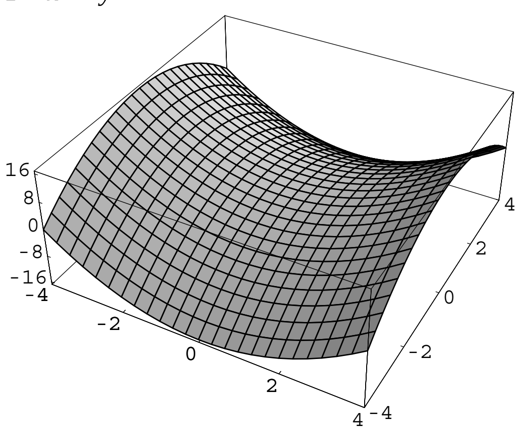
2. $x^2 + z^2 = 4$



3. $z = x^2 + 4y^2$



4. $z = x^2 - y^2$



5. **a.** $\frac{d}{dx}2x^3 = 6x^2$

b. $\frac{d}{dx}5x^3 = 15x^2$

c. $\frac{d}{dx}kx^3 = 3kx^2$

d. $\frac{d}{dx}ax^3 = 3ax^2$

6. **a.** $\frac{d}{dx}\sin 2x = 2\cos 2x$

b. $\frac{d}{dt}\sin 17t = 17\cos 17t$

c. $\frac{d}{dt}\sin at = a\cos at$

d. $\frac{d}{dt}\sin bt = b\cos bt$

7. a. $\frac{d}{da}\sin 2a = 2\cos 2a$

b. $\frac{d}{da}\sin 17a = 17\cos 17a$

c. $\frac{d}{da}\sin ta = t\cos ta$

d. $\frac{d}{da}\sin sa = s\cos sa$

8. a. $\frac{d}{dt}e^{4t+1} = 4e^{4t+1}$

b. $\frac{d}{dx}e^{-7x+4} = -7e^{-7x+4}$

c. $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$

d. $\frac{d}{dx}e^{tx+s} = te^{tx+s}$

9. $f(x) = \frac{1}{x^2-1}$ is both continuous and differentiable at $x = 2$ since rational functions are continuous and differentiable at every real number in their domain.

10. $f(x) = \tan x$ is not continuous at $x = \pi/2$ since $f(\pi/2)$ is undefined.

11. $f(x) = |x-4|$ is continuous at $x = 4$ since $\lim_{x\to 4}|x-4| = f(4) = 0.$ f is not differentiable at $x = 4$.

12. As x approaches 0 from the right, $1/x$ approaches $+\infty$, so $\sin(1/x)$ oscillates between -1 and 1 and $\lim_{x\to 0} f(x)$ does not exist. Since the limit at $x = 0$ does not exist, f is not continuous at $x = 0$. Consequently, f is not differentiable at $x = 0$.

13. $f'(x) = 3-3(x-1)^2$; $f'(x) = 0$ when $x = 0, 2$; $f''(x) = -6(x-1)$; Since $f''(2) = -6$, a local maximum occurs at $x = 2$. Since $f(0) = -1$, $f(2) = 5$, and $f(4) = -15$, the maximum value of f on [0, 4] is 5 while the minimum value is -15.

14. $f'(x) = 4x^3 - 54x^2 + 226x - 288$

$= 2(2x-9)(x^2-9x+16)$

$f'(x) = 0$ when $x = \frac{9}{2}$ or $x = \frac{9\pm\sqrt{17}}{2}$ (using the quadratic formula).

In $[2,6]$, $f'(x) = 0$ when $x = \frac{9-\sqrt{17}}{2} \approx 2.438$ or $x = \frac{9}{2} = 4.5$. $f''(x) = 12x^2 - 108x + 226$. Since $f''(2.438) > 0$, a local minimum occurs at $x = \frac{9-\sqrt{17}}{2}$. Since $f''(4.5) < 0$, a local maximum occurs at $x = 4.5$.

$f(2) = f(6) = 0$, $f\left(\frac{9-\sqrt{17}}{2}\right) = -4$, and $f(4.5) = 14.0625$. Thus, the minimum value of f on $[2,6]$ is -4 and the maximum value of f on $[2,6]$ is 14.0625.

15. $S = 2\pi r^2 + 2\pi rh$

Since the volume is to be 8 cubic feet, we have

$V = 8$

$\pi r^2 h = 8$

$h = \frac{8}{\pi r^2}$

Substituting for h in our surface area equation gives us

$S = 2\pi r^2 + 2\pi r\left(\frac{8}{\pi r^2}\right) = 2\pi r^2 + \frac{16}{r}$

Thus, we can write S as a function of r:

$S(r) = 2\pi r^2 + \frac{16}{r}$

16. The area of two of the sides of the box will be $l\cdot h$. Two other sides will have area $w\cdot h$. The area of the base is $l\cdot w$. Thus, the total cost, C, of the box will be $C = 2lh + 2wh + 3lw$, where C is in dollars. Since $h = \frac{27}{lw}$, $C = \frac{54}{w} + \frac{54}{l} + 3lw$.