CHAPTER 12

Derivatives for Functions of Two or More Variables

12.1 Concepts Review

1. real-valued function of two real variables

2. level curve; contour map

3. concentric circles

4. parallel lines

Problem Set 12.1

1. **a.** 5

   b. 0

   c. 6

   d. $a^6 + a^2$

   e. $2x^2,\ \ x \neq 0$

   f. Undefined

   The natural domain is the set of all (x, y) such that y is nonnegative.

2. **a.** 4

   b. 17

   c. $\frac{17}{16}$

   d. $1 + a^2, a \neq 0$

   e. $x^3 + x, x \neq 0$

   f. Undefined

   The natural domain is the set of all (x, y) such that x is nonzero.

3. **a.** $\sin(2\pi) = 0$

   b. $4\sin\left(\frac{\pi}{6}\right) = 2$

   c. $16\sin\left(\frac{\pi}{2}\right) = 16$

   d. $\pi^2 \sin(\pi^2) \approx -4.2469$

4. **a.** 6

   b. 12

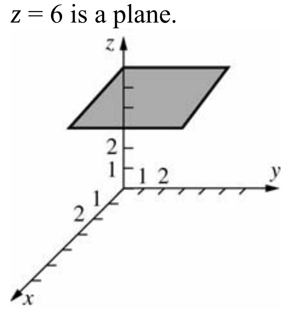
   c. 2

   d. $(3\cos 6)^{1/2} + 1.44 \approx 3.1372$

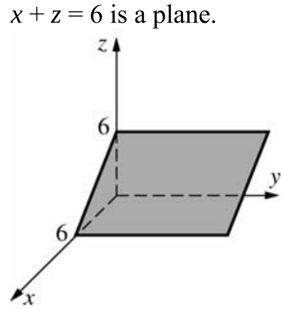
5. $F(t\cos t,\ \sec^2 t) = t^2 \cos^2 t \sec^2 t = t^2,\ \ \cos\ t \neq 0$

6. $F(f(t),\ g(t)) = F(\ln t^2,\ e^{t/2})$
   $= \exp(\ln t^2) + (e^{t/2})^2 = t^2 + e^t, t \neq 0$

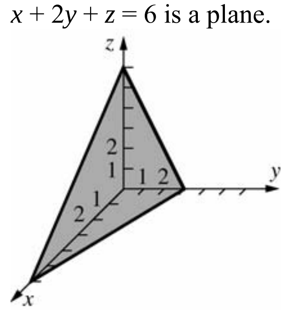
7. $z = 6$ is a plane.



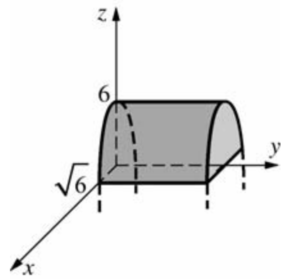
8. $x + z = 6$ is a plane.



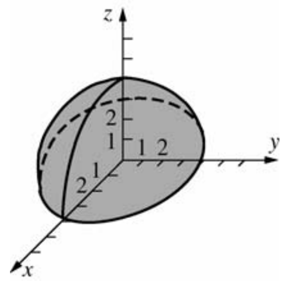
9. $x + 2y + z = 6$ is a plane.



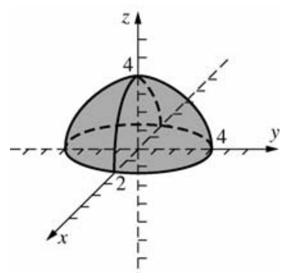
10. $z = 6 - x^2$ is a parabolic cylinder.



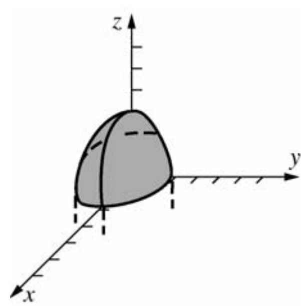
11. $x^2 + y^2 + z^2 = 16$, $z \ge 0$ is a hemisphere.



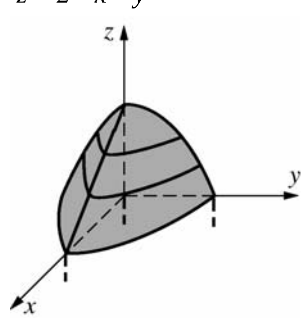
12. $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{16} = 1$, $z \ge 0$ is a hemi-ellipsoid.



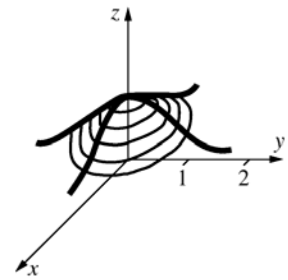
13. $z = 3 - x^2 - y^2$ is a paraboloid.



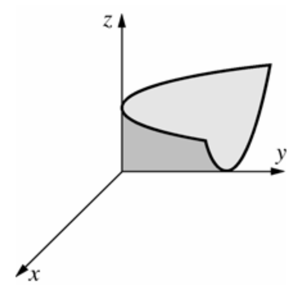
14. $z = 2 - x - y^2$



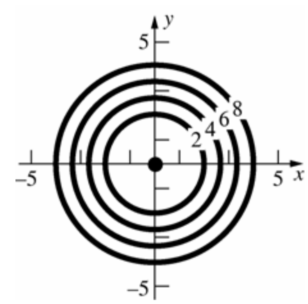
15. $z = \exp[-(x^2 + y^2)]$



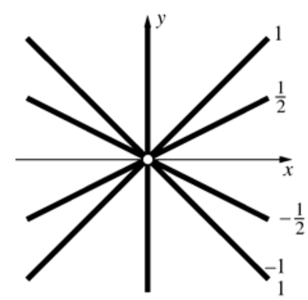
16. $z = \dfrac{x^2}{y}$, $y > 0$



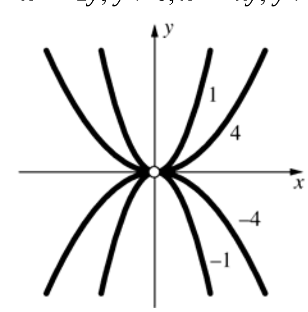
17. $x^2 + y^2 = 2z; x^2 + y^2 = 2k$



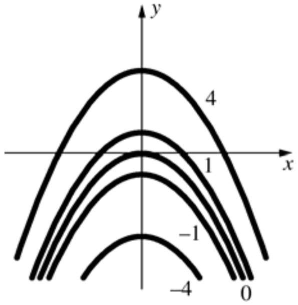
18. $x = zy, y \ne 0$; $x = ky,\ y \ne 0$



19. $x^2 = zy, y \ne 0; x^2 = ky, y \ne 0$



20. $x^2 = -(y - z); x^2 = -(y - k)$



21. $z = \dfrac{x^2 + 1}{x^2 + y^2}, k = 1, 2, 4$

$k = 1$: $y^2 = 1$ or $y = \pm 1$;
two parallel lines

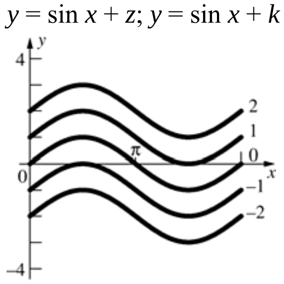
$k = 2$: $2x^2 + 2y^2 = x^2 + 1$

$\dfrac{x^2}{1} + \dfrac{y^2}{\frac{1}{2}} = 1$; ellipse

$k = 4$: $4x^2 + 4y^2 = x^2 + 1$

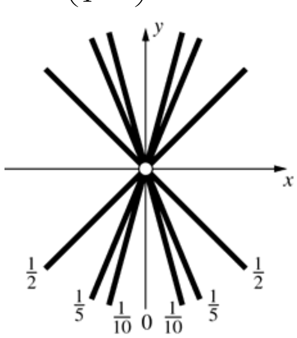
$\dfrac{x^2}{\frac{1}{3}} + \dfrac{y^2}{\frac{1}{4}} = 1$; ellipse

22. $y = \sin x + z; y = \sin x + k$

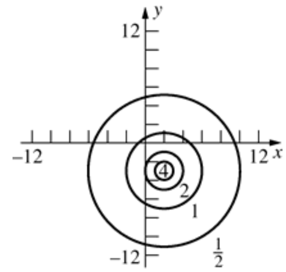


23. $x = 0$, if $T = 0$:

$y^2 = \left(\dfrac{1}{T} - 1\right)x^2$, if $y \neq 0$.



24. $(x - 2)^2 + (y + 3)^2 = \dfrac{16}{V^2}$



25. **a.** San Francisco and St. Louis had a temperature between 70 and 80 degrees Fahrenheit.

b. Drive northwest to get to cooler temperatures, and drive southeast to get warmer temperatures.

c. Since the level curve for 70 runs southwest to northeast, you could drive southwest or northeast and stay at about the same temperature.

26. **a.** The lowest barometric pressure, 1000 millibars and under, occurred in the region of the Great Lakes, specifically near Wisconsin. The highest barometric pressure, 1025 millibars and over, occurred on the east coast, from Massachusetts to South Carolina.

b. Driving northwest would take you to lower barometric pressure, and driving southeast would take you to higher barometric pressure.

c. Since near St. Louis the level curves run southwest to northeast, you could drive southwest or northeast and stay at about the same barometric pressure.

27. $x^2 + y^2 + z^2 \geq 16$; the set of all points on and outside the sphere of radius 4 that is centered at the origin

28. The set of all points inside (the part containing the z-axis) and on the hyperboloid of one sheet;
$\dfrac{x^2}{9} + \dfrac{y^2}{9} - \dfrac{z^2}{9} = 1.$

29. $\dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{1} \leq 1$; points inside and on the ellipsoid

30. Points inside (the part containing the z-axis) or on the hyperboloid of one sheet, $\frac{x^2}{9}+\frac{y^2}{9}-\frac{z^2}{16}=1$, excluding points on the coordinate planes

31. Since the argument to the natural logarithm function must be positive, we must have $x^2+y^2+z^2>0$. This is true for all (x,y,z) except $(x,y,z)=(0,0,0)$. The domain consists all points in $\mathbb{R}^3$ except the origin.

32. Since the argument to the natural logarithm function must be positive, we must have $xy>0$. This occurs when the ordered pair (x,y) is in the first quadrant or the third quadrant of the xy-plane. There is no restriction on z. Thus, the domain consists of all points (x,y,z) such that x and y are both positive or both negative.

33. $x^2+y^2+z^2=k,\ k>0$; set of all spheres centered at the origin

34. $100x^2+16y^2+25z^2=k,\ k>0$;
$\frac{x^2}{\frac{k}{100}}+\frac{y^2}{\frac{k}{16}}+\frac{z^2}{\frac{k}{25}}=1$; set of all ellipsoids centered at origin such that their axes have ratio $\left(\frac{1}{10}\right):\left(\frac{1}{4}\right):\left(\frac{1}{5}\right)$ or 2:5:4.

35. $\frac{x^2}{\frac{1}{16}}+\frac{y^2}{\frac{1}{4}}-\frac{z^2}{1}=k$; the elliptic cone
$\frac{x^2}{9}+\frac{y^2}{9}=\frac{z^2}{16}$ and all hyperboloids (one and two sheets) with z-axis for axis such that a:b:c is $\left(\frac{1}{4}\right):\left(\frac{1}{4}\right):\left(\frac{1}{3}\right)$ or 3:3:4.

36. $\frac{x^2}{\frac{1}{9}}-\frac{y^2}{\frac{1}{4}}-\frac{z^2}{1}=k$; the elliptical cone
$\frac{y^2}{9}+\frac{z^2}{36}=\frac{x^2}{4}$ and all hyperboloids (one and two sheets) with x-axis for axis such that a:b:c is $\left(\frac{1}{3}\right):\left(\frac{1}{2}\right):1$ or 2:3:6

37. $4x^2-9y^2=k,\ k$ in R; $\frac{x^2}{\frac{k}{4}}-\frac{y^2}{\frac{k}{9}}=1$, if $k\neq 0$; planes $y=\pm\frac{2x}{3}$ (for $k=0$) and all hyperbolic cylinders parallel to the z-axis such that the ratio a:b is $\left(\frac{1}{2}\right):\left(\frac{1}{3}\right)$ or 3:2 (where a is associated with the x-term)

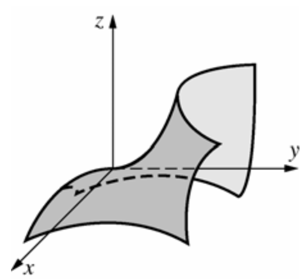
38. $e^{x^2+y^2+z^2}=k,\ k>0$
$x^2+y^2+z^2=\ln k$
concentric circles centered at the origin.

39. **a.** All (w,x,y,z) except $(0,0,0,0)$, which would cause division by 0.

b. All $(x_1,x_2,\ldots,x_n)$ in n-space.

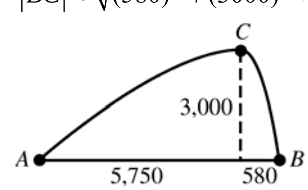
c. All $(x_1,x_2,\ldots,x_n)$ that satisfy $x_1^2+x_2^2+\cdots+x_n^2\leq 1$; other values of $(x_1,x_2,\ldots,x_n)$ would lead to the square root of a negative number.

40. If $z=0$, then $x=0$ or $x=\pm\sqrt{3}y$.



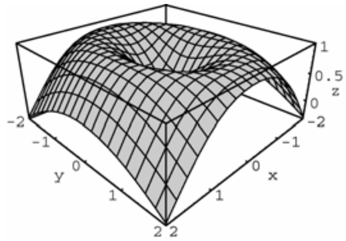
41. **a.** AC is the least steep path and BC is the most steep path between A and C since the level curves are farthest apart along AC and closest together along BC.

b. $|AC|\approx\sqrt{(5750)^2+(3000)^2}\approx 6490$ ft
$|BC|\approx\sqrt{(580)^2+(3000)^2}\approx 3060$ ft

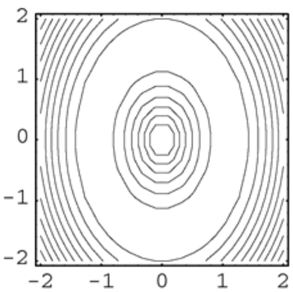


42. Completing the squares on x and y yields the equivalent equation
$f(x,y)+25.25=(x-0.5)^2+3(y+2)^2$, an elliptic paraboloid.

43.



$\sin\sqrt{2x^2 + y^2}$



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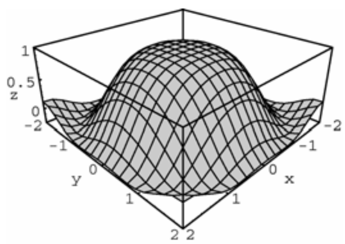
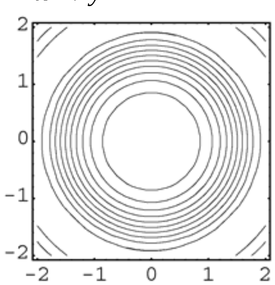


$\dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$



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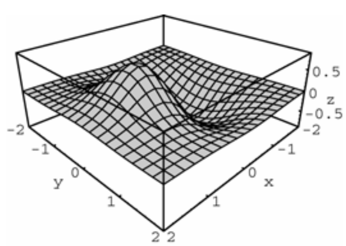
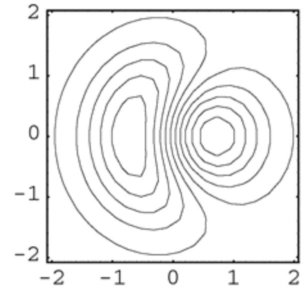


$(2x - y^2)\exp(-x^2 - y^2)$



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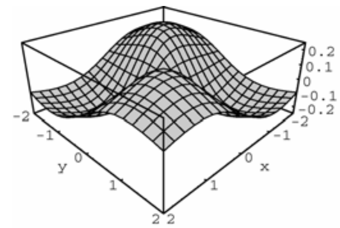
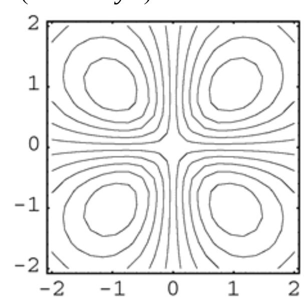


$\dfrac{\sin x \sin y}{(1 + x^2 + y^2)}$



12.2 Concepts Review

1. $\lim_{h \to 0} \dfrac{[(f(x_0 + h,\ y_0) - f(x_0,\ y_0)]}{h}$; partial derivative of f with respect to x

2. 5; 1

3. $\dfrac{\partial^2 f}{\partial y \partial x}$

4. 0

Problem Set 12.2

1. $f_x(x,\ y) = 8(2x - y)^3$; $f_y(x,\ y) = -4(2x - y)^3$

2. $f_x(x,\ y) = 6(4x - y^2)^{1/2}$;
$f_y(x,\ y) = -3y(4x - y^2)^{1/2}$

3. $f_x(x, y) = \frac{(xy)(2x) - (x^2 - y^2)(y)}{(xy)^2} = \frac{x^2 + y^2}{x^2 y}$

$f_y(x, y) = \frac{(xy)(-2y) - (x^2 - y^2)(x)}{(xy)^2}$

$= -\frac{(x^2 + y^2)}{xy^2}$

4. $f_x(x, y) = e^x \cos y$; $f_y(x, y) = -e^x \sin y$

5. $f_x(x, y) = e^y \cos x$; $f_y(x, y) = e^y \sin x$

6. $f_x(x, y) = \left(-\frac{1}{3}\right)(3x^2 + y^2)^{-4/3}(6x)$

$= -2x(3x^2 + y^2)^{-4/3}$;

$f_y(x, y) = \left(-\frac{1}{3}\right)(3x^2 + y^2)^{-4/3}(2y)$

$= \left(-\frac{2y}{3}\right)(3x^2 + y^2)^{-4/3}$

7. $f_x(x, y) = x(x^2 - y^2)^{-1/2}$;

$f_y(x, y) = -y(x^2 - y^2)^{-1/2}$

8. $f_u(u, v) = ve^{uv}$; $f_v(u, v) = ue^{uv}$

9. $g_x(x, y) = -ye^{-xy}$; $g_y(x, y) = -xe^{-xy}$

10. $f_s(s, t) = 2s(s^2 - t^2)^{-1}$;

$f_t(s, t) = -2t(s^2 - t^2)^{-1}$

11. $f_x(x, y) = 4[1 + (4x - 7y)^2]^{-1}$;

$f_y(x, y) = -7[1 + (4x - 7y)^2]^{-1}$

12. $F_w(w, z) = w\frac{1}{\sqrt{1 - \left(\frac{w}{z}\right)^2}}\left(\frac{1}{z}\right) + \sin^{-1}\left(\frac{w}{z}\right)$

$= \frac{\frac{w}{z}}{\sqrt{1 - \left(\frac{w}{z}\right)^2}} + \sin^{-1}\left(\frac{w}{z}\right)$;

$F_z = (w, z) = w\frac{1}{\sqrt{1 - \left(\frac{w}{z}\right)^2}}\left(-\frac{w}{z^2}\right) = \frac{-\left(\frac{w}{z}\right)^2}{\sqrt{1 - \left(\frac{w}{z}\right)^2}}$

13. $f_x(x, y) = -2xy\sin(x^2 + y^2)$;

$f_y(x, y) = -2y^2 \sin(x^2 + y^2) + \cos(x^2 + y^2)$

14. $f_s(s, t) = -2se^{t^2 - s^2}$; $f_s(s, t) = 2te^{t^2 - s^2}$

15. $F_x(x, y) = 2\cos x \cos y$; $F_y(x, y) = -2\sin x \sin y$

16. $f_r(r, \theta) = 9r^2 \cos 2\theta$; $f_\theta(r, \theta) = -6r^3 \sin 2\theta$

17. $f_x(x, y) = 4xy^3 - 3x^2 y^5$;

$f_{xy}(x, y) = 12xy^2 - 15x^2 y^4$

$f_y(x, y) = 6x^2 y^2 - 5x^3 y^4$;

$f_{yx}(x, y) = 12xy^2 - 15x^2 y^4$

18. $f_x(x, y) = 5(x^3 + y^2)^4(3x^2)$;

$f_{xy}(x, y) = 60x^2(x^3 + y^2)^3(2y)$

$= 120x^2 y(x^3 + y^2)^3$

$f_y(x, y) = 5(x^3 + y^2)^4(2y)$;

$f_{yx}(x, y) = 40y(x^3 + y^2)^3(3x^2)$

$= 120x^2 y(x^3 + y^2)^3$

19. $f_x(x, y) = 6e^{2x} \cos y$; $f_{xy}(x, y) = -6e^{2x} \sin y$

$f_y(x, y) = -3e^{2x} \sin y$; $f_{yx}(x, y) = -6e^{2x} \sin y$

20. $f_x(x, y) = y(1 + x^2 y^2)^{-1}$;

$f_{xy}(x, y) = (1 - x^2 y^2)(1 + x^2 y^2)^{-2}$

$f_x(x, y) = x(1 + x^2 y^2)^{-1}$;

$f_{xy}(x, y) = (1 - x^2 y^2)(1 + x^2 y^2)^{-2}$

21. $F_x(x, y) = \frac{(xy)(2) - (2x - y)(y)}{(xy)^2} = \frac{y^2}{x^2 y^2} = \frac{1}{x^2}$;

$F_x(3, -2) = \frac{1}{9}$

$F_y(x, y) = \frac{(xy)(-1) - (2x - y)(x)}{(xy)^2} = \frac{-2x^2}{x^2 y^2} = -\frac{2}{x^2}$;

$F_y(3, -2) = -\frac{1}{2}$

22. $F_x(x, y) = (2x + y)(x^2 + xy + y^2)^{-1}$;

$F_x(-1, 4) = \frac{2}{13} \approx 0.1538$

$F_y(x, y) = (x + 2y)(x^2 + xy + y^2)^{-1}$;

$F_y(-1, 4) = \frac{7}{13} \approx 0.5385$

23. $f_x(x, y) = -y^2(x^2 + y^4)^{-1}$;

$f_x\left(\sqrt{5}, -2\right) = -\frac{4}{21} \approx -0.1905$

$f_y(x, y) = 2xy(x^2 + y^4)^{-1}$;

$f_y\left(\sqrt{5}, -2\right) = -\frac{4\sqrt{5}}{21} \approx -0.4259$

24. $f_x(x, y) = e^y \sinh x$;

$f_x(-1, 1) = e\sinh(-1) \approx -3.1945$

$f_y(x, y) = e^y \cosh x$;

$f_y(-1, 1) = e\cosh(-1) \approx 4.1945$

25. Let $z = f(x, y) = \frac{x^2}{9} + \frac{y^2}{4}$.

$f_y(x, y) = \frac{y}{2}$

The slope is $f_y(3, 2) = 1$.

26. Let $z = f(x, y) = (1/3)(36 - 9x^2 - 4y^2)^{1/2}$.

$f_y(x, y) = \left(-\frac{4}{3}\right)y(36 - 9x^2 - y^2)^{-1/2}$

The slope is $f_y(1, -2) = \frac{8}{3\sqrt{11}} \approx 0.8040$.

27. $z = f(x, y) = \left(\frac{1}{2}\right)(9x^2 + 9y^2 - 36)^{1/2}$

$f_x(x, y) = \frac{9x}{2(9x^2 + 9y^2 - 36)^{1/2}}$

$f_x(2, 1) = 3$

28. $z = f(x, y) = \left(\frac{5}{4}\right)(16 - x^2)^{1/2}$.

$f_x(x, y) = \left(-\frac{5}{4}\right)x(16 - x^2)^{-1/2}$

$f_x(2, 3) = -\frac{5}{4\sqrt{3}} \approx -0.7217$

29. $V_r(r, h) = 2\pi rh$;

$V_r(6, 10) = 120\pi \approx 376.99$ in.^{2}

30. $T_y(x, y) = 3y^2$; $T_y(3, 2) = 12$ degrees per ft

31. $P(V, T) = \frac{kT}{V}$

$P_T(V, T) = \frac{k}{V}$;

$P_T(100, 300) = \frac{k}{100}$ lb/in.^{2} per degree

32. $V[P_V(V, T)] + T[P_T(V, T)]$

$= V(-kTV^{-2}) + T(kV^{-1}) = 0$

$P_V V_T T_P = \left(-\frac{kT}{V^2}\right)\left(\frac{k}{P}\right)\left(\frac{V}{k}\right) = -\frac{kT}{PV} = -\frac{PV}{PV} = -1$

33. $f_x(x, y) = 3x^2y - y^3$; $f_{xx}(x, y) = 6xy$;

$f_y(x, y) = x^3 - 3xy^2$; $f_{yy}(x, y) = -6xy$

Therefore, $f_{xx}(x, y) + f_{yy}(x, y) = 0$.

34. $f_x(x, y) = 2x(x^2 + y^2)^{-1}$;

$f_{xx}(x, y) = -2(x^2 - y^2)(x^2 + y^2)^{-1}$

$f_y(x, y) = 2y(x^2 + y^2)^{-1}$;

$f_{yy}(x, y) = 2(x^2 - y^2)(x^2 + y^2)^{-1}$

35. $F_y(x, y) = 15x^4y^4 - 6x^2y^2$;

$F_{yy}(x, y) = 60x^4y^3 - 12x^2y$;

$F_{yyy}(x, y) = 180x^4y^2 - 12x^2$

36. $f_x(x, y) = [-\sin(2x^2 - y^2)](4x)$

$= -4x\sin(2x^2 - y^2)$

$f_{xx}(x, y) = (-4x)[\cos(2x^2 - y^2)](4x)$
$\quad + [\sin(2x^2 - y^2)](-4)$

$f_{xxy}(x, y) = -16x^2[-\sin(2x^2 - y^2)](-2y)$
$\quad - 4[\cos(2x^2 - y^2)](-2y)$

$= -32x^2y\sin(2x^2 - y^2) + 8y\cos(2x^2 - y^2)$

37. **a.** $\frac{\partial^3 f}{\partial y^3}$

b. $\frac{\partial^3 y}{\partial y \partial x^2}$

c. $\frac{\partial^4 y}{\partial y^3 \partial x}$

38. a. f_{yxx}

b. f_{yyxx}

c. f_{yyxxx}

39. a. $f_x(x, y, z) = 6xy - yz$

b. $f_y(x, y, z) = 3x^2 - xz + 2yz^2$;
$f_y(0, 1, 2) = 8$

c. Using the result in a, $f_{xy}(x, y, z) = 6x - z$.

40. a. $12x^2(x^3 + y^2 + z)^3$

b. $f_y(x, y, z) = 8y(x^3 + y^2 + z)^3$;
$f_y(0, 1, 1) = 64$

c. $f_z(x, y, z) = 4(x^3 + y^2 + z)^3$;
$f_{zz}(x, y, z) = 12(x^2 + y^2 + z)^2$

41. $f_x(x, y, x) = -yze^{-xyz} - y(xy - z^2)^{-1}$

42. $f_x(x, y, z) = \left(\frac{1}{2}\right)\left(\frac{xy}{z}\right)^{-1/2}\left(\frac{y}{z}\right)$;

$f_x(-2, -1, 8) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^{-1/2}\left(-\frac{1}{8}\right) = -\frac{1}{8}$

43. If $f(x, y) = x^4 + xy^3 + 12$, $f_y(x, y) = 3xy^2$; $f_y(1, -2) = 12$. Therefore, along the tangent line $\Delta y = 1 \Rightarrow \Delta z = 12$, so $\langle 0, 1, 12\rangle$ is a tangent vector (since $\Delta x = 0$). Then parametric equations of the tangent line are $\left\{\begin{array}{c} x = 1 \\ y = -2 + t \\ z = 5 + 12t \end{array}\right\}$. Then the point of *xy*-plane at which the bee hits is (1, 0, 29) [since $y = 0 \Rightarrow t = 2 \Rightarrow x = 1, z = 29$].

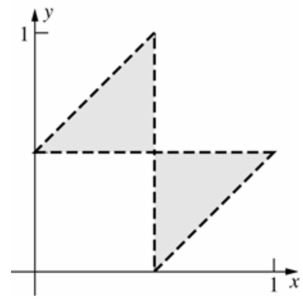
44. The largest rectangle that can be contained in the circle is a square of diameter length 20. The edge of such a square has length $10\sqrt{2}$, so its area is 200. Therefore, the domain of *A* is $\{(x, y) : 0 \le x^2 + y^2 < 400\}$, and the range is (0, 200].

45. Domain: (Case $x < y$)
The lengths of the sides are then x, $y - x$, and $1 - y$. The sum of the lengths of any two sides must be greater than the length of the remaining side, leading to three inequalities:

$x + (y - x) > 1 - y \Rightarrow y > \frac{1}{2}$

$(y - x) + (1 - y) > x \Rightarrow x < \frac{1}{2}$

$x + (1 - y) > y - x \Rightarrow y < x + \frac{1}{2}$



The case for $y < x$ yields similar inequalities (x and y interchanged). The graph of D_A, the domain of *A* is given above. In set notation it is

$$D_A = \left\{(x, y) : x < \frac{1}{2}, y > \frac{1}{2}, y < x + \frac{1}{2}\right\}$$
$$\cup \left\{(x, y) : x > \frac{1}{2}, y < \frac{1}{2}, x < y + \frac{1}{2}\right\}.$$

Range: The area is greater than zero but can be arbitrarily close to zero since one side can be arbitrarily small and the other two sides are bounded above. It seems that the area would be largest when the triangle is equilateral. An equilateral triangle with sides equal to $\frac{1}{3}$ has area $\frac{\sqrt{3}}{36}$. Hence the range of *A* is $\left(0, \frac{\sqrt{3}}{36}\right]$. (In Sections 8 and 9 of this chapter methods will be presented which will make it easy to prove that the largest value of *A* will occur when the triangle is equilateral.)

46. a. $u = \cos(x)\cos(ct)$:
$u_x = -\sin(x)\cos(ct)$; $u_t = -c\cos(x)\sin(ct)$
$u_{xx} = -\cos(x)\cos(ct)$
$u_{tt} = -c^2\cos(x)\cos(ct)$
Therefore, $c^2 u_{xx} = u_{tt}$.

$u = e^x\cosh(ct)$:
$u_x = e^x\cosh(ct), u_t = ce^x\sinh(ct)$
$u_{xx} = e^x\cosh(ct), u_{tt} = c^2e^x\cosh(ct)$
Therefore, $c^2 u_{xx} = u_{tt}$.

b. $u = e^{-ct}\sin(x)$:

$u_x = e^{-ct}\cos x$

$u_{xx} = -e^{-ct}\sin x$

$u_t = -ce^{ct}\sin x$

Therefore, $cu_{xx} = u_t$.

$u = t^{-1/2}e^{-x^2/4ct}$:

$u_x = t^{-1/2}e^{-x^2/4ct}\left(-\dfrac{x}{2ct}\right)$

$u_{xx} = \dfrac{(x^2 - 2ct)}{(4c^2t^{5/2}e^{x^2/4ct})}$

$u_t = \dfrac{(x^2 - 2ct)}{(4ct^{5/2}e^{x^2/4ct})}$

Therefore, $cu_{xx} = u_t$.

47. a. Moving parallel to the y-axis from the point (1, 1) to the nearest level curve and approximating $\dfrac{\Delta z}{\Delta y}$, we obtain

$$f_y(1, 1) = \frac{4-5}{1.25-1} = -4.$$

b. Moving parallel to the x-axis from the point (–4, 2) to the nearest level curve and approximating $\dfrac{\Delta z}{\Delta x}$, we obtain

$$f_x(-4, 2) \approx \frac{1-0}{-2.5-(-4)} = \frac{2}{3}.$$

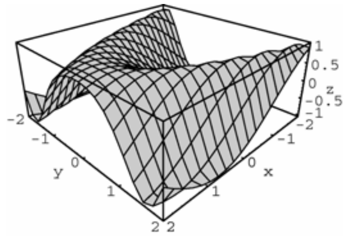
c. Moving parallel to the x-axis from the point (–5, –2) to the nearest level curve and approximately $\dfrac{\Delta z}{\Delta x}$, we obtain

$$f_x(-4, -5) \approx \frac{1-0}{-2.5-(-5)} = \frac{2}{5}.$$

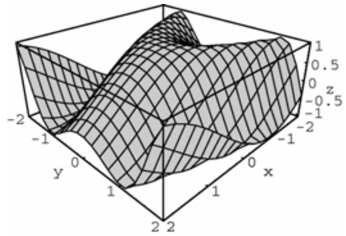
d. Moving parallel to the y-axis from the point (0, –2) to the nearest level curve and approximating $\dfrac{\Delta z}{\Delta y}$, we obtain

$$f_y(0, 2) \approx \frac{0-1}{\frac{-19}{8}-(-2)} = \frac{8}{3}.$$

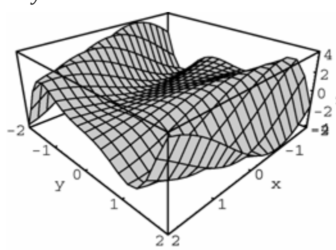
48. a. $\sin(x + y^2)$



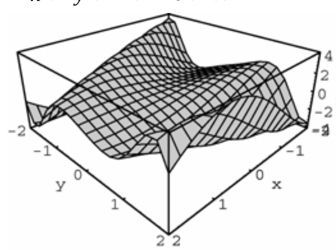
b. $D_x(\sin(x + y^2))$



c. $D_y(\sin(x + y^2))$



d. $D_x(D_y(\sin(x + y)^2))$



49. a. $f_y(x, y, z)$

$$= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

b. $f_z(x, y, z)$

$$= \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

c. $G_x(w, x, y, z)$

$$= \lim_{\Delta x \to 0} \frac{G(w, x + \Delta x, y, z) - G(w, x, y, z)}{\Delta x}$$

d. $\frac{\partial}{\partial z}\lambda(x,y,z,t)$

$= \lim_{\Delta z \to 0} \frac{\lambda(x,y,z+\Delta z,t)-\lambda(x,y,z,t)}{\Delta z}$

e. $\frac{\partial}{\partial b_2} S(b_0,b_1,b_2,\ldots,b_n) =$

$= \lim_{\Delta b_2 \to 0}\left(\frac{S(b_0,b_1,b_2+\Delta b_2,\ldots,b_n) - S(b_0,b_1,b_2,\ldots,b_n)}{\Delta b_2}\right)$

50. a. $\frac{\partial}{\partial w}(\sin w \sin x \cos y \cos z)$

$= \cos w \sin x \cos y \cos z$

b. $\frac{\partial}{\partial x}\left[x \ln(wxyz)\right] = x \cdot \frac{wyz}{wxyz} + 1 \cdot \ln(wxyz)$

$= 1 + \ln(wxyz)$

c. $\lambda_t(x,y,z,t)$

$= \frac{(1+xyzt)\cos x - t(\cos x)xyz}{(1+xyzt)^2}$

$= \frac{\cos x}{(1+xyzt)^2}$

12.3 Concepts Review

1. 3; (x, y) approaches $(1, 2)$.

2. $\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2)$

3. contained in S

4. an interior point of S; boundary points

Problem Set 12.3

1. –18

2. 3

3. $\lim_{(x,y)\to(2,\pi)}\left[x\cos^2 xy - \sin\left(\frac{xy}{3}\right)\right]$

$= 2\cos^2 2\pi - \sin\left(\frac{2\pi}{3}\right) = 2 - \frac{\sqrt{3}}{2} \approx 1.1340$

4. The limit does not exist because of Theorem A. The function is a rational function, but the limit of the denominator is 0, while the limit of the numerator is -1.

5. $-\frac{5}{2}$

6. -1

7. 1

8. $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{(x^2+y^2)}$

$= \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} \frac{1}{\cos(x^2+y^2)}$

$= (1)(1) = 1$

9. The limit does not exist since the function is not defined anywhere along the line $y = x$. That is, there is no neighborhood of the origin in which the function is defined everywhere except possibly at the origin.

10. $\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^2-y^2)}{x^2+y^2}$

$= \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0$

11. Changing to polar coordinates,

$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta \cdot r\sin\theta}{r}$

$= \lim_{r\to 0} r\cos\theta \cdot \sin\theta = 0$

12. If (x,y) approaches (0, 0) along the line $y = x$,

$\lim_{(x,x)\to(0,0)} \frac{x^2}{(x^2+x^2)^2} = \lim_{(x,x)\to(0,0)} \frac{1}{4x^2} = +\infty$

Thus, the limit does not exist.

13. Use polar coordinates.

$\frac{x^{7/3}}{x^2+y^2} = \frac{r^{7/3}(\cos\theta)^{7/3}}{r^2} = r^{1/3}(\cos\theta)^{7/3}$

$r^{1/3}(\cos\theta)^{7/3} \to 0$ as $r \to 0$, so the limit is 0.

14. Changing to polar coordinates,

$\lim_{r\to 0} r^2 \cos\theta \sin\theta \cdot \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$

$= \lim_{r\to 0} r^2 \cos\theta \sin\theta \cos 2\theta = 0$

15. $f(x, y) = \dfrac{r^4 \cos^2\theta \sin^2\theta}{r^2 \cos^2\theta + r^4 \sin^4\theta}$

$= r^2 \left(\dfrac{\cos^2\theta \sin^2\theta}{\cos^2\theta + r^2 \sin^4\theta} \right)$

If $\cos\theta = 0$, then $f(x, y) = 0$. If $\cos\theta \neq 0$, hen this converges to 0 as $r \to 0$. Thus the limit is 0.

16. As (x, y) approaches (0,0) along x = y^2,

$\lim\limits_{(x,x)\to(0,0)} \dfrac{y^4}{y^4 + y^4} = \dfrac{1}{2}$. Along the x-axis,

however, $\lim\limits_{(x,0)\to(0,0)} \dfrac{0}{x^2} = 0$. Thus, the limit does not exist.

17. $f(x, y)$ is continuous for all (x, y) since for all (x, y), $x^2 + y^2 + 1 \neq 0$.

18. $f(x, y)$ is continuous for all (x, y) since for all (x, y), $x^2 + y^2 + 1 > 0$.

19. Require $1 - x^2 - y^2 > 0; x^2 + y^2 < 1$. S is the interior of the unit circle centered at the origin.

20. Require $1 + x + y > 0; y > -x - 1$. S is the set of all (x, y) above the line $y = -x - 1$.

21. Require $y - x^2 \neq 0$. S is the entire plane except the parabola $y = x^2$.

22. The only points at which f might be discontinuous occur when $xy = 0$.

$\lim\limits_{(x,\, y)\to(a,\, 0)} \dfrac{\sin(xy)}{xy} = 1 = f(a, 0)$ for all nonzero a in $\mathbb{R}$, and then

$\lim\limits_{(x,\, y)\to(0,\, b)} \dfrac{\sin(xy)}{xy} = 1 = f(0, b)$ for all b in $\mathbb{R}$.

Therefore, f is continuous on the entire plane.

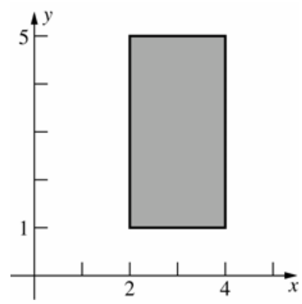
23. Require $x - y + 1 \geq 0; y \leq x + 1$. S is the region below and on the line $y = x + 1$.

24. Require $4 - x^2 - y^2 > 0; x^2 + y^2 < 4$. S is the interior of the circle of radius 2 centered at the origin.

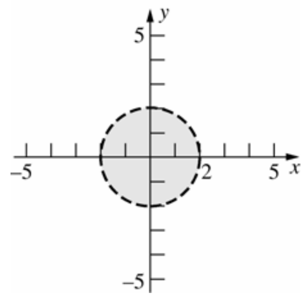
25. $f(x, y, z)$ is continuous for all $(x, y, z) \neq (0,0,0)$ since for all $(x, y, z) \neq (0,0,0)$, $x^2 + y^2 + z^2 > 0$.

26. Require $4 - x^2 - y^2 - z^2 > 0$; $x^2 + y^2 + z^2 < 4$. S is the space in the interior of the sphere centered at the origin with radius 2.

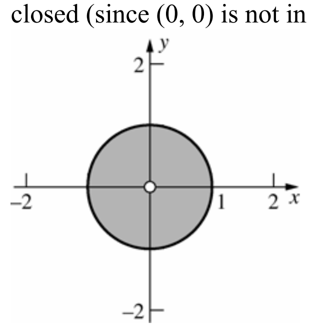
27. The boundary consists of the points that form the outer edge of the rectangle. The set is closed.



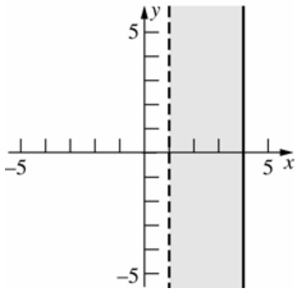
28. The boundary consists of the points of the circle shown. The set is open.



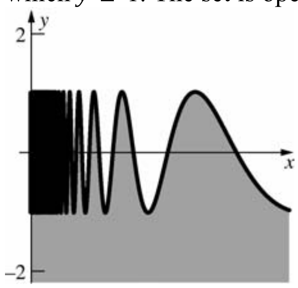
29. The boundary consists of the circle and the origin. The set is neither open (since, for example, (1, 0) is not an interior point), nor closed (since (0, 0) is not in the set).



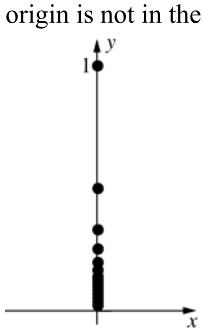
30. The boundary consists of the points on the line $x = 1$ along with the points on the line $x = 4$. The set is neither closed nor open.



31. The boundary consists of the graph of $y = \sin\left(\frac{1}{x}\right)$ along with the part of the y-axis for which $y \le 1$. The set is open.



32. The boundary is the set itself along with the origin. The set is neither open (since none of its points are interior points) nor closed (since the origin is not in the set).



33. $\dfrac{x^2 - 4y^2}{x - 2y} = \dfrac{(x+2y)(x-2y)}{x-2y} = x + 2y$ (if $x \neq 2y$)

If $x = 2y$, $x + 2y = 2x$. Take $g(x) = 2x$.

34. Let L and M be the latter two limits.

$[f(x, y) + g(x, y)] - [L + M]|$

$\le |f(x, y) - L| + |f(x, y) - M| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

for (x, y) in some δ-neighborhood of (a, b). Therefore,

$$\lim_{(x, y)\to(a, b)} [f(x, y) + g(x, y)] = L + M.$$

35. Along the x-axis ($y = 0$): $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{0}{x^2 + 0} = 0$.

Along $y = x$:

$$\lim_{(x, y)\to(0, 0)} \frac{x^2}{2x^2} = \lim_{(x, y)\to(0, 0)} \frac{1}{2} = \frac{1}{2}.$$

Hence, the limit does not exist because for some points near the origin $f(x, y)$ is getting closer to 0, but for others it is getting closer to $\frac{1}{2}$.

36. Along $y = 0$: $\displaystyle\lim_{x\to 0} \frac{0}{x^2 + 0} = 0.$ Along $y = x$:

$$\lim_{x\to 0} \frac{x^2 + x^3}{x^2 + x^2} = \lim_{x\to 0} \frac{1 + x}{2} = \frac{1}{2}.$$

37. a. $\displaystyle\lim_{x\to 0} \frac{x^2(mx)}{x^4 + (mx)^2} = \lim_{x\to 0} \frac{mx^3}{x^4 + m^2x^2}$

$\displaystyle= \lim_{x\to 0} \frac{mx}{x^2 + m^2} = 0$

b. $\displaystyle\lim_{x\to 0} \frac{x^2(x^2)}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$

c. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^2 y}{x^4 + y}$ does not exist.

38. f is discontinuous at each overhang. More interesting, f is discontinuous along the Continental Divide.

39. a. $\{(x, y, z) : x^2 + y^2 = 1, z \text{ in } [1, 2]\}$ [For $x^2 + y^2 < 1$, the particle hits the hemisphere and then slides to the origin (or bounds toward the origin); for $x^2 + y^2 = 1$, it bounces up; for $x^2 + y^2 > 1$, it falls straight down.]

b. $\{(x, y, z) : x^2 + y^2 = 1, z = 1\}$ (As one moves at a level of $z = 1$ from the rim of the bowl toward any position away from the bowl there is a change from seeing all of the interior of the bowl to seeing none of it.)

c. $\{(x, y, z): z = 1\}$ [$f(x, y, z)$ is undefined (infinite) at $(x, y, 1)$.]

d. ϕ (Small changes in points of the domain result in small changes in the shortest path from the points to the origin.)

40. f is continuous on an open set D and P_0 is in D implies that there is neighborhood of P_0 with radius r on which f is continuous. f is continuous at $P_0 \Rightarrow \lim\limits_{P \to P_0} f(P) = f(P_0)$. Now let $\varepsilon = f(P_0)$ which is positive. Then there is a δ such that $0 < \delta < r$ and $|f(p) - f(P_0)| < f(P_0)$ if P is in the δ-neighborhood of P_0. Therefore, $-f(P_0) < f(p) - f(P_0) < f(P_0)$, so $0 < f(p)$ (using the left-hand inequality) in that δ-neighborhood of P_0.

41. a. $f(x, y) = \begin{Bmatrix} (x^2 + y^2)^{1/2} + 1 & \text{if } y \neq 0 \\ |x-1| & \text{if } y = 0 \end{Bmatrix}$. Check discontinuities where $y = 0$.

As $y = 0$, $(x^2 + y^2)^{1/2} + 1 = |x| + 1$, so f is continuous if $|x| + 1 = |x - 1|$. Squaring each side and simplifying yields $|x| = -x$, so f is continuous for $x \le 0$. That is, f is discontinuous along the positive x-axis.

b. Let $P = (u, v)$ and $Q = (x, y)$.

$$f(u, v, x, y) = \begin{Bmatrix} |OP| + |OQ| & \text{if } P \text{ and } Q \text{ are not on same ray from the origin and neither is the origin} \\ |\mathrm{PQ}| & \text{otherwise} \end{Bmatrix}.$$

This means that in the first case one travels from P to the origin and then to Q; in the second case one travels directly from P to Q without passing through the origin, so f is discontinuous on the set $\{(u, v, x, y) : \langle u, v\rangle = k\langle x, y\rangle \text{ for some } k > 0, \langle u, v\rangle \neq \mathbf{0}, \langle x, y\rangle \neq \mathbf{0}\}$.

42. a. $$f_x(0, y) = \lim_{h \to 0}\left(\frac{\frac{hy(h^2 - y^2)}{h^2 + y^2} - 0}{h}\right) = \lim_{h \to 0} \frac{y(h^2 - y^2)}{h^2 + y^2} = -y$$

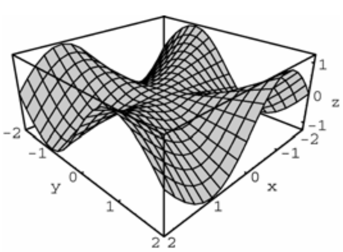
b. $$f_y(x, 0) = \lim_{h \to 0}\left(\frac{\frac{xh(x^2 - h^2)}{x^2 + h^2} - 0}{h}\right) = \lim_{h \to 0} \frac{y(x^2 - h^2)}{x^2 + y^2} = x$$

c. $$f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(0 + h, y) - f_y(0, y)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

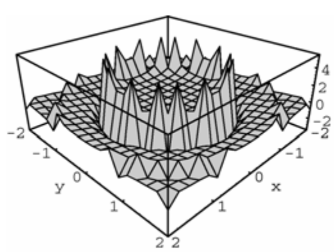
d. $$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_x(x, 0 + h) - f_x(x, 0)}{h} = \lim_{h \to 0} \frac{-h - 0}{h} = -1$$

Therefore, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

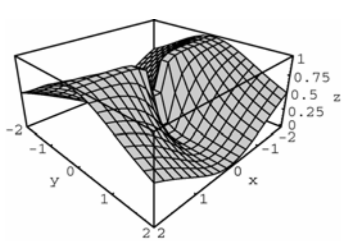
43.



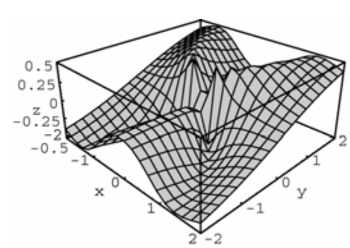
b.



44. a.



45.



46. A function *f* of three variables is continuous at a point (a,b,c) if $f(a,b,c)$ is defined and equal to the limit of $f(x,y,z)$ as (x,y,z) approaches (a,b,c). In other words,

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c).$$

A function of three variables is continuous on an open set *S* if it is continuous at every point in the interior of the set. The function is continuous at a boundary point *P* of *S* if $f(Q)$ approaches $f(P)$ as *Q* approaches *P* along any path through points in *S* in the neighborhood of *P*.

47. If we approach the point $(0,0,0)$ along a straight path from the point (x,x,x), we have

$$\lim_{(x,x,x)\to(0,0,0)} \frac{x(x)(x)}{x^3+x^3+x^3} = \lim_{(x,x,x)\to(0,0,0)} \frac{x^3}{3x^3} = \frac{1}{3}$$

Since the limit does not equal to $f(0,0,0)$, the function is not continuous at the point $(0,0,0)$.

48. If we approach the point $(0,0,0)$ along the x-axis, we get

$$\lim_{(x,0,0)\to(0,0,0)} (0+1)\frac{(x^2-0^2)}{(x^2+0^2)} = \lim_{(x,0,0)\to(0,0,0)} \frac{x^2}{x^2} = 1$$

Since the limit does not equal $f(0,0,0)$, the function is not continuous at the point $(0,0,0)$.

12.4 Concepts Review

1. gradient
2. locally linear
3. $\dfrac{\partial f}{\partial x}(\mathbf{p})\mathbf{i} + \dfrac{\partial f}{\partial y}(\mathbf{p})\mathbf{j}$; $\mathbf{y}^2\mathbf{i} + 2xy\mathbf{j}$
4. tangent plane

Problem Set 12.4

1. $\left\langle 2xy+3y,\ x^2+3x \right\rangle$
2. $\left\langle 3x^2y,\ x^3-3y^2 \right\rangle$
3. $\nabla f(x,\ y) = \left\langle (x)(e^{xy}y)+(e^{xy})(1),\ xe^{xy}x \right\rangle = e^{xy}\left\langle xy+1,\ x^2 \right\rangle$
4. $\left\langle 2xy\cos y,\ x^2(\cos y - y\sin y) \right\rangle$
5. $x(x+y)^{-2}\left\langle y(x+2),\ x^2 \right\rangle$
6. $\nabla f(x,\ y) = \left\langle 3[\sin^2(x^2y)][\cos(x^2y)](2xy),\ 3[\sin^2(x^2y)][\cos(x^2y)](x^2) \right\rangle = 3x\sin^2(x^2y)\cos(x^2y)\left\langle 2y,\ x \right\rangle$

7. $(x^2 + y^2 + z^2)^{-1/2} \langle x, y, z \rangle$

8. $\left\langle 2xy + z^2, x^2 + 2yz, y^2 + 2xz \right\rangle$

9. $\nabla f(x, y) = \left\langle (x^2 y)(e^{x-z}) + (e^{x-z})(2xy), x^2 e^{x-z}, x^2 y e^{x-z}(-1) \right\rangle = xe^{x-z} \langle y(x+2), x, -xy \rangle$

10. $\left\langle xz(x+y+z)^{-1} + z\ln(x+y+z), xz(x+y+z)^{-1}, xz(x+y+z)^{-1} + x\ln(x+y+z) \right\rangle$

11. $\nabla f(x, y) = \left\langle 2xy - y^2, x^2 - 2xy \right\rangle$; $\nabla f(-2, 3) = \langle -21, 16 \rangle$

$z = f(-2, 3) + \langle -21, 16 \rangle \cdot \langle x+2, y-3 \rangle = 30 + (-21x - 42 + 16y - 48)$

$z = -21x + 16y - 60$

12. $\nabla f(x, y) = \left\langle 3x^2 y + 3y^2, x^3 + 6xy \right\rangle$, so $\nabla f(2, -2) = (-12, -16)$.

Tangent plane:

$z = f(2, -2) + \nabla(2, -2) \cdot \langle x-2, y+2 \rangle = 8 + \langle -12, -16 \rangle \cdot \langle x-2, y+2 \rangle = 8 + (-12x + 24 - 16y - 32)$

$z = -12x - 16y$

13. $\nabla f(x, y) = \langle -\pi \sin(\pi x)\sin(\pi y), \pi\cos(\pi x)\cos(\pi y) + 2\pi\cos(2\pi y) \rangle$

$\nabla f\left(-1, \frac{1}{2}\right) = \langle 0, -2\pi \rangle$

$z = f\left(-1, \frac{1}{2}\right) + \langle 0, -2\pi \rangle \cdot \left\langle x+1, y - \frac{1}{2} \right\rangle = -1 + (0 - 2\pi y + \pi);$

$z = -2\pi y + (\pi - 1)$

14. $\nabla f(x, y) = \left\langle \frac{2x}{y}, -\frac{x^2}{y^2} \right\rangle; \nabla f(2, -1) = \langle -4, -4 \rangle$

$z = f(2, -1) + \langle -4, -4 \rangle \cdot \langle x-2, y+1 \rangle$

$= -4 + (-4x + 8 - 4y - 4)$

$z = -4x - 4y$

15. $\nabla f(x, y, z) = \left\langle 6x + z^2, -4y, 2xz \right\rangle$, so $\nabla f(1, 2, -1) = \langle 7, -8, -2 \rangle$

Tangent hyperplane:

$w = f(1, 2, -1) + \nabla f(1, 2, -1) \cdot \langle x-1, y-2, z+1 \rangle = -4 + \langle 7, -8, -2 \rangle \cdot \langle x-1, y-2, z+1 \rangle$

$= -4 + (7x - 7 - 8y + 16 - 2z - 2)$

$w = 7x - 8y - 2z + 3$

16. $\nabla f(x, y, z) = \langle yz + 2x, xz, xy \rangle$; $\nabla f(2, 0, -3) = \langle 4, -6, 0 \rangle$

$w = f(2, 0, -3) + \langle 4, -6, 0 \rangle \cdot \langle x-2, y, z+3 \rangle = 4 + (4x - 8 - 6y + 0)$

$w = 4x - 6y - 4$

17. $\nabla\left(\frac{f}{g}\right) = \frac{\langle gf_x - fg_x, gf_y - fg_y, gf_z - fg_z \rangle}{g^2} = \frac{g\langle f_x, f_y, f_z \rangle - f\langle g_x, g_y, g_z \rangle}{g^2} = \frac{g\nabla f - f\nabla g}{g^2}$

18. $\nabla(f^r) = \left\langle rf^{r-1} f_x, rf^{r-1} f_y, rf^{r-1} f_z \right\rangle = rf^{r-1} \langle f_x, f_y, f_z \rangle = rf^{r-1}\nabla f$

19. Let $F(x,y,z) = x^2 - 6x + 2y^2 - 10y + 2xy - z = 0$

$\nabla F(x,y,z) = \langle 2x-6+2y, 4y-10+2x, -1\rangle$

The tangent plane will be horizontal if $\nabla F(x,y,z) = \langle 0,0,k\rangle$, where $k \neq 0$. Therefore, we have the following system of equations:

$2x+2y-6=0$

$2x+4y-10=0$

Solving this system yields $x=1$ and $y=2$. Thus, there is a horizontal tangent plane at $(x,y)=(1,2)$.

20. Let $F(x,y,z) = x^3 - z = 0$

$\nabla F(x,y,z) = \langle 3x^2, 0, -1\rangle$

The tangent plane will be horizontal if $\nabla F(x,y,z) = \langle 0,0,k\rangle$, where $k \neq 0$. Therefore, we need only solve the equation $3x^2 = 0$. There is a horizontal tangent plane at $(x,y) = (0,y)$. (Note: there are infinitely many points since y can take on any value).

21. a. The point $(2,1,9)$ projects to $(2,1,0)$ on the *xy* plane. The equation of a plane containing this point and parallel to the *x*-axis is given by $y=1$. The tangent plane to the surface at the point $(2,1,9)$ is given by

$z = f(2,1) + \nabla f(2,1)\cdot\langle x-2, y-1\rangle$
$= 9 + \langle 12,10\rangle\langle x-2,y-1\rangle$
$= 12x+10y-25$

The line of intersection of the two planes is the tangent line to the surface, passing through the point $(2,1,9)$, whose projection in the *xy* plane is parallel to the x-axis. This line of intersection is parallel to the cross product of the normal vectors for the planes. The normal vectors are $\langle 12,10,-1\rangle$ and $\langle 0,1,0\rangle$ for the tangent plane and vertical plane respectively. The cross product is given by $\langle 12,10,-1\rangle \times \langle 0,1,0\rangle = \langle 1,0,12\rangle$

Thus, parametric equations for the desired tangent line are $x = 2+t$

$y = 1$

$z = 9+12t$

b. Using the equation for the tangent plane from the previous part, we now want the vertical plane to be parallel to the *y*-axis, but still pass through the projected point $(2,1,0)$. The vertical plane now has equation $x=2$. The normal equations are given by $\langle 12,10,-1\rangle$ and $\langle 1,0,0\rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:

$\langle 12,10,-1\rangle \times \langle 1,0,0\rangle = \langle 0,10,10\rangle$

Thus, parametric equations for the desired tangent line are $x=2$

$y = 1+10t$

$z = 9+10t$

c. Using the equation for the tangent plane from the first part, we now want the vertical plane to be parallel to the line $y=x$, but still pass through the projected point $(2,1,0)$. The vertical plane now has equation $y-x+1=0$. The normal equations are given by $\langle 12,10,-1\rangle$ and $\langle 1,-1,0\rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:

$\langle 12,10,-1\rangle \times \langle 1,-1,0\rangle = \langle -1,-1,-22\rangle$

Thus, parametric equations for the desired tangent line are $x = 2-t$

$y = 1-t$

$z = 9-22t$

22. a. The point $(3,2,72)$ on the surface is the point $(3,2,0)$ when projected into the *xy* plane. The equation of a plane containing this point and parallel to the x-axis is given by $y=2$. The tangent plane to the surface at the point $(3,2,72)$ is given by

$z = f(3,2) + \nabla f(3,2)\cdot\langle x-3,y-2\rangle$
$= 72 + \langle 48,108\rangle\langle x-3,y-2\rangle$
$= 48x+108y-288$

The line of intersection of the two planes is the tangent line to the surface, passing through the point $(3,2,72)$, whose projection in the *xy* plane is parallel to the x-axis. This line of intersection is parallel to the cross product of the normal vectors for the planes. The normal vectors are $\langle 48,108,-1\rangle$ and $\langle 0,2,0\rangle$ for the tangent plane and vertical plane respectively. The cross product is given by

$\langle 48,108,-1\rangle \times \langle 0,2,0\rangle = \langle 2,0,96\rangle$

Thus, parametric equations for the desired tangent line are

$x = 3+2t$

$y = 2$

$z = 72+96t$

b. Using the equation for the tangent plane from the previous part, we now want the vertical plane to be parallel to the y-axis, but still pass through the projected point $(3, 2, 72)$. The vertical plane now has equation $x = 3$. The normal equations are given by $\langle 48,108,-1\rangle$ and $\langle 3,0,0\rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:
$\langle 48,108,-1\rangle \times \langle 3,0,0\rangle = \langle 0,-3,-324\rangle$
Thus, parametric equations for the desired tangent line are
$x = 3$
$y = 2-3t$
$z = 72-324t$

c. Using the equation for the tangent plane from the first part, we now want the vertical plane to be parallel to the line $x = -y$, but still pass through the projected point $(3, 2, 72)$. The vertical plane now has equation $y + x - 5 = 0$. The normal equations are given by $\langle 48,108,-1\rangle$ and $\langle 1,1,0\rangle$ for the tangent and vertical planes respectively. Again we find the cross product of the normal vectors:
$\langle 48,108,-1\rangle \times \langle 1,1,0\rangle = \langle 1,-1,-60\rangle$
Thus, parametric equations for the desired tangent line are
$x = 3+t$
$y = 2-t$
$z = 72-60t$

23. $\nabla f(x,\, y) = \left\langle -10\left(\frac{1}{2\sqrt{|xy|}}\frac{|xy|}{xy}y\right),\, -10\left(\frac{1}{2\sqrt{|xy|}}\frac{|xy|}{xy}x\right)\right\rangle = \frac{-5xy}{|xy|^{3/2}}\langle y,\, x\rangle$ $\left[\text{Note that } \frac{|a|}{a} = \frac{a}{|a|}.\right]$

$\nabla f(1,\, -1) = \langle -5,\, 5\rangle$

Tangent plane:
$z = f(1,\, -1) + \nabla f(1,\, -1)\cdot\langle x-1,\, y+1\rangle = -10 + \langle -5,\, 5\rangle \cdot \langle x-1,\, y+1\rangle = -10 + (-5x+5+5y+5)$
$z = -5x + 5y$

24. Let **a** be any point of S and let **b** be any other point of S. Then for some c on the line segment between **a** and **b**:
$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(c)\cdot(\mathbf{b}-\mathbf{a}) = 0\cdot(\mathbf{b}-\mathbf{a}) = 0,$ so $f(\mathbf{b}) = f(\mathbf{a})$ (for all **b** in S).

25. $f(\mathbf{a}) - f(\mathbf{b}) = f\langle 2,1\rangle - f\langle 0,0\rangle = 4-9 = -5$

$\nabla f(x,y) = \langle -2x,-2y\rangle$; $\mathbf{b}-\mathbf{a} = \langle 2,1\rangle$

The value $\mathbf{c} = \langle c_x, c_y\rangle$ will be a solution to

$-5 = \langle -2c_x, -2c_y\rangle\langle 2,1\rangle$

$\mathbf{c} \in \{\langle c_x, c_y\rangle : 4c_x + 2c_y = 5\}$

In order for **c** to be between **a** and **b**, **c** must lie on the line $y = \frac{1}{2}x$. Consequently, **c** will be the solution to the following system of equations: $4c_x + 2c_y = 5$ and $c_y = \frac{1}{2}c_x$. The solution is $\mathbf{c} = \langle 1, \frac{1}{2}\rangle$.

26. $f(\mathbf{b}) - f(\mathbf{a}) = f\langle 2,6\rangle - f\langle 0,0\rangle = 0-2 = -2$

$\nabla f(x,y) = \left\langle \frac{-x}{\sqrt{4-x^2}}, 0\right\rangle$; $\mathbf{b}-\mathbf{a} = \langle 2,6\rangle$

The value $\mathbf{c} = \langle c_x, c_y\rangle$ will be the solution to

$-2 = \left\langle \frac{-c_x}{\sqrt{4-{c_x}^2}}, 0\right\rangle\langle 2,6\rangle$

$-2 = \frac{-2c_x}{\sqrt{4-{c_x}^2}} \Rightarrow c_x = \sqrt{2}$

Since **c** must be between **a** and **b**, **c** must lie on the line y = 3x. Since $c_x = \sqrt{2},\ c_y = 3\sqrt{2}$.

Thus, $\mathbf{c} = \langle \sqrt{2}, 3\sqrt{2}\rangle$.

27. $\nabla f(\mathbf{p}) = \nabla g(\mathbf{p}) \Rightarrow \nabla[f(\mathbf{p}) - g(\mathbf{p})] = \mathbf{0}$
$\Rightarrow f(\mathbf{p}) - g(\mathbf{p})$ is a constant.

28. $\nabla f(\mathbf{p}) = \mathbf{p} \Rightarrow \nabla f(x, y) = \langle x, y \rangle$

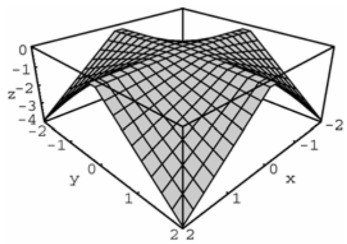
$\Rightarrow f_x(x, y) = x,\ f_y(x, y) = y$

$\Rightarrow f(x, y) = \frac{1}{2}x^2 + \alpha(y)$ for any function of y,

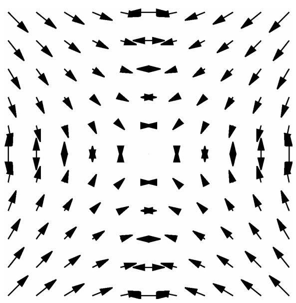
and $f(x, y) = \frac{1}{2}y^2 + \beta(x)$ for any function of x.

$\Rightarrow f(x, y) = \frac{1}{2}(x^2 + y^2) + C$ for any C in $\mathbb{R}$.

29.



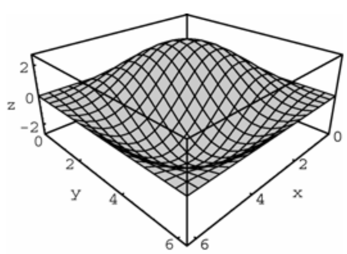
$-|xy|$



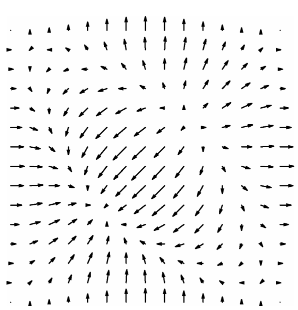
a. The gradient points in the direction of greatest increase of the function.

b. No. If it were, $|0+h| - |0| = 0 + |h|\delta(h)$ where $\delta(h) \to 0$ as $h \to 0$, which is possible.

30.



$\sin(x) + \sin(y) - \sin(x + y)$



31. a. (i)

$$\nabla[f+g] = \frac{\partial(f+g)}{\partial x}\mathbf{i} + \frac{\partial(f+g)}{\partial y}\mathbf{j} + \frac{\partial(f+g)}{\partial z}\mathbf{k}$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} + \frac{\partial g}{\partial z}\mathbf{k}$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} + \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}$$

$$= \nabla f + \nabla g$$

(ii) $$\nabla[\alpha f] = \frac{\partial[\alpha f]}{\partial x}\mathbf{i} + \frac{\partial[\alpha f]}{\partial y}\mathbf{j} + \frac{\partial[\alpha f]}{\partial z}\mathbf{k}$$

$$= \alpha\frac{\partial[f]}{\partial x}\mathbf{i} + \alpha\frac{\partial[f]}{\partial y}\mathbf{j} + \alpha\frac{\partial[f]}{\partial z}\mathbf{k}$$

$$= \alpha\nabla f$$

(iii)

$$\nabla[fg] = \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} + \frac{\partial(fg)}{\partial z}\mathbf{k}$$

$$= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\mathbf{j}$$

$$+ \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\mathbf{k}$$

$$= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)$$

$$+ g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)$$

$$= f\nabla g + g\nabla f$$

b. (i)

$$\nabla[f+g] = \frac{\partial(f+g)}{\partial x_1}\mathbf{i}_1 + \frac{\partial(f+g)}{\partial x_2}\mathbf{i}_2$$

$$+ \cdots + \frac{\partial(f+g)}{\partial x_n}\mathbf{i}_n$$

$$= \frac{\partial f}{\partial x_1}\mathbf{i}_1 + \frac{\partial g}{\partial x_1}\mathbf{i}_1 + \frac{\partial f}{\partial x_2}\mathbf{i}_2 + \frac{\partial g}{\partial x_2}\mathbf{i}_2$$

$$+ \cdots + \frac{\partial f}{\partial x_n}\mathbf{i}_n + \frac{\partial g}{\partial x_n}\mathbf{i}_n$$

$$= \frac{\partial f}{\partial x_1}\mathbf{i}_1 + \frac{\partial f}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{i}_n$$

$$+ \frac{\partial g}{\partial x_1}\mathbf{i}_1 + \frac{\partial g}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial g}{\partial x_n}\mathbf{i}_n$$

$$= \nabla f + \nabla g$$

(ii)

$$\nabla[\alpha f] = \frac{\partial[\alpha f]}{\partial x_1}\mathbf{i}_1 + \frac{\partial[\alpha f]}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial[\alpha f]}{\partial x_n}\mathbf{i}_n$$
$$= \alpha\frac{\partial[f]}{\partial x_1}\mathbf{i}_1 + \alpha\frac{\partial[f]}{\partial x_2}\mathbf{i}_2 + \cdots + \alpha\frac{\partial[f]}{\partial x_n}\mathbf{i}_n$$
$$= \alpha\nabla f$$

(iii)

$$\nabla[fg] = \frac{\partial(fg)}{\partial x_1}\mathbf{i}_1 + \frac{\partial(fg)}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial(fg)}{\partial x_n}\mathbf{i}_n$$
$$= \left(f\frac{\partial g}{\partial x_1} + g\frac{\partial f}{\partial x_1}\right)\mathbf{i}_1 + \left(f\frac{\partial g}{\partial x_2} + g\frac{\partial f}{\partial x_2}\right)\mathbf{i}_2$$
$$+ \cdots + \left(f\frac{\partial g}{\partial x_n} + g\frac{\partial f}{\partial x_n}\right)\mathbf{i}_n$$
$$= f\left(\frac{\partial g}{\partial x_1}\mathbf{i}_1 + \frac{\partial g}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial g}{\partial x_n}\mathbf{i}_n\right)$$
$$+ g\left(\frac{\partial f}{\partial x_1}\mathbf{i}_1 + \frac{\partial f}{\partial x_2}\mathbf{i}_2 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{i}_n\right)$$
$$= f\nabla g + g\nabla f$$

12.5 Concepts Review

1. $\dfrac{[f(\mathbf{p}+h\mathbf{u}) - f(\mathbf{p})]}{h}$

2. $u_1 f_x(x, y) + u_2 f_y(x, y)$

3. greatest increase

4. level curve

Problem Set 12.5

1. $D_\mathbf{u} f(x, y) = \left\langle 2xy, x^2\right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$; $D_\mathbf{u} f(1, 2) = \frac{8}{5}$

2. $D_\mathbf{u} f(x, y) = \left\langle x^{-1}y^2, 2y\ln x\right\rangle \cdot \left[\left(\frac{1}{\sqrt{2}}\right)\langle 1, -1\rangle\right]$;

   $D_\mathbf{u} f(1, 4) = 8\sqrt{2} \approx 11.3137$

3. $D_\mathbf{u} f(x, y) = f(x, y)\cdot\mathbf{u}$ $\left(\text{where } u = \frac{\mathbf{a}}{|\mathbf{a}|}\right)$

   $= \langle 4x + y, x - 2y\rangle \cdot \frac{\langle 1, -1\rangle}{\sqrt{2}}$;

   $D_\mathbf{u} f(3, -2) = \langle 10, 7\rangle \cdot \frac{\langle 1, -1\rangle}{\sqrt{2}} = \frac{3}{\sqrt{2}} \approx 2.1213$

4. $D_\mathbf{u} f(x, y)$

   $= \langle 2x - 3y, -3x + 4y\rangle \cdot \left[\left(\frac{1}{\sqrt{5}}\right)\langle 2, -1\rangle\right]$;

   $D_\mathbf{u} f(-1, 2) = -\frac{27}{\sqrt{5}} \approx -12.0748$

5. $D_\mathbf{u} f(x, y) = e^x\langle \sin y, \cos y\rangle \cdot \left[\left(\frac{1}{2}\right)\langle 1, \sqrt{3}\rangle\right]$;

   $D_\mathbf{u} f\left(0, \frac{\pi}{4}\right) = \frac{\left(\sqrt{2}+\sqrt{6}\right)}{4} \approx 0.9659$

6. $D_\mathbf{u} f(x, y) = \left\langle -ye^{-xy} - xe^{-xy}\right\rangle \cdot \frac{\langle -1, \sqrt{3}\rangle}{2}$

   $D_\mathbf{u} f(1, -1) = \langle e, -e\rangle \cdot \frac{\langle -1, \sqrt{3}\rangle}{2} = \frac{-e - e\sqrt{3}}{2}$

   ≈ -3.7132

7. $D_\mathbf{u} f(x, y, z) =$

   $= \left\langle 3x^2y, x^3 - 2yz^2, -2y^2z\right\rangle \cdot \left[\left(\frac{1}{3}\right)\langle 1, -2, 2\rangle\right]$;

   $D_\mathbf{u} f(-2, 1, 3) = \frac{52}{3}$

8. $D_\mathbf{u} f(x, y, z) = \langle 2x, 2y, 2z\rangle \cdot \left[\left(\frac{1}{2}\right)\left\langle\sqrt{2}, -1, -1\right\rangle\right]$;

   $D_\mathbf{u} f(1, -1, 2) = \sqrt{2} - 1 \approx 0.4142$

9. f increases most rapidly in the direction of the gradient. $\nabla f(x, y) = \left\langle 3x^2, -5y^4\right\rangle$;

   $\nabla f(2, -1) = \langle 12, -5\rangle$

   $\frac{\langle 12, -5\rangle}{13}$ is the unit vector in that direction. The rate of change of $f(x, y)$ in that direction at that point is the magnitude of the gradient.

   $|\langle 12, -5\rangle| = 13$

10. $\nabla f(x, y) = \left\langle e^y\cos x, e^y\sin x\right\rangle$;

    $\nabla f\left(\frac{5\pi}{6}, 0\right) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$, which is a unit vector.

    The rate of change in that direction is 1.

11. $\nabla f(x, y, z) = \left\langle 2xyz,\ x^2 z,\ x^2 y \right\rangle$;

$f(1, -1, 2) = \langle -4, 2, -1 \rangle$

A unit vector in that direction is

$\left(\frac{1}{\sqrt{21}}\right)\langle -4, 2, -1 \rangle$. The rate of change in that direction is $\sqrt{21} \approx 4.5826$.

12. f increases most rapidly in the direction of the gradient. $\nabla f(x, y, z) = \left\langle e^{yz},\ xze^{yz},\ xye^{yz} \right\rangle$;

$\nabla f(2, 0, -4) = \langle 1, -8, 0 \rangle$

$\frac{\langle 1, -8, 0 \rangle}{\sqrt{65}}$ is a unit vector in that direction.

$\left|\langle 1, -8, 0 \rangle\right| = \sqrt{65} \approx 8.0623$ is the rate of change of $f(x, y, z)$ in that direction at that point.

13. $-\nabla f(x, y) = 2\langle x, y \rangle$; $-\nabla f(-1, 2) = 2\langle -1, 2 \rangle$ is the direction of most rapid decrease. A unit vector in that direction is $\mathbf{u} = \left(\frac{1}{\sqrt{5}}\right)\langle -1, 2 \rangle$.

14. $-\nabla f(x, y) = \langle -3\cos(3x - y),\ \cos(3x - y) \rangle$;

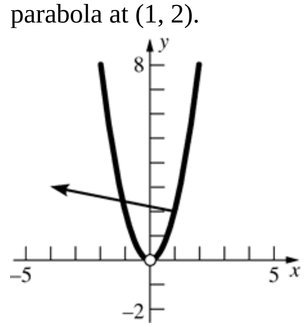
$-\nabla f\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)\langle -3, 1 \rangle$ is the direction of most rapid decrease. A unit vector in that direction is $\left(\frac{1}{\sqrt{10}}\right)\langle -3, 1 \rangle$.

15. The level curves are $\frac{y}{x^2} = k$. For $\mathbf{p} = (1, 2)$,

$\mathbf{k} = 2$, so the level curve through (1, 2) is $\frac{y}{x^2} = 2$ or $y = 2x^2$ $(x \neq 0)$.

$\nabla f(x, y) = \left\langle -2yx^{-3},\ x^{-2} \right\rangle$

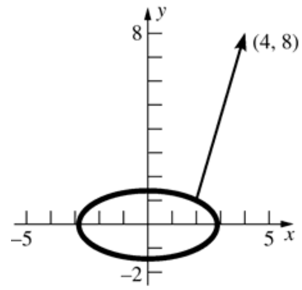
$\nabla f(1, 2) = \langle -4, 1 \rangle$, which is perpendicular to the parabola at (1, 2).



16. At (2, 1), $x^2 + 4y^2 = 8$ is the level curve.

$\nabla f(x, y) = \langle 2x, 8y \rangle$

$\nabla f(2, 1) = 4\langle 1, 2 \rangle$, which is perpendicular to the level curve at (2, 1).



17. $u = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

$D_{\mathbf{u}} f(x, y, z) = \langle y, x, 2z \rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

$D_{\mathbf{u}} f(1, 1, 1) = \frac{2}{3}$

18. $\left(0, \frac{\pi}{3}\right)$ is on the y-axis, so the unit vector toward the origin is $-\mathbf{j}$.

$D_{\mathbf{u}}(x, y) = \left\langle -e^{-x}\cos y,\ -e^{-x}\sin y \right\rangle \cdot \langle 0, -1 \rangle$

$= e^{-x}\sin y$;

$D_{\mathbf{u}}\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

19. **a.** Hottest if denominator is smallest; i.e., at the origin.

b. $\nabla T(x, y, z) = \frac{-200\langle 2x, 2y, 2z \rangle}{(5 + x^2 + y^2 + z^2)^2}$;

$\nabla T(1, -1, 1) = \left(-\frac{25}{4}\right)\langle 1, -1, 1 \rangle$

$\langle -1, 1, -1 \rangle$ is one vector in the direction of greatest increase.

c. Yes

20. $-\nabla V(x, y, z)$

$= -100e^{-(x^2+y^2+z^2)}\langle -2x, -2y, -2z \rangle$

$= 200e^{-(x^2+y^2+z^2)}\langle x, y, z \rangle$ is the direction of greatest decrease at (x, y, z), and it points away from the origin.

21. $\nabla f(x, y, z)$

$$= \left\langle x\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cos\sqrt{x^2+y^2+z^2},\right.$$

$$y\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cos\sqrt{x^2+y^2+z^2},$$

$$\left. z\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cos\sqrt{x^2+y^2+z^2}\right\rangle$$

$$= \left(\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \cos\sqrt{x^2+y^2+z^2}\right)\langle x, y, z\rangle$$

which either points towards or away from the origin.

22. Let $D = \sqrt{x^2+y^2+z^2}$ be the distance. Then we have

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right\rangle = \left\langle \frac{dT}{dD}\frac{\partial D}{\partial x}, \frac{dT}{dD}\frac{\partial D}{\partial y}, \frac{dT}{dD}\frac{\partial D}{\partial z}\right\rangle$$

$$= \left\langle \frac{dT}{dD} x\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}, \frac{dT}{dD} y\left(x^2+y^2+z^2\right)^{-\frac{1}{2}},\right.$$

$$\left. \frac{dT}{dD} z\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right\rangle$$

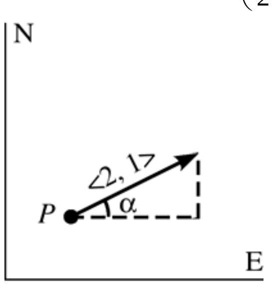
$$= \left(\frac{dT}{dD}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right)\langle x, y, z\rangle$$

which either points towards or away from the origin.

23. He should move in the direction of

$$-\nabla f(\mathbf{p}) = -\langle f_x(\mathbf{p}),\ f_y(\mathbf{p})\rangle = -\left\langle -\frac{1}{2}, -\frac{1}{4}\right\rangle$$

$= \left(\frac{1}{4}\right)\langle 2, 1\rangle$. Or use $\langle 2, 1\rangle$. The angle α formed with the East is $\tan^{-1}\left(\frac{1}{2}\right) \approx 26.57°$ (N63.43°E).



24. The unit vector from (2, 4) toward (5, 0) is $\left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$. Then

$$D_{\mathbf{u}} f(2, 4) = \langle -3, 8\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle = -8.2.$$

25. The climber is moving in the direction of $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\langle -1, 1\rangle$. Let

$$f(x, y) = 3000e^{-(x^2+2y^2)/100}.$$

$$\nabla f(x, y) = 3000e^{-(x^2+2y^2)/100}\left\langle -\frac{x}{50}, -\frac{y}{25}\right\rangle;$$

$$f(10, 10) = -600e^{-3}\langle 1, 2\rangle$$

She will move at a slope of

$$D_u(10, 10) = -600e^{-3}\langle 1, 2\rangle \cdot \left(\frac{1}{\sqrt{2}}\right)\langle -1, 1\rangle$$

$$= \left(-300\sqrt{2}\right)e^{-3} \approx -21.1229\ .$$

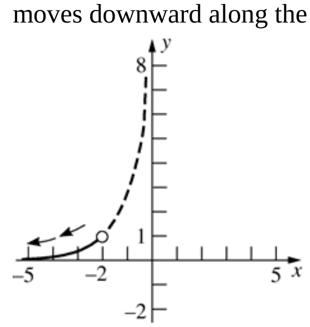
She will descend. Slope is about –21.

26. $\dfrac{\frac{dx}{dt}}{2x} = \dfrac{\frac{dy}{dt}}{-2y}; \dfrac{dx}{x} = \dfrac{dy}{-y}; \ln|x| = -\ln|y| + C$

At $t = 0$: $\ln|-2| = -\ln|1| + C \Rightarrow C = \ln 2.$

$$\ln|x| = -\ln|y| + \ln 2 = \ln\left|\frac{2}{y}\right|; |x| = \left|\frac{2}{y}\right|; |xy| = 2$$

Since the particle starts at (–2, 1) and neither x nor y can equal 0, the equation simplifies to $xy = -2$. $\nabla T(-2, 1) = \langle -4, -2\rangle$, so the particle moves downward along the curve.



27. $\nabla T(x, y) = \langle -4x, -2y\rangle$

$$\frac{dx}{dt} = -4x, \frac{dy}{dt} = -2y$$

$\dfrac{\frac{dx}{dt}}{-4x} = \dfrac{\frac{dy}{dt}}{-2y}$ has solution $|x| = 2y^2$. Since the particle starts at (–2, 1), this simplifies to $x = -2y^2$.

28. $f(1, -1) = 5\langle -1, 1\rangle$

$D_{\langle u_1, u_2\rangle} f(1, -1) = \langle u_1, u_2\rangle \cdot \langle -5, 5\rangle = -5u_1 + 5u_2$

a. $\langle -1, 1\rangle$ (in the direction of the gradient); $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\langle -1, 1\rangle$.

b. $\pm\langle 1, 1\rangle$ (direction perpendicular to gradient); $\mathbf{u} = \left(\pm\frac{1}{\sqrt{2}}\right)\langle 1, 1\rangle$

c. Want $D_{\mathbf{u}} f(1, -1) = 1$ where $|\mathbf{u}| = 1$. That is, want $-5u_1 + 5u_2 = 1$ and $u_1^2 + u_2^2 = 1$. Solutions are $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$ and $\left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$.

29. a. $\nabla T(x, y, z) = \left\langle -\frac{10(2x)}{(x^2+y^2+z^2)^2}, -\frac{10(2y)}{(x^2+y^2+z^2)^2}, -\frac{10(2z)}{(x^2+y^2+z^2)^2}\right\rangle$

$= -\frac{20}{(x^2+y^2+z^2)^2}\langle x, y, z\rangle$

$r(t) = \langle t\cos \pi t, t\sin \pi t, t\rangle$, so $r(1) = \langle -1, 0, 1\rangle$. Therefore, when $t = 1$, the bee is at (–1, 0, 1), and $\nabla T(-1, 0, 1) = -5\langle -1, 0, 1\rangle$.

$\mathbf{r}'(t) = \langle \cos \pi t - \pi t \sin \pi t, \sin \pi t + \pi t \cos \pi t, 1\rangle$, so $\mathbf{r}'(1) = \langle -1, -\pi, 1\rangle$.

$U = \frac{r'(1)}{|r'(1)|} = \frac{\langle -1, -\pi, 1\rangle}{2+\pi^2}$ is the unit tangent vector at (–1, 0, 1).

$D_{\mathbf{u}}T(-1, 0, 1) = \mathbf{u}\cdot\nabla T(-1, 0, 1)$

$= \frac{\langle -1, -\pi, 1\rangle \cdot \langle 5, 0, -5\rangle}{\sqrt{2+\pi^2}} = -\frac{10}{\sqrt{2+\pi^2}} \approx -2.9026$

Therefore, the temperature is decreasing at about 2.9°C per meter traveled when the bee is at (–1, 0, 1); i.e., when $t = 1$ s.

b. Method 1: (First express T in terms of t.)

$T = \frac{10}{x^2+y^2+z^2} = \frac{10}{(t\cos \pi t)^2 + (t \sin \pi t)^2 + (t)^2} = \frac{10}{2t^2} = \frac{5}{t^2}$

$T(t) = 5t^{-2}; T'(t) = -10t^{-3}; t'(1) = -10$

Method 2: (Use Chain Rule.)

$D_tT(t) = \frac{dT}{ds}\frac{ds}{dt} = (D_{\mathbf{u}}T)(|\mathbf{r}'(t)|)$, so $D_tT(t) = [D_{\mathbf{u}}T(-1, 0, 1)](|\mathbf{r}'(1)|) = -\frac{10}{\sqrt{2+\pi^2}}\left(\sqrt{2+\pi^2}\right) = -10$

Therefore, the temperature is decreasing at about 10°C per second when the bee is at (–1, 0, 1); i.e., when $t = 1$ s.

30. a. $D_{\mathbf{u}} f = \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle \cdot \langle f_x \cdot f_y\rangle = -6$, so

$3f_x - 4f_y = -30$.

$D_{\mathbf{v}} f = \left\langle \frac{4}{5}, \frac{3}{5}\right\rangle \cdot \langle f_x, f_y\rangle = 17$, so

$4f_x + 3f_y = 85$.

The simultaneous solution is

$f_x = 10$, $f_y = 15$, so $\nabla f = \langle 10, 15\rangle$.

b. Without loss of generality, let $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$. If θ and ϕ are the angles between u and ∇f, and between $\mathbf{v}$ and ∇f, then:

1. $\theta + \phi = \frac{\pi}{2}$ (if ∇f is in the 1st quadrant).

2. $\theta = \frac{\pi}{2} + \phi$ (if ∇f is in the 2nd quadrant).

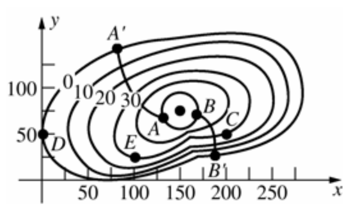
3. $\phi + \theta = \frac{3\pi}{2}$ (if ∇f is in the 3rd quadrant).

4. $\phi = \frac{\pi}{2} + \theta$ (if ∇f is in the 4th quadrant).

In each case $\cos\phi = \sin\theta$ or $\cos\phi = -\sin\theta$, so $\cos^2\phi = \sin^2\theta$. Thus,

$$(D_{\mathbf{u}} f)^2 + (D_{\mathbf{v}} f)^2 = (\mathbf{u} \cdot \nabla f)^2 + (\mathbf{v} \cdot \nabla f)^2$$
$$= |\nabla f|^2 \cos^2\theta + |\nabla f|^2 \cos^2\phi$$
$$= |\nabla f|^2 (\cos^2\theta + \cos^2\phi)$$
$$= |\nabla f|^2 \cos^2\theta + \sin^2\theta = |\nabla f|^2.$$

31.



a. $A'(100, 120)$

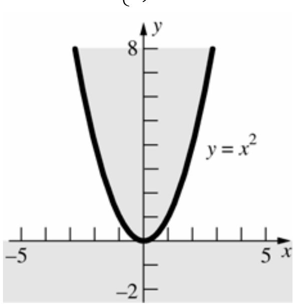
b. $B'(190, 25)$

c. $f_x(C) \approx \frac{20-30}{230-200} = -\frac{1}{3}$; $f_y(D) = 0$;

$D_{\mathbf{u}} f(E) \approx \frac{40-30}{25} = \frac{2}{5}$

32. Graph of domain of f

$$f(x, y) = \begin{Bmatrix} 0, \text{ in shaded region} \\ 1, \text{ elsewhere} \end{Bmatrix}$$



$\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist since $(x, y) \to (0, 0)$:

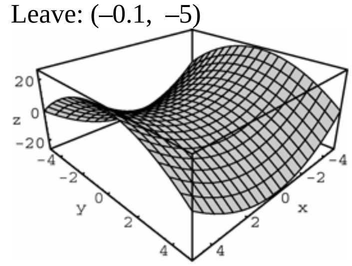
along the y-axis, $f(x, y) = 0$, but along the $y = x^4$ curve, $f(x, y) = 1$.

Therefore, f is not differentiable at the origin. But $D_{\mathbf{u}} f(0, 0)$ exists for all u since

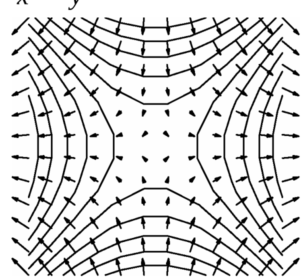
$$f_x(0, 0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0-0}{h}$$
$$= \lim_{h\to 0}(0) = 0, \text{ and}$$

$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0-0}{h}$$
$$= \lim_{h\to 0}(0) = 0, \text{ so } \nabla f(0, 0) = \langle 0, 0 \rangle = \mathbf{0}. \text{ Then}$$
$$D_{\mathbf{u}} f(0, 0) = \nabla f(0, 0) \cdot \mathbf{u} = \mathbf{0} \cdot \mathbf{u} = 0.$$

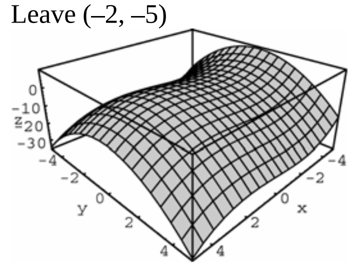
33. Leave: (–0.1, –5)



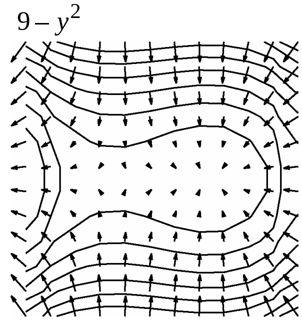
$x^2 - y^2$



34. Leave (–2, –5)



$\frac{x - x^3}{9 - y^2}$



35. Leave: (3, 5)

36. (4.2, 4.2)

12.6 Concepts Review

1. $\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$

2. $y^2\cos t+2xy(-\sin t)$
$=\cos^3 t-2\sin^2 t\cos t$

3. $\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

4. 12

Problem Set 12.6

1. $\frac{dw}{dt}=(2xy^3)(3t^2)+(3x^2y^2)(2t)$
$=(2t^9)(3t^2)+(3t^{10})(2t)=12t^{11}$

2. $\frac{dw}{dt}=(2xy-y^2)(-\sin t)+(x^2-2xy)(\cos t)$
$=(\sin t+\cos t)(1-3\sin t\cos t)$

3. $\frac{dw}{dt}=(e^x\sin y+e^y\cos x)(3)+(e^x\cos y+e^y\sin x)(2)=3e^{3t}\sin 2t+3e^{2t}\cos 3t+2e^{3t}\cos 2t+2e^{2t}\sin 3t$

4. $\frac{dw}{dt}=\left(\frac{1}{x}\right)\sec^2 t+\left(-\frac{1}{y}\right)(2\sec^2 t\tan t)=\frac{\sec^2 t}{\tan t}-2\tan t=\frac{\sec^2 t-2\tan^2 t}{\tan t}=\frac{1-\tan^2 t}{\tan t}$

5. $\frac{dw}{dt}=[yz^2(\cos(xyz^2)](3t^2)+[xz^2\cos(xyz^2)](2t)+[2xyz\cos(xyz^2)](1)$
$=(3yz^2t^2+2xz^2t+2xyz)\cos(xyz^2)=(3t^6+2t^6+2t^6)\cos(t^7)=7t^6\cos(t^7)$

6. $\frac{dw}{dt}=(y+z)(2t)+(x+z)(-2t)+(y+x)(-1)=2t(2-t-t^2)-2t(1-t+t^2)-(1)=-4t^3+2t-1$

7. $\frac{\partial w}{\partial t}=(2xy)(s)+(x^2)(-1)=2st(s-t)s-s^2t^2=s^2t(2s-3t)$

8. $\frac{\partial w}{\partial t}=(2x-x^{-1}y)(-st^{-2})+(-\ln x)(s^2)=s^2\left[1-2t^{-3}-\ln\left(\frac{s}{t}\right)\right]$

9. $\frac{\partial w}{\partial t}=e^{x^2+y^2}(2x)(s\cos t)+e^{x^2+y^2}(2y)(\sin s)=2e^{x^2+y^2}(xs\cos t+y\sin s)$
$=2(s^2\sin t\cos t+t\sin^2 s)\exp(s^2\sin^2 t+t^2\sin^2 s)$

10. $\frac{\partial w}{\partial t}=[(x+y)^{-1}-(x-y)^{-1}](e^s)+[(x+y)^{-1}+(x-y)^{-1}](se^{st})=\frac{2e^{s(t+1)}(st-1)}{t^2e^{2s}-e^{2st}}$

11. $\frac{\partial w}{\partial t} = \frac{x(-s\sin st)}{(x^2+y^2+z^2)^{1/2}} + \frac{y(s\cos st)}{(x^2+y^2+z^2)^{1/2}} + \frac{z(s^2)}{(x^2+y^2+z^2)^{1/2}} = s^4 t(1+s^4t^2)^{-1/2}$

12. $\frac{\partial w}{\partial t} = (e^{xy+z}y)(1) + (e^{xy+z}x)(-1) + (e^{xy+z})(2t) = e^{xy+z}(y-x+2t) = e^{s^2}(0) = 0$

13. $\frac{\partial z}{\partial t} = (2xy)(2) + (x^2)(-2st) = 4(2t+s)(1-st^2) - 2st(2t+s)^2; \left(\frac{\partial z}{\partial t}\right)\bigg|_{(1,\,-2)} = 72$

14. $\frac{\partial z}{\partial s} = (y+1)(1) + (x+1)(rt) = 1 + rt(1+2s+r+t); \left(\frac{\partial z}{\partial s}\right)\bigg|_{(1,\,-1,\,2)} = 5$

15. $\frac{dw}{dx} = (2u - \tan v)(1) + (-u\sec^2 v)(\pi) = 2x - \tan \pi x - \pi x \sec^2 \pi x$

$\frac{dw}{dx}\bigg|_{x=\frac{1}{4}} = \left(\frac{1}{2}\right) - 1 - \left(\frac{\pi}{2}\right) = -\frac{1+\pi}{2} \approx -2.0708$

16. $\frac{\partial w}{\partial \theta} = (2xy)(-\rho\sin\theta\sin\phi) + (x^2)(\rho\cos\theta\sin\phi) + (2z)(0) = \rho^3\cos\theta\sin^3\phi(-2\sin^2\theta + \cos^2\theta);$

$\left(\frac{\partial w}{\partial \theta}\right)\bigg|_{(2,\,\pi,\,\frac{\pi}{2})} = -8$

17. $V(r, h) = \pi r^2 h,\ \frac{dr}{dt} = 0.5$ in./yr,

$\frac{dh}{dt} = 8$ in./yr

$\frac{dV}{dt} = (2\pi rh)\left(\frac{dr}{dt}\right) + (\pi r^2)\left(\frac{dh}{dt}\right);$

$\left(\frac{dV}{dt}\right)\bigg|_{(20,\,400)} = 11200\pi$ in.^{3}/yr

$= \frac{11200\pi \text{ in.}^3}{1 \text{ yr}} \times \frac{1 \text{ board ft}}{144 \text{ in.}^3} \approx 244.35$ board ft/yr

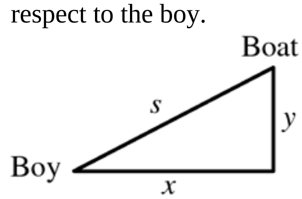
18. Let $T = e^{-x-3y}$.

$\frac{dT}{dt} = e^{-x-3y}(-1)\frac{dx}{dt} + e^{-x-3y}(-3)\frac{dy}{dt}$

$= e^{-x-3y}(-1)(2) + e^{-x-3y}(-3)(2) = -8e^{-x-3y}$

$\frac{dT}{dt}\bigg|_{(0,\,0)} = -8$, so the temperature is decreasing at 8°/min.

19. The stream carries the boat along at 2 ft/s with respect to the boy.



$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 4,\ s^2 = x^2 + y^2$

$2s\left(\frac{ds}{dt}\right) = 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right)$

$\frac{ds}{dt} = \frac{(2x+4y)}{s}$

When $t = 3$, $x = 6$, $y = 12$, $s = 6\sqrt{5}$. Thus,

$\left(\frac{ds}{dt}\right)\bigg|_{t=3} = \sqrt{20} \approx 4.47$ ft/s

20. $V(r, h) = \left(\frac{1}{3}\right)\pi r^2 h,\ \frac{dh}{dt} = 3$ in./min,

$\frac{dr}{dt} = 2$ in./min

$\frac{dV}{dt} = \left(\frac{2}{3}\right)\pi rh\left(\frac{dr}{dt}\right) + \left(\frac{1}{3}\right)\pi r^2\left(\frac{dh}{dt}\right);$

$\left(\frac{dV}{dt}\right)\bigg|_{(40,\,100)} = \frac{20{,}800\pi}{3} \approx 21{,}782$ in.^{3}/min

21. Let $F(x, y) = x^3 + 2x^2y - y^3 = 0$.

Then $\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{(3x^2 + 4xy)}{2x^2 - 3y^2} = \frac{3x^2 + 4xy}{3y^2 - 2x^2}$.

22. Let $F(x, y) = ye^{-x} + 5x - 17 = 0$.

$$\frac{dy}{dx} = -\frac{(-ye^{-x} + 5)}{e^{-x}} = y - 5e^x$$

23. Let $F(x, y) = x \sin y + y \cos x = 0$.

$$\frac{dy}{dx} = -\frac{(\sin y - y \sin x)}{x \cos y + \cos x} = \frac{y \sin x - \sin y}{x \cos y + \cos x}$$

24. Let $F(x, y) = x^2 \cos y - y^2 \sin x = 0$.

Then $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-(2x \cos y - y^2 \cos x)}{-x^2 \sin y - 2y \sin x}$

$= \frac{2x \cos y - y^2 \cos x}{x^2 \sin y + 2y \sin x}$.

25. Let $F(x, y, z) = 3x^2z + y^3 - xyz^3 = 0$.

$$\frac{\partial z}{\partial x} = -\frac{(6xz - yz^3)}{3x^2 - 3xyz^2} = \frac{yz^3 - 6xz}{3x^2 - 3xyz^2}$$

26. Let $f(x, y, z) = ye^{-x} + z \sin x = 0$.

$$\frac{\partial x}{\partial z} = \frac{-\sin x}{-ye^{-x} + z \cos x} = \frac{\sin x}{ye^{-x} - z \cos x}$$

27. $\frac{\partial T}{\partial s} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial T}{\partial w}\frac{\partial w}{\partial s}$

28. We use the z_r notation for $\frac{\partial z}{\partial r}$.

$z_r = z_x x_r + z_y y_r = z_x \cos\theta + z_y \sin\theta$

$z_\theta = z_x x_\theta + z_y y_\theta = z_x(-r \sin\theta) + z_y(r \cos\theta)$, so

$r^{-1}z_\theta = -z_x \sin\theta + z_y \cos\theta$. Thus,

$(z_r)^2 + (r^{-2})(z_\theta)^2 = (z_x \cos\theta + z_y \sin\theta)^2$

$\qquad + (-z_x \sin\theta + z_y \cos\theta)^2$

$= (z_x)^2 + (z_y)^2$ (expanding and using

$\sin^2\theta + \cos^2\theta = 1$).

29. $y = \left(\frac{1}{2}\right)[f(u) + f(v)]$,

where $u = x - ct$, $v = x + ct$.

$y_x = \left(\frac{1}{2}\right)[f'(u)(1) + f'(v)(1)] = \left(\frac{1}{2}\right)[f'(u) + f'(v)]$

$y_{xx} = \left(\frac{1}{2}\right)[f''(u)(1) + f''(v)(1)]$

$= \left(\frac{1}{2}\right)[f''(u) + f''(v)]$

$y_t = \left(\frac{1}{2}\right)[f'(u)(-c) + f'(v)(c)]$

$= \left(-\frac{c}{2}\right)[f'(u) - f'(u)]$

$y_{tt} = \left(-\frac{c}{2}\right)[f''(u)(-c) - f''(v)(c)]$

$= \left(\frac{c^2}{2}\right)[f''(u) + f''(v)] = c^2 y_{xx}$

30. Let $w = f(x, y, z)$ where $x = r - s$, $y = s - t$, $z = t - r$. Then

$w_r + w_s + w_t = (w_x x_r + w_x x_s) + (w_y y_s + w_y y_t)$

$\qquad + (w_z z_t + w_z z_r)$

$= [w_x(1) + w_x(-1)] + [w_y(1) + w_y(-1)]$

$\qquad + [w_z(1) + w_z(-1)]$

$= 0$

31. Let $w = \int_x^y f(u)du = -\int_y^x f(u)du$, where $x = g(t)$, $y = h(t)$.

Then

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = -f(x)g'(t) + f(y)h'(t)$

$= f(h(t))h'(t) - f(g(t))g'(t)$.

Thus, for the particular function given

$F'(t) = \sqrt{9 + (t^2)^4}\,(2t)$

$\qquad - \sqrt{9 + \left(\sin\sqrt{2}\pi t\right)^4}\left(\sqrt{2}\pi \cos\sqrt{2}\pi t\right)$;

$F'\left(\sqrt{2}\right) = (5)\left(2\sqrt{2}\right) - (3)\left(\sqrt{2}\pi\right)$

$= 10\sqrt{2} - 3\sqrt{2}\pi \approx 0.8135$.

32. If $f(tx, ty) = tf(x, y)$, then

$\frac{d}{dt}[f(tx, ty)] = \frac{d}{dt}[tf(x, y)]$.

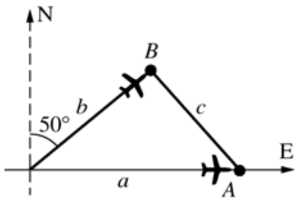
That is,

$[f_{tx}(tx, ty)][x] + [f_{ty}(tx, ty)][y] = f(x, y)$.

Letting $t = 1$ yields the desired result.

33. $c^2 = a^2 + b^2 - 2ab\cos 40°$ (Law of Cosines) where a, b, and c are functions of t.

$2cc' = 2aa' + 2bb' - 2(a'b + ab')\cos 40°$ so $c' = \dfrac{aa' + bb' - (a'b + ab')\cos 40°}{c}$.



When $a = 200$ and $b = 150$, $c^2 = (200)^2 + (150)^2 - 2(200)(150)\cos 40° = 62{,}500 - 60{,}000\cos 40°$.

It is given that $a' = 450$ and $b' = 400$, so at that instant,

$$c' = \frac{(200)(450) + (150)(400) - [(450)(150) + (200)(400)]\cos 40°}{\sqrt{62{,}500 - 60{,}000\cos 40°}} \approx 288.$$

Thus, the distance between the airplanes is increasing at about 288 mph.

34. $r = \langle x, y, z \rangle$, so $r^2 = |r|^2 = x^2 + y^2 + z^2$.

$$F = \frac{GMm}{x^2 + y^2 + z^2}, \text{ so}$$

$$F'(t) = F_m m'(t) + F_x x'(t) + F_y y'(t) + F_z z'(t)$$

$$= \frac{GMm'(t)}{x^2 + y^2 + z^2} - \frac{2GMmxx'(t)}{(x^2 + y^2 + z^2)^2} - \frac{2GMmyy'(t)}{(x^2 + y^2 + z^2)^2} + \frac{2GMmzz'(t)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{GM[(x^2 + y^2 + z^2)m'(t) - 2m(xx'(t) + yy'(t) + zz'(t)]}{(x^2 + y^2 + z^2)^2}.$$

12.7 Concepts Review

1. perpendicular

2. $\langle 3, 1, -1 \rangle$

3. $x - +4(y - 1) + 6(z - 1) = 0$

4. $\dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

Problem Set 12.7

1. $\nabla F(x, y, z) = 2\langle x, y, z \rangle$;

$\nabla F\left(2, 3, \sqrt{3}\right) = 2\left\langle 2, 3, \sqrt{3} \right\rangle$

Tangent Plane:

$2(x - 2) + 3(y - 3) + \sqrt{3}\left(z - \sqrt{3}\right) = 0$, or

$2x + 3y + \sqrt{3}z = 16$

2. $\nabla F(x, y, z) = 2\langle 8x, y, 8z \rangle$;

$\nabla F\left(1, 2, \dfrac{\sqrt{2}}{2}\right) = 4\left\langle 4, 1, 2\sqrt{2} \right\rangle$

Tangent Plane:

$4(x - 1) + 1(y - 2) + 2\sqrt{2}\left(z - \dfrac{\sqrt{2}}{2}\right)$, or

$4x + y + 2\sqrt{2}z = 8$.

3. Let $F(x, y, z) = x^2 - y^2 + z^2 + 1 = 0$.

$\nabla F(x, y, z) = \langle 2x, -2y, 2z \rangle = 2\langle x, -y, z \rangle$

$\nabla F\left(1, 3, \sqrt{7}\right) = 2\left\langle 1, -3, \sqrt{7} \right\rangle$, so $\left\langle 1, -3, \sqrt{7} \right\rangle$ is normal to the surface at the point. Then the tangent plane is

$1(x - 1) - 3(y - 3) + \sqrt{7}\left(z - \sqrt{7}\right) = 0$, or more simply, $x - 3y + \sqrt{7}z = -1$.

4. $\nabla f(x, y, z) = 2\langle x, y, -z\rangle$;
$\nabla f(2, 1, 1) = 2\langle 2, 1, -1\rangle$
Tangent plane:
$2(x-2)+1(y-1)-1(z-1)=0$, or $2x+y-z=4$.

5. $\nabla f(x, y) = \left(\frac{1}{2}\right)\langle x, y\rangle; \nabla f(2, 2) = \langle 1, 1\rangle$
Tangent plane: $z-2=1(x-2)+1(y-2)$, or
$x+y-z=2$.

6. Let $f(x, y) = xe^{-2y}$.
$\nabla f(x, y) = \left\langle e^{-2y}, -2xe^{-2y}\right\rangle$
$\nabla f(1,0) = \langle 1, -2\rangle$
Then $\langle 1, -2, -1\rangle$ is normal to the surface at
(1, 0, 1), and the tangent plane is
$1(x-1)-2(y-0)-1(z-1)=0$, or $x-2y-z=0$.

7. $\nabla f(x, y) = \left\langle -4e^{3y}\sin 2x, 6e^{3y}\cos 2x\right\rangle$;
$\nabla f\left(\frac{\pi}{3}, 0\right) = \left\langle -2\sqrt{3}, -3\right\rangle$
Tangent plane: $z+1=-2\sqrt{3}\left(x-\frac{\pi}{3}\right)-3(y-0)$,
or $2\sqrt{3}x+3y+z=\frac{\left(2\sqrt{3}\pi-3\right)}{3}$.

8. $\nabla f(x, y) = \left(\frac{1}{2}\right)\left\langle \frac{1}{\sqrt{x}}, \frac{1}{\sqrt{y}}\right\rangle$; $\nabla f(1, 4) = \left\langle \frac{1}{2}, \frac{1}{4}\right\rangle$
Tangent plane: $z-3=\left(\frac{1}{2}\right)(x-1)+\left(\frac{1}{4}\right)(y-4)$,
or $\frac{1}{2}x+\frac{1}{4}y-z=-\frac{3}{2}$.

9. Let $z = f(x, y) = 2x^2y^3$;
$dz = 4xy^3dx + 6x^2y^2dy$. For the points given,
$dx = -0.01$, $dy = 0.02$,
$dz = 4(-0.01)+6(0.02) = 0.08$.
$\Delta z = f(0.99, 1.02) - f(1, 1)$
$= 2(0.99)^2(1.02)^3 - 2(1)^2(1)^3 \approx 0.08017992$

10. $dz = (2x-5y)dx + (-5x+1)dy$
$= (-11)(0.03)+(-9)(-0.02) = -0.15$
$\Delta z = f(2.03, 2.98) - f(2,3) = -0.1461$

11. $dz = 2x^{-1}dx + y^{-1}dy = (-1)(0.02)+\left(\frac{1}{4}\right)(-0.04)$
$= -0.03$
$\Delta z = f(-1.98, 3.96) - f(-2.4)$
$= \ln[(-1.98)^2(3.96)] - \ln 16 \approx -0.030151$

12. Let $z = f(x, y) = \tan^{-1} xy$;
$dz = \frac{y}{1+x^2y^2}dx + \frac{x}{1+x^2y^2}dy$;
$= \frac{(-0.5)(-0.03)+(-2)(-0.01)}{1+(4)(0.25)} = 0.0175.$
$\Delta z = f(-2.03, -0.51) - f(-2, -0.5) \approx 0.017342$

13. Let
$F(x, y, z) = x^2 - 2xy - y^2 - 8x + 4y - z = 0$;
$\nabla F(x, y, z) = \langle 2x-2y-8, -2x-2y+4, -1\rangle$
Tangent plane is horizontal if $\nabla F = \langle 0, 0, k\rangle$ for any $k \neq 0$.
$2x - 2y - 8 = 0$ and $-2x - 2y + 4 = 0$ if $x = 3$ and $y = -1$. Then $z = -14$. There is a horizontal tangent plane at (3, –1, –14).

14. $\langle 8, -3, -1\rangle$ is normal to $8x - 3y - z = 0$.
Let $F(x,y,z) = 2x^2 + 3y^2 - z$.
$\nabla F(x, y, z) = \langle 4x, 6y, -1\rangle$ is normal to
$z = 2x^2 + 3y^2$ at (x, y, z). $4x = 8$ and
$6y = -3$, if $x = 2$ and $y = -\frac{1}{2}$; then
$z = 8.75$ at $\left(2, -\frac{1}{2}, 8.75\right)$.

15. For $F(x, y, z) = x^2 + 4y + z^2 = 0$,
$\nabla F(x, y, z) = \langle 2x, 4, 2z\rangle = 2\langle x, 2, z\rangle$.
$F(0, -1, 2) = 0$, and
$\nabla F(0, -1, 2) = 2\langle 0, 2, 2\rangle = 4\langle 0, 1, 1\rangle$.
For $G(x, y, z) = x^2 + y^2 + z^2 - 6z + 7 = 0$,
$\nabla G(x, y, z) = \langle 2x, 2y, 2z-6\rangle = 2\langle x, y, z-3\rangle$.
$G(0, -1, 2) = 0$, and
$\nabla G(0, -1, 2) = 2\langle 0, -1, -1\rangle = -2\langle 0, 1, 1\rangle$.
$\langle 0, 1, 1\rangle$ is normal to both surfaces at
(0, –1, 2) so the surfaces have the same tangent plane; hence, they are tangent to each other at (0, –1, 2).

16. (1, 1, 1) satisfies each equation, so the surfaces intersect at (1, 1, 1). For

$z = f(x, y) = x^2 y : \nabla f(x, y) = \left\langle 2xy, x^2 \right\rangle;$

$\nabla f(1, 1) = \langle 2, 1 \rangle$, so $\langle 2, 1, -1 \rangle$ is normal at (1, 1, 1).

For $f(x, y, z) = x^2 - 4y + 3 = 0;$

$\nabla f(x, y, z) = \langle 2, -4, 0 \rangle;$

$\nabla f(1, 1, 1) = \langle 2, -4, 0 \rangle$ so $\langle 2, -4, 0 \rangle$ is normal at (1, 1, 1).

$\langle 2, 1, -1 \rangle \cdot \langle 2, -4, 0 \rangle = 0$, so the normals, hence tangent planes, and hence the surfaces, are perpendicular at (1, 1, 1).

17. Let $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 12 = 0;$

$\nabla F(x, y, z) = 2\langle x, 2y, 3z \rangle$ is normal to the plane.

A vector in the direction of the line, $\langle 2, 8, -6 \rangle = 2\langle 1, 4, -3 \rangle$, is normal to the plane.

$\langle x, 2y, 3z \rangle = k\langle 1, 4, -3 \rangle$ and (x, y, z) is on the surface for points (1, 2, –1) [when k =1] and (–1, –2, 1) [when k = –1].

18. Let $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

$$\nabla F(x, y, z) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

$$\nabla F(x_0, y_0, z_0) = 2\left\langle \frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2} \right\rangle$$

The tangent plane at (x_0, y_0, z_0) is

$$\frac{x_0(x - x_0)}{a^2} + \frac{y_0(y - y_0)}{b^2} + \frac{z_0(z - z_0)}{c^2} = 0.$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \right) = 0$$

Therefore, $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} + \dfrac{z_0 z}{c^2} = 1$, since

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1.$$

19. $\nabla f(x, y, z) = 2\langle 9x, 4y, 4z \rangle;$

$\nabla f(1, 2, 2) = 2\langle 9, 8, 8 \rangle$

$\nabla g(x, y, z) = 2\langle 2x, -y, 3z \rangle;$

$\nabla f(1, 2, 2) = 4\langle 1, -1, 3 \rangle$

$\langle 9, 8, 8 \rangle \times \langle 1, -1, 3 \rangle = \langle 32, -19, -17 \rangle$

Line: $x = 1 + 32t, y = 2 - 19t, z = 2 - 17t$

20. Let $f(x, y, z) = x - z^2$, and $g(x, y, z) = y - z^3$.

$\nabla f(x, y, z) = \langle 1, 0, -2z \rangle$ and

$\nabla g(x, y, z) = \left\langle 0, 1, -3z^2 \right\rangle$

$\nabla f(1, 1, 1) = \langle 1, 0, -2 \rangle$ and

$\nabla g(1, 1, 1) = \langle 0, 1, -3 \rangle$

$\langle 1, 0, -2 \rangle \times \langle 0, 1, -3 \rangle = \langle 2, 3, 1 \rangle$

Line: $x = 1 + 2t, y = 1 + 3t, z = 1 + t$

21. $dS = S_A dA + S_W dW$

$$= -\frac{W}{(A - W)^2} dA + \frac{A}{(A - W)^2} dW = \frac{-W dA + A dW}{(A - W)^2}$$

At W = 20, A = 36:

$$dS = \frac{-20 dA + 36 dW}{256} = \frac{-5 dA + 9 dW}{64}.$$

Thus, $|dS| \le \dfrac{5|dA| + 9|dW|}{64} \le \dfrac{5(0.02) + 9(0.02)}{64}$

$= 0.004375$

22. $V = lwh$, $dl = dw = \dfrac{1}{2}, dh = \dfrac{1}{4}$, l = 72, w = 48, h = 36

$dV = whdl + lhdw + lwdh = 3024$ in.^{3} (1.75 ft^3)

23. $V = \pi r^2 h, dV = 2\pi rh\, dr + \pi r^2 dh$

$|dV| \le 2\pi rh|dr| + \pi r^2 |dh| \le 2\pi rh(0.02r) + \pi r^2 (0.03h)$

$= 0.04\pi r^2 h + 0.03\pi r^2 h = 0.07V$

Maximum error in V is 7%.

24. $T = f(L, g) = 2\pi\sqrt{\dfrac{L}{g}}$

$dT = f_L dL + f_g dg$

$$= 2\pi\left(\frac{1}{2\sqrt{\frac{L}{g}}} \right)\left(\frac{1}{g} \right) dL + 2\pi\left(\frac{1}{2\sqrt{\frac{L}{g}}} \right)\left(-\frac{L}{g^2} \right) dg$$

$$= \frac{\pi(gdL - Ldg)}{g^2\sqrt{\frac{L}{g}}}, \text{ so}$$

$$\frac{dT}{T} = \frac{\pi(gdL - Ldg)}{\left(2\pi\sqrt{\frac{L}{g}}\right)\left(g^2\sqrt{\frac{L}{g}}\right)} = \frac{gdL - Ldg}{2gL}$$

$$= \frac{1}{2}\left(\frac{dL}{L} - \frac{dg}{g} \right).$$

Therefore,

$$\left|\frac{dT}{T}\right| \le \frac{1}{2}\left(\left|\frac{dL}{L}\right| + \left|\frac{dg}{g}\right| \right) = \frac{1}{2}(0.5\% + 0.3\%) = 0.4\%.$$

25. Solving for R, $R = \dfrac{R_1R_2}{R_1+R_2}$, so

$$\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1+R_2)^2} \text{ and } \frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1+R_2)^2}.$$

Therefore, $dR = \dfrac{R_2^2 dR_1 + R_1^2 dR_2}{(R_1+R_2)^2}$;

$|dR| \le \dfrac{R_2^2|dR_1| + R_1^2|dR_2|}{(R_1+R_2)^2}$. Then at $R_1 = 25$,

$R_2 = 100, dR_1 = dR_2 = 0.5$, $R = \dfrac{(25)(100)}{25+100} = 20$

and $|dR| \le \dfrac{(100)^2(0.5)+(25)^2(0.5)}{(125)^2} = 0.34$.

26. Let $F(x, y, z) = x^2 + y^2 + 2z^2$.

$\nabla F(x, y, z) = \langle 2x, 2y, 4z\rangle$;

$\nabla F(1, 2, 1) = 2\langle 1, 2, 2\rangle$; $\dfrac{\nabla F}{|\nabla F|} = \left\langle \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right\rangle$

Thus, $\mathbf{u} = \left\langle \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right\rangle$ is the unit vector in the direction of flight, and

$\langle x, y, z\rangle = \langle 1, 2, 1\rangle + 4t\left\langle \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right\rangle$ is the location of the bee along its line of flight t seconds after takeoff. Using the parametric form of the line of flight to substitute into the equation of the plane yields $t = 3$ as the time of intersection with the plane. Then substituting this value of t into the equation of the line yields $x = 5$, $y = 10$, $z = 9$ so the point of intersection is (5, 10, 9).

27. Let $F(x, y, z) = xyz = k$; let (a, b, c) be any point on the surface of F.

$\nabla F(x, y, z) = \langle yz, xz, xy\rangle = \left\langle \dfrac{k}{x}, \dfrac{k}{y}, \dfrac{k}{z}\right\rangle$

$= k\left\langle \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}\right\rangle$

$\nabla F(a, b, c) = k\left\langle \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}\right\rangle$

An equation of the tangent plane at the point is

$\left(\dfrac{1}{a}\right)(x-a) + \left(\dfrac{1}{b}\right)(x-b) + \left(\dfrac{1}{c}\right)(x-c) = 0$, or

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 3$.

Points of intersection of the tangent plane on the coordinate axes are $(3a, 0, 0)$, $(0, 3b, 0)$, and $(0, 0, 3c)$.

The volume of the tetrahedron is

$\left(\dfrac{1}{3}\right)$(area of base)(altitude)$= \dfrac{1}{3}\left(\dfrac{1}{2}|3a||3b|\right)(|3c|)$

$= \dfrac{9|abc|}{2} = \dfrac{9|k|}{2}$ (a constant).

28. If $F(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$, then $\nabla F(x, y, z) = 0.5\left\langle \dfrac{1}{\sqrt{x}}, \dfrac{1}{\sqrt{y}}, \dfrac{1}{\sqrt{z}}\right\rangle$. The equation of the tangent is

$0.5\left\langle \dfrac{1}{\sqrt{x_0}}, \dfrac{1}{\sqrt{y_0}}, \dfrac{1}{\sqrt{z_0}}\right\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$, or $\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = a$.

Intercepts are $a\sqrt{x_0}, a\sqrt{y_0}, a\sqrt{z_0}$; so the sum is $a\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = a^2$.

29. $f(x,\ y) = (x^2 + y^2)^{1/2}$; $f(3,\ 4) = 5$

$f_x(x,\ y) = x(x^2 + y^2)^{-1/2}$; $f_x(3,\ 4) = \frac{3}{5} = 0.6$; $f_y = (x,\ y) = y(x^2 + y^2)^{-1/2}$; $f_x(3,\ 4) = \frac{4}{5} = 0.8$

$f_{xx}(x,\ y) = y^2(x^2 + y^2)^{-3/2}$; $f_x(3,\ 4) = \frac{16}{125} = 0.128$; $f_{xy}(x,\ y) = -xy(x^2 + y^2)^{-3/2}$;

$f_{xy}(3,\ 4) = -\frac{12}{125} = -0.096$

$f_{yy} = x^2(x^2 + y^2)^{-3/2}$; $f_{xx}(3,\ 4) = \frac{9}{125} = 0.072$

Therefore, the second order Taylor approximation is

$f(x,\ y) = 5 + 0.6(x-3) + 0.8(y-4) + 0.5[0.128(x-3)^2 + 2(-0.096)(x-3)(y-4) + 0.072(y-4)^2]$

a. First order Taylor approximation: $f(x, y) = 5 + 0.6(x - 3) + 0.8(y - 4)$.
Thus, $f(3.1, 3.9) \approx 5 + 0.6(0.1) + 0.8(-0.1) = 4.98$.

b. $f(3.1, 3.9) \approx 5 + 0.6(-0.1) + 0.8(0.1) + 0.5[0.128(0.1)^2 + 2(-0.096)(0.1)(-0.1) + 0.072(-0.1)^2] = 4.98196$

c. $f(3.1,\ 3.9) \approx 4.9819675$

30. $f(x, y) = \tan\left(\frac{x^2 + y^2}{64}\right)$; $f(0, 0) = 0$

$f_x(x, y) = \frac{x}{32} \cdot \sec^2\left(\frac{x^2 + y^2}{64}\right)$; $f_x(0, 0) = 0$

$f_y(x, y) = \frac{y}{32} \cdot \sec^2\left(\frac{x^2 + y^2}{64}\right)$; $f_y(0, 0) = 0$

$f_{xx}(x, y) = \frac{2x^2}{32^2}\sec^2\left(\frac{x^2 + y^2}{64}\right)\tan\left(\frac{x^2 + y^2}{64}\right) + \frac{1}{32}\sec^2\left(\frac{x^2 + y^2}{64}\right)$; $f_{xx}(0, 0) = \frac{1}{32}$

$f_{yy}(x, y) = \frac{2y^2}{32^2}\sec^2\left(\frac{x^2 + y^2}{64}\right)\tan\left(\frac{x^2 + y^2}{64}\right) + \frac{1}{32}\sec^2\left(\frac{x^2 + y^2}{64}\right)$; $f_{yy}(0, 0) = \frac{1}{32}$

When computed, each term of $f_{xy}(x, y)$ will contain either *x* or *y*, resulting in $f_{xy}(0, 0) = 0$. Therefore, the second-order Taylor approximation is

$f(x, y) = 0 + 0 \cdot x + 0 \cdot y + \frac{1}{2}\left[\frac{1}{32}x^2 + 2 \cdot 0 \cdot x \cdot y + \frac{1}{32}y^2\right] = \frac{1}{64}x^2 + \frac{1}{64}y^2$

a. The first-order Taylor approximation is $f(x, y) = 0 + 0 \cdot x + 0 \cdot y = 0$; Thus, $f(0.2, -0.3) \approx 0$.

b. $f(0.2, -0.3) \approx \frac{1}{64}(0.2)^2 + \frac{1}{64}(-0.3)^3 = 0.00203125$

c. $f(0.2, -0.3) \approx 0.0020312528$

12.8 Concepts Review

1. closed bounded

2. boundary; stationary; singular

3. $\nabla f(x_0, y_0) = \mathbf{0}$

4. $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$

Problem Set 12.8

1. $\nabla f(x, y) = \langle 2x-4, 8y \rangle = \langle 0, 0 \rangle$ at (2, 0), a stationary point.
$D = f_{xx} f_{yy} - f_{xy}^2 = (2)(8) - (0)^2 = 16 > 0$ and $f_{xx} = 2 > 0.$ Local minimum at (2, 0).

2. $\nabla f(x, y) = \langle 2x-2, 8y+8 \rangle = \langle 0, 0 \rangle$ at (1, –1), a stationary point. $D = f_{xx} f_{yy} - f_{xy}^2 = (2)(8) - (0)2 = 16 > 0$ and $f_{xx} = 2 > 0.$ Local minimum at (1, –1).

3. $\nabla f(x, y) = \left\langle 8x^3 - 2x, 6y \right\rangle = \left\langle 2x(4x^2-1), 6y \right\rangle$
$= \langle 0, 0 \rangle,$ at $(0,0), (0.5,0), (-0.5,0)$ all stationary points.
$f_{xx} = 24x^2 - 2;$ $D = f_{xx} f_{yy} - f_{xy}^2 = (24x^2-2)(6) - (0)^2 = 12(12x^2-1).$
At $(0,0)$: $D = -12$, so (0, 0) is a saddle point.
At $(0.5, 0)$ and $(-0.5, 0)$: $D = 24$ and $f_{xx} = 6$, so local minima occur at these points.

4. $\nabla f(x, y) = \left\langle y^2 - 12x, 2xy - 6y \right\rangle = \langle 0, 0 \rangle$ at stationary points $(0,0), (3,-6)$ and $(3,6).$
$D = f_{xx} f_{yy} - f_{xy}^2 = (-12)(2x-6) - (2y)^2 = -4(y^2 + 6x - 18),$ $f_{xx} = -12$
At (0, 0): D = 72, and $f_{xx} = -12,$ so local maximum at (0, 0).
At $(3, \pm 6)$: $D = -144,$ so $(3, \pm y)$ are saddle points.

5. $\nabla f(x, y) = \langle y, x \rangle = \langle 0, 0 \rangle$ at (0, 0), a stationary point.
$D = f_{xx} f_{yy} - f_{xy}^2 = (0)(0) - (1)^2 = -1,$ so (0, 0) is a saddle point.

6. Let $\nabla f(x, y) = \left\langle 3x^2 - 6y, 3y^2 - 6x \right\rangle = \langle 0, 0 \rangle.$ Then $3x^2 - 6y = 0$ and $3y^2 - 6x = 0.$

$$3x^2 - 6y = 0 \to 3x^2 = 6y \to x^2 = 2y \to x^4 = 4y^2 \to \frac{1}{4}x^4 = y^2$$

$$3y^2 - 6x = 0 \to 3\left(\frac{1}{4}x^4\right) - 6x = 0 \to \frac{3}{4}x^4 - 6x = 0 \to \frac{3}{4}x\left(x^3 - 8\right) = 0 \to \frac{3}{4}x(x-2)\left(x^2 + 2x + 4\right) = 0 \to x = 0, x = 2$$

$x = 0$: $3x^2 - 6y = 0 \to 3(0) - 6y = 0 \to -6y = 0 \to y = 0$

$x = 2$: $3x^2 - 6y = 0 \to 3(2)^2 - 6y \to 12 - 6y = 0 \to 12 = 6y \to 2 = y$

Solving simultaneously, we obtain the solutions (0, 0) and (2, 2).
$f_{xx} = 6x;$ $D = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6y) - (-6)^2 = 36\,(xy - 1)$; At (0, 0): $D = -36 < 0$, so (0, 0) is a saddle point.
At (2, 2): $D = 108 > 0$, $f_{xx} > 0,$ so a local minimum occurs at $(2, 2)$.

7. $\nabla f(x, y) = \left\langle \frac{x^2y-2}{x^2}, \frac{xy^2-4}{y^2} \right\rangle = \langle 0, 0\rangle$ at (1, 2).

$D = f_{xx}f_{yy} - f_{xy}^2 = (4x-3)(8y-3)-(1)^2 = 32x^{-3}y^{-3}-1,\ f_{xx} = 4x^{-3}$

At (1, 2): $D = 3 > 0$, and $f_{xx} > 0,$ so a local minimum at (1, 2).

8. $\nabla f(x, y) = -2\exp(-x^2-y^2+4y)\langle x, y-2\rangle = \langle 0, 0\rangle$ at $(0,2)$.

$D = f_{xx}f_{yy} - f_{xy}^2 = \exp 2(-x^2-y^2+4y)[(4x^2-2)(4y^2-16y+14)-(4xy-8x)^2],$

$f_{xx} = (4x^2-2)\exp(-x^2-y^2+4y)$

At $(0,2)$: $D > 0$, and $f_{xx} < 0,$ so local maximum at (0, 2).

9. Let $\nabla f(x, y)$

$= \langle -\sin x - \sin(x+y), -\sin y - \sin(x+y)\rangle = \langle 0, 0\rangle$

Then $\begin{pmatrix} -\sin x - \sin(x+y) = 0 \\ \sin y + \sin(x+y) = 0 \end{pmatrix}$. Therefore,

$\sin x = \sin y$, so $x = y = \frac{\pi}{4}.$ However, these values satisfy neither equation. Therefore, the gradient is defined but never zero in its domain, and the boundary of the domain is outside the domain, so there are no critical points.

10. $\nabla f(x, y) = \langle 2x - 2a\cos y, 2ax\sin y\rangle = \langle 0, 0\rangle$ at $\left(0, \pm\frac{\pi}{2}\right), (a, 0)$

$D = f_{xx}f_{yy} - f_{xy}^2 = (2)(2ax\cos y) - (2a\sin y)^2,$

$f_{xx} = 2$

At $\left(0, \pm\frac{\pi}{2}\right)$: $D = -4a^2 < 0,$ so $\left(0, \pm\frac{\pi}{2}\right)$ are saddle points. At $(a, 0)$: $D = 4a^2 > 0$ and $f_{xx} > 0,$ so local minimum at $(a, 0)$.

11. We do not need to use calculus for this one. $3x$ is minimum at 0 and $4y$ is minimum at –1. (0, –1) is in S, so $3x + 4y$ is minimum at (0, –1); the minimum value is –4. Similarly, $3x$ and $4y$ are each maximum at 1. (1, 1) is in S, so $3x + 4y$ is maximum at (1, 1); the maximum value is 7. (Use calculus techniques and compare.)

12. We do not need to use calculus for this one. Each of x^2 and y^2 is minimum at 0 and (0, 0) is in S, so $x^2 + y^2$ is minimum at (0, 0); the minimum value is 0. Similarly, x^2 and y^2 are maximum at $x = 3$ and $y = 4$, respectively, and (3, 4) is in S, so $x^2 + y^2$ is maximum at (3, 4); the maximum value is 25. (Use calculus techniques and compare.)

13. $\nabla f(x, y) = \langle 2x, -2y\rangle = \langle 0, 0\rangle$ at (0, 0).

$D = f_{xx}f_{yy} - f_{xy}^2 = (2)(-2) - (0)2 < 0,$ so (0, 0) is a saddle point. A parametric representation of the boundary of S is $x = \cos t$, $y = \sin t$, t in $[0, 2\pi]$.

$f(x, y) = f(x(t), y(t)) = \cos^2 t - \sin^2 t + 1$

$= \cos 2t - 1$

$\cos 2t - 1$ is maximum if $\cos 2t = 1$, which occurs for $t = 0,\ \pi,\ 2\pi$. The points of the curve are $(\pm 1, 0)$. $f(\pm 1, 0) = 2$

$f(x, y) = \cos 2t - 1$ is minimum if $\cos 2t = -1$, which occurs for $t = \frac{\pi}{2}, \frac{3\pi}{2}.$ The points of the curve are $(0, \pm 1)$. $f(0, \pm 1) = 0$. Global minimum of 0 at (0, ± 1); global maximum of 2 at (±1, 0).

14. $\nabla f(x, y) = \langle 2x - 6, 2y - 8\rangle = \langle 0, 0\rangle$ at (3, 4), which is outside S, so there are no stationary points. There are also no singular points. $x = \cos t, y = \sin t$, t in $[0, 2\pi]$ is a parametric representation of the boundary of S.

$f(x, y) = f(x(t), y(t))$

$= \cos^2 t - 6\cos t + \sin^2 t - 8\sin t + 7$

$= 8 - 6\cos t - 8\sin t = F(t)$

$F'(t) = 6\sin t - 8\cos t = 0$ if $\tan t = \frac{4}{3}.$ t can be in the 1st or 3rd quadrants. The corresponding points of the curve are $\left(\pm\frac{3}{5}, \pm\frac{4}{5}\right)$.

$f\left(-\frac{3}{5}, -\frac{4}{5}\right) = 18;\ f\left(\frac{3}{5}, \frac{4}{5}\right) = -2$

Global minimum of –2 at $\left(\frac{3}{5}, \frac{4}{5}\right)$; global maximum of 18 at $\left(-\frac{3}{5}, -\frac{4}{5}\right)$.

15. Let x, y, z denote the numbers, so $x + y + z = N$. Maximize

$P = xyz = xy(N - x - y) = Nxy - x^2y - xy^2.$

Let $\nabla P(x, y) = \left\langle Ny - 2xy - y^2, Nx - x^2 - 2xy \right\rangle$

$= \langle 0, 0 \rangle.$

Then $\begin{pmatrix} Ny - 2xy - y^2 = 0 \\ Nx - x^2 - 2xy = 0 \end{pmatrix}.$

$N(x, -y) = x^2 - y^2 = (x+y)(x-y).$ $x = y$ or $N = x + y.$

Therefore, $x = y$ (since $N = x + y$ would mean that $P = 0$, certainly not a maximum value).

Then, substituting into $Nx - x^2 - 2xy = 0,$ we obtain $Nx - x^2 - 2x^2 = 0,$ from which we obtain $x(N - 3x) = 0$, so $x = \frac{N}{3}$ (since $x = 0 \Rightarrow P = 0$).

$P_{xx} = -2y;$

$D = P_{xx}P_{yy} - P_{xy}^2$

$= (-2y)(-2x) - (N - 2x - 2y)^2$

$= 4xy - (N - 2x - 2y)^2$

At $x = y = \frac{N}{3} : D = \frac{N^2}{3} > 0, P_{xx} = -\frac{2N}{3} < 0$ (so local maximum)

If $x = y = \frac{N}{3},$ then $z = \frac{N}{3}.$

Conclusion: Each number is $\frac{N}{3}$. (If the intent is to find three distinct numbers, then there is no maximum value of P that satisfies that condition.)

16. Let s be the distance from the origin to (x, y, z) on the plane. $s^2 = x^2 + y^2 + z^2$ and $x + 2y + 3z = 12.$ Minimize

$s^2 = f(y, z) = (12 - 2y - 3z)^2 + y^2 + z^2.$

$\nabla f(y, z) = \langle -48 + 12x + 10y, -72 + 12y + 20z \rangle$

$= \langle 0, 0 \rangle$ at $\left(\frac{12}{7}, \frac{18}{7} \right).$

$D = f_{yy}f_{zz} - f_{yz}^2 = 56 > 0$ and $f_{yy} = 10 > 0;$

local maximum at $\left(\frac{12}{7}, \frac{18}{7} \right)$

$s^2 = \frac{504}{49},$ so the shortest distance is

$s = \frac{6\sqrt{14}}{7} \approx 3.2071.$

17. Let S denote the surface area of the box with dimensions x, y, z.

$S = 2xy + 2xz + 2yz$ and $V_0 = xyz,$ so

$S = 2(xy + V_0 y^{-1} + V_0 x^{-1}).$

Minimize $f(x, y) = xy + V_0 y^{-1} + V_0 x^{-1}$ subject to $x > 0, y > 0.$

$\nabla f(x, y) = \left\langle y - V_0 x^{-2}, x - V_0 y^{-2} \right\rangle = \langle 0, 0 \rangle$ at $(V_0^{1/3}, V_0^{1/3}).$

$D = f_{xx}f_{yy} - f_{xy}^2 = 4V_0^2 x^{-3} y^{-3} - 1,$

$f_{xx} = 2V_0 x^{-3}.$

At $\left(V_0^{1/3}, V_0^{1/3} \right)$: $D = 3 > 0,$ $f_{xx} = 2 > 0,$ so local minimum.

Conclusion: The box is a cube with edge $V_0^{1/3}.$

18. Let L denote the sum of edge lengths for a box of dimensions x, y, z. Minimize $L = 4x + 4y + 4z$, subject to $V_0 = xyz.$

$L(x, y) = 4x + 4y + \frac{4V_0}{xy}, x > 0, y > 0$

Let

$\nabla L(x, y) = 4x^{-1}y^{-1} \left\langle x^{-1}(x^2y - V_0), y^{-1}(xy^2 - V_0) \right\rangle$

$= \langle 0,0 \rangle$.

Then $x^2y = V_0$ and $xy^2 = V_0,$ from which it follows that $x = y$. Therefore $x = y = z = V_0^{1/3}.$

$L_{xx} = \frac{8V_0}{x^3y};$

$D = L_{xx}L_{yy} - L_{xy}^2 = \left(\frac{8V_0}{x^3y} \right)\left(\frac{8V_0}{xy^3} \right) - \left(\frac{4V_0}{x^2y^2} \right)^2$

At $\left(V_0^{1/3}, V_0^{1/3} \right)$: $D > 0,$ $L_{xx} > 0$ (so local minimum).

There are no other critical points, and as $(x, y) \to$ boundary, $L \to \infty$. Hence, the optimal box is a cube with edge $V_0^{1/3}.$

19. Let S denote the area of the sides and bottom of the tank with base l by w and depth h.

$S = lw + 2lh + 2wh$ and $lwh = 256.$

$S(l, w) = lw + 2l\left(\frac{256}{lw} \right) + 2w\left(\frac{256}{lw} \right),$ $w > 0, l > 0.$

$S(l\ w) = \left\langle w - 5121^{-2}, l - 512w^{-2} \right\rangle = \langle 0, 0 \rangle$ at

(8, 8). $h = 4$ there. At (8, 8) $D > 0$ and $S_{11} > 0,$ so local minimum. Dimensions are 8’ $\times$ 8’ $\times$ 4’.

20. Let V denote the volume of the box and (x, y, z) denote its 1st octant vertex.

$V = (2x)(2y)(2z) = 8xyz$ and $24x^2 + y^2 + z^2 = 9$.

$$V^2 = 64\left[\left(\frac{1}{24}\right)(9 - y^2 - z^2)\right]y^2z^2$$

Maximize $f(y, z) = (9 - y^2 - z^2)y^2z^2$, $y > 0$,
$z > 0$.

$\nabla f(y, z) = 2\left\langle yz^2(9 - 2y^2 - z^2),\ y^2z(9 - y^2 - 2z^2)\right\rangle = \langle 0, 0\rangle$ at $(\sqrt{3}, \sqrt{3})$. $x = \dfrac{\sqrt{2}}{4}$

At $\left(\sqrt{3}, \sqrt{3}\right)$, $D = f_{yy}f_{zz} - f_{yz}^2 > 0$ and $f_{yy} < 0$, so local maximum. The greatest possible volume is

$8\left(\dfrac{\sqrt{2}}{4}\right)\left(\sqrt{3}\right)\left(\sqrt{3}\right) = 6\sqrt{2}.$

21. Let $\langle x, y, z\rangle$ denote the vector; let S be the sum of its components.

$x^2 + y^2 + z^2 = 81$, so $z = (81 - x^2 - y^2)^{1/2}$.

Maximize $S(x, y) = x + y + (81 - x^2 - y^2)^{1/2}$, $0 \le x^2 + y^2 \le 9$.

Let $\nabla S(x, y) = \left\langle 1 - x(81 - x^2 - y^2)^{-1/2}, 1 - y(81 - x^2 - y^2)^{-1/2}\right\rangle = \langle 0, 0\rangle$.

Therefore, $x = (81 - x^2 - y^2)^{1/2}$ and $y = (81 - x^2 - y^2)^{1/2}$. We then obtain $x = y = 3\sqrt{3}$ as the only stationary point. For these values of x and y, $z = 3\sqrt{3}$ and $S = 9\sqrt{3} \approx 15.59$.
The boundary needs to be checked. It is fairly easy to check each edge of the boundary separately. The largest value of S at a boundary point occurs at three places and turns out to be $\dfrac{18}{\sqrt{2}} \approx 12.73$.

Conclusion: the vector is $3\sqrt{3}\langle 1, 1, 1\rangle$.

22. Let $P(x, x, z)$ be any point in the plane $2x + 4y + 3z = 12$. The square of the distance between the origin and P is $d^2 = x^2 + y^2 + z^2$. Consequently, $d^2 = f(x, y) = x^2 + y^2 + (12 - 2x - 4y)^2/9$. To find the critical points, set $f_x(x, y) = 2x + \frac{2}{9}(12 - 2x - 4y)(-2) = 0$ and $f_y(x, y) = 2y + \frac{2}{9}(12 - 2x - 4y)(-4) = 0$ The resulting system of equations is $13x + 8y = 24$ and $8x + 25y = 48$, which leads to a critical point of $\left(\frac{24}{29}, \frac{48}{29}\right)$. Since $f_{xx}(x, y) = \frac{26}{9}$, $f_{yy}(x, y) = \frac{50}{9}$, and $f_{xy}(x, y) = \frac{16}{9}$, $D\left(\frac{24}{29}, \frac{48}{29}\right) = \frac{116}{9}$ Since $D\left(\frac{24}{29}, \frac{48}{29}\right) > 0$ and $f_{xx}\left(\frac{24}{29}, \frac{48}{29}\right) > 0$, $\left(\frac{24}{29}, \frac{48}{29}\right)$ yields a minimum distance. The point on the plane $2x + 4y + 3z = 12$ that is closest to the origin is $\left(\frac{24}{29}, \frac{48}{29}, \frac{36}{29}\right)$ and this minimum distance is approximately 2.2283.

23. Let $P(x, y, z)$ be any point on $z = x^2 + y^2$. The square of the distance between the point $(1, 2, 0)$ and P can be expressed as $d^2 = f(x, y) = (x - 1)^2 + (y - 2)^2 + z^2$. To find the critical points, set $f_x(x, y)$ $= 4x^3 + 2x + 4xy^2 - 2 = 0$ and $f_y(x, y) = 4y^3 + 2y + 4x^2y - 4 = 0$. Multiplying the first equation by y and the second equation by x and summing the results leads to the equation $-2y + 4x = 0$. Thus, $y = 2x$. Substituting into the first equation yields $10x^3 + x - 1 = 0$, whose solution is $x \approx 0.393$. Consequently, $y \approx 0.786$. $f_{xx}(x, y) = 2 + 12x^2 + 4y^2$, $f_{yy}(x, y) = 2 + 12y^2 + 4x^2$, and $f_{xy}(x, y) = 8xy$. The value of D for the critical point (0.393, 0.786) is approximately 57 and since $f_{xx}(0.393, 0.786) > 0$, (0.393, 0.786) yields a minimum distance. The point on the surface $z = x^2 + y^2$ is (0.393, 0.786, 0.772) and this minimum distance is approximately 1.56.

24. Let (x, y, z) denote a point on the cone, and s denote the distance between (x, y, z) and $(1, 2, 0)$.
$s^2 = (x-1)^2 + (y-2)^2 + z^2$ and $z^2 = x^2 + y^2$. Minimize $s^2 = f(x, y) = (x-1)^2 + (y-2)^2 + (x^2 + y^2)$, x, y in $\mathbb{R}$.
$\nabla f(x, y) = 2\langle 2x-1, 2y-2\rangle = \langle 0, 0\rangle$ at $\left(\frac{1}{2}, 1\right)$. At $\left(\frac{1}{2}, 1\right)$, $D > 0$ and $f_{xx} > 0$, so local minimum.

Conclusion: Minimum distance is $s = \sqrt{\frac{5}{2}} \approx 1.5811.$

25. $A = \left(\frac{1}{2}\right)[y + (y + 2x\sin\alpha)](x\cos\alpha)$ and

$2x + y = 12$. Maximize $A(x, \alpha) = 12x\cos\alpha - 2x^2\cos\alpha + \left(\frac{1}{2}\right)x^2\sin 2\alpha$, x in $(0, 6]$, a in $\left(0, \frac{\pi}{2}\right)$.

$A(x, \alpha) = \left\langle 12\cos\alpha - 4x\cos\alpha + 2x\sin\alpha\cos\alpha, -12x\sin\alpha + 2x^2\sin\alpha + x^2\cos 2\alpha\right\rangle = \langle 0, 0\rangle$ at $\left(4, \frac{\pi}{6}\right)$.

At $\left(4, \frac{\pi}{6}\right)$, $D > 0$ and $A_{xx} < 0$, so local maximum, and $A = 12\sqrt{3} \approx 20.78$. At the boundary point of $x = 6$, we get

$\alpha = \frac{\pi}{4}, A = 18$. Thus, the maximum occurs for width of turned-up sides = 4”, and base angle $= \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$.

26. The lines are skew since there are no values of s and t that simultaneously satisfy $t - 1 = 3s$, $2t = s + 2$, and $t + 3 = 2s - 1$. Minimize f, the square of the distance between points on the two lines.
$f(s, t) = (3s - t + 1)^2 + (s + 2 - 2t)^2 + (2s - 1 - t - 3)^2$
Let
$\nabla f(s, t) = \langle 2(3s - t + 1)(3) + 2(s - 2t + 2)(1) + 2(2s - t - 4)(2), 2(3s - t + 1)(-1) + 2(s - 2t + 2)(-2) + 2(2s - t - 4)(-1)\rangle$
$= \langle 28s - 14t - 6, -14s + 12t - 28\rangle = \langle 0, 0\rangle$.

Solve $28s - 14t - 6 = 0$, $-14s + 12t - 2 = 0$, obtaining $s = \frac{5}{7}, t = 1$.

$D = f_{ss}f_{tt} - f_{st}^2 = (28)(12) - (-14)^2 > 0$; $f_{ss} = 28 > 0$. (local minimum)
The nature of the problem indicates the global minimum occurs here.
$f\left(\frac{5}{7}, 1\right) = \left(\frac{15}{7}\right)^2 + \left(\frac{5}{7}\right)^2 + \left(-\frac{25}{7}\right)^2 = \frac{875}{49}$
Conclusion: The minimum distance between the lines is $\sqrt{875}/7 \approx 4.2258$.

27. Let M be the maximum value of $f(x, y)$ on the polygonal region, P. Then $ax + by + (c - M) = 0$ is a line that either contains a vertex of P or divides P into two subregions. In the latter case $ax + by + (c - M)$ is positive in one of the regions and negative in the other. $ax + by + (c - M) > 0$ contradicts that M is the maximum value of $ax + by + c$ on P. (Similar argument for minimum.)

a.

x	y	$2x + 3y + 4$
–1	2	8
0	1	7
1	0	6
–3	0	–2
0	–4	–8

Maximum at $(-1, 2)$

b.

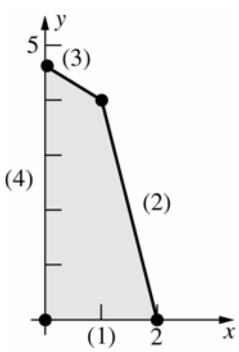
x	y	$-3x + 2y + 1$
–3	0	10
0	5	11
2	3	1
4	0	–11
1	–4	–10

Minimum at (4, 0)

28.

x	y	$2x + y$
0	0	0
2	0	4
1	4	6
0	$14/3$	$14/3$

Maximum of 6 occurs at $(1, 4)$

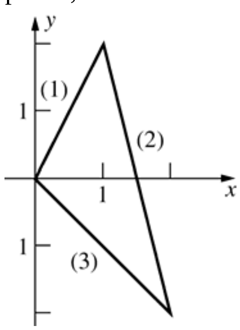


The edges of P are segments of the lines:
1. $y = 0$
2. $4x + y = 8$
3. $2x + 3y = 14$, and
4. $x = 0$

29. $z(x, y) = y^2 - x^2$

$z(x, y) = \langle -2x, 2y \rangle = \langle 0, 0 \rangle$ at (0, 0).

There are no stationary points and no singular points, so consider boundary points.



On side 1:

$y = 2x$, so $z = 4x^2 - x^2 = 3x^2$

$z'(x) = 6x = 0$ if $x = 0$.

Therefore, (0, 0) is a candidate.

On side 2:

$y = -4x + 6$, so

$z = (-4x+6)^2 - x^2 = 15x^2 - 48x + 36.$

$z'(x) = 30x - 48 = 0$ if $x = 1.6$.

Therefore, (1.6, –0.4) is a candidate.

On side 3:

$y = -x$, so $z = (-x)^2 - x^2 = 0.$

Also, all vertices are candidates.

x	y	z
0	0	0
1.6	–0.4	–2.4
2	–2	0
1	2	3

Minimum value of –2.4; maximum value of 3

30. a. $$\frac{\partial f}{\partial m} = \sum_{i=1}^{n} \frac{\partial}{\partial m}\left(y_i - mx_i - b\right)^2$$

$$= 2\sum_{i=1}^{n}\left(y_i - mx_i - b\right)(-x_i)$$

$$= -2\sum_{i=1}^{n}\left(x_i y_i - mx_i^2 - bx_i\right)$$

Setting this result equal to zero yields

$$0 = -2\sum_{i=1}^{n}\left(x_i y_i - mx_i^2 - bx_i\right)$$

$$0 = \sum_{i=1}^{n}\left(x_i y_i - mx_i^2 - bx_i\right)$$

or equivalently,

$$\sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i$$

$$\frac{\partial f}{\partial b} = \sum_{i=1}^{n} \frac{\partial}{\partial b}\left(y_i - mx_i - b\right)^2$$

$$= 2\sum_{i=1}^{n}\left(y_i - mx_i - b\right)(-1)$$

$$= -2\sum_{i=1}^{n}\left(y_i - mx_i - b\right)$$

Setting this result equal to zero yields

$$0 = -2\sum_{i=1}^{n}\left(y_i - mx_i - b\right)$$

$$0 = \sum_{i=1}^{n}\left(y_i - mx_i - b\right)$$

or equivalently,

$$m\sum_{i=1}^{n} x_i + nb = \sum_{i=1}^{n} y_i$$

b. $$nb = \sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i$$

Therefore,

$$b = \frac{\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i}{n}$$

$$\sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + \frac{\left(\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i}{n}$$

This simplifies into

$$m = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}$$

c. $\frac{\partial^2 f}{\partial m^2} = 2\sum_{i=1}^{n} x_i^2$

$\frac{\partial^2 f}{\partial b^2} = 2n$

$\frac{\partial^2 f}{\partial m \partial b} = 2\sum_{i=1}^{n} x_i$

Then, by Theorem C, we have

$$D = 4n\left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right).$$

Assuming that all the x_i are not the same, we find that $D > 0$ and $\frac{\partial^2 f}{\partial m^2} > 0$.

Thus, $f(m,b)$ is minimized.

31.

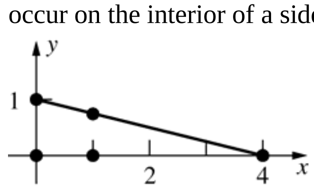
	x_i	y_i	x_i^2	$x_i y_i$
	3	2	9	6
	4	3	16	12
	5	4	25	20
	6	4	36	24
	7	5	49	35
$\Sigma_{i=1}^{5}$	25	18	135	97

$m(135) + b(25) = (97)$ and $m(25) + (5)b = (18)$.
Solve simultaneously and obtain $m = 0.7$, $b = 0.1$.
The least-squares line is $y = 0.7x + 0.1$.

32. $z = 2x^2 + y^2 - 4x - 2y + 5$, so

$\nabla z = \langle 4x - 4, 2y - 2\rangle = \mathbf{0}$.

$\nabla z = \mathbf{0}$ at (1, 1) which is outside the region. Therefore, extreme values occur on the boundary. Three critical points are the vertices of the triangle, (0, 0), (0, 1), and (4, 0). Others may occur on the interior of a side of the triangle.



On vertical side: $x = 0$

$z(y) = y^2 - 2y + 5$, $y = [0, 1]$. $z'(y) = 2y - 2$, so $z'(y) = 0$ if $y = 1$. Hence, no additional critical point.

On horizontal side: $y = 0$

$z(x) = 2x^2 - 4x + 5$, x in [0, 4]. $z'(x) = 4x - 4$, so $z'(x) = 0$ if $x = 1$. Hence, an additional critical point is (1, 0).

On hypotenuse: $x = 4 - 4y$

$z(y) = 2(4 - 4y)^2 + y^2 - 4(4 - 4y) - 2y + 5$

$= 33y^2 - 50y + 21$, y in [0, 1].

$z'(y) = 66y - 50$, so $z'(y) = 0$ if $y = \frac{25}{33}$.

Hence, an additional critical point is $\left(\frac{32}{33}, \frac{25}{33}\right)$.

x	y	z
0	0	5
4	0	21
0	1	4
1	0	3
32/33	25/33	2.06

Maximum value of z is 21; it occurs at (4, 0).
Minimum value of z is about 2.06; it occurs at $\left(\frac{32}{33}, \frac{25}{33}\right)$.

33. Let x and y be defined as shown in Figure 4 from Section 12.8. The total cost is given by

$C(x, y) = 400\sqrt{x^2 + 50^2} + 200(200 - x - y)$
$+300\sqrt{y^2 + 100^2}$

Taking partial derivatives and setting them equal to 0 gives

$C_x(x, y) = 200(x^2 + 50^2)^{-1/2}(2x) - 200 = 0$

$C_y(x, y) = 150(y^2 + 100^2)^{-1/2}(2y) - 200 = 0$

The solution of these equations is

$x = \frac{50}{\sqrt{3}} \approx 28.8675$ and $y = \frac{100}{\sqrt{1.25}} \approx 89.4427$

We now apply the second derivative test:

$$C_{xx}(x, y) = \frac{400\sqrt{x^2 + 50^2} - 400x^2 / \sqrt{x^2 + 50^2}}{x^2 + 50^2}$$

$$C_{yy}(x, y) = \frac{300\sqrt{y^2 + 100^2} - 300y^2 / \sqrt{y^2 + 100^2}}{y^2 + 100^2}$$

$C_{xy}(x, y) = 0$

Evaluated at $x = 50/\sqrt{3}$ and $y = 100/\sqrt{1.25}$,

$D \approx (5.196)(1.24) - 0^2 > 0$

Thus, a local minimum occurs with

$C\left(50/\sqrt{3}, 100/\sqrt{1.25}\right) \approx \$79,681$

We must also check the boundary. When x = 0,

$C_1(y) = C(0, y) = 200(200 - y) + 300\sqrt{y^2 + 100^2}$

and when y = 0,

$C_2(x) = C(x,0) = 400\sqrt{x^2+50^2} + 200(200-x)$

Using the methods from Chapter 3, we find that C_1 reaches a minimum of about \$82,361 when $y = \sqrt{8000}$ and C_2 reaches a minimum of about \$87,321 when $x = \sqrt{2500/3}$. Addressing the boundary x + y = 200, we find that

$C_3(x) = C(x,200-x) = 400\sqrt{x^2+50^2}$
$+300\sqrt{(200-x)^2+100^2}$ This function reaches a minimum of about \$82,214 when $x \approx 41.08$.
Thus, the minimum cost path is when $x = 50/\sqrt{3} \approx 28.8675$ ft and $y = 100/\sqrt{1.25} \approx 89.4427$ ft, which produces a cost of about \$79,681.

34. Let x and y be defined as shown in Figure 4 from Section 12.8. The total cost is given by

$C(x,y) = 500\sqrt{x^2+50^2} + 200(200-x-y)$
$+100\sqrt{y^2+100^2}$

Taking partial derivatives and setting them equal to 0 gives

$C_x(x,y) = 500(x^2+50^2)^{-1/2}(2x) - 200 = 0$

$C_y(x,y) = 100(y^2+100^2)^{-1/2}(2y) - 200 = 0$

There is, however, no solution to $C_y(x,y) = 0$

Now we check the boundary. When x = 0,

$C_1(y) = C(0,y) = 200(200-y) + 100\sqrt{y^2+100^2}$

There is, however, no solution to $C_1{}'(y) = 0$.

When $y = 0$,

$C_2(x) = C(x,0) = 500\sqrt{x^2+50^2} + 200(200-x)$

$C_2{}'(x) = 0$ yields $x = 100/\sqrt{21}$ and

$C\left(100/\sqrt{21}, 0\right) \approx \$72,913$

On the boundary $x + y = 200$, we find that

$C_3(x) = C(x,200-x) = 500\sqrt{x^2+50^2}$
$+100\sqrt{(200-x)^2+100^2}$ This function reaches a minimum of about \$46,961 when $x \approx 9.0016$.
Thus, the minimum cost path is when $x \approx 9.0016$ ft and $y \approx 190.9984$ ft, which produces a cost of about \$46,961.

35. $f(x,y) = 10 + x + y$

$\nabla f = \langle 1,1 \rangle \neq \mathbf{0}$; thus no interior critical points exist. Letting

$x = 3\cos t,\ y = 3\sin t,\ 0 \le t \le 2\pi,$

$g(t) = f(3\cos t, 3\sin t)$ and

$g'(t) = 3\cos t - 3\sin t$. Setting $g'(t) = 0$ yields $t = \pi/4$ or $5\pi/4$.

The critical points are $\left(3/\sqrt{2}, 3/\sqrt{2}\right)$ and $\left(-3/\sqrt{2}, -3/\sqrt{2}\right)$.

Since $f\left(3/\sqrt{2}, 3/\sqrt{2}\right) = 10 + 6/\sqrt{2}$ and $f\left(-3/\sqrt{2}, -3/\sqrt{2}\right) = 10 - 6/\sqrt{2}$, the minimum value of f on $x^2 + y^2 \le 3$ is $10 - 6/\sqrt{2}$ and the maximum value of f is $10 + 6/\sqrt{2}$.

36. $f(x,y) = x^2 + y^2;\ \nabla f = \langle 2x, 2y \rangle$.

$\nabla f = \mathbf{0}$ at (0,0).

$D(0,0) = 2 \cdot 2 - 0^2 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$,

Thus, $f(0,0) = 0$ is a minimum.

In order to optimize $g(t) = f(a\cos t, b\sin t)$ where $0 \le t \le 2\pi$, we find

$g'(t) = 2x(-a\sin t) + 2y(b\cos t)$

$= 2b^2 \sin t \cos t - 2a^2 \sin t \cos t$

$= (b^2 - a^2)\sin 2t$. Setting $g'(t) = \mathbf{0}$, we have $t = 0, \pi/2, \pi,$ or $3\pi/2$. The resulting critical points are $(a,0)$, $(0,b)$, $(-a,0)$, and $(0,-b)$.

$f(a,0) = f(-a,0) = a^2;\ \ f(0,b) = f(0,-b) = b^2.$

Since a > b, the maximum value of f on the given region is a^2 and the minimum value of f is 0.

37. The volume of the box can be expressed as $V(l,w,h) = lwh = 2$ and the surface area as $S(l,w,h) = 2lh + 2wh + lw + lw$. Since $h = \frac{2}{lw}$,

$S(l,w) = \frac{4}{w} + \frac{4}{l} + lw + lw$ When cost is factored,

$C(l,w) = \frac{1}{w} + \frac{1}{l} + 0.65lw$ with $w > 0, l > 0$

$C_l(l,w) = -\frac{1}{l^2} + 0.65w = 0$

$C_w(l,w) = -\frac{1}{w^2} + 0.65l = 0$

Solving this system of equations leads to

$w = \sqrt[3]{\frac{0.65}{0.4225}} \approx 1.1544$ and $l = w \approx 1.1544$.

Consequently, $h \approx 1.501$. Applying the second derivative test with $C_{ll}(l,w) = \frac{2}{l^3}$,

$C_{ww}(l,w) = \frac{2}{w^3}$ and $C_{lw}(l,w) = 0.65$,

$D \approx 1.268 > 0$. Thus, the minimum cost occurs when the length is approximately 1.1544 feet, the width is approximately 1.1544 feet and the height is approximately 1.501 feet.

38. The cost function in three variables is $C(l, w, h) = 4lw + 2lh + 2wh + 6l + 6w + 4h,$ where $lwh = 60.$ Substituting $h = \frac{60}{lw}$ yields

$C(l, w) = 4lw + \frac{120}{w} + 6l + 6w + \frac{240}{lw}$ with $l > 0$ and $w > 0.$

$C_l(l, w) = 4w - \frac{120}{l^2} + 6 - \frac{240}{wl^2} = 0$

$C_w(l, w) = 4l - \frac{120}{w^2} + 6 - \frac{240}{lw^2} = 0$

Multiplying both sides of the first equation by $wl^2,$ multiplying both sides of the second equation by $lw^2,$ and subtracting the resulting equations produces $-120w + 120l = 0$ or $l = w.$

Consequently, $4w - \frac{120}{w^2} + 6 - \frac{240}{w^3} = 0$ or

$2w^4 + 3w^3 - 60w - 120 = 0$ Using a CAS, this equation yields $w \approx 3.2134$

$C_{ll}(l, w) = \frac{240}{l^3} + \frac{480}{wl^3};$ $C_{ww}(l, w) = \frac{240}{w^3} + \frac{480}{lw^3};$

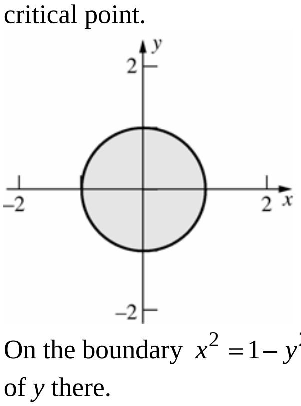
$C_{lw}(l, w) = 4 + \frac{240}{l^2 w^2};$ Using the critical point (3.2134, 3.2134), $D \approx 131.44 > 0$

Thus, $w = l \approx 3.2$ yields a minimum. The minimum cost involved with making this box is approximately \$177.79. This minimum cost occurs when the length and width are approximately 3.2 feet and the height is approximately 5.8 feet.

39. $T(x, y) = 2x^2 + y^2 - y$

$\nabla T = \langle 4x, 2y - 1 \rangle = \mathbf{0}$

If $x = 0$ and $y = \frac{1}{2},$ so $\left(0, \frac{1}{2}\right)$ is the only interior critical point.



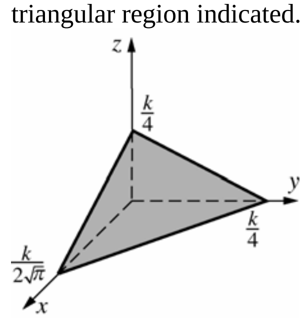
On the boundary $x^2 = 1 - y^2,$ so **T** is a function of *y* there.

$T(y) = 2(1 - y^2) + y^2 - y = 2 - y - y^2,$
$y = [-1, 1]$

$T'(y) = -1 - 2y = 0$ if $y = -\frac{1}{2},$ so on the boundary, critical points occur where *y* is $-1, -\frac{1}{2}, 1.$

Thus, points to consider are $\left(0, \frac{1}{2}\right),$ (0, –1), $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and (0, 1). Substituting these into $T(x, y)$ yields that the coldest spot is $\left(0, \frac{1}{2}\right)$ where the temperature is $-\frac{1}{4},$ and there is a tie for the hottest spot at $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ where the temperature is $\frac{9}{4}.$

40. Let x^2, y^2, z^2 denote the areas enclosed by the circle, and the two squares, respectively. Then the radius of the circle is $\frac{x}{\sqrt{\pi}}$, and the edges of the two squares are *y* and *z*, respectively. We wish to optimize $A(x, y, z) = x^2 + y^2 + z^2,$ subject to $2\pi\left(\frac{x}{\sqrt{\pi}}\right) + 4y + 4z = k,$ or equivalently $2\sqrt{\pi}x + 4y + 4z = k,$ with each of *x*, *y*, and *z* nonnegative. Geometrically: we seek the smallest and largest of all spheres with center at the origin and some point in common with the triangular region indicated.

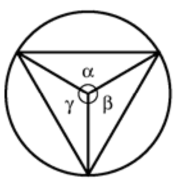


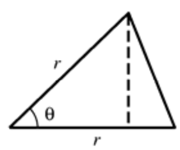
Since $\frac{k}{2\sqrt{\pi}} > \frac{k}{4},$ the largest sphere will intersect the region only at point $\left(\frac{k}{2\sqrt{\pi}}, 0, 0\right)$ and will thus have radius $\frac{k}{2\sqrt{\pi}}.$ Thus *A* will be maximum if $x = \frac{k}{2\sqrt{\pi}},$ $y = z = 0$ (all of the wire goes into the circle). The smallest sphere will be tangent to the triangle. The point of tangency is on the normal line through the origin,

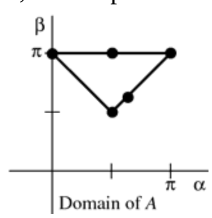
$\langle x, y, z\rangle = t\langle\sqrt{\pi}, 2, 2\rangle$. Substituting $x = \sqrt{\pi}$, $y = 2$, $z = 2$ into the equation of the plane yields the value $t = \frac{k}{2(\pi+8)}$, so the minimum value of A is obtained for the values of $x = \frac{k\sqrt{\pi}}{2(\pi+8)}$, $y = z = \frac{k}{\pi+8}$. Thus the circle will have radius $\frac{\left[\frac{k\sqrt{\pi}}{2(\pi+8)}\right]}{\sqrt{\pi}} = \frac{k}{2(\pi+8)}$, and the squares will each have sides $\frac{k}{(\pi+8)}$. Therefore, the circle will use $\frac{\pi k}{(\pi+8)}$ units and the squares will each use $\frac{4k}{(\pi+8)}$ units.

[Note: sum of the three lengths is k.]

41. Without loss of generality we will assume that $\alpha \le \beta \le \gamma$. We will consider it intuitively clear that for a triangle of maximum area the center of the circle will be inside or on the boundary of the triangle; i.e., α, β, and γ are in the interval $[0, \pi]$. Along with $\alpha + \beta + \gamma = 2\pi$, this implies that $\alpha + \beta \ge \pi$.







The area of an isosceles triangle with congruent sides of length r and included angle θ is $\frac{1}{2}r^2 \sin\theta$.

$$\begin{aligned}\text{Area}(\Delta ABC) &= \frac{1}{2}r^2 \sin\alpha + \frac{1}{2}r^2 \sin\beta + \frac{1}{2}r^2 \sin\gamma \\ &= \frac{1}{2}r^2(\sin\alpha + \sin\beta + \sin[2\pi - (\alpha+\beta)] \\ &= \frac{1}{2}r^2[\sin\alpha + \sin\beta - \sin(\alpha+\beta)]\end{aligned}$$

Area(ΔABC) will be maximum if (*) $A(\alpha, \beta) = \sin\alpha + \sin\beta - \sin(\alpha+\beta)$ is maximum.

Restrictions are $0 \le \alpha \le \beta \le \pi$, and $\alpha + \beta \ge \pi$.

Three critical points are the vertices of the triangular domain of $A: \left(\frac{\pi}{2}, \frac{\pi}{2}\right), (0, \pi)$, and (π, π). We will now search for others.

$\Delta A(\alpha, \beta) = \langle\cos\alpha - \cos(\alpha+\beta), \cos\beta - \cos(\alpha+\beta)\rangle = 0$ if

$\cos\alpha = \cos(\alpha+\beta) = \cos\beta$.

Therefore, $\cos\alpha = \cos\beta$, so $\alpha = \beta$ [due to the restrictions stated]. Then

$\cos\alpha = \cos(\alpha+\alpha) = \cos 2\alpha = 2\cos^2\alpha - 1$, so $\cos\alpha = 2\cos^2\alpha - 1$.

Solve for α: $2\cos^2\alpha - \cos\alpha - 1 = 0$; $(2\cos\alpha + 1)(\cos\alpha - 1) = 0$;

$\cos\alpha = -\frac{1}{2}$ or $\cos\alpha = 1$; $\alpha = \frac{2\pi}{3}$ or $\alpha = 0$.

(We are still in the case where $\alpha = \beta$.) $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ is a new critical point, but (0, 0) is out of the domain of *A*.

There are no critical points in the interior of the domain of *A*.

On the $\beta = \pi$ edge of the domain of *A*; $A(\alpha) = \sin\alpha - \sin(\alpha - \pi) = 2\sin\alpha$ so $A'(\alpha) = 2\cos\alpha$.

$A'(\alpha) = 0$ if $\alpha = \frac{\pi}{2}$. $\left(\frac{\pi}{2}, \pi\right)$ is a new critical point.

On the $\beta = \pi - \alpha$ edge of the domain of *A*:

$A(\alpha) = \sin\alpha + \sin(\pi - \alpha) - \sin(2\alpha - \pi) = 2\sin\alpha + \sin 2\alpha$, so

$A'(\alpha) = 2\cos\alpha + 2\cos 2\alpha = 2[\cos\alpha + (2\cos^2\alpha - 1)] = 2(2\cos\alpha - 1)(\cos\alpha + 1)$.

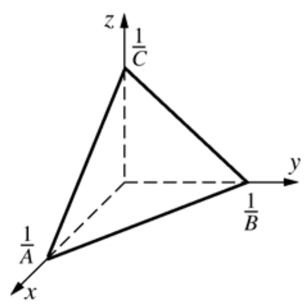
$A'(\alpha) = 0$ if $\cos\alpha = \frac{1}{2}$ or $\cos\alpha = -1$, so $\alpha = \frac{\pi}{3}$ or $\alpha = \pi$.

$\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$ and $(\pi, 0)$ are outside the domain of *A*.

(The critical points are indicated on the graph of the domain of *A*.)

α	β	A	
$\frac{\pi}{2}$	$\frac{\pi}{2}$	2	
0	π	0	
π	π	0	
$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$	Maximum value of *A*. The triangle is equilateral.
$\frac{\pi}{2}$	π	2	

42. If the plane through (*a*, *b*, *c*) is expressed as $Ax + By + Cz = 1$, then the intercepts are $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}$; volume of tetrahedron is $V = \left(\frac{1}{3}\right)\left[\left(\frac{1}{2}\right)\left(\frac{1}{A}\right)\left(\frac{1}{B}\right)\right]\left(\frac{1}{C}\right) = \frac{1}{6ABC}$.



To maximize *V* subject to $Aa + Bb + Cc = 1$ is equivalent to maximizing $z = ABC$ subject to $Aa + Bb + Cc = 1$.

$C = \frac{1 - aA - bB}{c}$, so $z = \frac{AB(1 - aA - bB)}{c}$.

$\nabla z = \left(\frac{1}{c}\right)\left\langle B - 2aAB - bB^2, A - 2bAB - aA^2\right\rangle = \mathbf{0}$ if $A = \frac{1}{3a}, B = \frac{1}{3b}\left[C = \frac{1}{3c}\right]$.

$\left(\frac{1}{3a}, \frac{1}{3b}\right)$ is the only critical point in the first quadrant. The second partials test yields that *z* is maximum at this point. The plane is $\frac{1}{3a}x + \frac{1}{3b}y + \frac{1}{3c}z = 1$, or $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

The volume of the first quadrant tetrahedron formed by the plane is $\frac{1}{\left[6\left(\frac{1}{3a}\right)\left(\frac{1}{3b}\right)\left(\frac{1}{3c}\right)\right]} = \frac{9abc}{2}$.

43. Local max: $f(1.75, 0) = 1.15$
Global max: $f(-3.8, 0) = 2.30$

44. Global max: $f(0, 1) = 0.5$
Global min: $f(0, -1) = -0.5$

45. Global min: $f(0, 1) = f(0, -1) = -0.12$

46. Global max: $f(0, 0) = 1$
Global min: $f(2, -2) = f(-2, 2) = e^{-9}$
≈ 0.00012341

47. Global max: $f(1.13, 0.79) = f(1.13, -0.79) = 0.53$
Global min: $f(-1.13, 0.79) = f(-1.13, -0.79)$
$= -0.53$

48. No global maximum or global minimum

49. Global max: $f(3,3) = f(-3,3) \approx 74.9225$
Global min: $f(1.5708, 0) = f(-1.5708, 0) = -8$

50. Global max: $f(1, 43, 0) = 0.13$
Global min: $f(-1.82, 0) = -0.23$

51. Global max: $f(0.67, 0) = 5.06$
Global min: $f(-0.75, 0) = -3.54$

52. Global max: $f(-5.12, -4.92) = 1071$
Global min: $f(5.24, -4.96) = -658$

53. Global max: $f(2.1, 2.1) = 3.5$
Global min: $f(4.2, 4.2) = -3.5$

54. **a.**
$$k(\alpha,\beta) = \frac{1}{2}[80\sin\alpha + 60\sin\beta + 48\sin(2\pi - \alpha - \beta)]$$
$$= 40\sin\alpha + 30\sin\beta - 24\sin(\alpha+\beta)$$
$$L(\alpha,\beta) = (164 - 160\cos\alpha)^{1/2} + (136 - 120\cos\beta)^{1/2}$$
$$+ (100 - 96\cos(\alpha+\beta))^{1/2}$$

b. (1.95, 2.04)

c. (2.26, 2.07)

12.9 Concepts Review

1. free; constrained

2. parallel

3. $g(x, y) = 0$

4. (2, 2)

Problem Set 12.9

1. $\langle 2x,\ 2y\rangle = \lambda\langle y,\ x\rangle$
$2x = \lambda y$, $2y = \lambda x$, $xy = 3$
Critical points are
$\left(\pm\sqrt{3}, \pm\sqrt{3}\right)$, $f\left(\pm\sqrt{3}, \pm\sqrt{3}\right) = 6.$
It is not clear whether 6 is the minimum or maximum, so take any other point on $xy = 3$, for example (1, 3). $f(1, 3) = 10$, so 6 is the minimum value.

2. $\langle y,\ x\rangle = \lambda\langle 8x,\ 18y\rangle$
$y = 8\lambda x$, $x = 18\lambda y$, $4x^2 + 9y^2 = 36$
Critical points are $\left(\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right).$
Maximum value of 3 occurs at $\left(\pm\frac{3}{\sqrt{2}}, \pm\frac{2}{\sqrt{2}}\right).$

3. Let $\nabla f(x,\ y) = \lambda\nabla g(x,\ y)$, where
$g(x,\ y) = x^2 + y^2 - 1 = 0.$
$\langle 8x - 4y, -4x + 2y\rangle = \lambda\langle 2x,\ 2y\rangle$
1. $4x - 2y = \lambda x$
2. $-2x + y = \lambda y$
3. $x^2 + y^2 = 1$
4. $0 = \lambda x + 2\lambda y$ (From equations 1 and 2)
5. $\lambda = 0$ or $x + 2y = 0$ (4)
$\lambda = 0$: 6. $y = 2x$ (1)
7. $x = \pm\frac{1}{\sqrt{5}}$ (6, 3)
8. $y = \pm\frac{2}{\sqrt{5}}$ (7, 6)
$x + 2y = 0$: 9. $x = -2y$
10. $y = \pm\frac{1}{\sqrt{5}}$ (9, 3)
11. $x = \frac{2}{\sqrt{5}}$ (10, 9)
Critical points: $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$, $\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$, $\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$, $\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
$f(x, y)$ is 0 at the first two critical points and 5 at the last two. Therefore, the maximum value of $f(x, y)$ is 5.

4. $\langle 2x+4y, 4x+2y \rangle = \lambda \langle 1, -1 \rangle$
$2x + 4y = \lambda$, $4x + 2y = -\lambda$, $x - y = 6$
Critical point is (3, –3).

5. $\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 3, -2 \rangle$
$2x = \lambda$, $2y = 3\lambda$, $2z = -2\lambda$, $x + 3y - 2z = 12$
Critical point is $\left(\frac{6}{7}, \frac{18}{7}, -\frac{12}{7}\right)$.
$f\left(\frac{6}{7}, \frac{18}{7}, -\frac{12}{7}\right) = \frac{72}{7}$ is the minimum.

6. Let $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, where
$g(x, y, z) = 2x^2 + y^2 - 3z = 0$.
$\langle 4, -2, 3 \rangle = \lambda \langle 4x, 2y, -3 \rangle$
1. $4 = 4\lambda x$
2. $-2 = 2\lambda y$
3. $3 = -3\lambda$
4. $2x^2 + y^2 - 3z = 0$
5. $\lambda = -1$ (3)
6. $x = -1$, $y = 1$ (5, 1, 2)
7. $z = 1$ (6, 4)

Therefore, (–1, 1, 1) is a critical point, and $f(-1, 1, 1) = -3$. (–3 is the minimum rather than maximum since other points satisfying $g = 0$ have larger values of f. For example, $g(1, 1, 1) = 0$, and $f(1, 1, 1) = 5$.)

7. Let l and w denote the dimensions of the base, h denote the depth. Maximize $V(l, w, h) = lwh$ subject to $g(l, w, h) = lw + 2lh + 2wh = 48$.
$\langle wh, lh, lw \rangle = \lambda \langle w+2h, l+2h, 2l+2w \rangle$
$wh = \lambda(w + 2h)$, $lh = \lambda(l + 2h)$, $lw = \lambda(2l + 2w)$, $lw + 2lh + 2wh = 48$
Critical point is (4, 4, 2).
$V(4, 4, 2) = 32$ is the maximum. ($V(11, 2, 1) = 22$, for example.)

8. Minimize the square of the distance to the plane, $f(x, y, z) = x^2 + y^2 + z^2$, subject to $x + 3y - 2z - 4 = 0$.
$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 3, -2 \rangle$
$2x = \lambda$, $2y = 3\lambda$, $2z = -2\lambda$, $x + 3y - 2z = 4$
Critical point is $\left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}\right)$. The nature of the problem indicates that this will give a minimum rather than a maximum. The least distance to the plane is $\left[f\left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}\right)\right]^{1/2} = \left(\frac{8}{7}\right)^{1/2} \approx 1.0690$.

9. Let l and w denote the dimensions of the base, h the depth. Maximize $V(l, w, h) = lwh$ subject to $0.601w + 0.20(lw + 2lh + 2wh) = 12$, which simplifies to $21w + lh + wh = 30$, or $g(l, w, h) = 2lw + lh + wh - 30$.
Let $\nabla V(l, w, h) = \lambda \nabla g(l, w, h)$;
$\langle wh, lh, lw \rangle = \lambda \langle 2w+h, 2l+h, l+w \rangle$.
1. $wh = \lambda(2w + h)$
2. $lh = \lambda(2l + h)$
3. $lw = \lambda(l + w)$
4. $2lw + lh + wh = 30$
5. $(w - l)h = 2\lambda(w - l)$ (1, 2)
6. $w = l$ or $h = 2\lambda$

$w = 1$:
7. $l = 2\lambda = w$ (3) Note: $w \neq 0$, for then $V = 0$.
8. $h = 4\lambda$ (7, 2)
9. $\lambda = \frac{\sqrt{5}}{2}$ (7, 8, 4)
10. $l = w = \sqrt{5}$, $h = 2\sqrt{5}$ (9, 7, 8)

$h = 2\lambda$:
11. $\lambda = 0$ (2)
12. $l = w = h = 0$ (11, 1 – 3)

(Not possible since this does not satisfy 4.)
$\left(\sqrt{5}, \sqrt{5}, 2\sqrt{5}\right)$ is a critical point and $V\left(\sqrt{5}, \sqrt{5}, 2\sqrt{5}\right) = 10\sqrt{5} \approx 22.36$ ft^3 is the maximum volume (rather than the minimum volume since, for example, $g(1, 1, 14) = 30$ and $V(1, 1, 14) = 14$ which is less than 22.36).

10. Minimize the square of the distance, $f(x, y, z) = x^2 + y^2 + z^2$, subject to $g(x, y, z) = x^2y - z^2 + 9 = 0$.
$\langle 2x, 2y, 2z \rangle = \lambda \langle 2xy, x^2, -2z \rangle$
$2x = 2\lambda xy$, $2y = \lambda x^2$, $2z = -2\lambda z$,
$x^2y - z^2 + 9 = 0$
Critical points are (0, 0, ±3) [case $x = 0$]; $\left(\pm\sqrt{2}, -1, \pm\sqrt{7}\right)$ [case $x \neq 0$, $\lambda = -1$]; and $\left(\pm 3\sqrt[6]{2/9}, -\sqrt[3]{9/2}, 0\right)$ [case $x \neq 0, \lambda \neq -1$].
Evaluating f at each of these eight points yields 9 (case $x = 0$), 10 (case $x \neq 0$, $\lambda = -1$), and $\frac{3}{2}\sqrt[3]{2}\left(\sqrt[3]{9}\right)^2$ (case $x \neq 0, \lambda \neq -1$). The latter is the smallest, so the least distance between the origin and the surface is $3\sqrt[6]{\frac{3}{4}} \approx 2.8596$.

11. Maximize $f(x, y, z) = xyz$, subject to

$$g(x, y, z) = b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 - a^2b^2c^2 = 0$$

$$\langle yz, xz, xy\rangle = \lambda\langle 2b^2c^2x, 2a^2c^2y, 2a^2b^2z\rangle$$

$$yz = 2b^2c^2x,\ xz = 2a^2c^2y,\ xy = 2a^2b^2z,$$

$$b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 = a^2b^2c^2$$

Critical point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

$V\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$, which is the maximum.

12. Maximize $V(x, y, z) = xyz$, subject to $g(x, y, z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$. Let $\nabla V(x, y, z) = \lambda \nabla g(x, y, z)$, so $\langle yz, xz, xy\rangle = \lambda\left\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\rangle$. Then

$\frac{\lambda x}{a} = \frac{\lambda y}{b} = \frac{\lambda z}{c}$ (each equals xyz).

$\lambda \neq 0$ since $\lambda = 0$ would imply $x = y = z = 0$ which would not satisfy the constraint.

Thus, $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. These along with the constraints yield $x = \frac{a}{3},\ y = \frac{b}{3},\ z = \frac{c}{3}$.

The maximum value of $V = \frac{abc}{27}$.

13. Maximize $f(x,y,z) = x + y + z$ with the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 81 = 0$. Let $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, so $\langle 1,1,1\rangle = \lambda\langle 2x, 2y, 2z\rangle$; Thus, $x = y = z$ and $3x^2 = 81$ or $x = y = z = \pm 3\sqrt{3}$.

The maximum value of f is $9\sqrt{3}$ when $\langle x, y, z\rangle = \langle 3\sqrt{3}, 3\sqrt{3}, 3\sqrt{3}\rangle$

14. Minimize $d^2 = f(x, y, z) = x^2 + y^2 + z^2$ with the constraint $g(x, y, z) = 2x + 4y + 3z - 12 = 0$

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$\langle 2x, 2y, 2z\rangle = \lambda\langle 2, 3, 4\rangle;$$

$2x = 2\lambda;\ 2y = 4\lambda;\ 2z = 3\lambda$ leads to a critical point of $\left(\frac{24}{29}, \frac{48}{29}, \frac{36}{29}\right)$ The nature of the problem indicates this will give a minimum rather than a maximum value. The minimum distance is

$$\sqrt{\frac{24}{29}^2 + \frac{48}{29}^2 + \frac{36}{29}^2} \approx 2.2283$$

15. Minimize $d^2 = f(x, y, z) = (x-1)^2 + (y-2)^2 + z^2$ with the constraint $g(x, y, z) = x^2 + y^2 - z = 0$;

$$\langle 2x-2, 2y-4, 2z\rangle = \lambda\langle 2x, 2y, -1\rangle$$

Setting up, solving each equation for λ, and substituting into equation $x^2 + y^2 - z = 0$ produces $\lambda \approx -1.5445$; The resulting critical point is approximately (0.393, 0.786, 0.772). The nature of this problem indicates this will give a minimum value rather than a maximum. The minimum distance is approximately 1.5616.

16. Minimize $d^2 = f(x, y, z) = (x-1)^2 + (y-2)^2 + z^2$ with the constraint $g(x, y, z) = x^2 + y^2 - z^2 = 0$

$$\langle 2x-2, 2y-4, 2z\rangle = \lambda\langle 2x, 2y, -2z\rangle$$

$\lambda = -1,\ x = \frac{1}{2},\ y = 1,\ z = \pm\frac{\sqrt{5}}{2}$; The critical points are $\left(\frac{1}{2}, 1, \frac{\sqrt{5}}{2}\right)$ and $\left(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2}\right)$ which both lead to a minimum distance of $\frac{\sqrt{10}}{2}$.

17. (See problem 37, section 12.8). Let the dimensions of the box be l, w, and h. Then the cost of the box is

$.25(2hl + 2hw + lw) + .4(lw)$ or

$C(l, w, h) = .5hl + .5hw + .65lw$.

We want to minimize C subject to the constraint $lhw = 2$; set $V(l, h, w) = lwh - 2$.

Now:

$$\nabla C(l, w, h) = (.5h + .65w)\mathbf{i} + (.5h + .65l)\mathbf{j} + .5(l + w)\mathbf{k}$$

and

$$\nabla V(l, w, h) = wh\,\mathbf{i} + lh\,\mathbf{j} + lw\,\mathbf{k}$$

Thus the Lagrange equations are

$$.5h + .65w = \lambda wh \qquad (1)$$
$$.5h + .65l = \lambda lh \qquad (2)$$
$$.5(l + w) = \lambda lw \qquad (3)$$
$$lwh = 2 \qquad (4)$$

Solving (4) for h and putting the result in (1) and (2), we get

$$\frac{1}{lw} + .65w = \frac{2\lambda}{l} \qquad (5)$$

$$\frac{1}{lw} + .65l = \frac{2\lambda}{w} \qquad (6)$$

Multiply (5) by l and (6) by w to get

$$\frac{1}{w} + .65lw = 2\lambda \qquad (7)$$

$$\frac{1}{l} + .65lw = 2\lambda \qquad (8)$$

from which we conclude that $l = w$. Putting this result into (3) we have

$$l = \lambda l^2 \qquad (9)$$

Since $V \neq 0$, $l \neq 0$ and (9) tells us that $l = \frac{1}{\lambda}$;

thus $l = \frac{1}{\lambda}$, $w = l = \frac{1}{\lambda}$, $h = \frac{2}{lw} = 2\lambda^2$.

Putting these results into equation (1), we conclude

$.5(2\lambda^2) + .65\left(\frac{1}{\lambda}\right) = \lambda\left(\frac{1}{\lambda}\right)(2\lambda^2)$ or

$.65\left(\frac{1}{\lambda}\right) = \lambda^2$.

Hence: $\lambda = \sqrt[3]{.65} \approx .866$, so the minimum cost is obtained when:

$l = w = \frac{1}{\lambda} \approx 1.154$ and $h = 2\lambda^2 \approx 1.5$

18. (See problem 38, section 12.8). Let the dimensions of the box be *l, w,* and *h*. Then the cost of the box is

$1(2hl + 2hw) + 4(lw) + 3(2l + 2w) + 4h$ or

$C(l, w, h) = 2hl + 2hw + 4lw + 6l + 6w + 4h$.

We want to minimize *C* subject to the constraint $lhw = 60$; set $V(l, h, w) = lwh - 60$.

Now:

$\nabla C(l, w, h) = (2h + 4w + 6)\mathbf{i} + (2h + 4l + 6)\mathbf{j}$ and $+ (2l + 2w + 4)\mathbf{k}$

$\nabla V(l, w, h) = wh\,\mathbf{i} + lh\,\mathbf{j} + lw\,\mathbf{k}$

Thus the Lagrange equations are

$2h + 4w + 6 = \lambda wh \qquad (1)$

$2h + 4l + 6 = \lambda lh \qquad (2)$

$2l + 2w + 4 = \lambda lw \qquad (3)$

$lwh = 60 \qquad (4)$

Solving (4) for $h = \frac{60}{lw}$ and putting the result in (1) and (2), we get

$\frac{120}{lw} + 4w + 6 = \frac{60\lambda}{l} \qquad (5)$

$\frac{120}{lw} + 4l + 6 = \frac{60\lambda}{w} \qquad (6)$

Multiply (5) by *l* and (6) by *w* to get

$\frac{120}{w} + 4lw + 6l = 60\lambda \qquad (7)$

$\frac{120}{l} + 4lw + 6w = 60\lambda \qquad (8)$

from which we conclude that

$\frac{120}{l} + 6w = \frac{120}{w} + 6l$ or $(l - w)(lw + 20) = 0$.

Since lw cannot be negative ($= -20$), we conclude that $l = w$; putting this result into equation (3), we have

$2w + 2w + 4 = \lambda w^2$ or $\lambda = 4\left(\frac{w+1}{w^2}\right)$.

Therefore, from equation (1), we have

$\frac{120}{w^2} + 4w + 6 = 4\left(\frac{w+1}{w^2}\right)w\left(\frac{60}{w^2}\right)$ or

(multiplying through by w^3 and simplifying)

$2w^4 + 3w^3 - 60w - 120 = 0$

Using one of several techniques available to solve, we conclude that $w = l = 3.213$ and

$h = \frac{60}{(3.213)^2} \approx 5.812$.

19. (See problem 40, section 12.8)

Let

c = circumference of circle

p = perimeter of first square

q = perimeter of second square

Then the sum of the areas is

$$A(c, p, q) = \frac{c^2}{4\pi} + \frac{p^2}{16} + \frac{q^2}{16} = \frac{1}{4}\left[\frac{c^2}{\pi} + \frac{p^2}{4} + \frac{q^2}{4}\right]$$

so we wish to maximize and minimize

$\tilde{A}(c, p, q) = \frac{c^2}{\pi} + \frac{p^2}{4} + \frac{q^2}{4}$ subject to the constraint $L(c, p, q) = c + p + q - k = 0$.

Now

$\nabla \tilde{A}(c, p, q) = \frac{2c}{\pi}\mathbf{i} + \frac{p}{2}\mathbf{j} + \frac{q}{2}\mathbf{k}$

$\nabla L(c, p, q) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

so the Lagrange equations are

$\frac{2c}{\pi} = \lambda \qquad (1)$

$\frac{p}{2} = \lambda \qquad (2)$

$\frac{q}{2} = \lambda \qquad (3)$

$c + p + q = k \qquad (4)$

Putting (1), (2) and (3) into (4) we get

$(4 + \frac{\pi}{2})\lambda = k$ or $\lambda = \frac{2k}{8+\pi}$

Therefore:

$c_0 = \frac{\pi k}{8 + \pi} \approx 0.282k$

$p_0 = \frac{4k}{8 + \pi} \approx 0.359k$

$q_0 = \frac{4k}{8 + \pi} \approx 0.359k$

Now $A(c_0, p_0, q_0) \approx 0.0224k^2$ while $A(k, 0, 0) \approx .079k^2$, so we conclude that

$A(c_0, p_0, q_0)$ is a minimum value. There is also a maximum value (see problem 40, section 12.8) but our Lagrange approach does not capture this. The reason is that the maximum exists because *c*, *p*, and *q* must all be ≥ 0. Our constraint, however, does not require this and allows negative values for any or all of the variables. Under these conditions, there is no global maximum.

20. (See problem 42, section 12.8). Let P be the plane $\frac{x}{A}+\frac{y}{B}+\frac{z}{C}=1$. This plane will cross the first octant, forming a triangle, T, in P; the vertices of this triangle occur where P intersects the coordinate axes. They are: $V_x=(A,0,0),\ V_y=(0,B,0),\ V_z=(0,0,C)$.

a. Define the vectors $\mathbf{g}=\langle -A,B,0\rangle$ and $\mathbf{h}=\langle -A,0,C\rangle$. From example 3 in 11.4, we know the area of T is

$$\frac{1}{2}|\mathbf{g}\times\mathbf{h}|=\frac{1}{2}\sqrt{(BC)^2+(AC)^2+(AB)^2}.$$

b. The height of the tetrahedron in question is the distance is the distance between (0,0,0) and P. By example 10 in 11.3, this distance is

$$h=\frac{1}{\sqrt{\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}}}=\sqrt{\frac{(ABC)^2}{(BC)^2+(AC)^2+(AB)^2}}$$

c. Finally, the volume of the tetrahedron is $\frac{1}{3}h(\text{area of } T)$, or

$$V(A,B,C)=\frac{1}{6}\sqrt{(BC)^2+(AC)^2+(AB)^2}\cdot\left[\frac{\sqrt{(ABC)^2}}{\sqrt{(BC)^2+(AC)^2+(AB)^2}}\right]$$

That is, $V(A,B,C)=\frac{1}{6}\sqrt{(ABC)^2}=\frac{1}{6}|ABC|$.

Hence we want to minimize $\tilde{V}(A,B,C)=|ABC|$ subject to the constraint $\frac{a}{A}+\frac{b}{B}+\frac{c}{C}=1$; define

$$g(A,B,C)=\frac{a}{A}+\frac{b}{B}+\frac{c}{C}-1.$$

Now

$$\nabla\tilde{V}(A,B,C)=\frac{A}{|ABC|}\mathbf{i}+\frac{B}{|ABC|}\mathbf{j}+\frac{C}{|ABC|}\mathbf{k}$$

and $\nabla g(A,B,C)=\frac{-a}{A^2}\mathbf{i}+\frac{-b}{B^2}\mathbf{j}+\frac{-c}{C^2}\mathbf{k}$. Thus the Lagrange equations are

$$\frac{A}{|ABC|}=\frac{-\lambda a}{A^2} \qquad (1)$$

$$\frac{B}{|ABC|}=\frac{-\lambda b}{B^2} \qquad (2)$$

$$\frac{C}{|ABC|}=\frac{-\lambda c}{C^2} \qquad (3)$$

$$\frac{a}{A}+\frac{b}{B}+\frac{c}{C}=1 \qquad (4)$$

From (1) – (3) we have

$$\lambda|ABC|=\frac{-A^3}{a}=\frac{-B^3}{b}=\frac{-C^3}{c} \qquad (5)$$

Solving in pairs we get

$$B=\left(\sqrt[3]{\frac{b}{a}}\right)A,\quad C=\left(\sqrt[3]{\frac{c}{a}}\right)A \qquad (6)$$

and putting these results into (4) we obtain

$$A=a+\sqrt[3]{ab^2}+\sqrt[3]{ac^2}$$
$$=\sqrt[3]{a}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)$$

Similarly, we have

$$B=\sqrt[3]{a^2b}+b+\sqrt[3]{bc^2}$$
$$=\sqrt[3]{b}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)$$

$$C=\sqrt[3]{a^2c}+\sqrt[3]{b^2c}+c$$
$$=\sqrt[3]{c}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)$$

Finally, the volume of the tetrahedron is

$$\frac{|ABC|}{6}=\frac{\sqrt[3]{abc}\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\right)^3}{6}.$$

21. Finding critical points on the interior first: $\frac{\partial f}{\partial x}=1\neq 0$ $\frac{\partial f}{\partial y}=1\neq 0$; There are no critical points on the interior. Finding critical points on the boundary: $\nabla f(x,y)=\lambda\nabla g(x,y)$; $\langle 1,1\rangle=\lambda\langle 2x,2y\rangle$; The solution to the system $1=\lambda\cdot 2x, 1=\lambda\cdot 2y, x^2+y^2=1$ is $\lambda=\pm\frac{1}{\sqrt{2}}$, $x=\pm\frac{1}{\sqrt{2}}$, $y=\pm\frac{1}{\sqrt{2}}$ The four critical points are $\left(\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$.

$f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=10+\sqrt{2}$ is the maximum value.

$f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)=10-\sqrt{2}$ is the minimum value.

22. Finding critical points on the interior:

$\frac{\partial f}{\partial x} = 1 - y = 0 \Rightarrow y = 1;$

$\frac{\partial f}{\partial y} = 1 - x = 0 \Rightarrow x = 1;$ The only critical point on the interior is $c_1 =$ (1,1). Finding critical points on the boundary: Solve the system of equations

$1 - y = \lambda \cdot 2x;\ 1 - x = \lambda \cdot 2y;\ x^2 + y^2 = 9$

Using substitution, it can be found that the critical points on the boundary are

$c_2 = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, $c_3 = \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$,

$c_4 =$ (2.56155, -1,56155),

$c_5 =$ (-1.56155, 2.56155)

The maximum value of 5 is obtained substituting either c_4 or c_5 into *f*. The minimum value of about -8.7426 is obtained by substituting c_3 into *f*.

23. Finding critical points on the interior:

$\frac{\partial f}{\partial x} = 2x + 3 - y = 0;\ \frac{\partial f}{\partial y} = 2y - x = 0$

The solution to this system is the only critical point on the interior, $c_1 = (-2,-1)$.

Critical points on the boundary will come from the solutions to the following system of equations:

$2x + 3 - y = \lambda \cdot 2x,\quad 2y - x = \lambda \cdot 2y,$

$x^2 + y^2 = 9$. From the solutions to this system, the critical points are $c_2 = (0,3)$,

$c_3 = \left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right),\ c_4 = \left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

$f(c_1) = -3,\quad f(c_2) = 9,\quad f(c_3) \approx 20.6913,$

$f(c_4) \approx -2.6913$ The max value of *f* is ≈ 20.6913 and the min value is -3.

24. $f(x,y) = \frac{x}{1+y^2}$ on the set $S = \left\{(x,y): \frac{x^2}{4} + \frac{y^2}{9} \le 1\right\}$

We first find the max and min for f on the set $\tilde{S} = \left\{(x,y): \frac{x^2}{4} + \frac{y^2}{9} < 1\right\}$ using the methods of section 12.8:

$\nabla f(x,y) = \frac{1}{1+y^2}\mathbf{i} + \frac{-2xy}{(1+y^2)^2}\mathbf{j}$ so setting $\nabla f(x,y) = \mathbf{0}$ we have $\frac{1}{1+y^2} = 0$ (impossible).

Thus f has no max or min on $\tilde{S}$.

We now look for the max and min of f on the boundary $\bar{S} = \left\{(x,y): \frac{x^2}{4} + \frac{y^2}{9} = 1\right\}$; this is done using Lagrange multipliers. Let

$g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1$; then

$\nabla f(x,y) = \frac{1}{1+y^2}\mathbf{i} + \frac{-2xy}{(1+y^2)^2}\mathbf{j}$ and

$\nabla g(x,y) = \frac{x}{2}\mathbf{i} + \frac{2y}{9}\mathbf{j}$

The Lagrange equations are

$$\frac{1}{1+y^2} = \frac{\lambda x}{2} \qquad (1)$$

$$\frac{-2xy}{(1+y^2)^2} = \frac{2\lambda y}{9} \qquad (2)$$

$$9x^2 + 4y^2 = 36 \qquad (3)$$

Putting (1) into (2) yields

$$\frac{-\lambda^2 x^3 y}{2} = \frac{2\lambda y}{9} \qquad (4)$$

One solution to (4) is $y = 0$ which yields, from (3), $x = \pm 2$. Thus $(2,0)$ and $(-2,0)$ are candidates for optimization points.

If $y \ne 0$, (4) can be reduced to

$$\frac{-\lambda^2 x^3}{2} = \frac{2\lambda}{9} \qquad (5)$$

so that $\lambda = \frac{-4}{9x^3}$. Putting this result into (1) yields $\frac{1}{1+y^2} = -\frac{2}{9x^2}$, which has no solutions (left side always +, right side always -).

Therefore the only two candidates for max/min are $(2,0)$ and $(-2,0)$. Since $f(2,0) = 2$ and $f(-2,0) = -2$ we conclude that the max value of f on S is 2 and the min value is -2.

25. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 2(1 + x + y) = 0 \Rightarrow x + y = -1$

There is no minimum or maximum value on the interior since there are an infinite number of critical points. The critical points on the boundary will come from the solutions to the following system of equations:

$2(1 + x + y) = \lambda \cdot \frac{1}{2}x$

$2(1 + x + y) = \lambda \cdot \frac{1}{8}y$

Solving these two equations for λ leads to $y = -x - 1$ or $y = 4x$. Together with the constraint $\frac{x^2}{4} + \frac{y^2}{16} - 1 = 0$ leads to the critical points on the boundary: $\left(\frac{-1-2\sqrt{19}}{5}, \frac{-4+2\sqrt{19}}{5}\right)$, $\left(\frac{-1+2\sqrt{19}}{5}, \frac{-4-2\sqrt{19}}{5}\right)$, $\left(-\frac{2}{\sqrt{5}}, -\frac{8}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, \frac{8}{\sqrt{5}}\right)$. Respectively, the maximum value is ≈ 29.9443 and the minimum value is 0.

26. It is clear that the maximum will occur for a triangle which contains the center of the circle. (With this observation in mind, there are additional constraints: $0 < \alpha < \pi$, $0 < \beta < \pi$, $0 > \gamma < \pi$.)
Note that in an isosceles triangle, the side opposite the angle θ which is between the congruent sides of length r has length $2r\sin\left(\frac{\theta}{2}\right)$. Then we wish to maximize

$$P(\alpha, \beta, \gamma) = 2r\left[\sin\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\beta}{2}\right) + \sin\left(\frac{\gamma}{2}\right)\right]$$

subject to $g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - 2\pi = 0 = 0$.

Let $r\left\langle \cos\left(\frac{\alpha}{2}\right), \cos\left(\frac{\beta}{2}\right), \cos\left(\frac{\gamma}{2}\right)\right\rangle = \lambda\langle 1, 1, 1\rangle$.

Then $\lambda = r\cos\left(\frac{\alpha}{2}\right) = r\cos\left(\frac{\beta}{2}\right) = r\cos\left(\frac{\gamma}{2}\right)$, so

$\alpha = \beta = \gamma$ $\left(\text{since } \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi\right)$.

$3\alpha a = 2\pi$, so $\alpha = \frac{2\pi}{3}$; then $\beta = \gamma = \frac{2\pi}{3}$.

27. Let $\alpha + \beta + \gamma = 1$, $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.
Maximize $P(x, y, z) = kx^{\alpha}y^{\beta}z^{\gamma}$, subject to $g(x, y, z) = ax + by + cz - d = 0$.

Let $\nabla P(x, y, z) = \lambda\nabla g(x, y, z)$. Then $\left\langle k\alpha x^{\alpha-1}y^{\beta}z^{\gamma}, k\beta x^{\alpha}y^{\beta-1}z^{\gamma}, k\gamma x^{\alpha}y^{\beta}z^{\gamma-1}\right\rangle = \lambda\langle a, b, c\rangle$.

Therefore, $\frac{\lambda ax}{\alpha} = \frac{\lambda by}{\beta} = \frac{\lambda cz}{\gamma}$ (since each equals $kx^{\alpha}y^{\beta}z^{\gamma}$).

$\lambda \neq 0$ since $\lambda = 0$ would imply $x = y = z = 0$ which would imply $P = 0$.

Therefore, $\frac{ax}{\alpha} = \frac{by}{\beta} = \frac{cz}{\gamma}$ (*).

The constraints $ax + by + cz = d$ in the form $\alpha\left(\frac{ax}{\alpha}\right) + \beta\left(\frac{by}{\beta}\right) + \gamma\left(\frac{cz}{\gamma}\right) = d$ becomes

$\alpha\left(\frac{ax}{\alpha}\right) + \beta\left(\frac{ax}{\alpha}\right) + \gamma\left(\frac{ax}{\alpha}\right) = d$, using (*).

Then $(\alpha + \beta + \gamma)\left(\frac{ax}{\alpha}\right) = d$, or $\frac{ax}{\alpha} = d$ (since $\alpha + \beta + \gamma = 1$).

$x = \frac{\alpha d}{a}$ (**); $y = \frac{\beta d}{b}$ and $z = \frac{\gamma d}{c}$ then following using (*) and (**).

Since there is only one interior critical point, and since P is 0 on the boundary, P is maximum when $x = \frac{\alpha d}{a}$, $y = \frac{\beta d}{b}$, $z = \frac{\gamma d}{c}$.

28. Let (x, y, z) denote a point of intersection. Let $f(x, y, z)$ be the square of the distance to the origin. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to $g(x, y, z) = x + y + z - 8 = 0$ and $h(x, y, z) = 2x - y + 3z - 28 = 0$.
Let $\nabla f(x, y, z) = \lambda\nabla g(x, y, z) + \mu\nabla h(x, y, z)$.

$\langle 2x, 2y, 2z\rangle = \lambda\langle 1,1,1\rangle + \mu\langle 2,-1,3\rangle$

1. $2x = \lambda + 2\mu$
2. $2y = \lambda - \mu$
3. $2z = \lambda + 3\mu$
4. $x + y + z = 8$
5. $2x - y + 3z = 28$
6. $3\lambda + 4\mu = 16$ (1, 2, 3, 4)
7. $2\lambda + 7\mu = 28$ (1, 2, 3, 5)
8. $\lambda = 0$, $\mu = 4$ (6, 7)
9. $x = 4$, $y = -2$, $z = 6$ (8, 1-3)

$f(4, -2, 6) = 56$, and the nature of the problem indicates this is the minimum rather than the maximum.

Conclusion: The least distance is $\sqrt{56} \approx 7.4833$.

29. $\langle -1,2,2\rangle = \lambda\langle 2x,2y,0\rangle + \mu\langle 0,1,2\rangle$

$-1 = 2\lambda x$, $2 = 2\lambda y + \mu$, $2 = 2\mu$, $x^2 + y^2 = 2$,
$y + 2z = 1$
Critical points are (–1, 1, 0) and (1, –1, 1).
$f(-1, 1, 0) = 3$, the maximum value;
$f(1, -1, 1) = -1$, the minimum value.

30. **a.** Maximize
$w(x_1, x_2, \ldots, x_n) = x_1x_2, \ldots, x_n, (x_i > 0)$
subject to the constraint
$g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n - 1 = 0.$ Le
t $\nabla w(x_1, x_2, \ldots, x_n) = \lambda\nabla g(x_1, x_2, \ldots, x_n).$
$\langle x_2 \ldots x_n,\ x_1x_3 \ldots x_n,\ x_1 \ldots x_{n-1}\rangle = \lambda\langle 1, 1, \ldots, 1\rangle.$
Therefore, $\lambda x_1 = \lambda x_2 = \ldots = \lambda x_n$ (since each equals $x_1x_2 \ldots x_n$). Then $x_1 = x_2 = \ldots = x_n$.
(If $\lambda = 0$, some $x_i = 0$, so $w = 0$.)

Therefore, $nx_i = 1$; $x_i = \dfrac{1}{n}$.

The maximum value of w is $\left(\dfrac{1}{n}\right)^n$, and

occurs when each $x_i = \dfrac{1}{n}$.

b. From part a we have that $x_1x_2 \ldots x_n \le \left(\dfrac{1}{n}\right)^n$.

Therefore, $\sqrt[n]{x_1x_2 \ldots x_n} \le \dfrac{1}{n}$.

If $x_i = \dfrac{a_i}{a_1 + \ldots + a_n} = \dfrac{a_i}{A}$ for each i, then

$\sqrt[n]{\dfrac{a_1}{A}\dfrac{a_2}{A} \ldots \dfrac{a_n}{A}} \le \dfrac{1}{n}$, so $\sqrt[n]{a_1a_2 \ldots a_n} \le \dfrac{A}{n}$, or

$\sqrt[n]{a_1a_2 \ldots a_n} \le \dfrac{a_1 + a_2 + \ldots a_n}{n}$.

31. Let $\langle a_1, a_2, \ldots a_n\rangle = \lambda\langle 2x_1, 2x_2, \ldots, 2x_n\rangle$.
Therefore, $a_i = 2\lambda x_i$, for each $i = 1, 2, \ldots, n$
(since $\lambda = 0$ implies $a_i = 0$, contrary to the hypothesis).

$\dfrac{x_i}{a_i} = \dfrac{x_j}{a_j}$ for all i, j $\left(\text{since each equals } \dfrac{1}{2\lambda}\right)$.

The constraint equation can be expressed

$$a_1^2\left(\frac{x_1}{a_1}\right)^2 + a_2^2\left(\frac{x_2}{a_2}\right)^2 + \ldots + a_n^2\left(\frac{x_n}{a_n}\right)^2 = 1.$$

Therefore, $\left(a_1^2 + a_2^2 + \ldots + a_n^2\right)\left(\dfrac{x_1}{a_1}\right)^2 = 1.$

$x_1^2 = \dfrac{a_1^2}{a_1^2 + \ldots + a_n^2}$; similar for each other x_i^2.

The function to be maximized in a hyperplane with positive coefficients and constant (so intercepts on all axes are positive), and the constraint is a hypersphere of radius 1, so the maximum will occur where each x_i is positive.

There is only one such critical point, the one obtained from the above by taking the principal square root to solve for x_i.

Then the maximum value of w is

$$a_1\left(\frac{a_1}{\sqrt{A}}\right) + a_2\left(\frac{a_2}{\sqrt{A}}\right) + \ldots + a_n\left(\frac{a_n}{\sqrt{A}}\right) = \frac{A}{\sqrt{A}} = \sqrt{A}$$

where $A = a_1^2 + a_2^2 + \ldots + a_n^2$.

32. Max: $f(-0.71, 0.71) = f(-0.71, -0.71) = 0.71$

33. Min: $f(4, 0) = -4$

34. Max: $f(1.41, 1.41) = f(-1.41, -1.44) = 0.037$

35. Min: $f(0, 3) = f(0, -3) = -0.99$

12.10 Chapter Review

Concepts Test

1. True: Except for the trivial case of $z = 0$, which gives a point.

2. False: Use $f(0, 0) = 0$; $f(x, y) = \dfrac{xy}{x^2 + y^2}$ elsewhere for counterexample.

3. True: Since $g'(0) = f_x(0, 0)$

4. True: It is the limit along the path, $y = x$.

5. True: Use "Continuity of a Product" Theorem.

6. True: Straight forward calculation of partial derivatives

7. False: See Problem 25, Section 12.4.

8. False: It is perpendicular to the level curves of f. The gradient of $F(x, y, z) = f(x, y) - z$ is perpendicular to the graph of $z = f(x, y)$.

9. True: Since $\langle 0,0,-1\rangle$ is normal to the tangent plane

10. False: C^{ex}: For the cylindrical surface $f(x, y) = y^3$, $f(\mathbf{p}) = \mathbf{0}$ for every $\mathbf{p}$ on the x-axis, but $f(\mathbf{p})$ is not an extreme value.

11. True: It will point in the direction of greatest increase of heat, and at the origin, $\nabla T(0, 0) = \langle 1, 0 \rangle$ is that direction.

12. True: It is nonnegative for all x, y, and it has a value of 0 at (0, 0).

13. True: Along the x-axis, $f(x, 0) \to \pm\infty$ as $x \to \pm\infty$.

14. False: $|D_{\mathbf{u}} f(x, y)| = |\langle 4, 4 \rangle \cdot \mathbf{u}| \le 4\sqrt{2}$ (equality if $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\langle 1,1 \rangle$)

15. True: $-D_{\mathbf{u}} f(x, y) = -[\nabla f(x, y) \cdot \mathbf{u}]$
$= \nabla f(x, y) \cdot (-\mathbf{u}) = D_{-\mathbf{u}} f(x, y)$

16. True: The set (call it S, a line segment) contains all of its boundary points because for every point P not in S (i.e., not on the line segment), there is an open neighborhood of P (i.e., a circle with P as center) that contains no point of S.

17. True: By the Min-Max Existence Theorem

18. False: (x_0, y_0) could be a singular point.

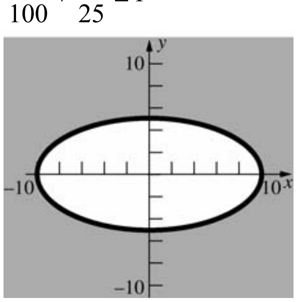
19. False: $f\left(\frac{\pi}{2}, 1\right) = \sin\left(\frac{\pi}{2}\right) = 1,$ the maximum value of f, and $(\pi/2, 1)$ is in the set.

20. False: The same function used in Problem 2 provides a counterexample.

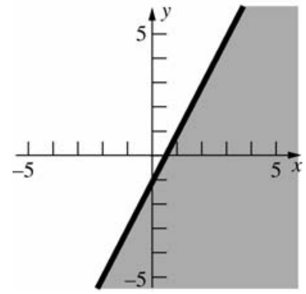
Sample Test Problems

1. a. $x^2 + 4y^2 - 100 \ge 0$

$\frac{x^2}{100} + \frac{y^2}{25} \ge 1$

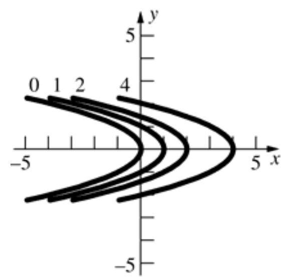


b. $2x - y - 1 \ge 0$



2. $x + y^2 = k$

$x - k = -y^2$



3. $f_x(x, y) = 12x^3y^2 + 14xy^7$

$f_{xx}(x, y) = 36x^2y^2 + 14y^7$

$f_{xy}(x, y) = 24x^3y + 98xy^6$

4. $f_x(x, y) = -2\cos x \sin x = -\sin 2x$

$f_{xx}(x, y) = -2x\cos 2x$

$f_{xy}(x, y) = 0$

5. $f_x(x, y) = e^{-y}\sec^2 x$

$f_{xx}(x, y) = 2e^{-y}\sec^2 x \tan x$

$f_{xy}(x, y) = -e^{-y}\sec^2 x$

6. $f_x(x, y) = -e^{-x}\sin y$

$f_{xx}(x, y) = e^{-x}\sin y$

$f_{xy}(x, y) = -e^{-x}\cos y$

7. $F_y(x, y) = 30x^3y^5 - 7xy^6$

$F_{yy}(x, y) = 150x^3y^4 - 42xy^5$

$F_{yyx}(x, y) = 450x^2y^4 - 42y^5$

8. $f_x(x, y, z) = y^3 - 10xyz^4$

$f_y(x, y, z) = 3xy^2 - 5x^2z^4$

$f_z(x, y, z) = -20x^2yz^3$

Therefore, $f_x(2, -1, 1) = 19$;
$f_y(2, -1, 1) = -14$; $f_z(2, -1, 1) = 80$

9. $z_y(x, y) = \frac{y}{2};\ \ z_y(2, 2) = \frac{2}{2} = 1$

10. Everywhere in the plane except on the parabola $x^2 = y$.

11. No. On the path $y = x$, $\lim_{x\to 0} \frac{x - x}{x + x} = 0$. On the path $y = 0$, $\lim_{x\to 0} \frac{x - 0}{x + 0} = 1$.

12. a. $\lim_{(x, y)\to(2, 2)} \frac{x^2 - 2y}{x^2 + 2y} = \frac{4 - 4}{4 + 4} = 0$

b. Does not exist since the numerator lends to 4 and the denominator to 0.

c. $\lim_{(x, y)\to(0, 0)} \frac{(x^2 + 2y^2)(x^2 - 2y^2)}{x^2 + 2y^2}$

$= \lim_{(x, y)\to(0, 0)} (x^2 - y^2) = 0$

13. a. $\nabla f(x, y, z) = \left\langle 2xyz^3, x^2z^3, 3x^2yz^2 \right\rangle$

$f(1, 2, -1) = \langle -4, -1, 6\rangle$

b. $\nabla f(x, y, z)$

$= \left\langle y^2 z\cos xz, 2y\sin xz, xy^2\cos xz \right\rangle$

$\nabla f(1, 2, -1) = -4\langle \cos(1), \sin(1), -\cos(1)\rangle$

$\approx \langle -2.1612, -3.3659, 2.1612\rangle$

14. $D_{\mathbf{u}} f(x, y) = \left\langle 3y(1 + 9x^2y^2)^{-1}, 3x(1 + 9x^2y^2)^{-1} \right\rangle \cdot \mathbf{u}$

$D_{\mathbf{u}} f(4, 2) = \left\langle \frac{6}{577}, \frac{12}{577} \right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = \frac{\left(3\sqrt{3} - 6\right)}{577}$

≈ -0.001393

15. $z = f(x, y) = x^2 + y^2$

$\left\langle 1, -\sqrt{3}, 0 \right\rangle$ is horizontal and is normal to the vertical plane that is given. By inspection, $\left\langle \sqrt{3}, 1, 0 \right\rangle$ is also a horizontal vector and is perpendicular to $\left\langle 1, -\sqrt{3}, 0 \right\rangle$ and therefore is parallel to the vertical plane. Then $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ is the corresponding 2-dimensional unit vector.

$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$

$= \langle 2x, 2y\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = \sqrt{3}x + y$

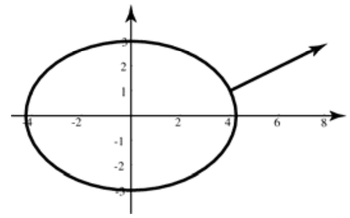
$D_{\mathbf{u}} f(1, 2) = \sqrt{3} + 2 \approx 3.7321$ is the slope of the tangent to the curve.

16. In the direction of $\nabla f(1, 2) = 4\langle 9, 4\rangle$

17. a. $f(4, 1) = 9$, so $\frac{x^2}{2} + y^2 = 9$, or $\frac{x^2}{18} + \frac{y^2}{9} = 1.$

b. $\nabla f(x, y) = \langle x, 2y\rangle$, so $f(4,1) = \langle 4, 2\rangle$.

c.



18. $F_x = F_u u_x + F_v v_x$

$= \frac{v}{1 + u^2v^2} \frac{y}{2\sqrt{xy}} + \frac{u}{1 + u^2v^2} \frac{1}{2\sqrt{x}}$

$= \frac{v\sqrt{y} + u}{2(1 + u^2v^2)\sqrt{x}}$

$F_y = F_u u_y + F_v v_y$

$= \frac{v}{1 + u^2v^2} \frac{x}{2\sqrt{xy}} + \frac{u}{1 + u^2v^2} \frac{-1}{2\sqrt{y}}$

$= \frac{v\sqrt{x} - u}{2(1 + u^2v^2)\sqrt{y}}$

19. $f_x = f_u u_x + f_v u_y = \left(\frac{1}{v}\right)(2x) + \left(-\frac{u}{v^2}\right)(yz)$

$= x^{-2}y^{-1}z^{-1}(x^2 + 3y - 4z)$

$f_y = f_u u_y + f_v v_y = \left(\frac{1}{v}\right)(-3) + \left(-\frac{u}{v^2}\right)(xz)$

$= -x^{-1}y^{-2}z^{-1}(x^2 + 4z)$

$f_z = f_u u_z + f_v v_z = \left(\frac{1}{v}\right)(4) + \left(-\frac{u}{v^2}\right)(xy)$

$= x^{-1}y^{-1}z^{-2}(3y - x^2)$

20. $\dfrac{dF}{dt}=\dfrac{dF}{dx}\dfrac{dx}{dt}+\dfrac{dF}{dy}\dfrac{dy}{dt}$

$=(3x^2-y^2)(-6\sin 3t)+(-2xy-4y^3)(3\cos t)$

$t=0 \Rightarrow x=2$ and $y=0$, so $\left.\left(\dfrac{dF}{dt}\right)\right|_{t=0}=0.$

21. $F_t=F_x x_t+F_y y_t+F_z z_t$

$=\left(\dfrac{10xy}{z^3}\right)\left(\dfrac{3t^{1/2}}{2}\right)+\left(\dfrac{5x^2}{z^3}\right)\left(\dfrac{1}{t}\right)+\left(-\dfrac{15x^2y}{z^4}\right)(3e^{3t})$

$=\dfrac{15xy\sqrt{t}}{z^3}+\dfrac{5x^2}{z^3t}-\dfrac{45x^2ye^{3t}}{z^4}$

22. $\dfrac{dc}{dt}=3, \dfrac{db}{dt}=-2, \dfrac{d\alpha}{dt}=0.1$

Area $=A(b, c, \alpha)=\left(\dfrac{1}{2}\right)c(b\sin\alpha)$

$\dfrac{dA}{dt}=\left[\left(\dfrac{b}{2}\right)(\sin\alpha)\left(\dfrac{dc}{dt}\right)+\left(\dfrac{c}{2}\right)(\sin\alpha)\left(\dfrac{db}{dt}\right)+\left(\dfrac{b}{2}\right)(bc\cos\alpha)\left(\dfrac{d\alpha}{dt}\right)\right]$

$\left.\left(\dfrac{dA}{dt}\right)\right|_{\left(8,\, 10,\, \frac{\pi}{6}\right)}=\dfrac{\left(7+4\sqrt{3}\right)}{2}\approx 6.9641\ \text{in.}^2/\text{s}$

23. Let $F(x, y, z)=9x^2+4y^2+9z^2-34=0$

$\nabla F(x, y, z)=\langle 18x, 8y, 18z\rangle$, so $\nabla f(1, 2, -1)=2\langle 9, 8, -9\rangle$.

Tangent plane is $9(x-1)+8(y-2)-9(z+1)=0$, or $9x+8y-9z=34$.

24. $V=\pi r^2h;\ \ dV=V_r\,dr+V_h\,dh=2\pi rh\,dr+\pi r^2\,dh$

If $r=10$, $|dr|\le 0.02$, $h=6$, $|dh|=0.01$, then

$|dV|\le 2\pi rh|dr|+\pi r^2|dh|\le 2\pi(10)(6)(0.02)+\pi(100)(0.01)=3.4\pi$

$V(10, 6)=\pi(100)(6)=600\pi$

Volume is $600\pi\pm 3.4\pi\approx 1884.96\pm 10.68$

25. $df=y^2(1+z^2)^{-1}dx+2xy(1+z^2)^{-1}dy-2xy^2z(1+z^2)^{-2}dz$

If $x=1$, $y=2$, $z=2$, $dx=0.01$, $dy=-0.02$, $dz=0.03$, then $df=-0.0272$.

Therefore, $f(1.01, 1.98, 2.03)\approx f(1, 2, 2)+df=0.8-0.0272=0.7728$

26. $\nabla f(x, y)=\left\langle 2xy-6x, x^2-12y\right\rangle=\langle 0, 0\rangle$

at $(0, 0)$ and $(\pm 6, 3)$.

$D=f_{xx}f_{yy}-f_{xy}^2=(2y-6)(-12)-(2x)^2$

$=4(18-6y-x^2);\ \ f_{xx}=2(y-3)$

At $(0, 0)$: $D=72>0$ and $f_{xx}<0$, so local maximum at $(0, 0)$.

At $(\pm 6, 3)$: $D<0$, so $(\pm 6, 3)$ are saddle points.

27. Let (x, y, z) denote the coordinates of the 1st octant vertex of the box. Maximize $f(x, y, z) = xyz$ subject to
$g(x, y, z) = 36x^2 + 4y^2 + 9z^2 - 36 = 0$
(where $x, y, z > 0$ and the box's volume is $V(x, y, z) = f(x, y, z)$).
Let $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.
$\langle yz, xz, xy \rangle 8 = \lambda \langle 72x, 8y, 18z \rangle$
1. $8yz = 72\lambda x$
2. $8xz = 8\lambda y$
3. $8xy = 18\lambda z$
4. $36x^2 + 4y^2 + 9z^2 = 36$
5. $\frac{yz}{xz} = \frac{72\lambda x}{8\lambda y}$, so $y^2 = 9x^2$. (1, 2)
6. $\frac{yz}{xz} = \frac{72\lambda x}{18\lambda y}$, so $z^2 = 4x^2$. (1, 3)
7. $36x^2 + 36x^2 + 36x^2 = 36$, so $x = \frac{1}{\sqrt{3}}$. (5, 6, 4)
8. $y = \frac{3}{\sqrt{3}}, z = \frac{2}{\sqrt{3}}$ (7, 5, 6)

$$V\left(\frac{1}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = 8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)$$
$$= \frac{16}{\sqrt{3}} \approx 9.2376$$

The nature of the problem indicates that the critical point yields a maximum value rather than a minimum value.

28. $\langle y, x \rangle = \lambda \langle 2x, 2y \rangle$

$y = 2\lambda x, x = 2\lambda y, \; x^2 + y^2 = 1$

Critical points are $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Maximum of $\frac{1}{2}$ at $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$; minimum of $-\frac{1}{2}$ at $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

29. Maximize $V(r, h) = \pi r^2 h$, subject to

$S(r, h) = 2\pi r^2 + 2\pi rh - 24\pi = 0$.

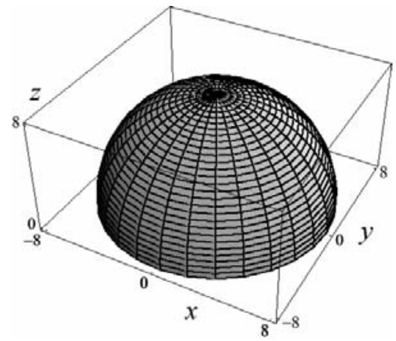
$\langle 2\pi rh, \pi r^2 \rangle = \lambda \langle 4\pi r + 2\pi h, 2\pi r \rangle$

$rh = \lambda(2r + h), r = 2\lambda, \; r^2 + rh = 12$

Critical point is (2, 4). The nature of the problem indicates that the critical point yields a maximum value rather than a minimum value. Conclusion: The dimensions are radius of 2 and height of 4.

Review and Preview Problems

1.



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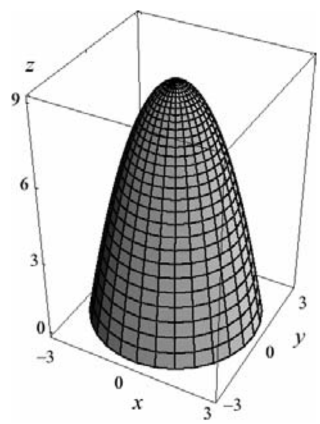

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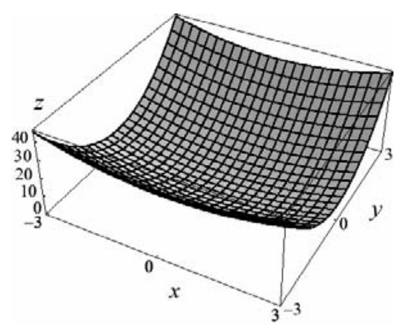

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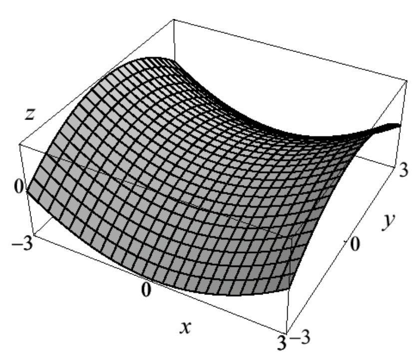

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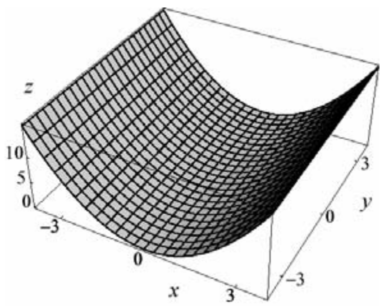
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x
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3
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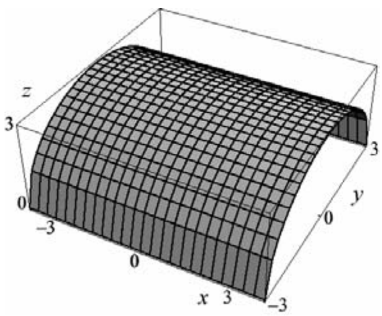
y
−3

6.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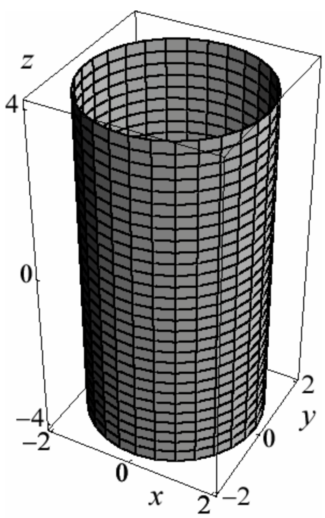
7.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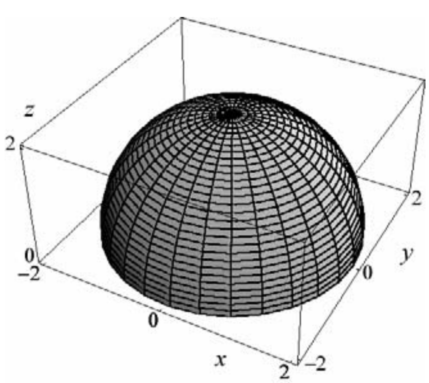
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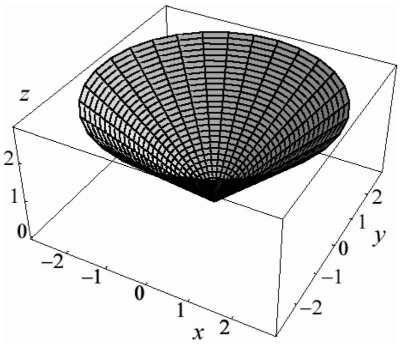
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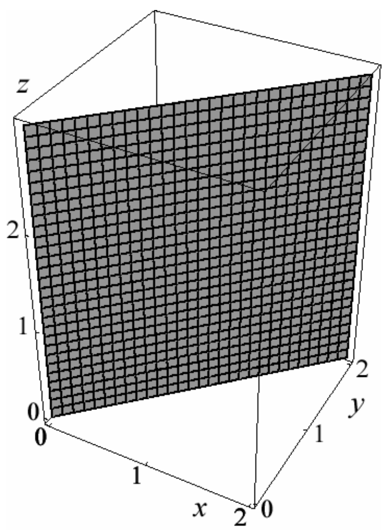
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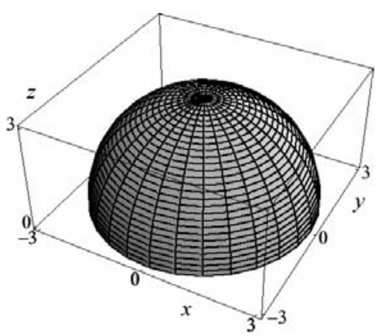
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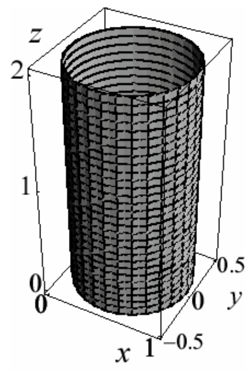
12.



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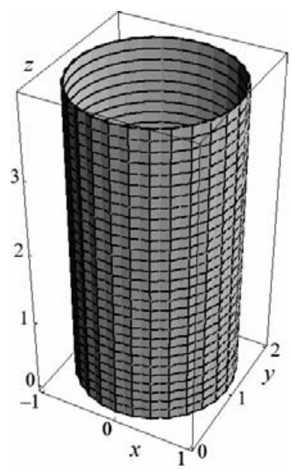

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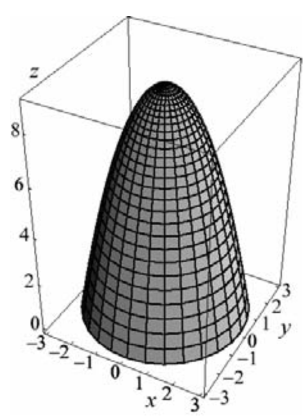

15. $\int e^{-2x}dx = -\frac{1}{2}e^{-2x} + C$

16. $\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx + C$

$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

17. $\int_{-a/2}^{a/2}\cos\left(\frac{x\pi}{a}\right)dx = \frac{a}{\pi}\sin\left(\frac{x\pi}{a}\right)_{-a/2}^{a/2} = \frac{2a}{\pi}$

18. $\int_0^2 \left(a + bx + c^2x^2\right)dx = \left[ax + \frac{1}{2}bx^2 + \frac{1}{3}c^2x^3\right]_0^2$

$= 2a + 2b + \frac{8}{3}c^2$

19. $\int_0^{\pi} \sin^2 x\,dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}$

20. $\int_{1/4}^{3/4} \frac{1}{1-x^2}dx = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|_{1/4}^{3/4} = \frac{1}{2}\ln\left(\frac{21}{5}\right)$

21. $\int_{x=1/4}^{3/4} \frac{1}{1+u}du = \frac{1}{2}\ln\left|1+x^2\right|_{1/4}^{3/4} = \frac{1}{2}\ln 2$

22. $\int_{x=0}^{4} \frac{1}{1+u^2}du = \left[\tan^{-1} e^x\right]_0^4 \approx 0.7671$

23. $u = 4r^2 + 1;\ du = 8r\,dr$

$\int_0^3 r\sqrt{4r^2+1}\,dr = \frac{1}{8}\int_{r=0}^{3}\sqrt{u}\,du$

$= \left[\frac{1}{8}\cdot\frac{2}{3}\left(4r^2+1\right)^{3/2}\right]_0^3 = \frac{-1+37^{3/2}}{12}$

24. $u = a^2 - r^2;\ du = -2r\,dr$

$\int_0^{a/2}\frac{ar}{\sqrt{a^2-r^2}}dr = -\frac{a}{2}\int_{r=0}^{a/2}\frac{1}{\sqrt{u}}du$

$= -\frac{a}{2}\left[\frac{1}{2}\sqrt{a^2-r^2}\right]_0^{a/2} = \frac{a^2\left(2-\sqrt{3}\right)}{8}$

25. $\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$

$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{4}$

26. $\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)^2 d\theta$

$= \int_0^{\pi/2}\left(\frac{1}{4}+\frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\right)^2 d\theta$

$= \int_0^{\pi/2}\left(\frac{3}{8}+\frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right)d\theta$

$= \left[\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_0^{\pi/2} = \frac{3\pi}{16}$

27. $2\pi\left(\sqrt{a^2-b^2} - \sqrt{a^2-c^2}\right)$

θ is not part of the integrand.

28. The area is an equilateral triangle of length $\sqrt{2}$.

$A = \frac{1}{2}\sqrt{2}\,\frac{\sqrt{6}}{2} = \frac{\sqrt{3}}{2}$

29. The solid is half of a right circular cylinder of radius 3 and height 8.

$V = \frac{1}{2}\pi r^2 h = \frac{\pi}{2}(9)(8) = 36\pi$

30. The solid is a sphere of radius 7.

$V = \frac{4}{3}\pi r^3 = \frac{4\pi}{3}7^3 = \frac{1372\pi}{3} \approx 1436.8$ **31.**

The solid looks similar to a football.

$V = \pi\int_0^{\pi} \sin^2 x\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{2}$

32. The solid is a right circular cylinder of radius 7 and height 100.

$V = \pi r^2 h = 4900\pi$

33. The solid is half an elliptic paraboloid.
In the xz-plane, we can consider rotating the graph of $z = 9 - x^2$ around the z-axis for $0 \le x \le 3$. Using the Shell Method, we would get

$V = 2\pi\int_0^3 x\left(9 - x^2\right)dx$

$= 2\pi\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = 2\pi\left[\frac{81}{2} - \frac{81}{4}\right] = \frac{81\pi}{2}$

34. The solid is half of a hollow sphere of radius 1 inside half of a solid sphere of radius 4.

$V = \frac{1}{2}\left(\frac{4}{3}\pi 4^3 - \frac{4}{3}\pi 1^3\right) = 42\pi$