CHAPTER 13 Multiple Integrals

13.1 Concepts Review

1. $\sum_{k=1}^{n} f(\overline{x}_k, \overline{y}_k)\Delta A_k$

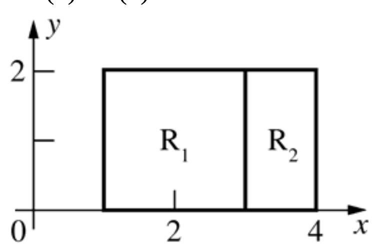
2. the volume of the solid under $z = f(x, y)$ and above R

3. continuous

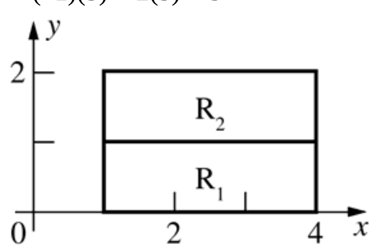
4. 12

Problem Set 13.1

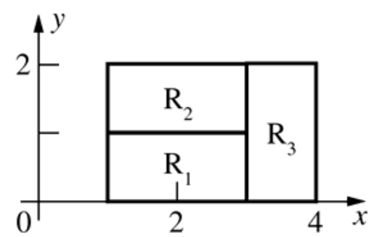
1. $\iint_{R_1} 2\,dA + \iint_{R_2} 3\,dA = 2A(R_1) + 3A(R_2)$
$= 2(4) + 3(2) = 14$



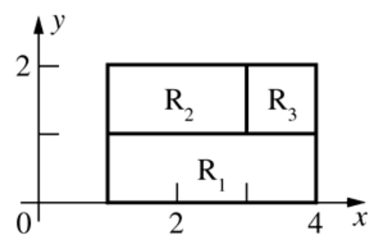
2. $\iint_{R_1} (-1)dA + \iint_{R_2} 2\,dA = (-1)A(R_1) + 2A(R_2)$
$= (-1)(3) + 2(3) = 3$



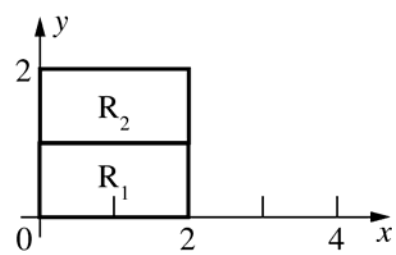
3. $\iint_R f(x, y)dA = \iint_{R_1} 2\,dA + \iint_{R_2} 1\,dA + \iint_{R_3} 3\,dA$
$= 2A(R_1) + 1A(R_2) + 3A(R_3)$
$= 2(2) + 1(2) + 3(2) = 12$



4. $\iint_{R_1} 2\,dA + \iint_{R_2} 3\,dA + \iint_{R_3} 1\,dA$
$= 2A(R_1) + 3A(R_2) + 1A(R_3)$
$= 2(3) + 3(2) + 1(1) = 13$



5. $3\iint_R f(x, y)dA - \iint_R g(x, y)dA = 3(3) - (5) = 4$

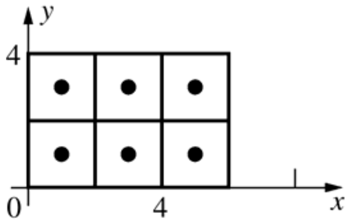


6. $2\iint_R f(x, y)dA + 5\iint_R g(x, y)dA$
$= 2(3) + 5(5) = 31$

7. $\iint_R g(x, y)dA - \iint_{R_1} g(x, y)dA = (5) - (2) = 3$

8. $2\iint_{R_1} g(x, y)dA + \iint_{R_1} 3\,dA = 2(2) + 3A(R_1)$
$= 4 + 3(2) = 10$

9.



$[f(1, 1) + f(3, 1) + f(5, 1) + f(1, 3) + f(3, 3)$
$+ f(5, 3)](4) = [(10) + (8) + (6) + (8) + (6)$
$+ (4)](4) = 168$

10. $4(9 + 9 + 9 + 1 + 1 + 1) = 120$

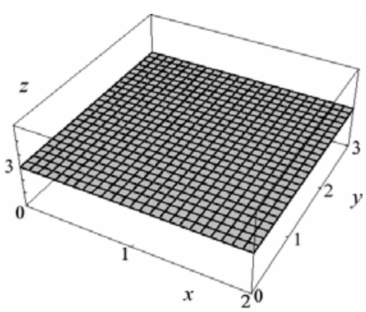
11. $4(3 + 11 + 27 + 19 + 27 + 43) = 520$

12. $\left[\left(\frac{41}{6}\right)+\left(\frac{33}{6}\right)+\left(\frac{25}{6}\right)+\left(\frac{35}{6}\right)+\left(\frac{27}{6}\right)+\left(\frac{19}{6}\right)\right](4)$
$= 120$

13. $4\left(\sqrt{2}+\sqrt{4}+\sqrt{6}+\sqrt{4}+\sqrt{6}+\sqrt{8}\right)\approx 52.5665$

14. $4(e+e^3+e^5+e^3+e^9+e^{15})\approx 13109247$

15.



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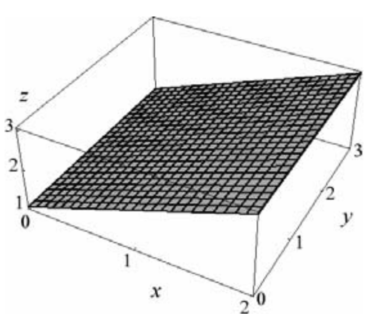

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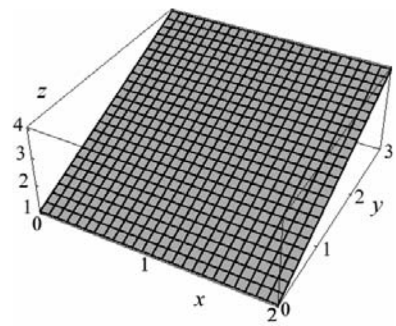

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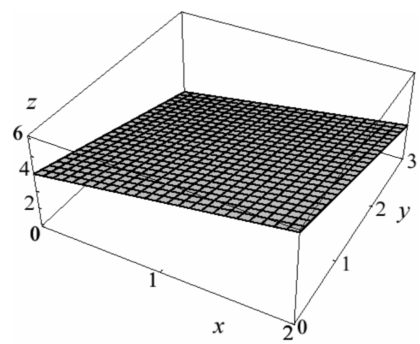

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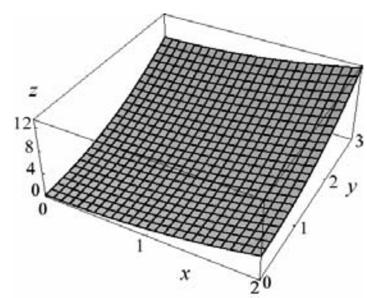

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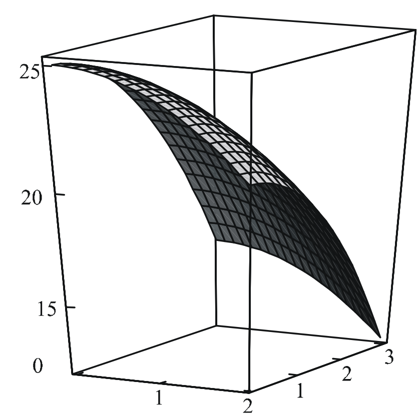

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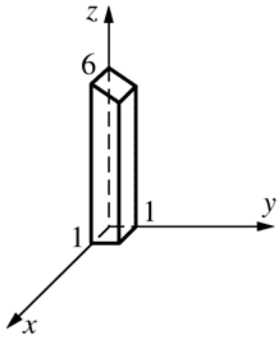
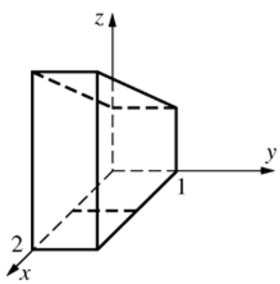


$z = 6 - y$ is a plane parallel to the x-axis. Let T be the area of the front trapezoidal face; let D be the distance between the front and back faces.

$\iint_R (6-y)dA =$ volume of solid $= (T)(D)$

$= \left[\left(\frac{1}{2}\right)(6+5)\right](1) = 5.5$

22.



$z = 1 + x$ is a plane parallel to the y-axis.

$\iint_R (1+x)dA$ is the product of the area of a trapezoidal side face and the distance between the side faces.

$= \left[\left(\frac{1}{2}\right)(1+3)(2)\right](1) = 4$

23. $\iint_R 0\,dA = 0A(R) = 0$

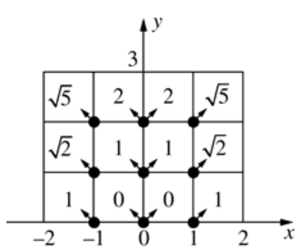
The conclusion follows.

24. $\iint_R m\,dA < \iint_R f(x,y)dA < \iint_R M\,dA$

(Comparison property)

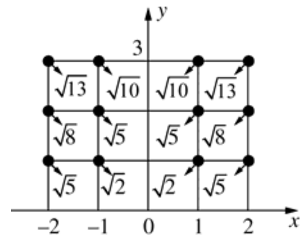
Therefore, $mA(R) < \iint_R f(x,y)dA < MA(R)$

25.



For c, take the sample point in each square to be the point of the square that is closest to the origin.

Then $c = 2\sqrt{5} + 2\sqrt{2} + 2(2) + 4(1) \approx 15.3006$



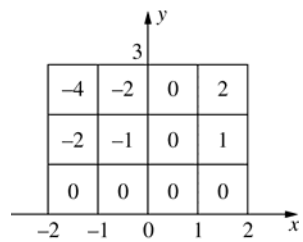
For C, take the sample point in each square to be the point of the square that is farthest from the origin. Then,

$C = 2\sqrt{13} + 2\sqrt{10} + 2\sqrt{8} + 4\sqrt{5} + 2\sqrt{2} \approx 30.9652.$

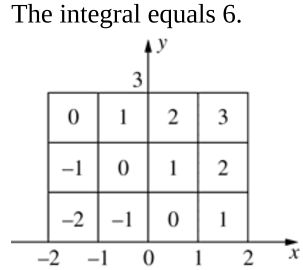
26. The integrand is symmetric with respect to the y-axis (i.e. an odd function), so the value of the integral is 0.

27. The values of $[\![x]\!][\![y]\!]$ and $[\![x]\!]+[\![y]\!]$ are indicated in the various square subregions of R. In each case the value of the integral on R is the sum of the values in the squares since the area of each square is 1.

a. The integral equals –6.



b. The integral equals 6.



28. Mass of the plate in grams

29. Total rainfall in Colorado in 2005; average rainfall in Colorado in 2005.

30. For each partition of R, each subrectangle contains some points at which $f(x, y) = 0$ and some points at which $f(x, y) = 1$. Therefore, for each partition there are sample points for which the Riemann sum is 0 and others for which the Riemann sum is (1)[Area (R)] = 12.

31. To begin, we divide the region R (we will use the outline of the contour plot) into 16 equal squares. Then we can approximate the volume by

$$V = \iint_R f(x, y)dA \approx \sum_{k=1}^{16} f(\bar{x}_k, \bar{y}_k)\Delta A_k .$$

Each square will have $\Delta A = (1\cdot 1) = 1$ and we will use the height at the center of each square as $f(\bar{x}_k, \bar{y}_k)$. Therefore, we get

$$V \approx \sum_{k=1}^{16} f(\bar{x}_k, \bar{y}_k) = 20+21+24+29+22+23+26+32+26+27+30+35+32+33+36+42$$
$$= 458 \text{ cubic units}$$

13.2 Concepts Review

1. iterated

2. $\int_{-1}^{2}\left[\int_0^2 f(x,\ y)dy\right]dx$; $\int_0^2\left[\int_{-1}^{2} f(x,\ y)dx\right]dy$

3. signed; plus; minus

4. is below the *xy*-plane

Problem Set 13.2

1. $\int_0^2 [9y - xy]_0^3\, dx = \int_0^2 [27 - 3x]dx$
$= \left[27x - \frac{3}{2}x^2\right]_0^2 = 48$

2. $\int_{-2}^{2}\left[9y - yx^2\right]_0^1 dx = \int_{-2}^{2}\left[9 - x^2\right]dx$
$= \left[9x - \frac{1}{3}x^3\right]_{-2}^{2} = \frac{92}{3}$

3. $\int_0^2 \left[\left(\frac{1}{2}\right)x^2y^2\right]_{y=1}^{3} dx = \int_0^2 4x^2 dx = \frac{32}{3}$

4. $\int_{-1}^{4}\left[xy + \left(\frac{1}{3}\right)y^3\right]_{y=1}^{2} dx = \int_{-1}^{4}\left(x + \frac{7}{3}\right)dx = \frac{115}{6}$

5. $\int_1^2 \left[\frac{x^2y}{2} + xy^2\right]_{x=0}^{3} dx = \int_1^2\left(\frac{9y}{2} + 3y^2\right)dy$
$= \left[\frac{9y^2}{4} + y^3\right]_1^2 = 17 - \frac{13}{4} = \frac{55}{4} = 13.75$

6. $\int_{-1}^{1}\left[\left(\frac{1}{3}\right)x^3 + xy^2\right]_{x=1}^{2} dy = \int_{-1}^{1}\left(\frac{7}{3} + y^2\right)dy = \frac{16}{3}$

7. $\int_0^{\pi}\left[\left(\frac{1}{2}\right)x^2 \sin y\right]_{x=0}^{1} dy = \int_0^{\pi}\left(\frac{1}{2}\right)\sin y\, dy = 1$

8. $\int_0^{\ln 3}\int_0^{\ln 2} e^x e^y dy\, dx = \int_0^{\ln 3}[e^x e^y]_{y=0}^{\ln 2} dx$
$= \int_0^{\ln 3}[e^x(2) - e^x(1)]dx = \int_0^{\ln 3} e^x dx$
$= [e^x]_0^{\ln 3} = 3 - 1 = 2$

9. $\int_0^{\pi/2}[-\cos xy]_{y=0}^{1} dx = \int_0^{\pi/2}(1 - \cos x)dx$
$= \frac{\pi}{2} - 1 \approx 0.5708$

10. $\int_0^1 [e^{xy}]_{y=0}^{1} dx = \int_0^1 (e^x - 1)dx = e - 2 \approx 0.7183$

11. $\int_0^3 \left[\frac{2(x^2+y)^{3/2}}{3}\right]_{x=0}^{1} dy$
$= \int_0^3 \frac{2[(1+y)^{3/2} - y^{3/2}]}{3} dy$
$= \left[\frac{4[(1+y)^{5/2} - y^{5/2}]}{15}\right]_0^3 = \frac{4\left(32 - 9\sqrt{3}\right) - 4}{15}$
$= \frac{4\left(31 - 9\sqrt{3}\right)}{15} \approx 4.1097$

12. $\int_0^1 [-(xy+1)^{-1}]_{x=0}^{1} dy = \int_0^1\left(1 - \frac{1}{y+1}\right)dy$
$= 1 - \ln 2 \approx 0.3069$

13. $\int_0^{\ln 3}\left[\left(\frac{1}{2}\right)\exp(xy^2)\right]_{y=0}^{1} dx = \int_0^{\ln 3}\left(\frac{1}{2}\right)(e^x - 1)dx$
$= 1 - \left(\frac{1}{2}\right)\ln 3 \approx 0.4507$

14. $\int_0^1 \left[\frac{y^2}{2(1+x^2)}\right]_{y=0}^{2} dx = \int_0^1 \frac{2}{1+x^2}dx$
$= [2\tan^{-1} x]_0^1 = 2\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{2}$

15. $\int_0^{\pi}\left[\frac{1}{2}y^2\cos^2 x\right]_0^3 dx = \int_0^{\pi}\frac{9}{2}\cos^2 x\,dx$

$= \left[\frac{9}{4}x + \frac{9}{8}\cos 2x\right]_0^{\pi} = \frac{9\pi}{4}$

16. $\frac{1}{2}\int_{-1}^{1}\left[e^{x^2}\right]_0^1 dy = \frac{1}{2}\int_{-1}^{1}(e-1)\,dy$

$= \frac{1}{2}\left[y(e-1)\right]_{-1}^{1} = e-1$

17. $\int_0^1 0\,dx = 0$ (since xy^3 defines an odd function in y).

18. $\int_{-1}^{1}\left[x^2y + \left(\frac{1}{3}\right)y^3\right]_{y=0}^{2} dx = \int_{-1}^{1}\left(2x^2 + \frac{8}{3}\right)dx$

$= \left[\left(\frac{2}{3}\right)x^3 + \left(\frac{8}{3}\right)x\right]_{-1}^{1} = \frac{20}{3}$

19. $\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dx\,dy$

$= \int_0^{\pi/2}[-\cos(x+y)]_{x=0}^{\pi/2}dy$

$= \int_0^{\pi/2}\left[-\cos\left(\frac{\pi}{2}+y\right) + \cos y\right]dy$

$= \int_0^{\pi/2}(\sin y + \cos y)\,dy = [-\cos y + \sin y]_0^{\pi/2}$

$= (0 + 1) - (-1 + 0) = 2$

20. $\int_1^2\left[\left(\frac{1}{3}\right)y(1+x^2)^{3/2}\right]_{x=0}^{\sqrt{3}} dy = \int_1^2\left(\frac{7}{3}\right)y\,dy$

$= \left[\left(\frac{7}{6}\right)y^2\right]_1^2 = 3.5$

21. $V = \int_0^2\int_0^3(20 - x - y)\,dy\,dx$

$= \int_0^2\left(20y - xy - \frac{1}{2}y^2\right)_0^3 dx$

$= \int_0^2\left(60 - 3x - \frac{9}{2}\right)dy = 105$

22. $V = \int_0^2\int_0^3\left(25 - x^2 - y^2\right)dy\,dx$

$= \int_0^2\left(25y - x^2y - \frac{1}{3}y^3\right)_0^3 dx$

$= \int_0^2\left(75 - 3x^2 - 9\right)dx = 124$

23. $V = \int_0^3\int_0^4\left(1 + x^2 + y^2\right)dx\,dy$

$= \int_0^3\left(x + \frac{1}{3}x^3 + xy^2\right)_0^3 dy$

$= \int_0^3\left(4 + \frac{64}{3} + 4y^2\right)dy = 112$

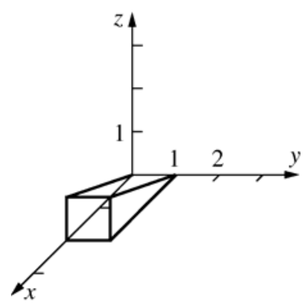
24. $V = \int_0^3\int_0^2 5xye^{-x^2}\,dx\,dy$

$= \int_0^3 5y\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right)_0^2 dy$

$= \int_0^3 5y\left[-\frac{5}{4}e^{-4} - \frac{1}{4}\right]dy = \frac{45\left(e^4 - 1\right)}{4e^4} \approx 11.0439$

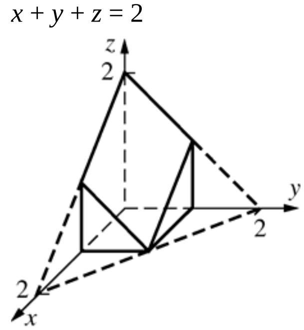
25. $z = \frac{x}{2}$ is a plane.

$x - 2z = 0$

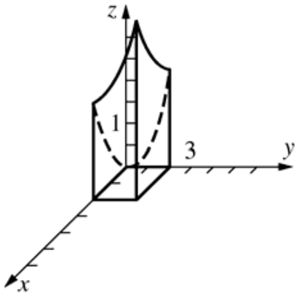


26. $z = 2 - x - y$ is a plane.

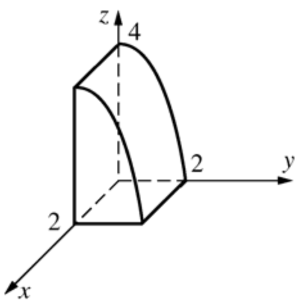
$x + y + z = 2$



27. $z = x^2 + y^2$ is a paraboloid opening upward with z-axis.



28. $z = 4 - y^2$ is a parabolic cylinder parallel to the x-axis.



29. $\int_1^3 \int_0^1 (x+y+1)\,dx\,dy = \int_1^3 \left[\left(\frac{1}{2}\right)x^2 + yx + x\right]_{x=0}^{1} dy$

$= \int_1^3 \left(y + \frac{3}{2}\right) dy = 7$

30. $\int_1^2 \int_0^4 (2x+3y)\,dy\,dx = \int_1^2 \left[2xy + \left(\frac{3}{2}\right)y^2\right]_{y=0}^{4} dx$

$= \int_1^2 (8x+24)\,dx = 36$

31. $x^2 + y^2 + 2 > 1$

$\int_{-1}^1 \int_0^1 [(x^2+y^2+2)-1]\,dy\,dx$

$= \int_{-1}^1 \left[x^2 y + \left(\frac{1}{3}\right)y^3 + y\right]_{y=0}^{1} dx$

$= \int_{-1}^1 \left(x^2 + \frac{4}{3}\right) dx = \frac{10}{3}$

32. $\int_0^2 \int_0^2 (4-x^2)\,dx\,dy = \int_0^2 \left[4x - \frac{x^3}{3}\right]_0^2 dy$

$= \int_0^2 \left(\frac{16}{3}\right) dy = \left[\frac{16y}{3}\right]_0^2 = \frac{32}{3}$

33. $\int_a^b \int_c^d g(x)h(y)\,dy\,dx = \int_a^b g(x) \int_c^d h(y)\,dy\,dx$

$= \int_c^d h(y)\,dy \int_a^b g(x)\,dx$

(First step used linearity of integration with respect to y; second step used linearity of integration with respect to x; now commute.)

34. $\int_0^{\sqrt{\ln 2}} xe^{x^2}\,dx \int_0^1 y(1+y^2)^{-1}\,dy$

$= \left[\left(\frac{1}{2}\right)e^{x^2}\right]_0^{\sqrt{\ln 2}} \left[\left(\frac{1}{2}\right)\ln(1+y^2)\right]_0^1$

$= \left[\frac{1}{2}\right]\left[\left(\frac{1}{2}\right)\ln 2\right] = \left(\frac{1}{4}\right)\ln 2 \approx 0.1733$

35. $\int_0^1 \int_0^1 xye^{x^2} e^{y^2}\,dy\,dx = \left(\int_0^1 xe^{x^2}\,dx\right)\left(\int_0^1 ye^{y^2}\,dy\right)$

$= \left(\int_0^1 xe^{x^2}\,dx\right)^2$ (Changed the dummy variable y to the dummy variable x.)

$= \left(\left[\frac{e^{x^2}}{2}\right]_0^1\right)^2 = \left(\frac{e-1}{2}\right)^2 \approx 0.7381$

36. $V = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\cos x \cos y|\,dx\,dy$

$= \int_{-\pi}^{\pi} |\cos x|\,dx \int_{-\pi}^{\pi} |\cos y|\,dy = \left(\int_{-\pi}^{\pi} |\cos x|\,dx\right)^2$

$= \left(4\int_0^{\pi/2} |\cos x|\,dx\right)^2 = \left(4[\sin x]_0^{\pi/2}\right)^2 = 16$

37. $\int_{-2}^2 x^2\,dx \int_{-1}^1 |y^3|\,dy = \left(2\int_0^2 x^2\,dx\right)\left(2\int_0^1 y^3\,dy\right)$

$= 2\left(\frac{8}{3}\right)2\left(\frac{1}{4}\right) = \frac{8}{3}$

38. $\int_{-2}^2 [\![x^2]\!]\,dx \int_{-1}^1 y^3\,dy = 0$ (since the second integral equals 0).

39. $\int_{-2}^2 [\![x^2]\!]\,dx \int_{-1}^1 |y^3|\,dy = 2\int_0^2 [\![x^2]\!]\,dx\, 2\int_0^1 y^3\,dy$

$= 2\left[\int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^2 3\,dx\right]\left[2\left(\frac{1}{4}\right)\right]$

$= 2\left[0 + \left(\sqrt{2}-1\right) + 2\left(\sqrt{3}-\sqrt{2}\right) + 3\left(2-\sqrt{3}\right)\right]\left[\frac{1}{2}\right]$

$= 5 - \sqrt{3} - \sqrt{2} \approx 1.8537$

40. $\int_0^1\int_0^{\sqrt{3}} 8x(x^2+y^2+1)^{-2}\,dx\,dy = \int_0^1 [-4(x^2+y^2+1)^{-1}]_{x=0}^{\sqrt{3}}\,dy \;\; = 4\int_0^1\left[\frac{-1}{4+y^2}+\frac{1}{1+y^2}\right]dy$

$= 4\left[-\frac{1}{2}\arctan\left(\frac{y}{2}\right)+\arctan(y)\right]_0^1 \;\; = 4\left[\left(-\frac{1}{2}\arctan\left(\frac{1}{2}\right)+\frac{\pi}{4}\right)-0\right] \;\; = \pi - 2\arctan\left(\frac{1}{2}\right) \approx 2.2143$

41. $0 \le \int_a^b\int_a^b [f(x)g(y)-f(y)g(x)]^2\,dx\,dy \;\; = \int_a^b\int_a^b [f^2(x)g^2(y)-2f(x)g(x)f(y)g(y)+f^2(y)g^2(x)]dx\,dy$

$= \int_a^b f^2(x)dx\int_a^b g^2(y)dy - 2\int_a^b f(x)g(x)dx\int_a^b f(y)g(y)dy + \int_a^b f^2(y)dy\int_a^b g^2(x)dx$

$= 2\int_a^b f^2(x)dx\int_a^b g^2(x)dx - 2\left[\int_a^b f(x)g(x)dx\right]^2$

Therefore, $\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx.$

42. Since f is increasing, $[y-x][f(y)-f(x)] > 0$. Therefore,

$0 < \int_a^b\int_a^b [y-x][f(y)-f(x)]dx\,dy \;\; = \int_a^b\int_a^b yf(y)dx\,dy - \int_a^b\int_a^b yf(x)dx\,dy - \int_a^b\int_a^b xf(y)dx\,dy + \int_a^b\int_a^b xf(x)dx\,dy$

$= (b-a)\int_a^b yf(y)dy - \frac{b^2-a^2}{2}\int_a^b f(x)dx - \frac{b^2-a^2}{2}\int_a^b f(y)dy + (b-a)\int_a^b xf(x)dx$

$= 2(b-a)\int_a^b xf(x)dx - (b^2-a^2)\int_a^b f(x)dx \;\; = (b-a)\left[2\int_a^b xf(x)dx - (b+a)\int_a^b f(x)dx\right]$

Therefore, $(b+a)\int_a^b f(x)dx < 2\int_a^b xf(x)dx.$ Now divide each side by the positive number $2\int_a^b f(x)dx$ to obtain the desired result.

Interpretation:
If f is increasing on $[a, b]$ and $f(x) \ge 0$, then the x-coordinate of the centroid (of the region between the graph of f and the x-axis for x in $[a, b]$) is to the right of the midpoint between a and b.

Another interpretation:
If $f(x)$ is the density at x of a wire and the density is increasing as x increases for x in $[a, b]$, then the center of mass of the wire is to the right of the midpoint of $[a, b]$.

13.3 Concepts Review

1. A rectangle containing S; 0

2. $\phi_1(x) \le y \le \phi_2(x)$

3. $\int_a^b\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy\,dx$

4. $\int_0^1\int_0^{1-x} 2x\,dy\,dx; \frac{1}{3}$

Problem Set 13.3

1. $\int_0^1 [x^2y]_{y=0}^{3x}dx = \int_0^1 3x^3dx = \frac{3}{4}$

2. $\int_1^2\left[\left(\frac{1}{2}\right)y^2\right]_{y=0}^{x-1}dx = \int_1^2\left(\frac{1}{2}\right)(x-1)^2dx = \frac{1}{6}$

3. $\int_{-1}^3\left[\frac{x^3}{3}+y^2x\right]_{x=0}^{3y}dy = \int_{-1}^3(9y^3+3y^3)dy$

$= [3y^4]_{-1}^3 = 243-3 = 240$

4. $\int_{-3}^1\left[x^2y-\left(\frac{1}{4}\right)y^4\right]_{y=0}^{x}dx = \int_{-3}^1\left[x^3-\left(\frac{1}{4}\right)x^4\right]dx$

$= -32.2$

5. $\int_1^3\left[\left(\frac{1}{2}\right)x^2\exp(y^3)\right]_{x=-y}^{2y}dy = \int_1^3\left(\frac{3}{2}\right)y^2\exp(y^3)dy$

$= \left(\frac{1}{2}\right)(e^{27}-e) \approx 2.660\times 10^{11}$

6. $$\int_1^5 \left[\frac{3}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_{y=0}^{x} dx = \int_1^5 \frac{3}{x}\frac{\pi}{4}dx$$

$$= \left[\frac{3\pi \ln x}{4}\right]_1^5 = \frac{3\pi\ln 5}{4} \approx 3.7921$$

7. $$\int_{1/2}^{1}[y\cos(\pi x^2)]_{y=0}^{2x}dx = \int_{1/2}^{1} 2x\cos(\pi x^2)dx$$

$$= -\frac{\sqrt{2}}{2\pi} \approx -0.2251$$

8. $$\int_0^{\pi/4}\left[\left(\frac{1}{2}\right)r^2\right]_{r=\sqrt{2}}^{\sqrt{2}\cos\theta} d\theta = \int_0^{\pi/4}(\cos^2\theta - 1)d\theta$$

$$= \frac{(2-\pi)}{8} \approx -0.1427$$

9. $$\int_0^{\pi/9}[\tan\theta]_{\theta=\pi/4}^{3r}dr = \int_0^{\pi/9}(\tan 3r - 1)dr$$

$$= \left[-\frac{\ln|\cos 3r|}{3} - r\right]_0^{\pi/9}$$

$$= \left(\frac{-\ln\left(\frac{1}{2}\right)}{3} - \frac{\pi}{9}\right) - \left(-\frac{\ln(1)}{3} - 0\right)$$

$$= \frac{3\ln 2 - \pi}{9} \approx -0.1180$$

10. $$\int_0^2\left[ye^{-x^2}\right]_{-x}^{x}dx = \int_0^2 2xe^{-x^2}\,dx = \left[-e^{-x^2}\right]_0^2$$

$$= 1 - e^{-4}$$

11. $$\int_0^{\pi/2}[e^x\cos y]_{x=0}^{\sin y}dy = \int_0^{\pi/2}(e^{\sin y}\cos y - \cos y)dy$$

$$= e - 2 \approx 0.7183$$

12. $$\int_1^2\left[\frac{y^3}{3x}\right]_0^{x^2}dx = \int_1^2 \frac{x^5}{3}dx = \left[\frac{1}{18}x^6\right]_1^2 = 3.5$$

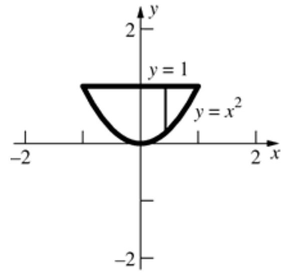
13. $$\int_0^2\left[xy + \left(\frac{1}{2}\right)y^2\right]_{y=0}^{\sqrt{4-x^2}}dx$$

$$= \int_0^2\left[x(4-x^2)^{1/2} + 2 - \left(\frac{1}{2}\right)x^2\right]dx = \frac{16}{3}$$

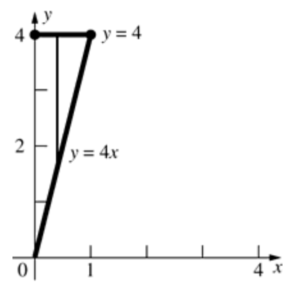
14. $$\int_{\pi/6}^{\pi/2}[3r^2\cos\theta]_{r=0}^{\sin\theta}d\theta = \int_{\pi/6}^{\pi/2}3\sin^2\theta\cos\theta\, d\theta$$

$$= [\sin^3\theta]_{\pi/6}^{\pi/2} = \frac{7}{8} = 0.875$$

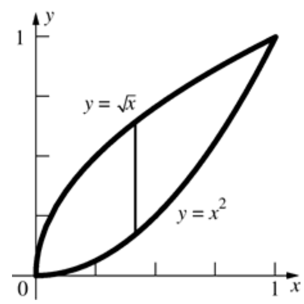
15. $$\int_{-1}^{1}\int_{x^2}^{1} xy\,dy\,dx = 0$$



16. $$\int_0^1\int_{4x}^{4}(x+y)dy\,dx = 6$$



17.



$$\int_0^1\int_{x^2}^{\sqrt{x}}(x^2+2y)dy\,dx = \int_0^1[x^2y + y^2]_{y=x^2}^{\sqrt{x}}dx$$

$$= \int_0^1[(x^{5/2}+x) - (x^4+x^4)]dx$$

$$= \left[\frac{2x^{7/2}}{7} + \frac{x^2}{2} - \frac{2x^5}{5}\right]_0^1 = \frac{2}{7} + \frac{1}{2} - \frac{2}{5}$$

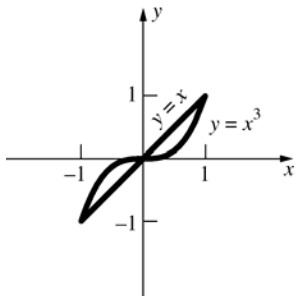
$$= \frac{27}{70} \approx 0.3857$$

18. $$\int_0^2\int_x^{3x-x^2}(x^2 - xy)dy\,dx$$

$$= \int_0^2 \frac{-x^3\left(x^2-4x+4\right)}{2}dx = -\frac{8}{15}$$

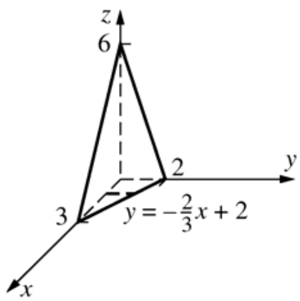
19. $$\int_0^2\int_x^2 2(1+x^2)^{-1}dy\,dx = 4\tan^{-1}2 - \ln 5 \approx 2.8192$$

20.



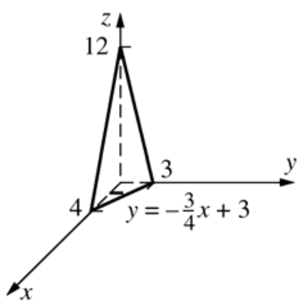
Since S is symmetric with respect to the origin and the integrand is an odd function in x, the value of the integral is 0.

21.



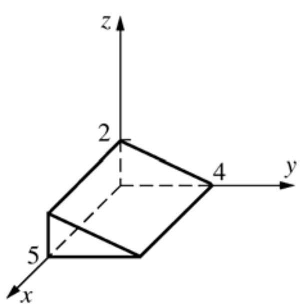
$$\int_0^3 \int_0^{(-2/3)x+2} (6 - 2x - 3y)\,dy\,dx = 6$$

22.



$$\int_0^4 \int_0^{(-3/4)x+3} (12 - 3x - 4y)\,dy\,dx = 24$$

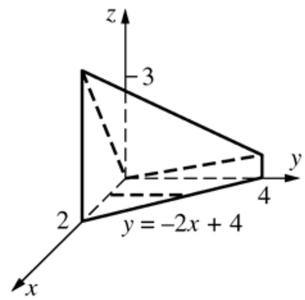
23.



$$\int_0^5 \int_0^4 \frac{4-y}{2}\,dy\,dx = \left(\int_0^5 1\,dx\right)\left(\int_0^4 \frac{4-y}{2}\,dy\right)$$

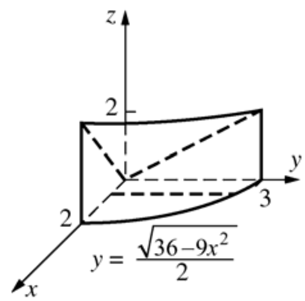
$$= 5\left[2y - \frac{y^2}{4}\right]_0^4 = 5(8-4) = 20$$

24.



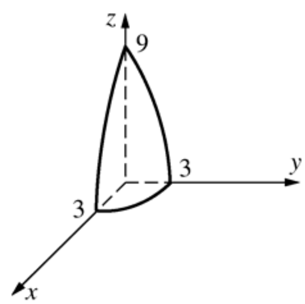
$$\int_0^2 \int_0^{-2x+4} \left[2x + \left(\frac{1}{4}\right)y\right] dy\,dx = \frac{20}{3}$$

25.



$$\int_0^2 \int_0^{(1/2)\sqrt{36-9x^2}} \left(\frac{1}{6}\right)(9x + 4y)\,dy\,dx = 10$$

26.



$$\int_0^3 \int_0^{\sqrt{9-x^2}} (9 - x^2 - y^2)\,dy\,dx$$

$$= \int_0^3 \left[(9-x^2)y - \frac{y^3}{3}\right]_{y=0}^{\sqrt{9-x^2}} dx = \int_0^3 \frac{2(9-x^2)^{3/2}}{3}\,dx$$

$$= \int_0^{\pi/2} 18\cos^3 t \, 3\cos t\,dt = \int_0^{\pi/2} 54\cos^4 t\,dt$$

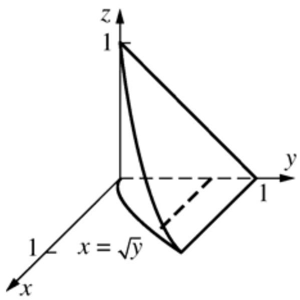
$$= \int_0^{\pi/2} \left(\frac{81}{4} + 27\cos 2t + \frac{27\cos 4t}{4}\right) dt$$

$$= \left[\frac{81t}{4} + \frac{27\sin 2t}{2} + \frac{27\sin 4t}{16}\right]_0^{\pi/2}$$

$$= \frac{81\pi}{8} \approx 31.8086$$

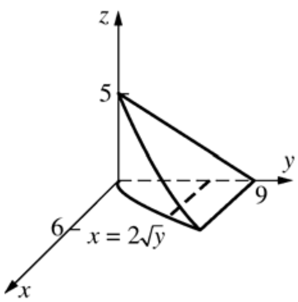
(At the third step, the substitution $x = 3\sin t$ was used. At the 5th step the identity $\cos^2 A = \left(\frac{1}{2}\right)(1 + \cos 2A)$ was used a few times.)

27.

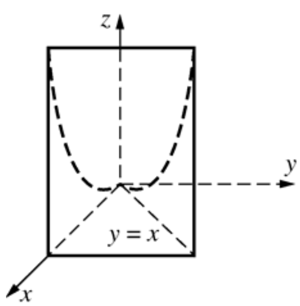


$$\int_0^1 \int_0^{\sqrt{y}} (1-y)\,dx\,dy = \frac{4}{15}$$

28. $\int_0^9 \int_0^{2\sqrt{y}} \left[5 - \left(\frac{5}{9}\right)y\right] dx\,dy = 72$



29.

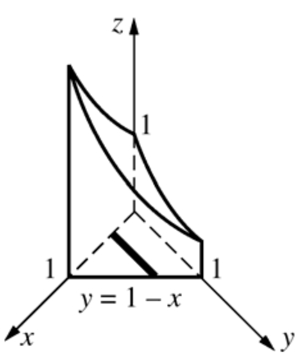


$$\int_0^1 \int_0^x \tan x^2\,dy\,dx = \int_0^1 [y \tan x^2]_{y=0}^x\,dx$$

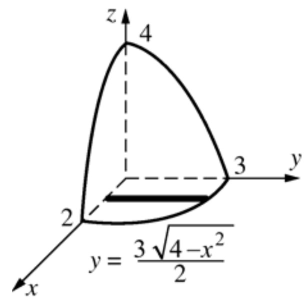
$$= \int_0^1 x \tan x^2\,dx = \left[-\frac{\ln\left|\cos x^2\right|}{2}\right]_0^1 = \left(-\frac{1}{2}\right)\ln(\cos 1)$$

$\approx$ 0.3078

30. $\int_0^1 \int_0^{1-x} e^{x-y}\,dy\,dx = \left(\frac{1}{2}\right)(e + e^{-1} - 2) \approx 0.5431$



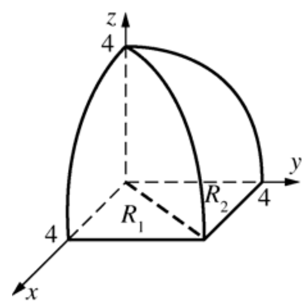
31.



$$\int_0^2 \int_0^{(3/2)\sqrt{4-x^2}} \left[4 - x^2 - \left(\frac{4}{9}\right)y^2\right] dy\,dx = 3\pi$$

≈ 9.4248

32.



Making use of symmetry, the volume is

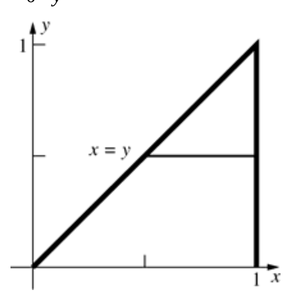
$$2\iint_{R_1} (16 - x^2)^{1/2}\,dA = 2\int_0^4 \int_0^x (16 - x^2)^{1/2}\,dy\,dx$$

$$= 2\int_0^4 [(16 - x^2)^{1/2}\,y]_{y=0}^x\,dx$$

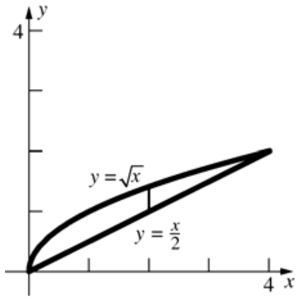
$$= 2\int_0^4 (16 - x^2)^{1/2}\,x\,dx = \left[\frac{-2(16 - x^2)^{3/2}}{3}\right]_0^4$$

$$= 0 + \frac{2(64)}{3} = \frac{128}{3} \approx 42.6667$$

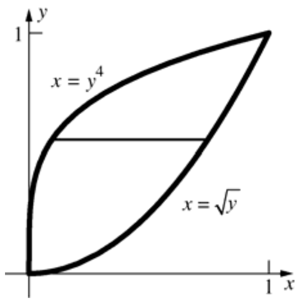
33. $\int_0^1 \int_y^1 f(x, y)\,dx\,dy$



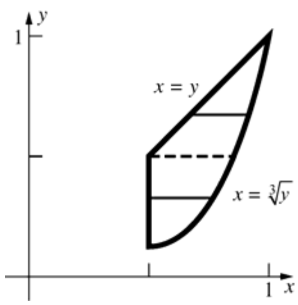
34. $\int_0^4 \int_{x/2}^{\sqrt{x}} f(x, y)\,dy\,dx$



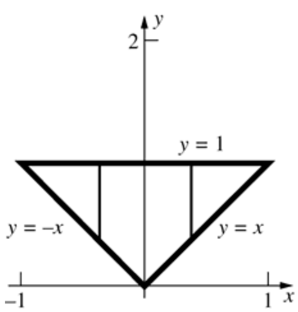
35. $\int_0^1 \int_{y^4}^{\sqrt{y}} f(x, y)\,dx\,dy$



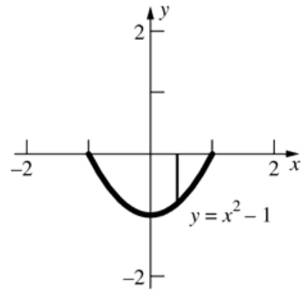
36. $\int_{1/8}^{1/2} \int_{1/2}^{y^{1/3}} f(x, y)\,dx\,dy + \int_{1/2}^{1} \int_{y}^{y^{1/3}} f(x, y)\,dx\,dy$



37. $\int_{-1}^0 \int_{-x}^1 f(x, y)\,dy\,dx + \int_0^1 \int_x^1 f(x, y)\,dy\,dx$

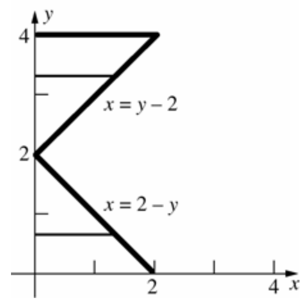


38. $\int_{-1}^1 \int_{x^2-1}^0 f(x, y)\,dy\,dx$



39. $\int_0^2 \int_0^{2-y} xy^2\,dx\,dy + \int_2^4 \int_0^{y-2} xy^2\,dx\,dy = \frac{256}{15}$

≈ 17.0667



40. $\int_{-2}^1 \int_{x^2}^{-x+2} xy\,dy\,dx - \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1/2} xy\,dy\,dx$

$= -\frac{45}{8} = -5.625$

41. The integral over S of $x^4 y$ is 0 since this is an odd function of y. Therefore,

$\iint_S (x^2 + x^4 y)\,dA = \iint_S x^2\,dA$

$= 4\left(\iint_{S_1} x^2\,dA + \iint_{S_2} x^2\,dA\right)$

$= 4\left(\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x^2\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x^2\,dy\,dx\right)$

$= 4\left(\int_0^2 x^2\sqrt{4-x^2}\,dx - \int_0^1 x^2\sqrt{1-x^2}\,dx\right)$

$= 4\left(16\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta - \int_0^{\pi/2} \sin^2\phi\cos^2\phi\,d\phi\right)$

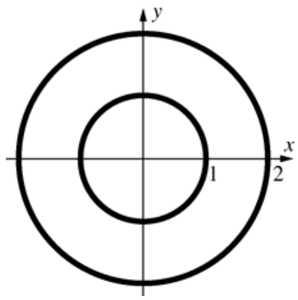
(using $x = 2\sin\theta$ in 1st integral; $x = \sin\phi$ in 2nd)

$= 60\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta = \frac{15\pi}{4}$

(See work in Problem 42.)

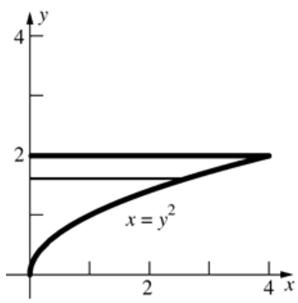
≈ 11.7810

42.



$z = f(x, y) = \sin(xy^2)$ is symmetric with respect to the x-axis, as is the annulus. Therefore, the integral equals 0.

43.

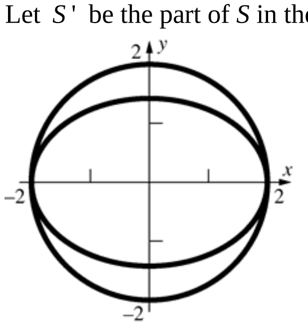


$$\int_0^2 \int_0^{y^2} \sin(y^3)\,dx\,dy = \int_0^2 [x \sin(y^3)]_{x=0}^{y^2}\,dy$$

$$= \int_0^2 y^2 \sin(y^3)\,dy = \left[-\frac{\cos(y^3)}{3}\right]_0^2$$

$$= \frac{1-\cos 8}{3} \approx 0.3818$$

44. Let S' be the part of S in the first quadrant.



$$\iint_{S'} x^2\,dA = \int_0^2 \int_{\sqrt{4-x^2}/\sqrt{2}}^{\sqrt{4-x^2}} x^2\,dy\,dx$$

$$= \int_0^2 x^2 \left(\sqrt{4-x^2} - \frac{\sqrt{4-x^2}}{\sqrt{2}}\right) dx$$

$$= \int_0^2 x^2 \sqrt{4-x^2} \left(1 - \frac{1}{\sqrt{2}}\right) dx$$

$$= \frac{\left(2-\sqrt{2}\right)}{2} \int_0^2 x^2 \sqrt{4-x^2}\,dx$$

Let $x = 2 \sin \theta$, θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Then $dx = 2 \cos \theta\, d\theta$

$x = 2 \Rightarrow \theta = \frac{\pi}{2}$

$x = 0 \Rightarrow \theta = 0$

$$= \frac{\left(2-\sqrt{2}\right)}{2} \int_0^{\pi/2} (2\sin\theta)^2 (2\cos\theta) 2\cos\theta\,d\theta$$

$$= 8\left(2-\sqrt{2}\right) \int_0^{\pi/2} \sin^2\theta \cos^2\theta\,d\theta$$

$$* = 8\left(2-\sqrt{2}\right)\left(\frac{\pi}{16}\right) = \frac{\pi\left(2-\sqrt{2}\right)}{2}$$

Therefore,

$$\iint_S x^2\,dA = 4\left[\frac{\pi\left(2-\sqrt{2}\right)}{2}\right] = 2\pi\left(2-\sqrt{2}\right) \approx 3.6806$$

$$* = \int_0^{\pi/2} \sin^2\theta \cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2}(1-\cos 2\theta)\right]\left[\frac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} (1-\cos^2 2\theta)\,d\theta$$

$$= \frac{1}{4}\frac{\pi}{2} - \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1+\cos 4\theta)\,d\theta$$

$$= \frac{\pi}{8} - \frac{1}{8}\frac{\pi}{2} + \frac{1}{8}\left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{\pi}{8} - \frac{\pi}{16} + 0 = \frac{\pi}{16}$$

45. We first slice the river into eleven 100' sections parallel to the bridge. We will assume that the cross-section of the river is roughly the shape of an isosceles triangle and that the cross-sectional area is uniform across a slice. We can then approximate the volume of the water by

$$V \approx \sum_{k=1}^{11} A_k(y_k)\Delta y = \sum_{k=1}^{11} \frac{1}{2}(w_k)(d_k)100$$

$$= 50 \sum_{k=1}^{11} (w_k)(d_k)$$

where w_k is the width across the river at the left side of the kth slice, and d_k is the center depth of the river at the left side of the kth slice. This gives

$$V \approx 50[300\cdot 40 + 300\cdot 39 + 300\cdot 35 + 300\cdot 31$$
$$+290\cdot 28 + 275\cdot 26 + 250\cdot 25 + 225\cdot 24$$
$$+205\cdot 23 + 200\cdot 21 + 175\cdot 19]$$
$$= 4{,}133{,}000 \text{ ft}^3$$

46. Since f is continuous on the closed and bounded set R, it achieves a minimum m and a maximum M on R. Suppose (x_1, y_1) and (x_2, y_2) are such that $f(x_1, y_1) = m$ and $f(x_2, y_2) = M$. Then,

$$m \le f(x,y) \le M$$

$$\iint_R m\,dA \le \iint_R f(x,y)\,dA \le \iint_R M\,dA$$

$$mA(R) \le \iint_R f(x,y)\,dA \le MA(R)$$

$$m \le \frac{1}{A(R)}\iint_R f(x,y)\,dA \le M$$

Let C be a continuous curve in the plane from (x_1, y_1) to (x_2, y_2) that is parameterized by $x = x(t),\ \ y = y(t),\ c \le t \le d$. Let $h(t) = f(x(t), y(t))$. Since f is continuous, so is h. By the Intermediate Value Theorem, there exists a t_0 in (c,d) such that

$h(t_0) = \frac{1}{A(R)}\iint_R f(x,y)\,dA$. But $h(t_0) = f(x(t_0), y(t_0)) = f(a,b)$, where $a = x(t_0)$ and $b = y(t_0)$. Thus,

$$f(a,b) = \frac{1}{A(R)}\iint_R f(x,y)\,dA$$

or, $\iint_R f(x,y)\,dA = f(a,b)\cdot A(R)$.

13.4 Concepts Review

1. $a \le r \le b; \alpha \le \theta \le \beta$

2. $r\,dr\,d\theta$

3. $\int_0^{\pi}\int_0^2 r^3 dr\,d\theta$

4. 4π

Problem Set 13.4

1. $\int_0^{\pi/2}\left[\left(\frac{1}{3}\right)r^3\sin\theta\right]_{r=0}^{\cos\theta} d\theta$

$= \int_0^{\pi/2}\left(\frac{1}{3}\right)\cos^3\theta\sin\theta\,d\theta$

$= \frac{1}{12} \approx 0.0833$

2. $\int_0^{\pi/2}\left[\left(\frac{1}{2}\right)r^2\right]_{r=0}^{\sin\theta} d\theta = \int_0^{\pi/2}\left(\frac{1}{2}\right)\sin^2\theta\,d\theta$

$= \frac{\pi}{8} \approx 0.3927$

3. $\int_0^{\pi}\left[\frac{r^3}{3}\right]_{r=0}^{\sin\theta} d\theta = \int_0^{\pi}\frac{\sin^3\theta}{3}d\theta$

$= \int_0^{\pi}\frac{(1-\cos^2\theta)\sin\theta}{3}d\theta$

$= \left[\frac{-\cos\theta}{3} + \frac{\cos^3\theta}{9}\right]_0^{\pi}$

$= \left(\frac{1}{3} - \frac{1}{9}\right) - \left(-\frac{1}{3} + \frac{1}{9}\right) = \frac{4}{9}$

4. $\int_0^{\pi}\left[\left(\frac{1}{2}\right)r^2\sin\theta\right]_{r=0}^{1-\cos\theta} d\theta$

$= \int_0^{\pi}\left(\frac{1}{2}\right)(1-\cos\theta)^2\sin\theta\,d\theta$

$= \frac{4}{3}$

5.

$\int_0^{\pi}\left[\frac{1}{2}r^2\cos\frac{\theta}{4}\right]_0^2 d\theta = \int_0^{\pi} 2\cos\frac{\theta}{4}d\theta$

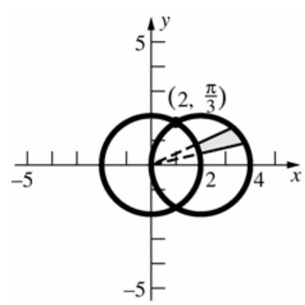
$= \left[8\sin\frac{\theta}{4}\right]_0^{\pi} = 4\sqrt{2}$

6.

$\int_0^{2\pi}\left[\frac{1}{2}r^2\right]_0^{\theta} d\theta = \int_0^{2\pi}\frac{1}{2}\theta^2\,d\theta = \left[\frac{1}{6}\theta^3\right]_0^{2\pi}$

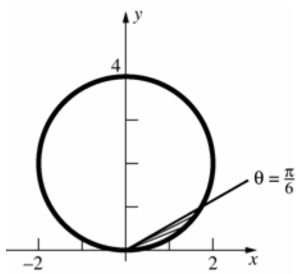
$= \frac{4\pi^3}{3}$

7.



$2\int_0^{\pi/3}\int_2^{4\cos\theta} r\,dr\,d\theta = 2\left[\frac{2\pi}{3} + \sqrt{3}\right] \approx 7.6529$

8.

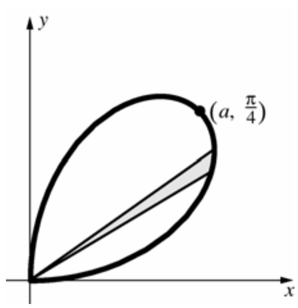


$$\int_0^{\pi/6}\int_0^{4\sin\theta} r\,dr\,d\theta = \int_0^{\pi/6}\left[\frac{r^2}{2}\right]_0^{4\sin\theta} d\theta$$

$$= \int_0^{\pi/6} 8\sin^2\theta\,d\theta = \int_0^{\pi/6} 4(1-\cos 2\theta)d\theta$$

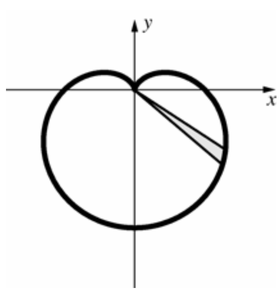
$$= [4\theta - 2\sin 2\theta]_0^{\pi/6} = \frac{2\pi}{3} - \sqrt{3} \approx 0.3623$$

9.



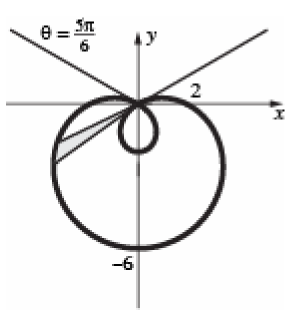
$$\int_0^{\pi/2}\int_0^{a\sin 2\theta} r\,dr\,d\theta = \frac{a^2\pi}{8}$$

10.



$$\int_0^{2\pi}\int_0^{6-6\sin\theta} r\,dr\,d\theta = 54\pi \approx 169.6460$$

11.



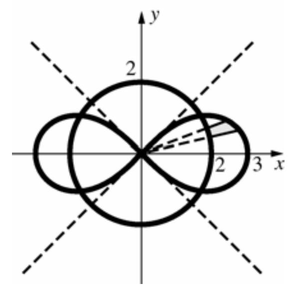
$$2\int_{5\pi/6}^{3\pi/2}\int_0^{2-4\sin\theta} r\,dr\,d\theta = 2\int_{5\pi/6}^{3\pi/2}\left[\frac{r^2}{2}\right]_0^{2-4\sin\theta} d\theta$$

$$= 2\int_{5\pi/6}^{3\pi/2}(6 - 8\sin\theta - 4\cos 2\theta)d\theta$$

$$= 2[6\theta + 8\cos\theta - 2\sin 2\theta]_{5\pi/6}^{3\pi/2}$$

$$= 2\left(4\pi + 3\sqrt{3}\right) \approx 35.525$$

12.

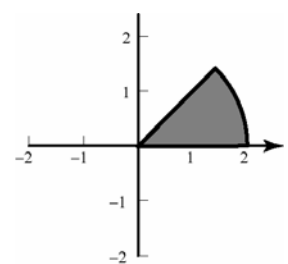


$$4\int_0^{(1/2)\cos^{-1}(4/9)}\int_2^{3\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

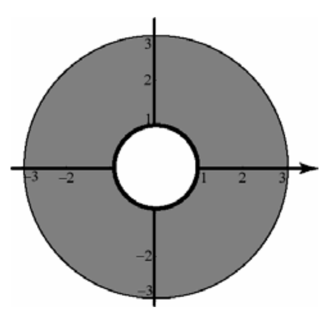
$$= \sqrt{65} - 4\cos^{-1}\left(\frac{4}{9}\right)$$

$$\approx 3.6213$$

13. $\int_0^{\pi/4}\left[\frac{1}{2}r^2\right]_0^2 d\theta = \int_0^{\pi/4} 2d\theta = \frac{\pi}{2}$

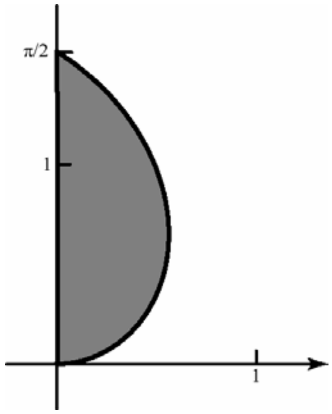


14. $\int_0^{2\pi}\left[\frac{1}{2}r^2\right]_1^3 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi$



15. $\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_0^{\theta} d\theta = \int_0^{\pi/2}\frac{1}{2}\theta^2\, d\theta$

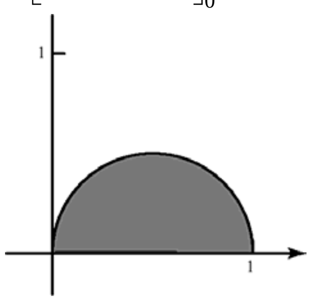
$$= \left[\frac{1}{6}\theta^3\right]_0^{\pi/2} = \frac{\pi^3}{48}$$



16. $\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_0^{\cos\theta} d\theta = \int_0^{\pi/2}\frac{1}{2}\cos^2\theta\, d\theta$

$$= \int_0^{\pi/2}\frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\cos 2\theta\right]d\theta$$

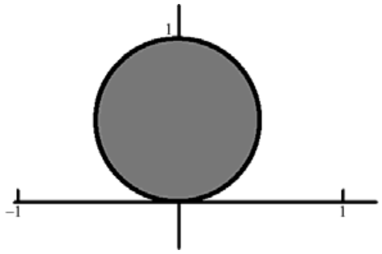
$$= \left[\frac{1}{4}\theta+\frac{1}{8}\cos 2\theta\right]_0^{\pi/2} = \frac{\pi}{8}$$



17. $\int_0^{\pi}\left[\frac{1}{2}r^2\right]_0^{\sin\theta} d\theta = \int_0^{\pi}\frac{1}{2}\sin^2\theta\, d\theta$

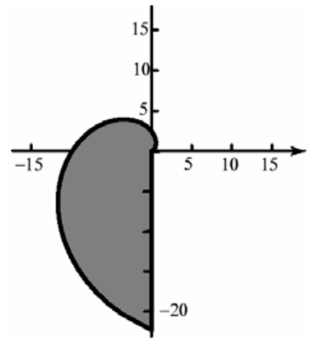
$$= \int_0^{\pi}\frac{1}{2}\left[\frac{1}{2}-\frac{1}{2}\sin 2\theta\right]d\theta$$

$$= \left[\frac{1}{4}\theta+\frac{1}{8}\cos 2\theta\right]_0^{\pi} = \frac{\pi}{4}$$

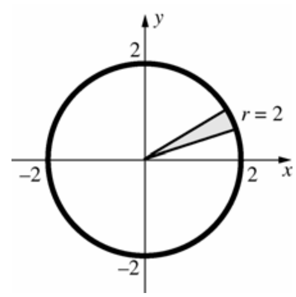


18. $\int_0^{3\pi/2}\left[\frac{1}{2}r^2\right]_0^{\theta^2} d\theta = \int_0^{3\pi/2}\frac{1}{2}\theta^4\, d\theta$

$$= \left[\frac{1}{10}\theta^5\right]_0^{3\pi/2} = \frac{243\pi^5}{320}$$

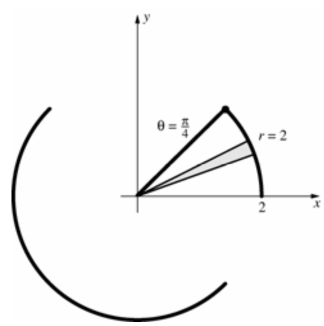


19.



$$2\int_0^{\pi}\int_0^{2} e^{r^2} r\,dr\,d\theta = \pi(e^4-1) \approx 168.3836$$

20.



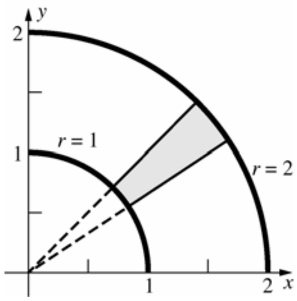
$$\int_0^{\pi/4}\int_0^{2}(4-r^2)^{1/2} r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\left[\frac{(4-r^2)^{3/2}}{-3}\right]_0^{2} d\theta$$

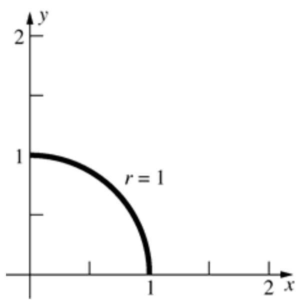
$$= \int_0^{\pi/4}\left(\frac{8}{3}\right)d\theta = \left[\frac{8\theta}{3}\right]_0^{\pi/4} = \frac{2\pi}{3} \approx 2.0944$$

21. $\int_0^{\pi/4}\int_0^{2}(4+r^2)^{-1} r\,dr\,d\theta = \left(\frac{\pi}{8}\right)\ln 2 \approx 0.2722$

22. $\int_0^{\pi/2}\int_1^2 r\sin\theta r\,dr\,d\theta = \frac{7}{3}$

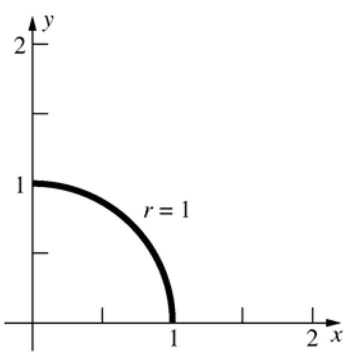


23.

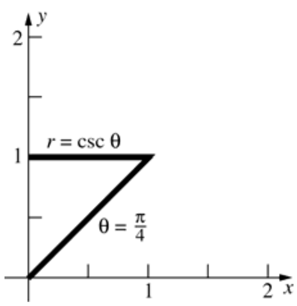


$$\int_0^{\pi/2}\int_0^1 (4-r^2)^{-1/2} r\,dr\,d\theta$$
$$= \int_0^{\pi/2}[-(4-r^2)^{1/2}]_0^1 d\theta$$
$$= \int_0^{\pi/2}\left(-\sqrt{3}+2\right)d\theta = \left(-\sqrt{3}+2\right)\left(\frac{\pi}{2}\right) \approx 0.4209$$

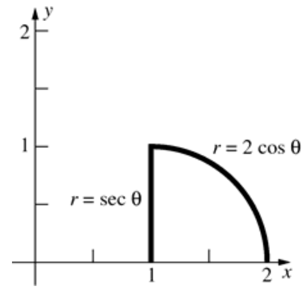
24. $\int_0^{\pi/2}\int_0^1 [\sin(r^2)]r\,dr\,d\theta = \left(\frac{\pi}{4}\right)(1-\cos 1)$
≈ 0.3610



25. $\int_{\pi/4}^{\pi/2}\int_0^{\csc\theta} r^2\cos^2\theta\, r\,dr\,d\theta = \frac{1}{12} \approx 0.0833$

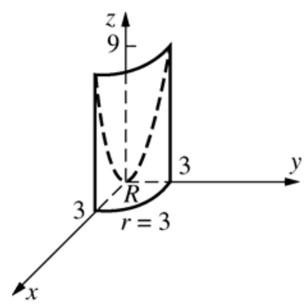


26.



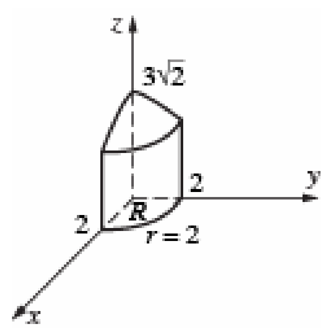
$$\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} r^{-1} r\,dr\,d\theta = \int_0^{\pi/4}[r]_{\sec\theta}^{2\cos\theta}\,d\theta$$
$$= \int_0^{\pi/4}(2\cos\theta - \sec\theta)d\theta$$
$$= \left[2\sin\theta - \ln\left|\sec\theta+\tan\theta\right|\right]_0^{\pi/4}$$
$$= \left[\sqrt{2} - \ln\left(\sqrt{2}+1\right)\right] - [0-\ln(1+0)]$$
$$= \sqrt{2} - \ln\left(\sqrt{2}+1\right) \approx 0.5328$$

27.



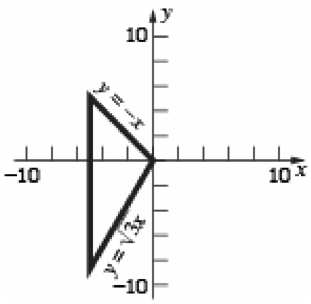
$$\iint_R (x^2+y^2)dA = \int_0^{\pi/2}\int_0^3 r^2 r\,dr\,d\theta$$
$$= \frac{81\pi}{8} \approx 31.8086$$

28.



$$4\iint_R (18-2x^2-2y^2)^{1/2}dA$$
$$= 4\int_0^{\pi/2}\int_0^2 (18-2r^2)^{1/2} r\,dr\,d\theta$$
$$= \left(\frac{\pi}{3}\right)(18^{3/2}-10^{3/2}) \approx 46.8566$$

29.

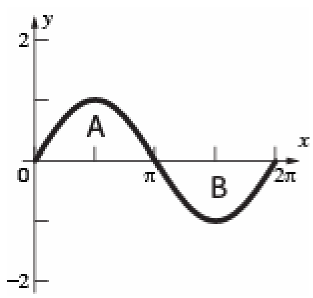


$$\int_{-5}^{0}\int_{\sqrt{3}x}^{-x}(y^2)\,dy\,dx = \int_{-5}^{0}\left[\frac{y^3}{3}\right]_{\sqrt{3}x}^{-x} dx$$

$$= \int_{-5}^{0}\frac{-1-3\sqrt{3}}{3}x^3 dx = \left[\frac{\left(-1-3\sqrt{3}\right)x^4}{12}\right]_{-5}^{0}$$

$$= \frac{\left(1+3\sqrt{3}\right)625}{12} \approx 322.7163$$

30. **a.** The solid bounded by the xy-plane and $z = \sin\sqrt{x^2+y^2}$ for $x^2+y^2 \le 4\pi^2$ is the solid of revolution obtained by revolving about the z-axis the region in the xz-plane that is bounded by the x-axis and the graph of $z = \sin x$ for $0 \le x \le 2\pi$.



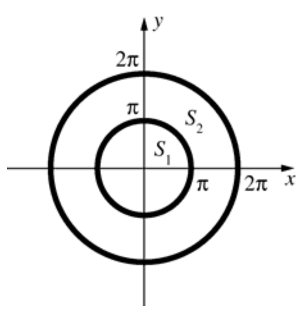
Regions A and B are congruent but region B is farther from the origin, so it generates a larger solid than region A generates. Therefore, the integral is negative.

b. $V = \int_0^{2\pi}\int_0^{2\pi}(\sin r)r\,dr\,d\theta = 2\pi\int_0^{2\pi}(\sin r)r\,dr$

Now use integration by parts.

$= 2\pi(-2\pi) = -4\pi^2 \approx -39.4784$

c.



$$W = \iint_{S_1}\sin\sqrt{x^2+y^2}\,dA + \iint_{S_2}-\sin\sqrt{x^2+y^2}\,dA$$

$$= \int_0^{2\pi}\left[\int_0^{\pi}(\sin r)r\,dr - \int_{\pi}^{2\pi}(\sin r)r\,dr\right]d\theta$$

$$= 2\pi[(\pi)-(-3\pi)] = 8\pi^2 \approx 78.9568$$

31. This can be done by the methods of this section, but an easier way to do it is to realize that the intersection is the union of two congruent segments (of one base) of the spheres, so (see Problem 20, Section 5.2, with $d = h$ and $a = r$) the volume is $2\left[\left(\frac{1}{3}\right)\pi d^2(3a-d)\right] = 2\pi d^2\frac{(3a-d)}{3}$.

32. $100 = \int_0^{2\pi}\int_0^{10}ke^{-r/10}r\,dr\,d\theta = 2\pi\int_0^{10}ke^{-r/10}r\,dr$

Let $u = r$ and $dv = e^{-r/10}dr$.

Then $du = dr$ and $v = -10e^{-r/10}$.

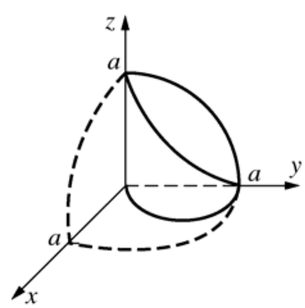
$$= 2\pi k\left(\left[-10re^{-r/10}\right]_0^{10} + \int_0^{10}10e^{-r/10}dr\right)$$

$$= 2\pi k\left(-100e^{-1} - \left[100e^{-r/10}\right]_0^{10}\right)$$

$$= 2\pi k(-100e^{-1} - 100e^{-1} + 100)$$

$$= 200\pi k(1-2e^{-1}),\ \text{so}\ k = \frac{e}{2\pi(e-2)} \approx 0.6023.$$

33.

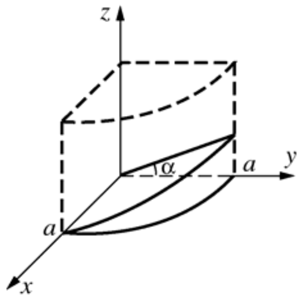


$$\text{Volume} = 4\int_0^{\pi/2}\int_0^{a\sin\theta}\sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\left(-\frac{1}{3}\right)(a^3\cos^3\theta - a^3)\right]d\theta$$

$$= \left(-\frac{4}{3}\right)a^3\left[\frac{2}{3}-\frac{\pi}{2}\right] = \left(\frac{2}{9}\right)a^3(3\pi-4)$$

34. Normal vector to plane is $\langle 0, -\sin a, \cos a\rangle$. Therefore, an equation of the plane is $(-\sin\alpha)y + (\cos\alpha)z = 0$, or $z = (\tan\alpha)y$, or $z = (\tan\alpha)(r\sin\theta)$.



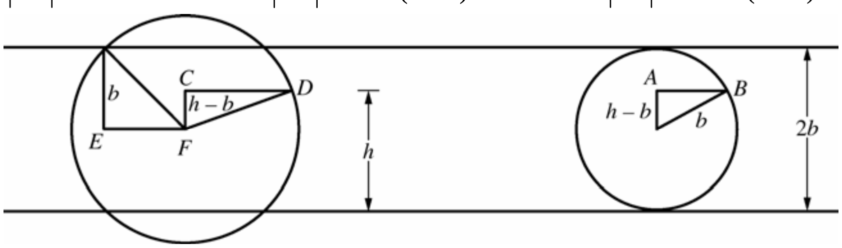
$$\text{Volume} = 2\int_0^{\pi/2}\int_0^a (\tan\alpha) r\sin\theta\, r\,dr\,d\theta = 2(\tan\alpha)\int_0^{\pi/2}\sin\theta\,d\theta\int_0^a r^2dr = 2(\tan\alpha)[1]\left[\frac{a^3}{3}\right] = \left(\frac{2}{3}\right)a^3\tan\alpha$$

35. Choose a coordinate system so the center of the sphere is the origin and the axis of the part removed is the z-axis.
Volume (Ring) = Volume (Sphere of radius a) – Volume (Part removed)

$$= \frac{4}{3}\pi a^3 - 2\int_0^{2\pi}\int_0^{\sqrt{a^2-b^2}}\sqrt{a^2-r^2}\,r\,dr\,d\theta = \frac{4}{3}\pi a^3 - 2(2\pi)\int_0^{\sqrt{a^2-b^2}}(a^2-r^2)^{1/2}r\,dr$$

$$= \frac{4}{3}\pi a^3 + 4\pi\left[\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{\sqrt{a^2-b^2}} = \frac{4}{3}\pi a^3 + 4\pi\frac{1}{3}(b^3-a^3) = \frac{4}{3}\pi b^3$$

36. $|EF|^2 = a^2 - b^2$ $\qquad |CD| = a^2-(h-b)^2$ $\qquad |AB|^2 = b^2-(h-b)^2$



Area of left cross-sectional region $= \pi[a^2-(h-b)^2] - \pi[a^2-b^2]$

$= \pi[b^2-(h-b)^2] =$ area of right cross-sectional region

$$\text{Volume} = \left(\frac{4}{3}\right)\pi b^3 - \left(\frac{1}{3}\right)\pi(2b-h)^2[3b-(2b-h)] = \left(\frac{1}{3}\right)\pi h^2(3b-h)$$

Alternative: $V = \int_0^h \pi\left[b^2-(t-b)^2\right]dt = \frac{1}{3}\pi h^2(3b-h)$

37. $$\int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^b (1+r^2)^{-2}r\,dr\right]d\theta = \int_0^{\pi/2}\left(\lim_{b\to\infty}\left[\left(-\frac{1}{2}\right)(1+b^2)^{-1} - \left(-\frac{1}{2}\right)\right]\right)d\theta = \int_0^{\pi/2}\left(\frac{1}{2}\right)d\theta = \frac{\pi}{4} \approx 0.7854$$

38. $$A = \frac{1}{2}r_2^2(\theta_2-\theta_1) - \frac{1}{2}r_1^2(\theta_2-\theta_1)$$

$$= \frac{1}{2}(\theta_1-\theta_2)(r_2^2-r_1^2)$$

$$= \frac{1}{2}(\theta_2-\theta_1)(r_2-r_1)(r_2+r_1)$$

$$= \frac{r_1+r_2}{2}(r_2-r_1)(\theta_2-\theta_1)$$

39. Using the substitution $u = \dfrac{x-\mu}{\sigma\sqrt{2}}$ we get

$du = \dfrac{dx}{\sigma\sqrt{2}}$. Our integral then becomes

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du$$

Using the result from Example 4, we see that

$\int_0^{\infty} e^{-u^2} du = \dfrac{\sqrt{\pi}}{2}$. Thus we have

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1.$$

13.5 Concepts Review

1. $\iint_S x^2 y^4 dA$

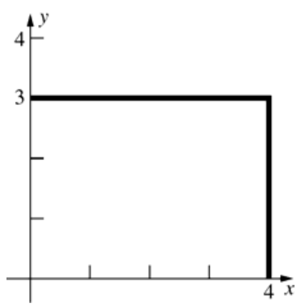
2. $\iint_S \dfrac{x^2 y^5 dA}{m}$

3. $\iint_S x^4 y^4 dA$

4. greater

Problem Set 13.5

1.



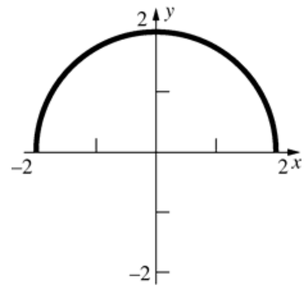
$$m = \int_0^3 \int_0^4 (y+1) dx\ dy = 30$$

$$M_y = \int_0^3 \int_0^4 x(y+1) dx\, dy = 60$$

$$M_x = \int_0^3 \int_0^4 y(y+1) dx\, dy = 54$$

$(\overline{x}, \overline{y}) = (2, 1.8)$

2.



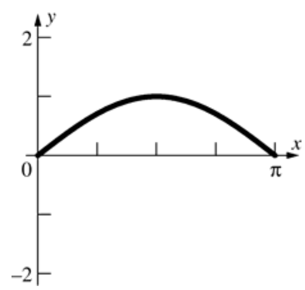
$$m = \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} y\, dy\, dx = \frac{16}{3}$$

$M_y = 0$ (symmetry)

$$M_x = \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} yy\, dx\, dy = 2\pi$$

$$(\overline{x}, \overline{y}) = \left(0, \frac{3\pi}{8}\right)$$

3.



$$m = \int_0^{\pi} \int_0^{\sin x} y\, dy\, dx = \int_0^{\pi} \left[\frac{y^2}{2}\right]_0^{\sin x} dx$$

$$= \int_0^{\pi} \frac{\sin^2 x}{2} dx = \int_0^{\pi} \frac{1-\cos 2x}{4} dx$$

$$= \left[\frac{x}{4} - \frac{\sin 2x}{8}\right]_0^{\pi} = \frac{\pi}{4}$$

$$M_x = \int_0^{\pi} \int_0^{\sin x} yy\, dy\, dx = \int_0^{\pi} \left[\frac{y^3}{3}\right]_0^{\sin x} dx$$

$$= \int_0^{\pi} \frac{\sin^3 x}{3} dx = \frac{1}{3} \int_0^{\pi} (1-\cos^2 x) \sin x\, dx$$

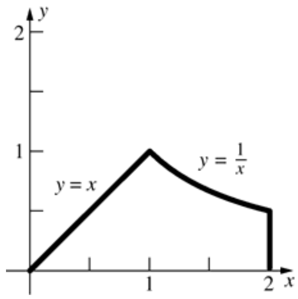
$$= \frac{1}{3}\left[-\cos x + \frac{\cos^3 x}{3}\right]_0^{\pi} = \frac{4}{9}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\frac{4}{9}}{\frac{\pi}{4}} = \frac{16}{9\pi} \approx 0.5659;$$

$\overline{x} = \dfrac{\pi}{2}$ (by symmetry)

Thus , $M_y = \overline{x} \cdot m = \dfrac{\pi}{4} \cdot \dfrac{\pi}{2} = \dfrac{\pi^2}{8}$

4.



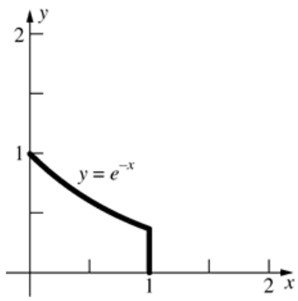
$$m = \int_0^1 \int_0^x x\,dy\,dx + \int_1^2 \int_0^{1/x} x\,dy\,dx = \frac{4}{3}$$

$$M_y = \int_0^1 \int_0^x x^2 dy\,dx + \int_1^2 \int_0^{1/x} x^2 dy\,dx = \frac{7}{4}$$

$$M_x = \int_0^1 \int_0^x xy\,dy\,dx + \int_1^2 \int_0^{1/x} xy\,dy\,dx = \left(\frac{1}{8}\right)(1+4\ln 2)$$

$$(\bar{x}, \bar{y}) = \left(\frac{21}{16}, \left(\frac{3}{32}\right)(1+4\ln 2)\right) \approx (1.3125, 0.3537)$$

5.



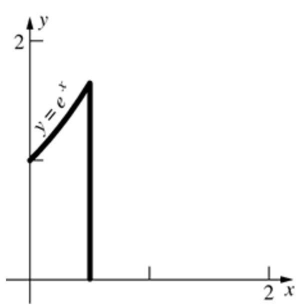
$$m = \int_0^1 \int_0^{e^{-x}} y^2 dy\,dx = \left(\frac{1}{9}\right)(1-e^{-3})$$

$$M_x = \int_0^1 \int_0^{e^{-x}} y^3 dy\,dx = \left(\frac{1}{16}\right)(1-e^{-4}) \approx 0.0614$$

$$M_y = \int_0^1 \int_0^{e^{-x}} xy^2 dy\,dx = \left(\frac{1}{27}\right)(1-4e^{-3}) \approx 0.0297$$

$$(\bar{x}, \bar{y}) = \left(\left(\frac{1}{3}\right)(e^3-4)(e^3-1)^{-1}, \left(\frac{9}{16}\right)e^{-1}(e^4-1)(e^3-1)^{-1}\right) \approx (0.2809, 0.5811)$$

6.



$$m = \int_0^1\int_0^{e^x}(2-x+y)dy\,dx = \int_0^1\left[(2-x)y+\frac{y^2}{2}\right]_{y=0}^{e^x}dx = \int_0^1\left(2e^x - xe^x + \frac{e^{2x}}{2}\right)dx$$

$$= \left[2e^x-(xe^x-e^x)+\frac{e^{2x}}{4}\right]_0^1 = \frac{e^2+8e-13}{4}$$

$$M_x = \int_0^1\int_0^{e^x}(2-x+y)y\,dy\,dx = \int_0^1\left[y^2-\frac{xy^2}{2}+\frac{y^3}{3}\right]_{y=0}^{e^x}dx = \int_0^1\left[e^{2x}-\frac{xe^{2x}}{2}+\frac{e^{3x}}{3}\right]dx$$

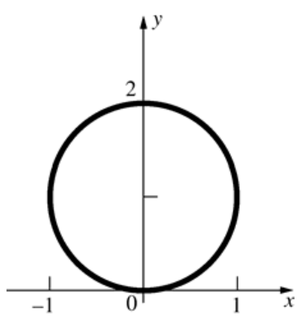
$$= \left[\frac{e^{2x}}{2}-\left(\frac{xe^{2x}}{4}-\frac{e^{2x}}{8}\right)+\frac{e^{3x}}{9}\right]_0^1 = \left(\frac{e^2}{2}-\frac{e^2}{4}+\frac{e^2}{8}+\frac{e^3}{9}\right)-\left(\frac{1}{2}-0+\frac{1}{8}+\frac{1}{9}\right) = \frac{8e^3+27e^2-53}{72}$$

$$M_y = \int_0^1\int_0^{e^x}(2-x+y)x\,dy\,dx = \int_0^1\left[2xy-x^2y+\frac{xy^2}{2}\right]_{y=0}^{e^x}dx = \int_0^1\left(2xe^x-x^2e^x+\frac{xe^{2x}}{2}\right)dx$$

$$= \left[(2xe^x-2e^x)-(x^2e^x-2xe^x+2e^x)+\left(\frac{xe^{2x}}{4}-\frac{e^{2x}}{8}\right)\right]_0^1 = \frac{e^2-8e+33}{8}$$

$$\bar{x} = \frac{M_y}{m} = \frac{e^2-8e+33}{2(e^2+8e-13)} \approx 0.5777;\ \bar{y} = \frac{M_x}{m} = \frac{8e^3+27e^2-53}{18(e^2+8e-13)} \approx 1.0577$$

7.



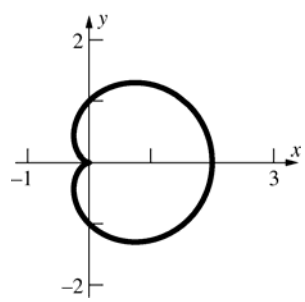
$$m = \int_0^{\pi}\int_0^{2\sin\theta} r\,r\,dr\,d\theta = \frac{32}{9}$$

$$M_x = \int_0^{\pi}\int_0^{2\sin\theta}(r\sin\theta)r\,r\,dr\,d\theta = \frac{64}{15}$$

$M_y = 0$ (symmetry)

$(\bar{x}, \bar{y}) = (0, 1.2)$

8.



$$m = 2\int_0^{\pi}\int_0^{1+\cos\theta} r\,r\,dr\,d\theta = \frac{5\pi}{3}$$

$$M_y = 2\int_0^{\pi}\int_0^{1+\cos\theta}(r\cos\theta)r\,r\,dr\,d\theta = \frac{7\pi}{4}$$

$M_x = 0$ (symmetry)

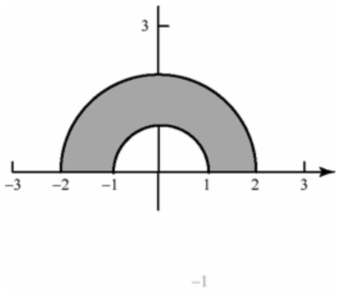
$(\bar{x}, \bar{y}) = (1.05, 0)$

9. $m = \int_0^{\pi}\int_1^2 \frac{1}{r} r\, dr d\theta = \int_0^{\pi}\int_1^2 dr d\theta = \pi$

$M_x = \int_0^{\pi}\int_1^2 \frac{1}{r} r\cos\theta\, r\, dr d\theta = \int_0^{\pi} \frac{3}{2}\cos\theta d\theta = 3$

$M_y = 0$ by symmetry

$(\bar{x}, \bar{y}) = \left(0, \frac{3}{\pi}\right)$



10. $m = \int_0^{2\pi}\int_0^{2+2\cos\theta} r\, r\, dr\, d\theta$

$= \int_0^{2\pi}\left[\frac{1}{3}r^3\right]_0^{2+2\cos\theta} d\theta$

$= \int_0^{2\pi}\frac{1}{3}\left[8 + 24\cos\theta + 24\cos^2\theta + 8\cos^3\theta\right]d\theta$

$= \frac{1}{3}\left[0 + 24\pi + 16\pi\right] = \frac{40\pi}{3}$

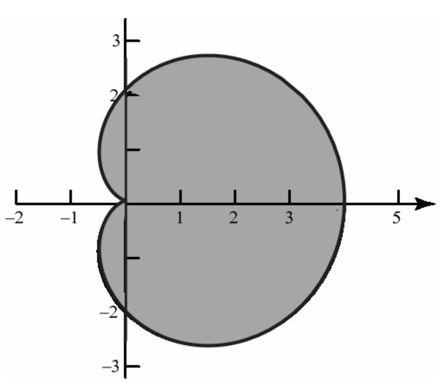
$M_x = 0$ by symmetry

$M_y = \int_0^{2\pi}\int_0^{2+2\cos\theta} r\left(r\cos\theta\right) r\, dr\, d\theta$

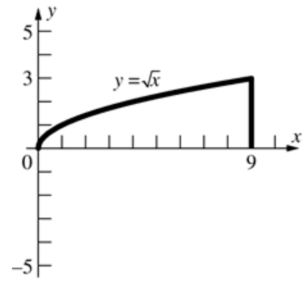
$= \int_0^{2\pi}\left[\frac{1}{4}r^4\cos\theta\right]_0^{2+2\cos\theta} d\theta$

$M_y = 28\pi$

$(\bar{x}, \bar{y}) = \left(\frac{28\pi}{40\pi/3}, 0\right) = \left(\frac{21}{10}, 0\right)$



11.



$I_x = \int_0^3\int_{y^2}^9 y^2(x+y)dx\, dy$

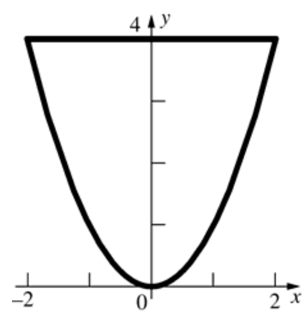
$= \int_0^3\left(\frac{81y^2}{2} + 9y^3 - \frac{y^6}{2} - y^5\right)dy = \frac{7533}{28} \approx 269$

$I_y = \int_0^9\int_0^{\sqrt{x}} x^2(x+y)dy\, dx = \int_0^9\left(x^{7/2} + \frac{x^3}{2}\right)dx$

$= \frac{41553}{8} \approx 5194$

$I_z = I_x + I_y = \frac{305937}{56} \approx 5463$

12.

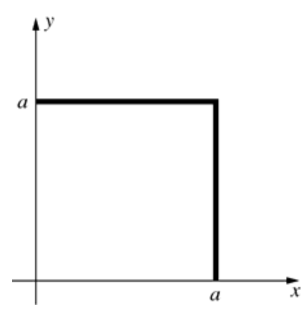


$I_x = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} y^3 dx\, dy = \frac{2048}{9} \approx 227.56$

$I_y = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} x^2 y\, dx\, dy = \frac{512}{21} \approx 24.38$

$I_z = I_x + I_y = \frac{15872}{63} \approx 251.94$

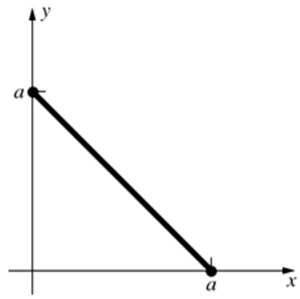
13.



$I_x = \int_0^a\int_0^a (x+y)y^2 dx\, dy = \left(\frac{5}{12}\right)a^5$

$I_y = \left(\frac{5}{12}\right)a^5; \quad I_z = \left(\frac{5}{6}\right)a^5$

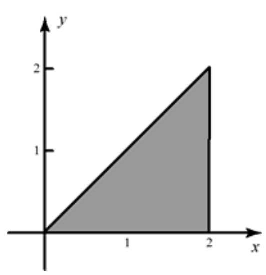
14.



$$I_x = \int_0^a \int_0^{a-y} (x^2+y^2)y^2 \, dx\, dy$$
$$= \frac{1}{3}\int_0^a (a^3y^2 - 3a^2y + 6ay^2 - 4y^5)dy \;\; = \frac{7a^6}{180}$$
$$I_y = \frac{7a^6}{180};\, I_z = \frac{7a^6}{90} \text{ (Same result for } a < 0)$$

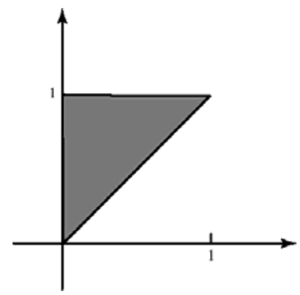
15. The density is constant , $\delta(x,y) = k$.

$$m = \int_0^2 kx\, dx = \left[\frac{k}{2}x^2\right]_0^2 = 2k$$
$$M_x = \int_0^2 \int_0^x ky\, dy\, dx = \int_0^2 \left[\frac{k}{2}y^2\right]_0^x dx$$
$$= \int_0^2 \frac{k}{2}x^2\, dx = \left[\frac{k}{6}x^3\right]_0^2 = \frac{4k}{3}$$
$$M_y = \int_0^2 \int_0^x kx\, dy\, dx = \int_0^2 [kxy]_0^x\, dx$$
$$= \int_0^2 kx^2\, dx = \left[\frac{k}{3}x^3\right]_0^2 = \frac{8k}{3}$$
$$(\bar{x},\bar{y}) = \left(\frac{8k/3}{2k}, \frac{4k/3}{2k}\right) = \left(\frac{4}{3}, \frac{2}{3}\right)$$



16. The density is proportional to the distance from the x-axis, $\delta(x,y) = ky$.

$$m = \int_0^1 \left[\frac{k}{2}y^2\right]_x^1 dx = \int_0^1 \frac{k}{2}\left(1 - x^2\right) = \frac{k}{3}$$
$$M_x = \int_0^1 \int_0^x ky^2\, dy\, dx = \int_0^1 \left[\frac{k}{3}y^3\right]_x^1 dx$$
$$= \frac{k}{3}\int_0^1 \left(1 - x^3\right) dx = \frac{k}{4}$$
$$M_y = \int_0^1 \int_x^1 kxy\, dy\, dx = \int_0^1 \left[\frac{kxy2}{2}\right]_x^1 dx$$
$$= \frac{k}{2}\int_0^1 \left(x - x^3\right) dx = \frac{k}{8}$$
$$(\bar{x},\bar{y}) = \left(\frac{k/8}{k/3}, \frac{k/4}{k/3}\right) = \left(\frac{3}{8}, \frac{3}{4}\right)$$

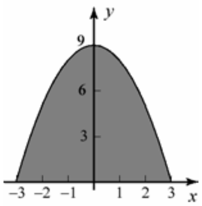


17. The density is proportional to the squared distance from the origin, $\delta(x,y) = k\left(x^2 + y^2\right)$.

$$m = \int_{-3}^3 \int_0^{9-x^2} k\left(x^2+y^2\right) dy\, dx$$
$$= \int_{-3}^3 k\left[x^2y + \frac{1}{3}y^3\right]_0^{9-x^2} dx$$
$$= \int_{-3}^3 k\left[246 - 72x^2 + 8x^4 - \frac{1}{3}x^6\right] dx$$
$$= k\left[246x - 24x^3 + \frac{8}{5}x^5 - \frac{1}{21}x^7\right]_{-3}^3 = \frac{25596k}{35}$$
$$M_x = \int_{-3}^3 \int_0^{9-x^2} ky\left(x^2+y^2\right) dy\, dx$$
$$= \int_{-3}^3 k\left[\frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right]_0^{9-x^2} dx$$
$$= \int_{-3}^3 k\left[\frac{6561}{4} - \frac{1377x^2}{2} + \frac{225}{2}x^4 - \frac{17x^6}{2} + \frac{x^8}{4}\right] dx$$
$$= \frac{29160k}{7}$$

$M_y = 0$ by symmetry

$$(\bar{x}, \bar{y}) = \left(0, \frac{29160k/7}{25596k/35}\right) = \left(0, \frac{450}{79}\right)$$



18. The density is constant, $\delta(x, y) = k$.

$$m = \int_{-\pi/2}^{\pi/2} k\cos x\, dx = [k\sin x]_{-\pi/2}^{\pi/2} = 2k$$

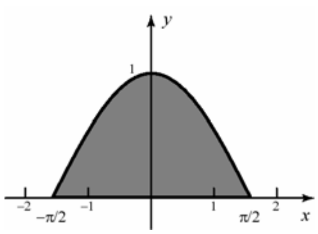
$$M_x = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} ky\, dy\, dx$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{k}{2}y^2\right]_0^{\cos x} dx$$

$$= \frac{k}{2}\int_{-\pi/2}^{\pi/2}\cos^2 x\, dx = \frac{k\pi}{4}$$

$M_y = 0$ by symmetry

$$(\bar{x}, \bar{y}) = \left(0, \frac{k\pi/4}{2k}\right) = \left(0, \frac{\pi}{8}\right)$$



19. The density is proportional to the distance from the origin, $\delta(r, \theta) = k \cdot r$.

$$m = \int_0^{\pi}\int_1^3 kr^2 dr\, d\theta = \int_0^{\pi}\left[\frac{k}{3}r^3\right]_1^3 d\theta$$

$$= \int_0^{\pi}\frac{26}{3}k\, d\theta = \frac{26k\pi}{3}$$

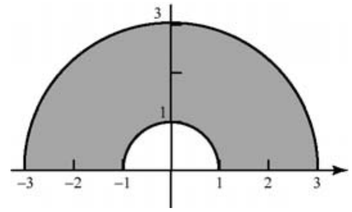
$$M_x = \int_0^{\pi}\int_1^3 kr^2\, r\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi}\left[\frac{k}{4}r^4\sin\theta\right]_1^3 d\theta = \int_0^{\pi} 20k\sin\theta\, d\theta$$

$$= [-20k\cos\theta]_0^{\pi} = 40k$$

$M_y = 0$ by symmetry

$$(\bar{x}, \bar{y}) = \left(0, \frac{40k}{26k\pi/3}\right) = \left(0, \frac{60}{13\pi}\right)$$



20. The density is constant, $\delta(r, \theta) = k$.

$$m = \int_0^{\pi/2}\int_0^{\theta} k\, r\, dr\, d\theta = \int_0^{\pi/2}\left[\frac{k}{2}r^2\right]_0^{\theta} d\theta$$

$$= \int_0^{\pi/2}\frac{k}{2}\theta^2\, d\theta = \left[\frac{k}{6}\theta^3\right]_0^{\pi/2} = \frac{k\pi^3}{48}$$

$$M_x = \int_0^{\pi/2}\int_0^{\theta} kr^2\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2}\left[\frac{k}{3}r^3\sin\theta\right]_0^{\theta} d\theta = \int_0^{\pi/2}\frac{k}{3}\theta^3\sin\theta\, d\theta$$

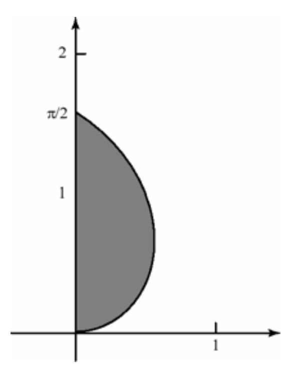
$$= \frac{k\left(\pi^2 - 8\right)}{4}$$

$$M_y = \int_0^{\pi/2}\int_0^{\theta} kr^2\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2}\left[\frac{k}{3}r^3\cos\theta\right]_0^{\theta} d\theta = \int_0^{\pi/2}\frac{k}{3}\theta^3\cos\theta\, d\theta$$

$$= \frac{k\left(\pi^3 - 24\pi + 48\right)}{24}$$

$$(\bar{x}, \bar{y}) = \left(\frac{2\left(\pi^3 - 24\pi + 48\right)}{\pi^3}, \frac{12\left(\pi^2 - 8\right)}{\pi^3}\right)$$



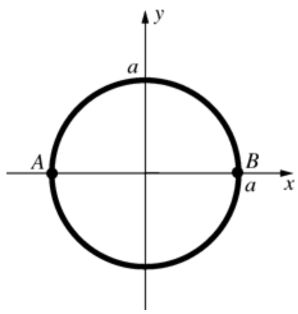
21. $m = \int_0^a \int_0^a (x+y)\,dx\,dy = a^3$

$$\bar{r} = \left(\frac{I_x}{m}\right)^{1/2} = \left(\frac{5}{12}\right)^{1/2} a \approx 0.6455a$$

22. $m = \int_0^a \int_0^{a-y} (x^2+y^2)\,dxdy = \left(\frac{1}{6}\right)a^4$

$$\bar{r} = \left(\frac{I_y}{m}\right)^{1/2} = \left(\frac{7}{30}\right)^{1/2} a \approx 0.4830a$$

23.



$m = \delta \pi a^2$

The moment of inertia about diameter AB is

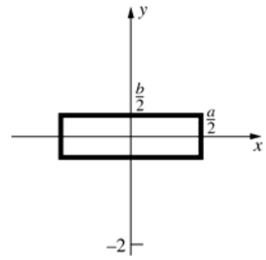
$$I = I_x = \int_0^{2\pi} \int_0^a \delta r^2 \sin^2\theta\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{\delta a^4 \sin^2\theta}{4}\,d\theta = \frac{\delta a^4}{8}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta$$

$$= \frac{\delta a^4}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\delta a^4 \pi}{4}$$

$$\bar{r} = \left(\frac{I}{m}\right)^{1/2} = \left(\frac{\frac{\delta a^4 \pi}{4}}{\delta \pi a^2}\right)^{1/2} = \frac{a}{2}$$

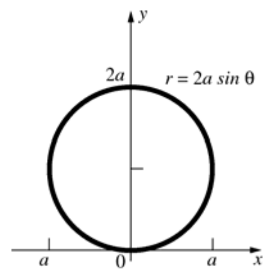
24.



$$I = I_z = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(x^2+y^2)k\,dx\,dy$$

$$= \left(\frac{k}{12}\right)(a^3 b + ab^3)$$

25.



$$I_x = \iint_S \delta y^2\,dA$$

$$= 2\delta \int_0^{\pi/2}\int_0^{2a\sin\theta}(r\sin\theta)^2 r\,dr\,d\theta$$

$$= 2\delta \int_0^{\pi/2} 4a^4 \sin^6\theta\,d\theta$$

$$= 8a^4 \delta \frac{(1)(3)(5)}{(2)(4)(6)}\frac{\pi}{2} = \frac{5a^4\delta\pi}{4}$$

26. $\bar{x} = 0$ (by symmetry)

$$M_x = \iint_S 1y\,dA = 2k\int_{-\pi/2}^{\pi/2}\int_0^{a(1+\sin\theta)}(r\sin\theta)r\,dr\,d\theta = 2k\int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\sin\theta\right]_{r=0}^{a(1+\sin\theta)} d\theta$$

$$= \frac{2ka^3}{3}\int_{-\pi/2}^{\pi/2}(1+\sin\theta)^3 \sin\theta\,d\theta$$

$$= \frac{2ka^3}{3}\int_{-\pi/2}^{\pi/2}(\sin\theta + 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta)\,d\theta$$

$$= \frac{4ka^3}{3}\int_0^{\pi/2}(3\sin^2\theta + \sin^4\theta)\,d\theta$$ (using the symmetry property for odd and even functions.)

$$= \frac{4ka^3}{3}\left(3\frac{1}{2}\frac{\pi}{2} + \frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2}\right) = \frac{5\pi k a^3}{4}$$ (using Formula 113)

Therefore, $\bar{y} = \frac{M_x}{m} = \frac{5a}{6}$.

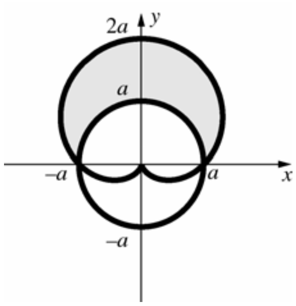
$$I_x = \iint_S ky^2 dA = 2k\int_{-\pi/2}^{\pi/2}\int_0^{a(1+\sin\theta)} (r\sin\theta)^2 r\,dr\,d\theta \ = 2k\int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\sin^2\theta\right]_{r=0}^{1(1+\sin\theta)} d\theta$$

$$= \frac{ka^4}{2}\int_{-\pi/2}^{\pi/2}(1+\sin\theta)^4\sin^2\theta\,d\theta \ = \frac{ka^4}{2}\int_{-\pi/2}^{\pi/2}(\sin^2\theta + 4\sin^3\theta + 6\sin^4\theta + 4\sin^5\theta + \sin^6\theta)d\theta$$

$$= ka\int_0^{\pi/2}(\sin^2\theta + 6\sin^4\theta + \sin^6\theta)d\theta \text{ (symmetry property for odd and even functions)}$$

$$= ka^4\left[\frac{1}{2}\frac{\pi}{2} + 6\frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{\pi}{2}\right] = \frac{49\pi ka^4}{32} \quad \text{(using Formula 113)}$$

27.



$\overline{x} = 0$ (by symmetry)

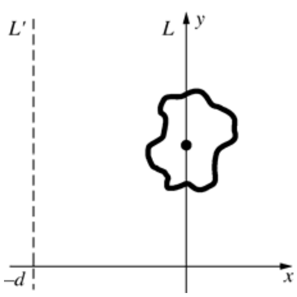
$$M_x = \iint_S ky\,dA = 2k\int_{-\pi/2}^{\pi/2}\int_0^{a(1+\sin\theta)}(r\sin\theta)r\,dr\,d\theta \ = 2k\int_0^{\pi/2}\left(\frac{a^3}{3}\right)(3\sin^2\theta + 3\sin^3\theta + \sin^4\theta)d\theta$$

$$= \left(\frac{2}{3}\right)ka^3\left[\frac{(15\pi+32)}{16}\right] = \left(\frac{1}{24}\right)ka^3(15\pi+32)$$

$$m = \iint_S k\,dA = 2k\int_{-\pi/2}^{\pi/2}\int_0^{a(1+\sin\theta)} r\,dr\,d\theta \ = 2k\int_0^{\pi/2}\left(\frac{1}{2}\right)a^2(2\sin\theta + \sin^2\theta)d\theta \ = ka^2\left[\frac{(8+\pi)}{4}\right] = \left(\frac{1}{4}\right)ka^2(\pi+8)$$

Therefore, $\overline{y} = \frac{M_x}{m} = \frac{\left(\frac{1}{24}\right)ka^3(15\pi+32)}{\left(\frac{1}{4}\right)ka^2(\pi+8)} = \frac{a(15\pi+32)}{6(\pi+8)} \approx 1.1836a$

28.

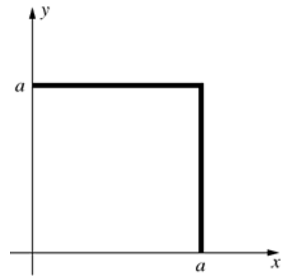


$$I' = \iint_S (x+d)^2\delta(x, y)dA = \iint_S (x^2 + 2xd + d^2)\delta(x, y)dA$$

$$= \iint_S x^2\delta(x, y)dA + \iint_S 2xd\,\delta(x, y)dA + \iint_S d^2\delta(x, y)dA$$

$$= I + M_y + d^2 m = I + 0 + d^2 m = I + d^2 m$$

29. a.

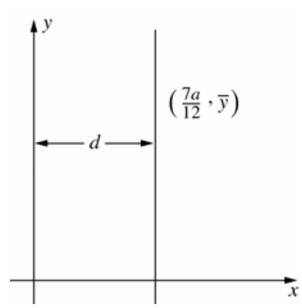


$$m = \iint_S (x+y)\,dA = \int_0^a \int_0^a (x+y)\,dx\,dy$$
$$= \int_0^a \left(\left[\frac{x^2}{2} + xy \right]_{x=0}^{a} \right) dy = \int_0^a \left(\frac{a^2}{2} + ay \right) dy$$
$$= \left[\frac{a^2 y}{2} + \frac{ay^2}{2} \right]_0^a = a^3$$

b. $$M_y = \iint_S x(x+y)\,dA = \int_0^a \int_0^a (x^2 + xy)\,dy\,dx$$
$$= \int_0^a \left(\left[x^2 y + \frac{xy^2}{2} \right]_{y=0}^{a} \right) dx = \int_0^a \left(ax^2 + \frac{a^2 x}{2} \right) dx$$
$$= \left[\frac{ax^3}{3} + \frac{a^2 x^2}{4} \right]_0^a = \frac{7a^4}{12}$$

Therefore, $\bar{x} = \frac{M_y}{m} = \frac{7a}{12}$.

c.

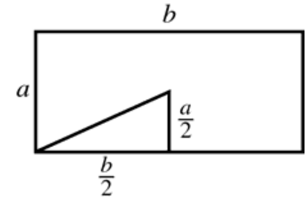


$$I_y = I_L + d^2 m, \text{ so } \frac{5a^5}{12} = I_L + \left(\frac{7a}{12} \right)^2 (a^3);$$
$$I_L = \frac{11a^5}{144}$$

30. $$I_{25} = I_{23} + md^2 = 0.25\delta a^4 \pi + (\delta \pi a^2) a^2$$
$$= 1.25 \delta a^4 \pi$$

31. $$I_x = 2[I_{23}] = \frac{ka^4 \pi}{2}; \quad I_y = 2[I_{23} + md^2]$$
$$= 2[0.25 a^4 \pi + (k\pi a^2)(2a)^2] = 8.5 k a^4 \pi$$
$$I_z = I_x + I_y = 9ka^4 \pi$$

32.



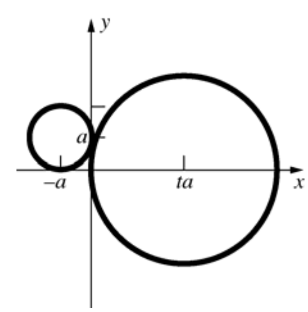
The square of the distance of the corner from the center of mass is $d^2 = \frac{a^2 + b^2}{4}$.

$$I = I(\text{Prob. 16}) + md^2$$
$$= \frac{k(a^3 b + ab^3)}{12} + (kab)\frac{a^2 + b^2}{4} = \frac{k(a^3 b + ab^3)}{3}$$

33. $$M_y = \iint_{S_1 \cup S_2} x\delta(x, y)\,dA$$
$$= \iint_{S_1} x\delta(x, y)\,dA + \iint_{S_2} x\delta(x, y)\,dA$$
$$= \frac{m_1 \iint_{S_1} x\delta(x, y)\,dA}{m_1} + \frac{m_2 \iint_{S_2} x\delta(x, y)\,dA}{m_2}$$
$$= m_1 \bar{x}_1 + m_2 \bar{x}_2$$

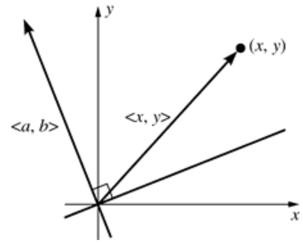
Thus, $\bar{x} = \frac{M_y}{m} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ which is equal to what we are to obtain and which is what we would obtain using the center of mass formula for two point masses. (Similar result can be obtained for $\bar{y}$.)

34.



$$\bar{x} = \frac{(-a)(\delta \pi a^2) + (ta)[\delta \pi (ta)^2]}{\delta \pi a^2 + \delta \pi (ta)^2} = \frac{a(t^3 - 1)}{t^2 + 1}$$
$$\bar{y} = \frac{(a)(\delta \pi a^2) + (0)}{\delta \pi a^2 + \delta \pi (ta)^2} = \frac{a}{t^2 + 1}$$

35.

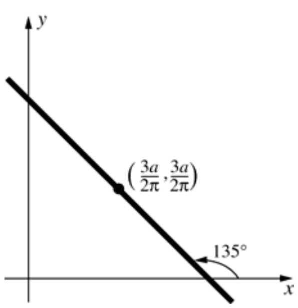


$\langle a,b\rangle$ is perpendicular to the line $ax + by = 0$. Therefore, the (signed) distance of (x, y) to L is the scalar projection of $\langle x,y\rangle$ onto $\langle a,b\rangle$, which is $d(x,\ y)=\dfrac{\langle x,\ y\rangle\cdot\langle a,\ b\rangle}{|\langle a,\ b\rangle|}=\dfrac{ax+by}{|\langle a,\ b\rangle|}$.

$$M_L=\iint_S d(x,\ y)\delta(x,\ y)dA$$
$$=\iint_S \frac{ax+by}{|\langle a,\ b\rangle|}\delta(x,\ y)dA$$
$$=\frac{a}{|\langle a,\ b\rangle|}\iint_S x\delta(x,\ y)dA+\frac{b}{|\langle a,\ b\rangle|}\iint_S y\delta(x,\ y)dA$$
$$=\frac{a}{|\langle a,\ b\rangle|}(0)+\frac{b}{|\langle a,\ b\rangle|}(0)=0$$

[since $(\overline{x},\ \overline{y})=(0,\ 0)$]

36.



The equation has the form $x + y = b$.

$\dfrac{3a}{2\pi}+\dfrac{3a}{2\pi}=b$ so $b=\dfrac{3a}{\pi}$.

Therefore, the equation is $x+y=\dfrac{3a}{\pi}$, or $\pi x+\pi y=3a$.

13.6 Concepts Review

1. $\|\mathbf{u}\times\mathbf{v}\|$

2. $\displaystyle\iint_S \sqrt{f_x^2+f_y^2+1}\,dA$

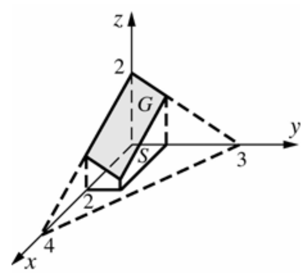
3. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\frac{a}{\sqrt{a^2-x^2-y^2}}\right)dy\,dx$

$$=\int_0^{2\pi}\int_0^a\left(\frac{ar}{\sqrt{a^2-r^2}}\right)dr\,d\theta;\quad 2\pi a^2$$

4. $2\pi ah$

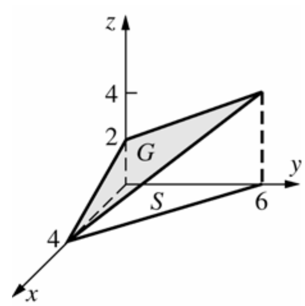
Problem Set 13.6

1.



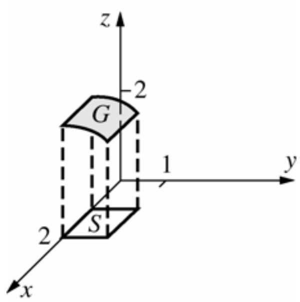
$$z=2-\frac{1}{2}x-\frac{2}{3}y$$
$$f_x(x,y)=-\frac{1}{2};\ f_y(x,y)=-\frac{2}{3}$$
$$A(G)=\int_0^2\int_0^1\sqrt{\frac{1}{4}+\frac{4}{9}+1}\,dy\,dx$$
$$=\frac{\sqrt{61}}{3}\approx 2.6034$$

2.



$$z=2-\frac{1}{2}x+\frac{1}{3}y$$
$$f_x(x,y)=-\frac{1}{2};\ f_y(x,y)=\frac{1}{3}$$
$$A(G)=\int_0^4\int_0^{-\frac{3}{2}x+6}\sqrt{\frac{1}{4}+\frac{1}{9}+1}dydx$$
$$=\frac{7}{6}\int_0^4\left(-\frac{3}{2}x+6\right)dx=\left(\frac{7}{6}\right)(12)=14$$

3.



$z = f(x, y) = (4 - y^2)^{1/2}$; $f_x(x, y) = 0$;

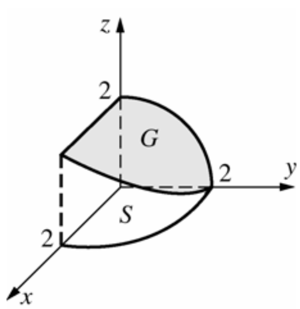
$f_y(x, y) = -y(4 - y^2)^{-1/2}$

$$A(G) = \int_0^1 \int_1^2 \sqrt{y^2(4-y^2)^{-1} + 1}\, dx\, dy$$

$$= \int_0^1 \int_1^2 \frac{2}{\sqrt{4-y^2}}\, dx\, dy = \int_0^1 \frac{2}{\sqrt{4-y^2}}\, dy$$

$$= \left[2\sin^{-1}\left(\frac{y}{2}\right)\right]_0^1 = 2\left(\frac{\pi}{6}\right) - 2(0) = \frac{\pi}{3} \approx 1.0472$$

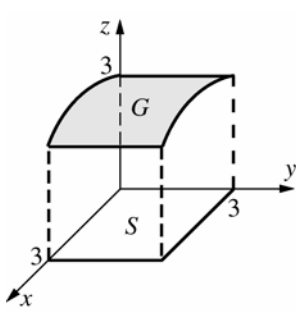
4.



$$A(G) = \int_0^2 \int_0^{\sqrt{4-y^2}} 2(4-y^2)^{-1/2}\, dx\, dy = 4$$

(See problem 3 for the integrand.)

5.



Let $z = f(x, y) = (9 - x^2)^{1/2}$.

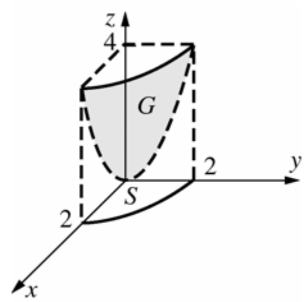
$f_x(x, y) = -x(9 - x^2)^{-1/2}$, $f_y(x, y) = 0$

$$A(G) = \int_0^2 \int_0^3 [x^2(9-x^2)^{-1} + 1] dy\, dx$$

$$= \int_0^2 \int_0^3 3(9-x^2)^{-1/2}\, dy\, dx = 9\sin^{-1}\left(\frac{2}{3}\right)$$

≈ 6.5675

6.



Let $z = f(x, y) = x^2 + y^2$; $f_x(x, y) = 2x$;

$f_y(x, y) = 2y$.

$$A(G) = 4\iint_S \sqrt{4x^2 + 4y^2 + 1}\, dy dx$$

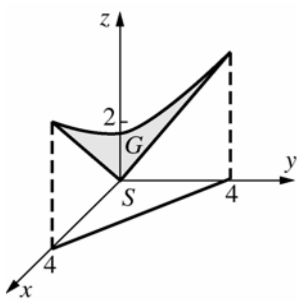
$$= 4\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4x^2 + 4y^2 + 1}\, dy\, dx$$

$$= 4\int_0^{\pi/2} \int_0^2 (4r^2 + 1)^{1/2} r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} \left[\frac{(4r^2+1)^{3/2}}{12}\right]_0^2 dr = \frac{(17^{3/2} - 1)}{3} \frac{\pi}{2}$$

≈ 36.1769

7.

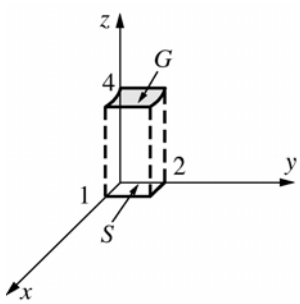


$$z = f(x, y) = (x^2 + y^2)^{1/2}$$

$$f_x(x, y) = x(x^2 + y^2)^{-1/2},\ f_y(x, y) = y(x^2 + y^2)^{-1/2}$$

$$A(G) = \int_0^4 \int_0^{4-x} [x^2(x^2 + y^2)^{-1} + y^2(x^2 + y^2)^{-1} + 1]^{1/2}\, dy\, dx \ = \int_0^4 \int_0^{4-x} \sqrt{2}\, dy\, dx = 8\sqrt{2}$$

8.



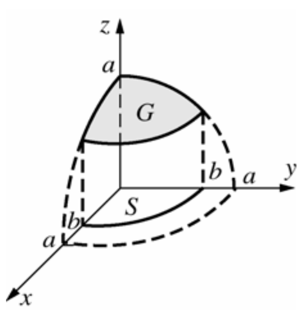
$$z = f(x, y) = \left(\frac{1}{4}\right)x^2 + 4$$

$$f_x = (x, y) = \frac{x}{2};\ f_y(x, y) = 0$$

$$A(G) = \int_0^1 \int_0^2 \left[\left(\frac{1}{4}\right)x^2 + 1\right]^{1/2} dy\, dx$$

$$= \frac{\sqrt{5}}{2} + 2\ln\left[\frac{\left(\sqrt{5}+1\right)}{2}\right] \approx 2.0805$$

9.



$$f_x(x, y) = \frac{x^2}{a^2 - x^2 - y^2};\ f_y(x, y) = \frac{y^2}{a^2 - x^2 - y^2}$$

(See Example 3.)

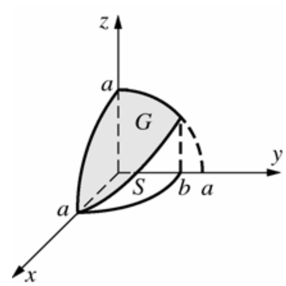
$$A(G) = 8\iint_S \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA$$

$$= 8\int_0^{\pi/2} \int_0^b \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$$= 8a\left(\frac{\pi}{2}\right)\int_0^b (a^2 - r^2)^{-1/2}\, r\, dr$$

$$= -4a\pi\left[(a^2 - r^2)^{1/2}\right]_0^b = 4\pi a\left(a - \sqrt{a^2 - b^2}\right)$$

10.



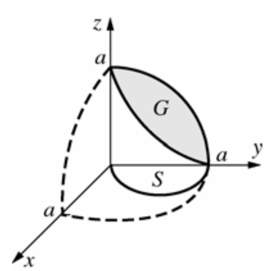
$$f_x(x, y) = \frac{x^2}{a^2 - x^2 - y^2};\ f_y(x, y) = \frac{y^2}{a^2 - x^2 - y^2}$$

(See Example 3.)

$$A(G) = 8\int_0^a \int_0^{(b/a)\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx$$

$$= 8a\int_0^a \sin^{-1}\left(\frac{b}{a}\right)dx = 8a^2 \sin^{-1}\left(\frac{b}{a}\right)$$

11.



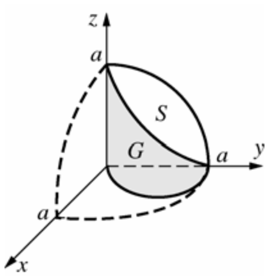
$$f_x(x,y) = \frac{x^2}{a^2 - x^2 - y^2};\ f_y(x,y) = \frac{y^2}{a^2 - x^2 - y^2}$$

(See Example 3.)

$$A(G) = 4\int_0^{\pi/2}\int_0^{a\sin\theta} \frac{a}{\sqrt{a^2 - r^2}}\, r\,dr\,d\theta$$

$$= 4a^2\int_0^{\pi/2}(1 - \cos\theta)d\theta = 2a^2(\pi - 2)$$

12.



Following the hint, treat this as a surface $x = f(y,z) = \sqrt{ay - yz}$.

$$f_y = \frac{a - 2y}{2\sqrt{y(a-y)}},\ \ f_z = 0$$

$$\sqrt{f_y^2 + f_z^2 + 1} = \frac{a}{2\sqrt{y(a-y)}}.$$

The region S in the yz-plane is a quarter circle.

$$A(G) = 4\iint_S \frac{a}{2\sqrt{y(a-y)}}\,dzdy$$

$$= 2a\int_0^a \int_0^{\sqrt{a^2-y^2}} \frac{1}{\sqrt{y(a-y)}}\,dzdy$$

$$= 2a\int_0^a \frac{\sqrt{a^2-y^2}}{\sqrt{y(a-y)}}\,dy = 2a\int_0^a \sqrt{\frac{a+y}{y}}\,dy$$

Make the substitution:

$y = a\tan^2 u$

$a + y = a\left(1 + \tan^2 u\right) = a\sec^2 u$

$$\sqrt{\frac{a+y}{y}} = \csc u$$

$dy = 2a\tan u \sec^2 u\,du$

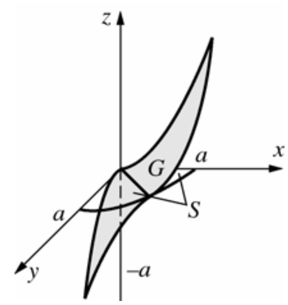
$$A(G) = 2a\int_0^{\pi/4} \csc u \cdot 2a\tan u \sec^2 u\,du$$

$$= 4a^2\int_0^{\pi/4} \sec^3 u\,du$$

$$= 4a^2\left[\tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln\left|\sec u + \tan u\right|\right]_0^{\pi/4}$$

$$= 2a^2\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right)$$

13.



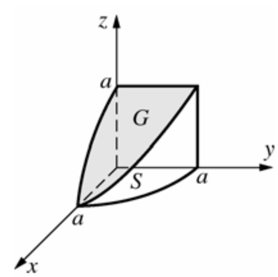
Let $f(x,y) = \dfrac{\left(x^2 - y^2\right)}{a}$.

$$f_x(x,y) = \frac{2x}{a};\ f_y(x,y) = \frac{-2y}{a}$$

$$A(G) = \int_0^{2\pi}\int_0^a \frac{\sqrt{4r^2 + a^2}}{a}\, r\,dr\,d\theta$$

$$= \frac{2\pi}{a}\int_0^a (4r^2 + a^2)^{1/2}\, r\,dr = \frac{\pi a^2\left(5\sqrt{5} - 1\right)}{6}$$

14.



$$f_x(x,y) = \frac{-x}{\sqrt{a^2 - x^2}};\ f_y(x,y) = 0$$

$$\sqrt{\left(f_x(x,y)\right)^2 + \left(f_y(x,y)\right) + 1} = \frac{a}{\sqrt{a^2 - x^2}}$$

$$= \frac{a}{\sqrt{a^2 - r^2\cos^2\theta}}$$

$$A(\text{all sides}) = 8\int_0^{\pi/2}\int_0^a \frac{a}{\sqrt{a^2 - r^2\cos^2\theta}}\,d\theta r dr$$

$$= 8\int_0^{\pi/2} \frac{a^2}{1 + \sin\theta}\,d\theta = 8a^2(1) = 8a^2$$

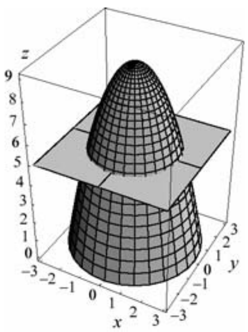
15. $f_x = -2x;\ f_y = -2y$

$$SA = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} \left[\left(4r^2 + 1\right)^{3/2} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{17^{3/2} - 1}{12}\, d\theta = \frac{\left(17^{3/2} - 1\right)\pi}{6} \approx 36.18$$



16. Using Formula 44 from the table of integrals,

$f_x = -2x;\ f_y = 0$

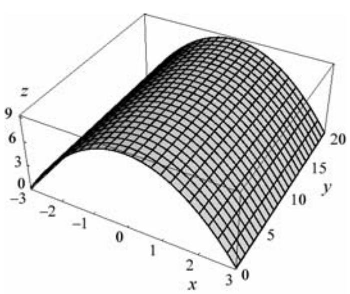
$$A(G) = \int_0^{20} \int_{-3}^{3} \sqrt{4x^2 + 1}\, dx\, dy$$

$$= 2\int_0^{20} \int_0^3 \sqrt{4x^2 + 1}\, dx\, dy$$

$$= \int_0^{20} \left[x\sqrt{4x^2 + 1} + \frac{1}{2}\ln\left|2x + \sqrt{4x^2 + 1}\right| \right]_0^3 dy$$

$$= \int_0^{20} \left[3\sqrt{37} + \frac{1}{2}\ln\left|6 + \sqrt{37}\right| - (0 + \ln 1) \right] dy$$

$$= 20\left[3\sqrt{37} + \frac{1}{2}\ln\left(6 + \sqrt{37}\right) \right] \approx 389.88$$



17. $z = f(x, y) = (D - Ax - By)/C$

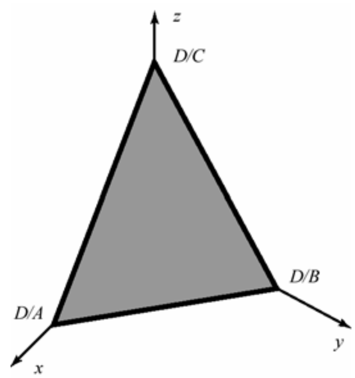
$$A(G) = \iint_R \sqrt{(A/C)^2 + (B/C)^2 + 1}\, dA$$

$$= \int_0^{D/A} \int_0^{D/B - (A/B)x} \sqrt{(A/C)^2 + (B/C)^2 + 1}\, dy\, dx$$

$$= \sqrt{(A/C)^2 + (B/C)^2 + 1} \times Area(Triangle)$$

$$= \frac{1}{2} \frac{D}{A} \frac{D}{B} \sqrt{(A/C)^2 + (B/C)^2 + 1}$$

$$= \frac{D^2}{2ABC} \sqrt{A^2 + B^2 + C^2}$$



18. Let $z = C - x$ be the equation of the plane that defines the roof, where C is a constant. Thus, $f_x = -1$ and $f_y = 0$.

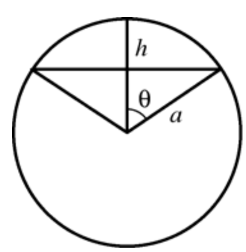
$$A(G) = \iint_R \sqrt{(-1)^2 + 0^2 + 1}\, dA$$

$$= \iint_R \sqrt{2}\, dA = \pi (18)^2 \sqrt{2} \approx 1440 \text{ sq.ft.}$$

19. $\overline{x} = \overline{y} = 0$ (by symmetry)

Let $h = \dfrac{h_1 + h_2}{2}$. Planes $z = h_1$ and $z = h$ cut out the same surface area as planes $z = h$ and $z = h_2$. Therefore, $\overline{z} = h$, the arithmetic average of h_1 and h_2.

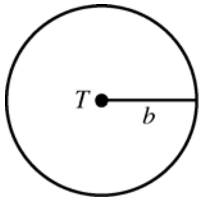
20.



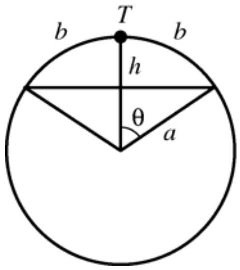
Area = $2\pi a h$

$= 2\pi a(a - a\cos\phi) = 2\pi a^2(1 - \cos\phi)$

21. **a.** $A = \pi b^2$



b.



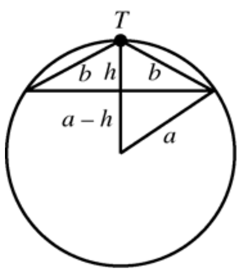
$B = 2\pi a^2(1-\cos\phi)$ (Problem 20)

$= 2\pi a^2\left[1-\cos\left(\frac{b}{a}\right)\right]$

$= 2\pi a^2\left[\frac{b^2}{2!a^2}-\frac{b^4}{4!a^4}+\frac{b^6}{6!a^4}+\dots\right]$

$= \pi b^2\left[1-\frac{b^2}{12a^2}+\frac{b^4}{360a^4}-\dots\right] \le \pi b^2$

c.

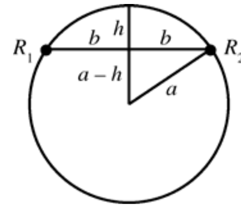


$a^2-(a-h)^2 = b^2-h^2$, so $h = \frac{b^2}{2a}$.

Thus, $C = 2\pi ah$

$= 2\pi a\left(\frac{b^2}{2a}\right) = \pi b^2$.

d.



$D = 2\pi ah$

$= 2\pi a\left(a-\sqrt{a^2-b^2}\right) = \frac{2\pi a[a^2-(a^2-b^2)]}{a+\sqrt{a^2-b^2}}$

$= \frac{2\pi ab^2}{a+\sqrt{a^2-b^2}} > \pi b^2$

Therefore, $B < A = C < D$.

22. $[A(S_{yz})]^2+[A(S_{xz})]^2+[A(S_{xy})]^2$

$=[A(S)\cos\alpha]^2+[A(S)\cos\beta]^2+[A(S)\cos\gamma]^2$

(where α, β, and γ are direction angles for a normal to S.)

$=[A(S)]^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)=[A(S)]^2$

23. In the following, each double integral is over S_{xy}

$A(S_{xy})f(\bar{x}, \bar{y}) = A(S_{xy})(a\bar{x}+b\bar{y}+c)$

$=\iint dA\left[a\frac{\iint x\,dA}{\iint dA}+b\frac{\iint y\,dA}{\iint dA}+c\right]$

$=a\iint x\,dA+b\iint y\,dA+c\iint dA$

$=\iint (ax+by+c)dA$

= Volume of solid cylinder under S_{xy}

24. Because the slopes of both roofs are the same, the area of T_m will be the same for both roofs. (Essentially we will be integrating over a constant). Therefore, the area of the roofs will be the same.

25. Let G denote the surface of that part of the plane $z = Ax+By+C$ over the region S. First, suppose that S is the rectangle $a \le x \le b,\ c \le y \le d$. Then the vectors **u** and **v** that form the edge of the parallelogram G are $\mathbf{u} = (b-a)\mathbf{i}+0\mathbf{j}+A(b-a)\mathbf{k}$ and $\mathbf{v} = 0\mathbf{i}+(d-c)\mathbf{j}+B(d-c)\mathbf{k}$. The surface area of G is thus

$|\mathbf{u}\times\mathbf{v}| =$

$|-A(b-a)(d-c)\mathbf{i}-B(b-a)(d-c)\mathbf{j}+(b-a)(d-c)\mathbf{k}|$

$=(b-a)(d-c)\sqrt{A^2+B^2+1}$

A normal vector to the plane is $\mathbf{n} = -A\mathbf{i}-B\mathbf{j}+\mathbf{k}$. Thus,

$$\cos\gamma = \frac{\mathbf{n}\cdot\mathbf{k}}{|\mathbf{n}||\mathbf{k}|} = \frac{\langle -A,-B,1\rangle\cdot\langle 0,0,1\rangle}{\sqrt{A^2+B^2+1}\cdot 1}$$

$$= \frac{1}{\sqrt{A^2+B^2+1}}$$

$$\sec\gamma = \frac{1}{\cos\gamma} = \sqrt{A^2+B^2+1} = \frac{|\mathbf{u}\times\mathbf{v}|}{A(S)} = \frac{A(G)}{A(S)}.$$

If S is not a rectangle, then make a partition of S with rectangles $R_1, R_2, \ldots, R_n$. The Riemann sum will be $\sum_{m=1}^{n} A(G_m) = \sec\gamma \sum_{m=1}^{n} A(R_m)$.

As we take the limit as $|P| \to 0$ the sum converges to the area of S. Thus the surface area will be

$$A(G) = \lim_{|P|\to 0} \sec\gamma \sum_{m=1}^{n} A(R_m) = \sec\gamma\, A(S).$$

26. Let $\gamma = \gamma(x, y, f(x, y))$ be the acute angle between a unit vector $\mathbf{n}$ that is normal to the surface and makes an acute angle with the z-axis. Let $F(x, y, z) = z - f(x, y)$. Then the normal vector to the surface $F(x, y, z) = 0 = z - f(x, y)$ is parallel to the gradient $\nabla F(x, y, z) = -f_x\mathbf{i} - f_y\mathbf{j} + 1\mathbf{k}$. The unit normal vector is thus

$$\mathbf{n} = \left(-f_x\mathbf{i} - f_y\mathbf{j} + 1\mathbf{k}\right)/\sqrt{f_x^2 + f_y^2 + 1}$$

The cosine of the angle γ is thus

$$\cos\gamma = \frac{\mathbf{n}\cdot\mathbf{k}}{|\mathbf{n}||\mathbf{k}|} = \frac{-f_x\mathbf{i} - f_y\mathbf{j} + 1\mathbf{k}}{\sqrt{f_x^2+f_y^2+1}}\cdot\mathbf{k}$$

$$= \frac{1}{\sqrt{f_x^2+f_y^2+1}}$$

Hence, $\sec\gamma = \sqrt{f_x^2 + f_y^2 + 1}$.

27. a. $f_x = 2x,\ f_y = 2y$

$$A(G) = \iint_S \sqrt{4x^2+4y^2+1}\, dA$$

$$= \int_0^{\pi/2}\int_0^3 \sqrt{4r^2+1}\, r\,dr\,d\theta$$

$$= \frac{\pi}{2}\left[\frac{1}{12}\left(4r^2+1\right)^{3/2}\right]_0^3$$

$$= \frac{\pi}{24}\left(37^{3/2}-1\right) \approx 29.3297$$

b. $f_x = 2x,\ f_y = 2y$

$$A(G) = \int_0^3\int_0^{3-x} \sqrt{4x^2+4y^2+1}\, dy\,dx$$

Parabolic rule with $n = 10$ gives $SA \approx 15.4233$

28. a. $f_x = 2x,\ f_y = -2y$

$$A(G) = \iint_S \sqrt{4x^2+4y^2+1}\, dA$$

$$= \frac{\pi}{24}\left(37^{3/2}-1\right) \approx 29.3297$$

(same integral as problem 27a)

b. $f_x = 2x,\ f_y = -2y$

$$A(G) = \int_0^3\int_0^{3-x} \sqrt{4x^2+4y^2+1}\, dy\,dx$$

Parabolic rule with $n = 10$ gives $SA \approx 15.4233$

(same integral as problem 27b)

29. The surface area of a paraboloid and a hyperbolic paraboloid are the same over identical regions. So, the areas depend on the regions.
$E = F < A = B < C = D$

13.7 Concepts Review

1. volume

2. $\iiint_S |xyz|\, dV$

3. $y;\ \sqrt{y}$

4. 0

Problem Set 13.7

1. $\int_{-3}^{7}\int_0^{2x}(x-1-y)\,dy\,dx = \int_{-3}^{7} -2x\,dx = -40$

2. $\int_0^2\int_{-1}^{4}(3y+x)\,dy\,dx = \int_0^2\left(\frac{45}{2}+5x\right)dx = 55$

3. $\int_1^4\int_{z-1}^{2z}\int_0^{y+2z} dx\,dy\,dz = \int_1^4\int_{z-1}^{2z}(y+2z)\,dy\,dz$
$= \int_1^4\left[\frac{y^2}{2}+2yz\right]_{y=z-1}^{2z} dz$
$= \int_1^4\left(\frac{7z^2}{2}+3z-\frac{1}{2}\right)dz$
$= \left[\frac{7z^3}{6}+\frac{3z^2}{2}-\frac{z}{2}\right]_1^4 = \frac{189}{2} = 94.5$

4. $6\int_0^5 z^3 dz\int_{-2}^{4} y^2 dy\int_1^2 x\,dx = \left(\frac{625}{4}\right)(72)(3)$
$= 33{,}750$

5. $\int_4^{24}\int_0^{24-x}\left[\frac{1}{x}\left(yz+\frac{1}{2}z^2\right)\right]_0^{24-x-y} dy\,dx$
$= \int_4^{24}\int_0^{24-x}\left[\frac{(x+y-24)(x-y-24)}{2x}\right]dy\,dx$
$= \int_4^{24}\left[-\frac{y\left(y^2-3(x-24)^2\right)}{6x}\right]_0^{24-x} dx$
$= \int_4^{24} -\frac{(x-24)^3}{3x}\,dx$
$= \left[-576x+12x^2-\frac{x^3}{9}+4608\ln x\right]_4^{24} \approx 1927.54$

6. $\int_0^5\int_0^3\left[\frac{yzx^2}{2}\right]_{z^2}^{9} dz\,dy = \int_0^5\int_0^3\left[\frac{81yz-yz^5}{2}\right]dz\,dy$
$= \int_0^5\left[\frac{-z^2\left(z^4-243\right)y}{12}\right]_0^3 dy = \int_0^5\frac{243y}{2}\,dy$
$= \left[\frac{243}{4}y^2\right]_0^5 = 1518.75$

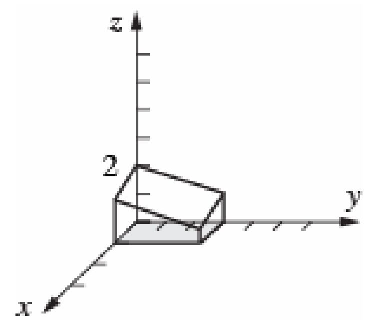
7. $\int_0^2\int_1^z x^2 dx\,dz = \int_0^2\left(\frac{1}{3}\right)(z^3-1)\,dz = \frac{2}{3}$

8. $\int_0^{\pi/2}\int_0^z\int_0^y \sin(x+y+z)\,dx\,dy\,dz$
$= \int_0^{\pi/2}\int_0^z[-\cos(2y+z)+\cos(y+z)]\,dy\,dz$
$= \int_0^{\pi/2}\left(-\frac{\sin 3z}{2}+\sin 2z-\frac{\sin z}{2}\right)dz = \frac{1}{3}$

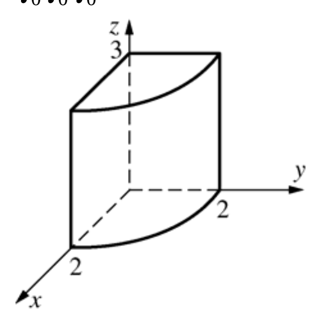
9. $\int_{-2}^{4}\int_{x-1}^{x+1} 3y^2 dy\,dx = \int_{-2}^{4}(6x^2+2)\,dx = 156$

10. $\int_0^{\pi/2}\int_{\sin 2z}^{0} y(1-\cos 2z)\,dy\,dz$
$= \int_0^{\pi/2}\left(-\frac{1}{2}\right)(\sin^2 2z)(1-\cos 2z)$
$= -\frac{\pi}{8} \approx -0.3927$

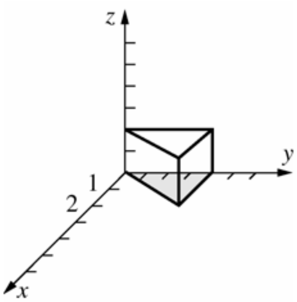
11. $\int_0^1\int_0^3\int_0^{(12-3x-2y)/6} f(x,\ y,\ z)\,dz\,dy\,dx$



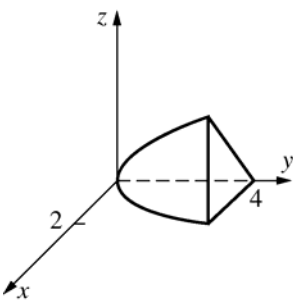
12. $\int_0^3\int_0^2\int_0^{\sqrt{4-y^2}} f(x,\ y,\ z)\,dx\,dy\,dz$



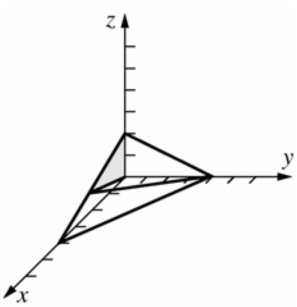
13. $\int_0^2 \int_0^4 \int_0^{y/2} f(x, y, z)dx\,dy\,dz$



14. $\int_0^4 \int_0^{\sqrt{y}} \int_0^{3x/2} f(x, y, z)dz\,dx\,dy$

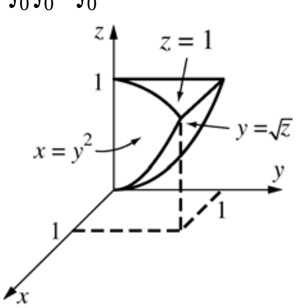


15. $\int_0^2 \int_0^{3z} \int_0^{4-x-2z} f(x, y, z)dy\ dx\ dz$

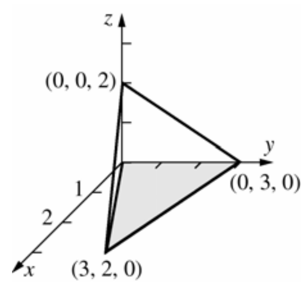


Alternate:
$\int_0^{12/5} \int_{x/3}^{(4-x)/2} \int_0^{4-x-2z} f(x, y, z)dy\ dz\ dx$

16. $\int_0^1 \int_0^{\sqrt{z}} \int_0^{y^2} f(x, y, z)dx\,dy\,dz$



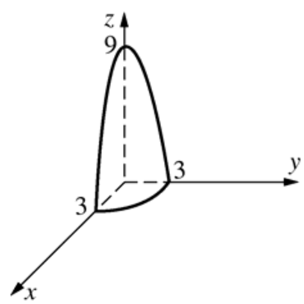
17.



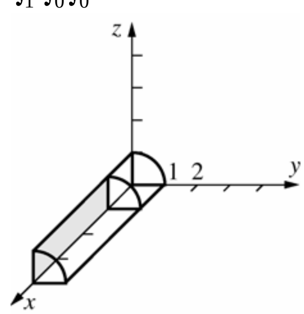
Using the cross product of vectors along edges, it is easy to show that $\langle 2,6,9 \rangle$ is normal to the upward face. Then obtain that its equation is $2x + 6y + 9z = 18$.

$\int_0^3 \int_{2x/3}^{(9-x)/3} \int_0^{(18-2x-6y)/9} f(x, y, z)dz\,dy\,dx$

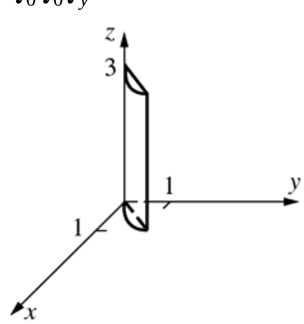
18. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} f(x, y, z)dz\,dy\,dx$



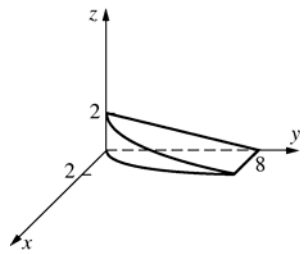
19. $\int_1^4 \int_0^1 \int_0^{\sqrt{1-y^2}} f(x, y, z)dz\,dy\,dx$



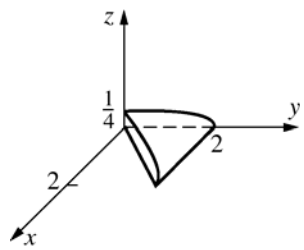
20. $\int_0^3 \int_0^1 \int_y^{\sqrt{2y-y^2}} f(x, y, z)dx\,dy\,dz$



21. $\int_0^2 \int_{2x^2}^8 \int_0^{2-y/4} 1\,dz\,dy\,dx = \frac{128}{15}$



22. $\int_0^2 \int_0^y \int_0^{\sqrt{4-y^2}/8} 1\,dz\,dx\,dy = \frac{1}{3}$



23. $V = 4\int_0^1 \int_0^{\sqrt{y}} \int_0^{\sqrt{y}} 1\,dz\,dx\,dy = 4\int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\,dx\,dy$

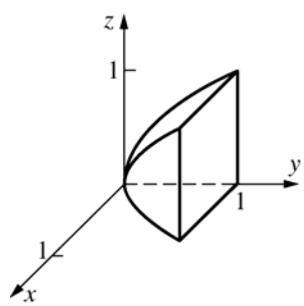
$= 4\int_0^1 \sqrt{y}\sqrt{y}\,dy = [2y^2]_0^1 = 2$

or

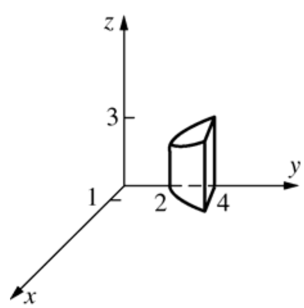
$V = 4\int_0^1 \int_{x^2}^1 \int_0^{\sqrt{y}} 1\,dz\,dy\,dx = 4\int_0^1 \int_{x^2}^1 \sqrt{y}\,dy\,dx$

$= 4\int_0^1 \left[\frac{2}{3}y^{3/2}\right]_{x^2}^1 dy = \frac{8}{3}\int_0^1 \left(1 - x^3\right)dx$

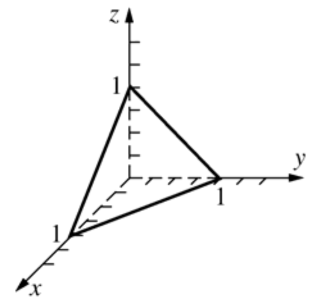
$= \frac{8}{3}\left[x - \frac{1}{4}x^4\right]_0^1 = \frac{8}{3}\left(\frac{3}{4}\right) = 2$



24. $2\int_0^{\sqrt{2}} \int_{x^2+2}^4 \int_0^{3y/4} 1\,dz\,dy\,dx = 32\frac{\sqrt{2}}{5} \approx 9.0510$



25. Let $\delta(x, y, z) = x + y + z$. (See note with next problem.)



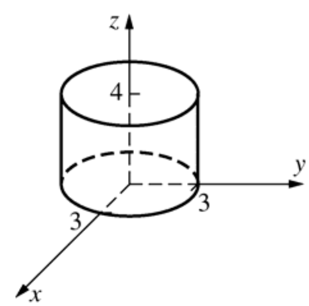
$m = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z)\,dz\,dy\,dx = \frac{1}{8}$

$M_{yz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x(x + y + z)\,dz\,dy\,dx = \frac{1}{30}$

$\overline{x} = \frac{4}{15}$; Then $\overline{y} = \overline{z} = \frac{4}{15}$ (symmetry).

26. $(x,\ y,\ z) = k(x^2 + y^2 + z^2)$

In evaluating the coordinates of the center of mass, k is a factor of the numerator and denominator and so may be canceled. Hence, for sake of convenience we may just let $k = 1$ when determining the center of mass. Note that this is not valid if we are concerned with values of moments or mass.



$m = 4\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^4 (x^2 + y^2 + z^2)\,dz\,dy\,dx$

$= 4\int_0^3 \int_0^{\sqrt{9-x^2}} \left[4(x^2 + y^2) + \frac{64}{3}\right] dy\,dx$

$= 4\int_0^{\pi/2} \int_0^3 \left(4r^2 + \frac{64}{3}\right) r\,dr\,d\theta$ (change to polar)

$= 4\int_0^{\pi/2} \left[r^4 + \frac{32r^2}{3}\right]_0^3 d\theta = 4\int_0^{\pi/2} 177\,d\theta = 354\pi$

$M_{xy} = 4\int_0^{\pi/2} \int_0^3 \int_0^4 z(r^2 + z^2)r\,dz\,dr\,d\theta$

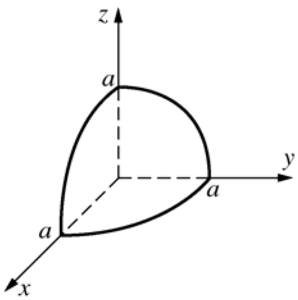
(polar coordinates)

$= 4\int_0^{\pi/2} \int_0^3 (8r^3 + 64r)\,dr\,d\theta$

$= 4\int_0^{\pi/2} 450\,d\theta = 900\pi$

$\overline{z} = \frac{900\pi}{354\pi} = \frac{150}{59} \approx 2.5425$; $\overline{x} = \overline{y} = 0$ (by symmetry)

27. Let $\delta(x, y, z) = 1$. (See note with previous problem.)



$$m = \left(\frac{1}{8}\right)(\text{volume of sphere}) = \left(\frac{\pi}{6}\right)a^3$$

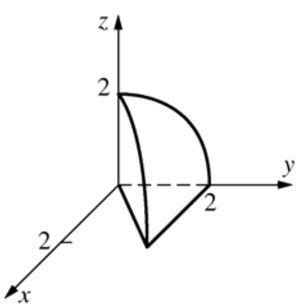
$$M_{xy} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z\, dz\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} zr\, dz\, dr\, d\theta = \left(\frac{\pi}{16}\right)a^4$$

$$\bar{z} = \left(\frac{3}{8}\right)a$$

$$\bar{x} = \bar{y} = \left(\frac{3}{8}\right)a \text{ (by symmetry)}$$

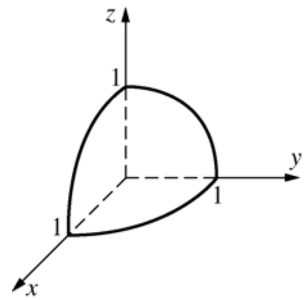
28. $y^2 + z^2$ is the distance of (x, y, z) from the x-axis.



$$I_x = \iiint_S (y^2 + z^2)\delta(x, y, z)dV$$

$$= \int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^y (y^2 + z^2) z\, dx\, dy\, dz = \frac{16}{3}$$

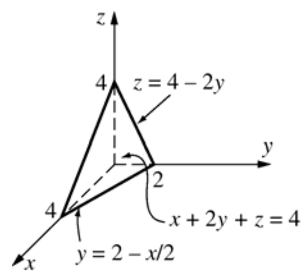
29.



The limits of integration are those for the first octant part of a sphere of radius 1.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x, y, z)dz\, dy\, dx$$

30.

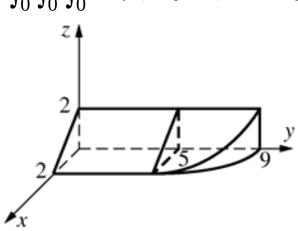


$$\int_0^4 \int_0^{2-x/2} \int_0^{4-x-2y} f(x, y, z)dz\, dy\, dx$$

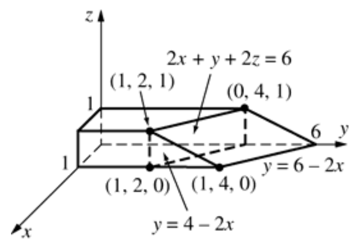
31. $\int_0^2 \int_0^{2-z} \int_0^{9-x^2} f(x, y, z)dy\, dx\, dz$

Figure is same as for Problem 32 except that the solid doesn't need to be divided into two parts.

32. $\int_0^5 \int_0^2 \int_0^{2-x} f(x, y, z)dz\, dx\, dy + \int_5^9 \int_0^{\sqrt{9-y}} \int_0^{2-x} f(x, y, z)dz\, dx\, dy$



33.



a. $\int_0^1\int_0^{4-2x}\int_0^1 dz\,dy\,dx + \int_0^1\int_{4-2x}^{6-2x}\int_0^{3-x-y/2} dz\,dy\,dx = 3+1 = 4$

b. $\int_0^1\int_0^1\int_0^{6-2x-2z} 1\,dy\,dx\,dz = 4$

c. $A(S_{xz})\,f(\overline{x},\,\overline{z})$ (S_{xz} is the unit square in the corner of xz-plane; and $(\overline{x},\,\overline{z}) = \left(\frac{1}{2},\frac{1}{2}\right)$ is the centroid of S_{xz}.)

$= (1)\left[6 - 2\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)\right] = 4$

34. The moment of inertia with respect to the y-axis is the integral (over the solid) of the function which gives the square of the distance of each point in the solid from the y-axis.

$\int_0^1\int_0^1\int_0^{6-2x-2z} k(x^2+z^2)dy\,dx\,dz = \frac{7}{3}k$

35. $\int_0^1\int_0^1\int_0^{6-2x-2z}(30-z)dy\,dx\,dz = \int_0^1\int_0^1(30-z)(6-2x-2z)dx\,dz = \int_0^1([(30-z)(6x-x^2-2xz)]_{x=0}^1)dz$

$= \int_0^1(30-z)(5-2z)dz = \int_0^1(150-65z+2z^2)dz = \left[150z - \frac{65z^2}{2} + \frac{2z^3}{3}\right]_0^1 = \frac{709}{6}$

The volume of the solid is 4 (from Problem 33).

Hence, the average temperature of the solid is $\frac{\frac{709}{6}}{4} = \frac{709}{24} \approx 29.54°$.

36. $T(x,y,z) = 30 - z = 29.54$. The set of all points whose temperature is the average temperature is the plane $z = 0.46$.

37. $M_{yz} = \int_0^1\int_0^1\int_0^{6-2x-2z} x\,dy\,dx\,dz = \int_0^1\int_0^1 x(6-2x-2z)dx\,dz = \int_0^1\left[3x^2 - \frac{2}{3}x^3 - x^2z\right]_{x=0}^1 dz$

$= \int_0^1\left(\frac{7}{3} - z\right)dz = \left[\frac{7}{3}z - \frac{1}{2}z^2\right]_0^1 = \frac{11}{6}$

$M_{xz} = \int_0^1\int_0^1\int_0^{6-2x-2z} y\,dy\,dx\,dz = \int_0^1\int_0^1\left[\frac{1}{2}(6-2x-2z)^2\right]dx\,dz = \int_0^1\left[18x - 6x^2 + \frac{2}{3}x^3 - 12xz + 2x^2z + 2xz^2\right]_{x=0}^1$

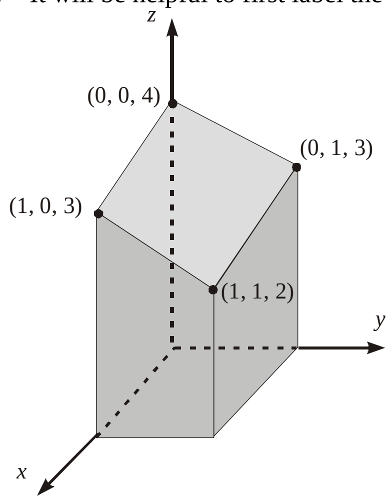
$= \int_0^1\left(\frac{38}{3} - 10z + 2z^2\right)dz = \frac{25}{3}$

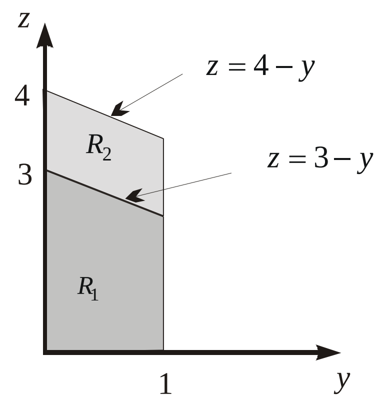
$M_{xy} = \int_0^1\int_0^1\int_0^{6-2x-2z} z\,dy\,dx\,dz = \int_0^1\int_0^1 z(6-2x-2z)dx\,dz = \int_0^1([z(6x-x^2-2xz)]_{x=0}^1)dz$

$= \int_0^1(5z-2z^2)dz = \left[\frac{5z^2}{2} \frac{2z^3}{3}\right]_0^1 = \frac{11}{6}$

Hence, $(\overline{x},\overline{y},\overline{z}) = \left(\frac{11/6}{4}, \frac{25/3}{4}, \frac{11/6}{4}\right) = \left(\frac{11}{24}, \frac{25}{12}, \frac{11}{24}\right)$

38. a. It will be helpful to first label the corner points at the top of the region.





Fixing z and y, we will be looking at the figure along the x-axis. The resulting projection is shown in the figure above and to the right. The possible values of x depends on where we are in the yz-plane. Therefore, we split up the solid into two parts. The volume of the solid will be the sum of these two smaller volumes. In the lower portion, x goes from 0 to 1, while in the upper portion, x goes from 0 to $4 - y - z$ (the plane that bounds the top of the square cylinder).

$$V = \int_0^1 \int_0^{3-y} \int_0^1 1\,dx\,dz\,dy + \int_0^1 \int_{3-y}^{4-y} \int_0^{4-y-z} 1\,dx\,dz\,dy = \frac{5}{2} + \frac{1}{2} = 3$$

b. $\int_0^1 \int_0^1 \int_0^{4-y-z} 1\,dx\,dy\,dz = 3$

c. $A(S_{xy})\,f(\overline{x}, \overline{y})$ (S_{xy} is the unit square in the corner of xy-plane; and $(\overline{x}, \overline{y}) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is the centroid of S_{xz}.)

$$= (1)\left[4 - \frac{1}{2} - \frac{1}{2}\right] = 3$$

39. $m = \int_0^1 \int_0^1 \int_0^{4-x-y} k\,dz\,dy\,dx = \int_0^1 \int_0^1 k(4 - x - y)\,dy\,dx = k\int_0^1 \left(\frac{7}{2} - x\right) dx = 3k$

$$M_{yz} = \int_0^1 \int_0^1 \int_0^{4-x-y} kx\,dz\,dy\,dx = k\int_0^1 \int_0^1 \left(4x - x^2 - xy\right) dy\,dx = k\int_0^1 \left(\frac{7}{2}x - x^2\right) dx = \frac{17k}{12}$$

$$M_{xz} = \int_0^1 \int_0^1 \int_0^{4-x-y} ky\,dz\,dy\,dx = k\int_0^1 \int_0^1 \left(4y - xy - y^2\right) dy\,dx = k\int_0^1 \left(\frac{5}{3}x - \frac{x}{2}\right) dx = \frac{17k}{12}$$

$$M_{xy} = \int_0^1 \int_0^1 \int_0^{4-x-y} kz\,dz\,dy\,dx = k\int_0^1 \int_0^1 \left(8 - 4x + \frac{x^2}{2} - 4y + xy + \frac{y^2}{2}\right) dy\,dx = k\int_0^1 \left(\frac{37}{6} - \frac{7x}{2} + \frac{x^2}{2}\right) dx = \frac{55k}{12}$$

$$\overline{x} = \frac{Myz}{m} = \frac{17k/12}{3k} = \frac{17}{36} \qquad \overline{y} = \frac{Mxz}{m} = \frac{17k/12}{3k} = \frac{17}{36} \qquad \overline{z} = \frac{M_{xy}}{m} = \frac{55k/12}{3k} = \frac{55}{36}$$

40. The temperature, as a function of (x, y, z) is $T(x, y, z) = 40 + 5z$. The average temperature is

$$\frac{1}{\text{Volume}}\int_0^1\int_0^1\int_0^{4-x-y} T(x,y,z)\,dz\,dy\,dx = \frac{1}{3}\int_0^1\int_0^1\int_0^{4-x-y}(40+5z)\,dz\,dy\,dx$$

$$= \frac{1}{3}\int_0^1\int_0^1\left(200 - 60x + \frac{5}{2}x^2 - 60y + 5xy + \frac{5y^2}{2}\right)dy\,dx$$

$$= \frac{1}{3}\int_0^1\left(\frac{1025}{18} - \frac{115x}{6} + \frac{5x^2}{6}\right)dx$$

$$= \frac{1715}{36} \approx 47.64$$

41.

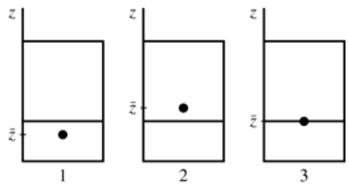


Figure 1: When the center of mass is in this position, it *will* go lower when a little more soda leaks out since mass above the center of mass is being removed.
Figure 2: When the center of mass is in this position, it *was* lower moments before since mass that was below the center of mass was removed, causing the center of mass to rise.
Therefore, the center of mass is lowest when it is at the height of the soda, as in Figure 3. The same argument would hold for a soda bottle.

42. The result obtained from a CAS is:

$$\int_0^c\int_0^{b\sqrt{1-z^2/c^2}}\int_0^{a\sqrt{1-y^2/b^2-z^2/c^2}} 8(xy+xz+yz)dxdydz = \frac{8}{15}a^2b^2c + \frac{8}{15}a^2bc^2 + \frac{8}{15}ab^2c^2 = \frac{8}{15}acb(ca+cb+ab)$$

43. a. $1 = \int_0^{12}\int_0^x ky\,dy\,dx = \int_0^{12}\frac{k}{2}x^2\,dx$

$= \left[\frac{k}{6}x^3\right]_0^{12} = 288k \Rightarrow k = \frac{1}{288}$

b. $P(Y > 4) = \int_4^{12}\int_4^x \frac{1}{288}y\,dydx$

$= \int_4^{12}\left[\frac{1}{576}y^2\right]_4^x dx = \int_4^{12}\frac{1}{576}\left(x^2 - 16\right)dx$

$= \left[\frac{1}{576}\left(x^3 - 16x\right)\right]_4^{12} = \frac{20}{27}$

c. $E[X] = \int_0^{12}\int_0^x x\frac{1}{288}y\,dy\,dx$

$= \int_0^{12}\left[\frac{1}{576}xy^2\right]_0^x dx = \int_0^{12}\frac{1}{576}x^3\,dx$

$= \left[\frac{1}{2304}x^4\right]_0^{12} = 9$

44. a. $1 = \int_0^2\int_0^4\int_0^y kxy\,dx\,dy\,dz$

$= \int_0^2\int_0^4\left[\frac{k}{2}x^2y\right]_0^y dy\,dz$

$= \int_0^2\int_0^4\frac{k}{2}y^3dy\,dz = \int_0^2\left[\frac{k}{8}y^4\right]_0^4 dz$

$= \int_0^2 32k\,dz = 64k \Rightarrow k = \frac{1}{64}$

b. $P(X > 2) = \int_0^2\int_2^4\int_x^4\frac{1}{64}xy\,dy\,dx\,dz$

$= \int_0^2\int_2^4\left[\frac{1}{128}xy^2\right]_x^4 dx\,dz$

$= \int_0^2\int_2^4\frac{1}{128}\left(16x - x^3\right)dx\,dz$

$= \int_0^2\frac{1}{128}\left[8x^2 - \frac{1}{4}x^4\right]_2^4 dz = \int_0^2\frac{9}{32}dz = \frac{9}{16}$

c. $E[X] = \int_0^2 \int_0^4 \int_0^y \frac{1}{64} x^2 y \, dx\, dy\, dz$

$= \int_0^2 \int_0^4 \left[\frac{1}{192} x^3 y \right]_0^y dy\, dz$

$\int_0^2 \int_0^4 \left[\frac{1}{192} y^4 \right] dy\, dz = \int_0^2 \left[\frac{1}{960} y^5 \right]_0^4 dz$

$= \int_0^2 \frac{16}{15} dz = \frac{32}{15}$

45. a. $P(X > 2) = \int_2^4 \int_x^4 \frac{3}{256}\left(x^2 + y^2\right) dy\, dx$

$= \int_2^4 \left[\frac{3}{256}\left(x^2 y + \frac{1}{3} y^3 \right) \right]_x^4 dx$

$= \int_2^4 \frac{1}{64}\left(-x^3 + 3x^2 + 16 \right) dx$

$= \left[\frac{1}{64}\left(-\frac{1}{4} x^4 + x^3 + 16x \right) \right]_2^4 = \frac{7}{16}$

b. $P(X + Y < 4) = \int_0^2 \int_x^{4-x} \frac{3}{256}\left(x^2 + y^2\right) dy\, dx$

$= \int_0^2 \left[\frac{3}{256}\left(x^2 y + \frac{1}{3} y^3 \right) \right]_x^{4-x} dx$

$= \int_0^2 \frac{1}{32}\left(-x^3 + 3x^2 - 6x + 8 \right) dx$

$= \frac{1}{32}\left[-\frac{1}{4} x^4 + x^3 - 3x^2 + 8x \right]_0^2 = \frac{1}{4}$

c. $E[X+Y]$

$= \int_0^4 \int_0^y (x+y) \frac{3}{256}\left(x^2 + y^2\right) dx\, dy$

$= \int_0^4 \left[\frac{3}{256}\left(\frac{1}{4} x^4 + \frac{1}{2} x^2 y^2 + \frac{1}{3} x^3 y + x y^3 \right) \right]_0^y dx$

$= \int_0^4 \frac{25}{1024} y^4 \, dy = \left[\frac{5}{1024} y^5 \right]_0^4 = 5$

46. a. $P(a < X < b) = \int_a^b \int_{-\infty}^{\infty} f(x,y)\, dy\, dx$

$= \int_a^b \left(\int_{-\infty}^{\infty} f(x,y)\, dy \right) dx$

$= \int_a^b f_X(x)\, dx$

b. $E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\, dy\, dx$

$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f(x,y)\, dy \right) dx$

$= \int_{-\infty}^{\infty} x f_X(x)\, dx$

47. $f_X(x) = \int_0^x \frac{1}{288} y\, dy$

$= \left[\frac{1}{576} y^2 \right]_0^x = \frac{1}{576} x^2;\ 0 \le x \le 12$

$E(X) = \int_0^{12} x \cdot f_X(x)\, dx = \frac{1}{576} \int_0^{12} x^3\, dx$

$= \frac{1}{576} \left[\frac{1}{4} x^4 \right]_0^{12} = 9$

48. $f_Y(y) = \int_c^d \int_a^b f(x,y,z)\, dx\, dz$

$= \int_0^2 \int_0^y \frac{1}{64} xy\, dx\, dz = \int_0^2 \left[\frac{1}{128} x^2 y \right]_0^y dz$

$= \int_0^2 \frac{1}{128} y^3 dz = \frac{y^3}{64};\ 0 \le y \le 4$

13.8 Concepts Review

1. $r\, dz\, dr\, d\theta;\ \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

2. $\int_0^{\pi/2} \int_0^1 \int_0^3 r^3 \cos\theta \sin\theta\, dz\, dr\, d\theta$

3. $\int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^4 \cos^2\phi \sin\phi\, d\rho\, d\theta\, d\phi$

4. $\frac{4\pi}{15}$

Problem Set 13.8

1. The region is a right circular cylinder about the z-axis with radius 3 and height 12.

$\int_0^{2\pi} \int_0^3 [rz]_0^{12}\, dr\, d\theta = \int_0^{2\pi} \int_0^3 12r\, dr\, d\theta$

$= \int_0^{2\pi} \left[6r^2 \right]_0^3 d\theta = \int_0^{2\pi} 54\, d\theta = 108\pi$

2. The region is a hollow right circular cylinder about the z-axis with inner radius 1, outer radius 3, and height 12.

$$\int_0^{2\pi}\int_1^3 [rz]_0^{12}\, dr\, d\theta = \int_0^{2\pi}\int_1^3 12r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[6r^2\right]_1^3 d\theta = \int_0^{2\pi} 48\, d\theta = 96\pi$$

3. The region is the region under the paraboloid $z = 9 - r^2$ above the xy-plane in that part of the first quadrant satisfying $0 \le \theta \le \frac{\pi}{4}$.

$$\int_0^{\pi/4}\int_0^3\left[\frac{1}{2}rz^2\right]_0^{9-r^2} dr\, d\theta$$

$$= \int_0^{\pi/4}\int_0^3\left[\frac{1}{2}r\left(81 - 18r + r^2\right)\right] dr\, d\theta$$

$$= \int_0^{\pi/4}\frac{1}{2}\left[\frac{81}{2}r^2 - 6r^3 + \frac{1}{4}r^4\right]_0^3 d\theta$$

$$= \int_0^{\pi/4}\frac{243}{4}\, d\theta = \frac{243\pi}{16}$$

4. The region is a right circular cylinder about the z-axis through the point $\left(0, \frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$ and height 2.

$$\int_0^{\pi}\int_0^{\sin\theta}[rz]_0^2\, dr\, d\theta = \int_0^{\pi}\int_0^{\sin\theta} 2r\, dr\, d\theta$$

$$= \int_0^{\pi}\left[r^2\right]_0^{\sin\theta} d\theta = \int_0^{\pi}\sin^2\theta\, d\theta$$

$$= \int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi}$$

$$= \frac{1}{2}\pi$$

5. The region is a sphere centered at the origin with radius a.

$$\int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{3}\rho^3\sin\varphi\right]_0^a d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}\frac{1}{3}a^3\sin\varphi\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\frac{2\pi}{3}a^3\sin\varphi\, d\varphi$$

$$= \left[-\frac{2\pi}{3}a^3\cos\varphi\right]_0^{\pi} = \frac{4\pi a^3}{3}$$

6. The region is one-eighth of a sphere in the first octant of radius a, centered at the origin.

$$\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{1}{3}\rho^3\cos^2\varphi\sin\varphi\right]_0^a d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{3}a^3\cos^2\varphi\sin\varphi\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\frac{\pi}{6}a^3\cos^2\varphi\sin\varphi\, d\varphi$$

$$= \left[-\frac{\pi}{6}a^3\cos\varphi\right]_0^{\pi/2} = \frac{\pi a^3}{18}$$

7. $\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\, dz\, dr\, d\theta = 8\pi \approx 25.1327$

8. $\int_0^{2\pi}\int_0^2\int_0^{\sqrt{9-r^2}} r\, dz\, dr\, d\theta = \left(\frac{2}{3}\right)\pi(27 - 5^{3/2})$

≈ 33.1326

9. $$V = \int_0^{2\pi}\int_0^3\int_4^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 [rz]_4^{\sqrt{25-r^2}} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3\left[r\sqrt{25-r^2} - 4r\right] dr d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}\left(25-r^2\right)^{3/2} - 2r^2\right]_0^3 d\theta$$

$$= \int_0^{2\pi}\frac{7}{3}\, d\theta = \frac{14\pi}{3}$$

10. $$V = \int_0^{2\pi}\int_0^4\int_0^{4+r\sin\theta} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^4 [rz]_0^{4+r\sin\theta} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^4\left[4r + r^2\sin\theta\right] dr\, d\theta$$

$$= \int_0^{2\pi}\left[2r^2 + \frac{1}{3}r^3\sin\theta\right]_0^4 d\theta$$

$$= \int_0^{2\pi}\left[32 + \frac{64}{3}\sin\theta\right]_0^4 d\theta$$

$$= 64\pi$$

11. $$\int_0^{2\pi}\int_0^2\int_{r^2/4}^{\sqrt{5-r^2}} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2\left[r(5-r^2)^{1/2} - \frac{r^3}{4}\right] dr\, d\theta$$

$$= \int_0^{2\pi}\frac{5^{3/2} - 4}{3}\, d\theta = \frac{2\pi(5^{3/2} - 4)}{3} \approx 15.0385$$

12. $2\cdot\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2\sin\theta\cos\theta} r\,dz\,dr\,d\theta=\frac{4}{3}$

13. Let $\delta(x, y, z) = 1$.
(See write-up of Problem 26, Section 13.7.)

$$m=\int_0^{2\pi}\int_0^2\int_{r^2}^{12-2r^2} r\,dz\,dr\,d\theta=24\pi$$

$$M_{xy}=\int_0^{2\pi}\int_0^2\int_{r^2}^{12-2r^2} zr\,dz\,dr\,d\theta=128\pi$$

$$\bar{z}=\frac{16}{3}$$

$\bar{x}=\bar{y}=0$ (by symmetry)

14. Let $\delta(x, y, z) = 1$.
(See comment at beginning of write-up of Problem 26 of the previous section.)

$$m=\int_0^{2\pi}\int_1^2\int_0^{12-r^2} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2 r(12-r^2)dr\,d\theta$$

$$=\int_0^{2\pi}\left[6r^2-\frac{r^4}{4}\right]_1^2 d\theta=\int_0^{2\pi}\left(\frac{57}{4}\right)d\theta=\frac{57\pi}{2}$$

$$M_{xy}=\int_0^{2\pi}\int_1^2\int_0^{12-r^2} z\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_1^2\frac{(12-r^2)^2(-2r)}{-4}dr\,d\theta$$

$$=\int_0^{2\pi}\frac{11^3-8^3}{12}d\theta=\frac{273\pi}{2}$$

Therefore, $\bar{z}=\frac{\frac{273\pi}{2}}{\frac{57\pi}{2}}=\frac{91}{19}\approx 4.7895.$

$\bar{x}=\bar{y}=0$ (by symmetry)

15. Let $\delta(x,\ y,\ z)=k\rho$

$$m=\int_0^{\pi}\int_0^{2\pi}\int_a^b k\rho\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=k\pi(b^4-a^4)$$

16. $8\int_{\pi/6}^{\pi/2}\int_0^{\pi/2}\int_{a\csc\phi}^{2a} k\rho^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$$=\left(\frac{56}{5}\right)k\pi a^5\sqrt{3}$$

17. Assume that the hemisphere lies above the xy-plane.
Let $\delta(x, y, z) = \rho$.
(Letting $k = 1$ - - see comment at the beginning of the write-up of Problem 26 of the previous section.)

$$m=\int_0^{\pi/2}\int_0^{2\pi}\int_0^a \rho^3\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{2\pi}\frac{a^4\sin\phi}{4}d\theta\,d\phi$$

$$=\int_0^{\pi/2}\frac{\pi a^4\sin\phi}{2}d\phi=\frac{\pi a^4}{2}$$

$$M_{xy}=\int_0^{\pi/2}\int_0^{2\pi}\int_0^a \rho^4\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi$$

$(z=\rho\cos\phi)$

$$=\int_0^{\pi/2}\int_0^{2\pi}\frac{a^5\sin 2\phi}{10}d\theta\,d\phi$$

$$=\int_0^{\pi/2}\frac{\pi a^5\sin 2\phi}{5}d\phi=\left(\frac{\pi}{5}\right)a^5$$

$\bar{z}=\frac{\frac{\pi a^5}{5}}{\frac{\pi a^4}{2}}=\frac{2}{5}a;\ \bar{x}=\bar{y}=0$ (by symmetry)

18. Assume that the hemisphere lies above the xy-plane.
$\delta(x, y, z) = \rho\sin\phi$ (letting $k = 1$)

$$m=\int_0^{\pi/2}\int_0^{2\pi}\int_0^a \rho^3\sin^2\phi\,d\rho\,d\theta\,d\phi=\left(\frac{1}{8}\right)\pi^2a^4$$

$$M_{xy}=\int_0^{\pi/2}\int_0^{2\pi}\int_0^a \rho^4\sin^2\phi\cos\phi\,d\rho\,d\theta\,d\phi$$

$$=\left(\frac{2}{15}\right)\pi a^5$$

$$\bar{z}=\frac{16a}{15\pi}\approx 0.3395a$$

$\bar{x}=\bar{y}=0$ (by symmetry)

19. $I_z=\iint_S(x^2+y^2)k(x^2+y^2)^{1/2}dV$

$$=\int_0^{\pi/2}\int_0^{2\pi}\int_0^a k\rho^5\sin^4\phi\,d\rho\,d\theta\,d\phi=\left(\frac{k}{16}\right)\pi^2a^6$$

20. Volume $=\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^4 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$$=\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\frac{64\sin\phi}{3}d\theta\,d\phi=\int_{\pi/4}^{\pi/2}\frac{128\pi\sin\phi}{3}d\phi$$

$$=\frac{64\sqrt{2}\pi}{3}\approx 94.7815$$

21. $\int_0^{\pi}\int_0^{\pi/6}\int_0^1 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\frac{\pi}{9}\approx 0.3491$

22. $\int_0^{\pi}\int_0^{2\pi}\int_0^3 \rho^3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=486\pi\approx 1526.81$

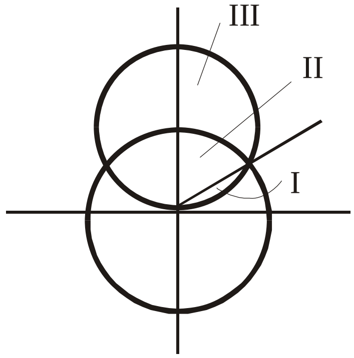
23. Volume $=\int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta} r\,dz\,dr\,d\theta$

$$=\int_0^{\pi}\int_0^{\sin\theta} r(r\sin\theta-r^2)dr\,d\theta=\int_0^{\pi}\frac{\sin^4\theta}{12}d\theta$$

$$=\frac{1}{48}\int_0^{\pi}\left[1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right]d\theta$$

$$=\frac{\pi}{32}\approx 0.0982$$

24. Consider the following diagram:



Method 1: (direct, requires 2 integrals)
$V = \text{I} + \text{II}$

$$= \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^{2\sqrt{2}\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi + \int_0^{\pi/4} \int_0^{2\pi} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \frac{2\sqrt{2}\pi}{3} + \frac{8\left(2-\sqrt{2}\right)\pi}{3} = \frac{2\left(8-3\sqrt{2}\right)\pi}{3} \approx 7.8694$$

Method 2: (indirect, requires 1 integral)
$V = \text{upper sphere volume} - \text{III}$

$$= \frac{8\sqrt{2}\pi}{3} - \int_0^{\pi/4} \int_0^{2\pi} \int_2^{2\sqrt{2}\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \frac{8\sqrt{2}\pi}{3} - \frac{2\left(7\sqrt{2}-8\right)\pi}{3} = \frac{2\left(8-3\sqrt{2}\right)\pi}{3} \approx 7.8694$$

25. **a.** Position the ball with its center at the origin. The distance of (x, y, z) from the origin is $(x^2+y^2+z^2)^{1/2} = \rho$.

$$\iiint_S (x^2+y^2+z^2)^{1/2} \, dV = 8\int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho(\sin\theta\rho^2) d\rho \, d\theta \, d\phi = \pi a^4$$

Then the average distance from the center is $\dfrac{\pi a^4}{\left[\left(\frac{4}{3}\right)\pi a^3\right]} = \dfrac{3a}{4}$.

b. Position the ball with its center at the origin and consider the diameter along the z-axis. The distance of (x, y, z) from the z-axis is $(x^2+y^2)^{1/2} = \rho\sin\phi$.

$$\iiint_S (x^2+y^2)^{1/2} \, dV = 8\int_0^{\pi/2} \int_0^{\pi/2} \int_0^a (\rho\sin\phi)(\rho^2\sin\theta) d\rho \, d\theta \, d\phi = \frac{a^4\pi^2}{4}$$

Then the average distance from a diameter is $\dfrac{\left[\frac{a^4\pi^2}{4}\right]}{\left[\left(\frac{4}{3}\right)\pi a^3\right]} = \dfrac{3\pi a}{16}$.

c. Position the sphere above and tangent to the xy-plane at the origin and consider the point on the boundary to be the origin. The equation of the sphere is $\rho = 2a\cos\phi$, and the distance of (x, y, z) from the origin is ρ.

$$\iiint_S (x^2+y^2+z^2)^{1/2} \, dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2a\cos\phi} \rho(\rho^2\sin\theta) d\rho \, d\theta \, d\phi = \frac{8\pi a^4}{5}$$

Then the average distance from the origin is $\dfrac{\left[\frac{8\pi a^4}{5}\right]}{\left[\left(\frac{4}{3}\right)\pi a^3\right]} = \dfrac{6a}{5}$.

26. Average value of $ax + by + cz + d$ on S is

$$\frac{\iiint_S (ax+by+cz+d)dV}{\iiint_S dV} = \frac{a\iiint_S kx\,dV + b\iiint_S ky\,dV + c\iiint_S kz\,dV + d\iiint_S k\,dV}{\iiint_S k\,dV}$$

$$= \frac{aM_{yz} + bM_{xz} + cM_{xy} + dm}{m} = a\bar{x} + b\bar{y} + c\bar{z} + d = f(\bar{x}, \bar{y}, \bar{z}).$$

27. a. $M_{yz} = \iiint_S kx\,dV \; = 4k\int_0^{\pi/2}\int_0^{\alpha}\int_0^{a}(\rho\sin\phi\cos\theta)(\rho^2\sin\phi)d\rho\,d\theta\,d\phi = ka^4\pi\frac{(\sin\alpha)}{4}$

$m = \iiint_S k\,dV = 4k\int_0^{\pi/2}\int_0^{\alpha}\int_0^{a}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{4a^3k\alpha}{3}$

Therefore, $\bar{x} = \frac{\left[\frac{ka^4\pi(\sin\alpha)}{4}\right]}{\left[\frac{4a^3k\alpha}{3}\right]} = \frac{3a\pi(\sin\alpha)}{16\alpha}.$

b. $\frac{3\pi a}{16}$ (See Problem 25b.)

28. a. $I_z = \iiint_S k[(x^2+y^2)^{1/2}]^2\,dV$

$= 8k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a}(\rho\sin\phi)^2(\rho^2\sin\phi)d\rho\,d\theta\,d\phi = \frac{8a^5\pi k}{15} = \frac{2a^2m}{5}$ (since $m = \left(\frac{4}{3}\right)\pi a^3k$)

b. $I' = I + d^2m = \frac{2a^2m}{5} + a^2m = \frac{7a^2m}{5}$

c. $I = 2\left[\frac{2a^2m}{5} + (a+b)^2m\right] = \frac{2m(7a^2+10ab+5b^2)}{5}$

29. Let m_1 and m_2 be the masses of the left and right balls, respectively. Then $m_1 = \frac{4}{3}\pi a^3k$ and $m_2 = \frac{4}{3}\pi a^3(ck)$, so

$m_2 = cm_1.$

$$\bar{y} = \frac{m_1(-a-b)+m_2(a+b)}{m_1+m_2}$$

$$= \frac{m_1(-a-b)+cm_1(a+b)}{m_1+cm_1} = \frac{-a-b+c(a+b)}{1+c}$$

$$= \frac{(a+b)(-1+c)}{1+c} = \frac{c-1}{c+1}(a+b)$$

(Analogue) $\bar{y} = \frac{m_1\bar{y}_1 + m_2\bar{y}_2}{m_1+m_2} = \bar{y}_1\frac{m_1}{m_1+m_2} + \bar{y}_2\frac{m_2}{m_1+m_2}$

13.9 Concepts Review

1. u-curve; v-curve

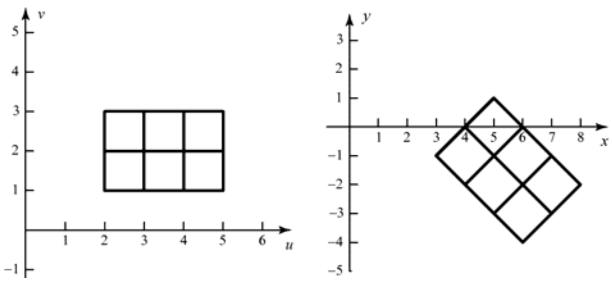
2. the integrand; the differential of $dx\,dy$; the region of integration

3. Jacobian

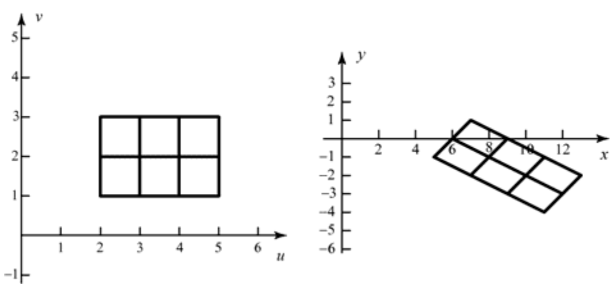
4. $|J(u,v)|$

Problem Set 13.9

1.



2.



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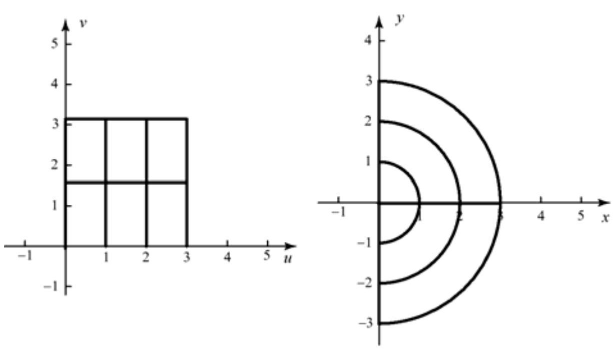
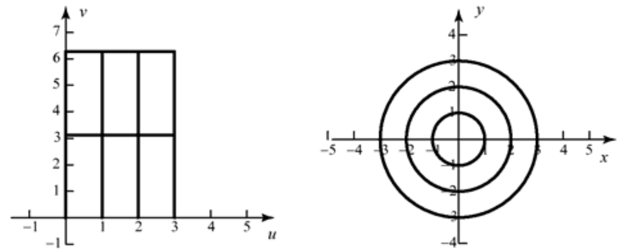
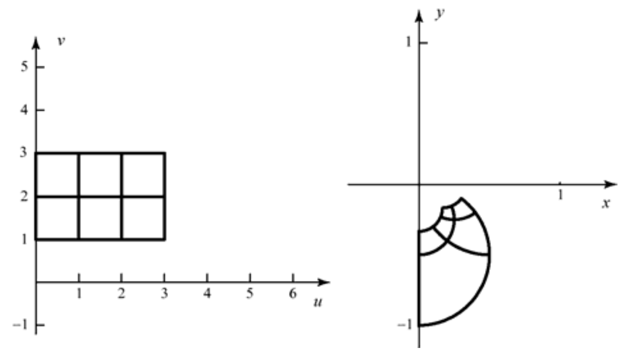


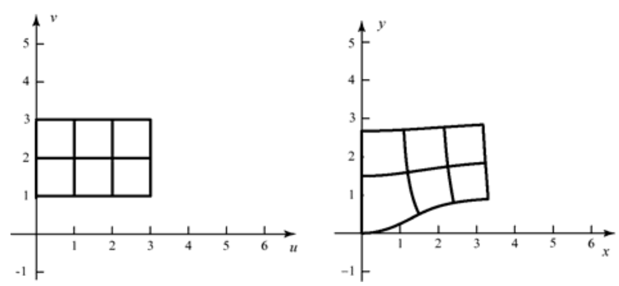
4.



5.



6.



7. $x = u + 2v; \quad y = u - 2v$

$G(0,0) = (0,0); \quad G(2,0) = (2,2)$
$G(2,1) = (4,0); \quad G(0,1) = (2,-2)$

The image is the square with corners $(0,0)$, $(2,2)$, $(4,0)$, and $(2,-2)$. The Jacobian is

$J = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4$.

8. $x = 2u + 3v; \quad y = u - v$

$G(0,0) = (0,0); \quad G(3,0) = (6,3)$
$G(3,1) = (9,2); \quad G(0,1) = (3,-1)$

The image is the parallelogram with vertices $(0,0)$, $(6,3)$, $(9,2)$, and $(3,-1)$. The Jacobian is $J = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -5$

9. $x = u^2 + v^2; \quad y = v$

$G(0,0) = (0,0); \quad G(1,0) = (1,0)$
$G(1,1) = (2,1); \quad G(0,1) = (1,1)$

Solving for u and v gives
$u = \sqrt{x - y^2}$ and $v = y$
The $u = 0$ curve is $0 = \sqrt{x - y^2} \Rightarrow x = y^2$.
The $u = 1$ curve is $1 = \sqrt{x - y^2} \Rightarrow x = y^2 + 1$.
The $v = 0$ curve is $y = 0$, and the $v = 1$ curve is $y = 1$. The image is the set of (x, y) that satisfy $y^2 \le x \le y^2 + 1, 0 \le y \le 1$. The Jacobian is
$J = \begin{vmatrix} 2u & 2v \\ 0 & 1 \end{vmatrix} = 2u$

10. $x = u; \quad y = u^2 - v^2$

Solving for u and v gives
$u = x$ and $v = \sqrt{x^2 - y}$
The $u = 0$ curve is $x = 0$, and the $u = 3$ curve is $x = 3$. The $v = 0$ curve is
$0 = \sqrt{x^2 - y} \Rightarrow y = x^2$ and the $v = 1$ curve is
$1 = \sqrt{x^2 - y} \Rightarrow y = x^2 - 1$. The image is therefore the set of (x, y) that satisfy
$x^2 - 1 \le y \le x^2; \ 0 \le x \le 3$. The Jacobian is
$J = \begin{vmatrix} 1 & 0 \\ 2u & -2v \end{vmatrix} = -2v$.

11. $u = x + 2y; \quad v = x - 2y$ Solving for x and y gives $x = u/2 + v/2; \quad y = u/4 - v/4$. The Jacobian is $J = \begin{vmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{vmatrix} = -\frac{1}{8} - \frac{1}{8} = -\frac{1}{4}$.

12. $u = 2x - 3y; \quad v = 3x - 2y$ Solving for x and y gives $x = -\frac{2}{5}u + \frac{3}{5}v; \quad y = -\frac{3}{5}u + \frac{2}{5}v$. The Jacobian is $J = \begin{vmatrix} -\frac{2}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{vmatrix} = -\frac{4}{25} + \frac{9}{25} = \frac{1}{5}$.

13. $u = x^2 + y^2; \quad v = x$. Solving for x and y gives $x = v; \quad y = \sqrt{u - v^2}$. The Jacobian is
$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{2\sqrt{u - v^2}} & \frac{-v}{\sqrt{u - v^2}} \end{vmatrix} = -\frac{1}{2\sqrt{u - v^2}}$

14. $u = x^2 - y^2; \quad v = x + y$. Solving for x and y gives $x = \frac{v^2 + u}{2v}; \quad y = \frac{v^2 - u}{2v}$. The Jacobian is $J = \begin{vmatrix} \frac{1}{2v} & \frac{1}{2} - \frac{u}{2v^2} \\ -\frac{1}{2v} & \frac{1}{2} + \frac{u}{2v^2} \end{vmatrix} = \frac{1}{2v}$

15. $u = xy; \quad v = x$. Solving for x and y gives $x = v; \quad y = u/v$. The Jacobian is
$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}$.

16. $u = x^2; \quad v = xy$. Solving for x and y gives $x = \sqrt{u}; \quad y = \frac{v}{\sqrt{u}}$. The Jacobian is
$J = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ \frac{-v}{2u^{3/2}} & \frac{1}{\sqrt{u}} \end{vmatrix} = \frac{1}{2u}$.

17. Let $u = x + y$, $v = x - y$. Solving for x and y gives $x = u/2 + v/2$ and $y = u/2 - v/2$. The Jacobian is $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$.

The region of integration gets transformed to the triangle in the uv-plane with vertices (1,1), (4,4), and (7,1). The integral in the uv-plane is more easily done by holding v fixed and integrating u. Thus,

$$\iint_R \ln \frac{x+y}{x-y}\, dA = \int_1^4 \int_v^{8-v} \ln \frac{u}{v} \left|\frac{1}{2}\right| du\, dv$$

$$= \frac{1}{2}\int_1^4 \left[-u + u \ln \frac{u}{v}\right]_v^{8-v} dv$$

$$= \frac{1}{2}\int_1^4 \left[-8 + 2v + (8-v)\ln \frac{8-v}{v}\right] dv$$

$$= \frac{1}{2}\left[3 - 64\ln 4 + \frac{49}{2}\ln 7 + 16 \ln 16\right]$$

$$\approx 3.15669$$

18. Let $u = x + y$, $v = x - y$. Solving for x and y gives $x = u/2 + v/2$ and $y = u/2 - v/2$. The Jacobian is $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$.

The region of integration gets transformed to the triangle in the uv-plane with vertices (1,1), (4,4), and (7,1). The integral in the uv-plane is more easily done by holding v fixed and integrating u. Thus, with the help of a CAS for the outer integral, we have

$$\iint_R \sqrt{\frac{x+y}{x-y}}\, dA = \int_1^4 \int_v^{8-v} \sqrt{\frac{u}{v}} \left|\frac{1}{2}\right| du\, dv$$

$$= \int_1^4 \left[\frac{1}{3}\frac{u^{3/2}}{v^{1/2}}\right]_v^{8-v} dv$$

$$= \frac{1}{3}\int_1^4 \left[\frac{(8-v)^{3/2}}{v^{1/2}} - \frac{v^{3/2}}{v^{1/2}}\right] dv$$

$$= \frac{49}{6} - \frac{19\sqrt{7}}{6} - 4\pi + 16\tan^{-1}\sqrt{7}$$

$$\approx 6.57295$$

19. Let $u = 2x - y$ and $v = y$. Then $x = u/2 + v/2$ and $y = v$. The Jacobian is $J = \begin{vmatrix} 1/2 & 1/2 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$. Thus,

$$\iint_R \sin(\pi(2x-y))\cos(\pi(y-2x))\, dA$$

$$= \int_0^3 \int_{v+2}^{8-v} \sin(\pi u)\cos(\pi(-u)) \left|\frac{1}{2}\right| du\, dv$$

$$= \frac{1}{2}\int_0^3 \int_{v+2}^{8-v} \frac{1}{2} 2\sin(\pi u)\cos(\pi u)\, du\, dv$$

$$= \frac{1}{4}\int_0^3 \int_{v+2}^{8-v} \sin(2\pi u)\, du\, dv$$

$$= \frac{1}{4}\int_0^3 \left[-\frac{\cos 2\pi u}{2\pi}\right]_{v+2}^{8-v} dv$$

$$= \frac{1}{8\pi}\int_0^3 \left[\cos(2\pi(v+2)) - \cos(2\pi(8-v))\right] dv$$

$$= \frac{1}{8\pi}\int_0^3 \left[\cos(2\pi v + 4\pi) - \cos(16\pi - 2\pi v)\right] dv$$

$$= \frac{1}{8\pi}\int_0^3 \left[\cos(2\pi v) - \cos(2\pi v)\right] dv = 0$$

20. Let $u = 2x - y$ and $v = y$. Then $x = u/2 + v/2$ and $y = v$. The Jacobian is $J = \begin{vmatrix} 1/2 & 1/2 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$. Thus,

$$\iint_R (2x-y)\cos(y-2x)\, dA$$

$$= \int_0^3 \int_{v+2}^{8-v} u\cos(-u)\frac{1}{2}\, du\, dv$$

$$= \frac{1}{2}\int_0^3 \left[u\sin u + \cos u\right]_{v+2}^{8-v} dv$$

$$= \frac{1}{2}(-2\cos 2 + 10\cos 5 - 8\cos 8$$
$$+ 2\sin 2 - 4\sin 5 + 2\sin 8)$$

$$\approx 6.23296$$

21. The transformation to spherical coordinates is

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - 0 \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta \end{vmatrix} + (-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left(-\rho^2\sin\phi\sin\theta\cos\phi\sin\theta - \rho^2\cos\phi\sin\phi\cos^2\theta\right) - \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$= -\rho^2\cos^2\phi\,\sin\phi\left(\sin^2\theta + \cos^2\theta\right) - \rho^2\sin^3\phi$$

$$= -\rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

$$= -\rho^2\sin\phi$$

22. Let $x = ua,\ y = vb,\ z = wc$. Then the Jacobian is $J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and the region of integration becomes the sphere with radius 1 centered at the origin. Thus,

$$V = \iiint\limits_{\text{ellipsoid}} 1\,dV = \iiint\limits_{\text{ellipsoid}} 1\,dz\,dy\,dx = \iiint\limits_{\substack{\text{unit}\\\text{sphere}}} 1\,abc\,dw\,dv\,du = \frac{4}{3}\pi abc$$

The moment of inertia about the z-axis is

$$M_z = \iiint\limits_{\text{ellipsoid}} \left(x^2 + y^2\right) dz\,dy\,dx$$

$$= \iiint\limits_{\substack{\text{unit}\\\text{sphere}}} \left(a^2u^2 + b^2v^2\right) abc\,dw\,dv\,du$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1}\left(a^2\rho^2\sin^2\phi\cos^2\theta + b^2\rho^2\sin^2\phi\sin^2\theta\right)\rho\sin^2\phi\,abc\,d\rho\,d\phi\,d\theta$$

$$= \frac{abc}{4}\int_0^{2\pi}\int_0^{\pi}\left(a^2\cos^2\theta\sin^4\phi + b^2\sin^2\theta\sin^4\phi\right) d\phi\,d\theta$$

$$= \frac{abc}{4}\int_0^{2\pi}\int_0^{\pi}\sin^4\phi\left(a^2\cos^2\theta + b^2\sin^2\theta\right) d\phi\,d\theta$$

$$= \frac{abc}{32}\int_0^{2\pi}\left(3a^2\pi\cos^2\theta + 3b^2\sin^2\theta\right) d\theta$$

$$= \frac{3abc\pi^2}{32}\left(a^2 + b^2\right)$$

23. Let $X = x(U,V)$ and $Y = y(U,V)$. If R is a region in the xy-plane with preimage S in the uv-plane, then

$P\big((X,Y) \in R\big) = P\big((U,V) \in S\big)$

Writing each of these as a double integral over the appropriate PDF and region gives

$$\iint_R f(x,y)\,dy\,dx = \iint_S g(u,v)\,dv\,du$$

Now, make the change of variable $x = x(u,v)$ and $y = y(u,v)$ in the integral on the left. Therefore,

$$\iint_S g(u,v)\,dv\,du = \iint_R f(x,y)\,dy\,dx = \iint_S f\left(x(u,v), y(u,v)\right)\left|J(u,v)\right|dv\,du$$

Thus probabilities involving (U,V) can be obtained by integrating $f\left(x(u,v), y(u,v)\right)\left|J(u,v)\right|$. Thus, $f\left(x(u,v), y(u,v)\right)\left|J(u,v)\right|$ is the joint PDF of (U,V).

24. Let $u = x + y,\ v = x - y$.

a. Solving for x and y gives $x = u/2 + v/2$ and $y = u/2 - v/2$. The Jacobian is

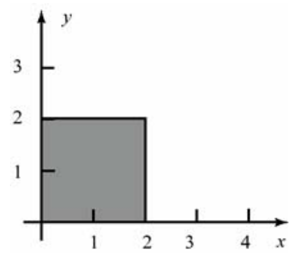
$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

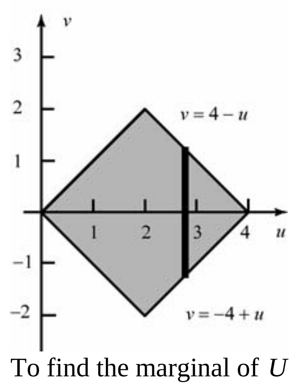
The joint PDF for (U,V) is therefore

$$g(u,v) = f\left(\frac{u}{2} + \frac{v}{2}, \frac{u}{2} - \frac{v}{2}\right)$$

$$= \begin{cases} \frac{1}{8}, & \text{if } 0 \le (u+v)/2 \le 2, 0 \le (u-v)/2 \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{8}, & \text{if } 0 \le u+v \le 4, 0 \le u-v \le 4 \\ 0, & \text{otherwise} \end{cases}$$





To find the marginal of U, we fix a u and integrate over all possible v. For $0 \le v \le 2$,

$g_U(u) = \int_{-u}^{u} \frac{1}{8}\,dv = \frac{u}{4}$ and for $2 < u \le 4$, $g_U(u) = \int_{-4+u}^{4-u} \frac{1}{8}\,dv = \frac{8-2u}{8} = 1 - \frac{u}{4}$.

Therefore,

$$g_U(u) = \begin{cases} u/4, & \text{if } 0 \le u \le 2 \\ 1 - u/4, & \text{if } 2 < u \le 4 \\ 0 & \text{otherwise} \end{cases}$$

25. **a.** Let $u = x + y,\ v = x$. Solving for x and y gives $x = v,\ y = u - v$. The Jacobian is

$$J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1.$$

Thus,

$$g(u,v) = f(v, u-v)\left|-1\right|$$

$$= \begin{cases} e^{-v-(u-v)}, & \text{if } 0 \le v, 0 \le u - v \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-u}, & \text{if } 0 \le v \le u \\ 0, & \text{otherwise} \end{cases}$$

b. The marginal PDF for U is obtained by integrating over all possible v for a fixed u. If $u \ge 0$, then

$$g_U(u) = \int_0^u e^{-u}\, dv = ue^{-u}$$

Thus, $g_U(u) = \begin{cases} ue^{-u}, & \text{if } 0 \le u \\ 0, & \text{otherwise} \end{cases}$

13.10 Chapter Review

Concepts Test

1. True: Use result of Problem 33, Section 13.2, and then change dummy variable *y* to dummy variable *x*.

2. False: Let *f*(*x*, *y*) = *x*. 1st integral is $\frac{1}{3}$; 2nd is $\frac{1}{6}$.

3. True: Inside integral is 0 since $\sin(x^3y^3)$ is an odd function in *x*.

4. True: Use Problem 33, Section 13.2. Each integrand, e^{x^2} and e^{2y^2}, determines and even function.

5. True: It is less than or equal to $\int_1^2 \int_0^2 1\, dx\, dy$ which equals 2.

6. True: $f(x, y) \ge \frac{f(x_0, y_0)}{2}$ in some neighborhood *N* of (x_0, y_0) due to the continuity. Then

$$\iint_R f(x, y)dA \ge \iint_N \left(\frac{1}{2}\right) f(x_0, y_0)dA$$

$$= \left(\frac{1}{2}\right) f(x_0, y_0)(\text{Area } N) > 0.$$

7. False: Let *f*(*x*, *y*) = *x*, $g(x, y) = x^2$, R = {(*x*, *y*): *x* in [0, 2], *y* in [0, 1]}. The inequality holds for the integrals but *f*(0.5, 0) > *g*(0.5, 0).

8. False: Let *f*(0, 0) = 1, *f*(*x*, *y*) = 0 elsewhere for $x^2 + y^2 \le 1$.

9. True: See the write-up of Problem 26, Section 13.7.

10. True: For each *x*, the density increases as *y* increases, so the top half of *R* is more dense than the bottom half. For each *y*, the density decreases as the *x* increases, so the right half of *R* is less dense than the left half.

11. True: The integral is the volume between concentric spheres of radii 4 and 1. That volume is 84π.

12. True: See Section 13.6.
A(*T*) = (Area of base)(sec 30°)

$$= \pi(1)^2\left(\frac{2}{\sqrt{3}}\right) = \frac{2\sqrt{3}\pi}{3}$$

13. False: There are 6.

14. False: The integrand should be *r*.

15. True: $\sqrt{f_x^2 + f_y^2 + 1} \le 9 = 3$

16. True: $J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

17. False: $J(u,v) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

Sample Test Problems

1. $\int_0^1 \left(\frac{1}{2}\right)(x^2 - x^3)dx = \frac{1}{24} \approx 0.0417$

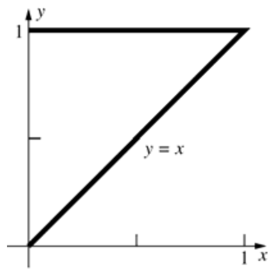
2. $\int_{-2}^{2} 0\, dy = 0$ (Integrand determines an odd function in *x*.)

3. $\int_0^{\pi/2} \left[\frac{r^2\cos\theta}{2}\right]_{r=0}^{2\sin\theta} d\theta = \int_0^{\pi/2} 2\sin^2\theta\cos\theta\, d\theta$

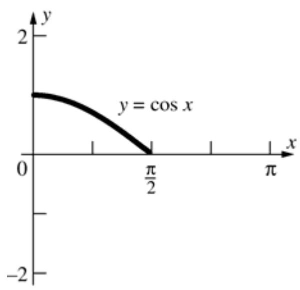
$$= \left[\frac{2\sin^3\theta}{3}\right]_0^{\pi/2} = \frac{2}{3}$$

4. $\int_1^2 \int_3^x \left(\frac{\pi}{3}\right) dy\,dx = \int_1^2 \left(\frac{\pi}{3}\right)(x-3)dx = -\frac{\pi}{2}$

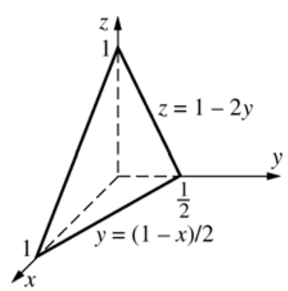
5. $\int_0^1 \int_0^y f(x, y)dx\,dy$



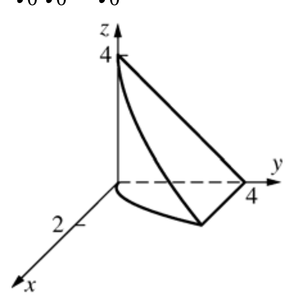
6. $\int_0^{\pi/2} \int_0^{\cos x} f(x, y)dy\,dx$



7. $\int_0^{1/2} \int_0^{1-2y} \int_0^{1-2y-z} f(x, y, z)dx\,dz\,dy$



8. $\int_0^4 \int_0^{4-z} \int_0^{\sqrt{y}} f(x, y, z)dx\,dy\,dz$

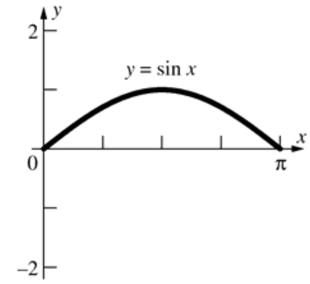


9. a. $8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$

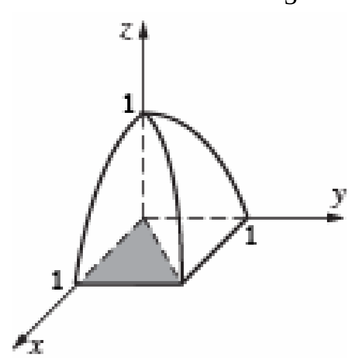
b. $8\int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

c. $8\int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

10. $\int_0^{\pi} \int_0^{\sin x} (x+y)dy\,dx = \frac{5\pi}{4} \approx 3.9270$



11. $8\int_0^1 \int_0^x \int_0^{1-x^2} z^2 dz\,dy\,dx = \frac{1}{3}$



12. $\int_0^{2\pi} \int_2^3 (r^{-2})r\,dr\,d\theta = \int_0^{2\pi} [\ln r]_2^3 d\theta$

$= \int_0^{2\pi} \ln\left(\frac{3}{2}\right) d\theta = 2\pi \ln\left(\frac{3}{2}\right) \approx 2.5476$

13. $m = \int_0^2 \int_1^3 xy^2 dx\,dy = \frac{32}{3}$

$M_x = \int_0^2 \int_1^3 xy^3 dx\,dy = 16$

$M_y = \int_0^2 \int_1^3 x^2 y^2 dx\,dy = \frac{208}{9}$

$(\overline{x}, \overline{y}) = \left(\frac{13}{6}, \frac{3}{2}\right)$

14. $I_x = \int_0^2 \int_1^3 xy^4 dx\,dy = \frac{128}{5} = 25.6$

15. $z = f(x, y) = (9-y^2)^{1/2};\ f_x(x, y) = 0;$

$f_y(x, y) = -y(9-y^2)^{-1/2}$

Area $= \int_0^3 \int_{y/3}^y \sqrt{y^2(9-y^2)^{-1}+1}\,dx\,dy$

$= \int_0^3 \int_{y/3}^y 3(9-y^2)^{-1/2} dx\,dy$

$= \int_0^3 (9-y^2)^{-1/2}(2y)dy = [-2(9-y^2)^{1/2}]_0^3 = 6$

16. a. $\int_0^{\pi/2}\int_0^2\int_0^3 r\,r\,dr\,dz\,d\theta = 9\pi \approx 28.2743$

b. $\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} z(4-r^2)^{1/2}\,r\,dz\,dr\,d\theta$

$= \left(\frac{8}{5}\right)\pi \approx 5.0265$

17. $\delta(x, y, z) = k\rho$

$m = \int_0^{\pi}\int_0^{2\pi}\int_1^3 k\rho\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = 80\pi k$

18. $m = \iint_R 1\,dA = \int_0^{2\pi}\int_0^{4(1+\sin\theta)} r\,dr\,d\theta$

$= 8\int_0^{2\pi}\left(1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right)d\theta = 24\pi$

$M_x = \iint_R y\,dA = \int_0^{2\pi}\int_0^{4(1+\sin\theta)} (r\sin\theta)r\,dr\,d\theta$

$= 80\pi$

$\bar{y} = \frac{80\pi}{24\pi} = \frac{10}{3};\ \bar{x} = 0$ (by symmetry)

19. $m = \int_0^a\int_0^{(b/a)(a-x)}\int_0^{(c/ab)(ab-bx-ay)} kx\,dz\,dy\,dx$

$= \left(\frac{k}{24}\right)a^2bc$

20. $\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} r\,dz\,dr\,d\theta = \frac{3\pi}{2} \approx 4.7124$

21. Let $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$. Then we have,

$\sin(x-y)\cos(x+y) = \sin u\cos v$ and

$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$.

Thus,

$\iint_R \sin(x-y)\cos(x+y)\,dxdy$

$= \int_0^{\pi}\int_0^{\pi}\frac{1}{2}\sin u\cos v\,du\,dv$

$= \frac{1}{2}\int_0^{\pi}\cos v\left[-\cos u\right]_0^{\pi}dv$

$= \int_0^{\pi}\cos v\,dv = 0$

Review and Preview Problems

1. Answers may vary. One solution is
$x = 3\cos t,\quad y = 3\sin t,\qquad 0 \le t < 2\pi$
Then: $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$ as required.

2. Answers may vary. One solution is
$x = \cos t + 2,\quad y = \sin t + 1,\qquad 0 \le t < 2\pi$
Then: $(x-2)^2 + (y-1)^2 = \cos^2 t + \sin^2 t = 1$ as required.

3. Answers may vary. One solution for the circle is
$x = 2\cos t,\quad y = 2\sin t$
To have the semicircle where $y > 0$, we need $\sin t > 0$, so we restrict the domain of t to $0 < t < \pi$.

4. Answers may vary. Consider
$x = a\cos(-t),\quad y = a\sin(-t),\qquad 0 \le t \le \pi$**;** this is a semicircle.

a. Since $\sin(-t) = -\sin t$, and since $\sin t \ge 0$ on $[0,\pi]$, $y \le 0$ on $[0,\pi]$.

b. As t goes from 0 to π, $-t$ goes from 0 to $-\pi$ so the orientation is clockwise.

5. Answers may vary. One solution is
$x = -2 + 5t,\ y = 2$ for $t \in [0,1]$.

6. Note that $x + y = 9$, so a simple parameterization is to let one variable be t and the other be $9 - t$. Since we are restricted to the first quadrant, we must have $t > 0$ and $9 - t > 0$; hence the domain is $t \in (0,9)$. Finally, since orientation is to be down and to the right, we want y to decrease and x to increase as t increases. Thus we use $x = t$ and $y = 9 - t$.

7. Note that $x + y = 9$, so a simple parameterization is to let one variable be t and the other be $9 - t$. Since we are restricted to the first quadrant, we must have $t > 0$ and $9 - t > 0$; hence the domain is $t \in (0,9)$. Finally, since orientation is to be up and to the left, we want y to increase and x to decrease as t increases. Thus we use $x = 9 - t$ and $y = t$.

8. Since we are restricting the parabola to the points where $y > 0$, a simple parameterization is
$x = t,\quad y = 9 - t^2,\qquad t \in [-3,3]$
Note that the orientation is left to right.

9. Since we are restricting the parabola to the points where $y > 0$, a simple parameterization is

$x = -t, \quad y = 9 - t^2, \qquad t \in [-3, 3]$

Note that the orientation is right to left.

10. $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. Using the parameterization in problem 6,

$x = t,\ y = 9 - t,\ a = 0,\ b = 9,\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = -1$ and

so $L = \int_0^9 \sqrt{1^2 + 1^2}\, dt = \left[\sqrt{2}\, t\right]_0^9 = 9\sqrt{2}$.

(Note: this can be verified by finding the distance between the points $(0,9)$ and $(9,0)$).

11. $\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$. Now if

$f(x,y) = x \sin x + y \cos y$, then

$f_x(x,y) = x\cos x + \sin x, \quad f_y(x,y) = \cos y - y \sin y$

Thus:

$\nabla f(x,y) = (x\cos x + \sin x)\,\mathbf{i} + (\cos y - y\sin y)\,\mathbf{j}$

12. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$. Now if

$f(x,y) = xe^{-xy} + ye^{xy}$, then

$$\begin{aligned} f_x(x,y) &= (-yxe^{-xy} + e^{-xy}) + (y^2 e^{xy}) \\ &= (1 - xy)e^{-xy} + y^2 e^{xy} \end{aligned}$$

and

$$\begin{aligned} f_y(x,y) &= -(x^2 e^{-xy}) + (xye^{xy} + e^{xy}) \\ &= -x^2 e^{-xy} + (1 + xy)e^{xy} \end{aligned}$$

Thus,

$$\begin{aligned} \nabla f(x,y) &= [(1 - xy)e^{-xy} + y^2 e^{xy}]\,\mathbf{i} \\ &\quad + [-x^2 e^{-xy} + (1 + xy)e^{xy}]\,\mathbf{j} \end{aligned}$$

13. $\nabla f(x,y,z) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. Now if

$f(x,y,z) = x^2 + y^2 + z^2$, then

$f_x(x,y,z) = 2x,\ f_y(x,y,z) = 2y,\ f_z(x,y,z) = 2z$

so

$\nabla f(x,y,z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$

14. $\nabla f(x,y,z) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. Now if

$f(x,y,z) = \frac{1}{x^2 + y^2 + z^2}$, then

$f_x = \frac{-2x}{(x^2 + y^2 + z^2)^2}, \quad f_y = \frac{-2y}{(x^2 + y^2 + z^2)^2},$

and $f_z = \frac{-2z}{(x^2 + y^2 + z^2)^2}$,

so

$$\nabla f(x,y,z) = \frac{-2x}{(x^2 + y^2 + z^2)^2}\,\mathbf{i} + \frac{-2y}{(x^2 + y^2 + z^2)^2}\,\mathbf{j} + \frac{-2z}{(x^2 + y^2 + z^2)^2}\,\mathbf{k}$$

15. $\nabla f(x,y,z) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. Now if

$f(x,y,z) = xy + xz + yz$, then $f_x(x,y,z) = y + z$,

$f_y(x,y,z) = x + z$, and $f_z(x,y,z) = x + y$ so

$\nabla f(x,y,z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

16. $\nabla f(x,y,z) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}$. Now if

$f(x,y,z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, then

$f_x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}},$

$f_y = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}},$ and

$f_z = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}},$

so

$$\nabla f(x,y,z) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

17.
$$\begin{aligned} \int_0^\pi \sin^2 t\, dt &= \int_0^\pi \frac{1 - \cos 2t}{2}\, dt \\ &= \frac{1}{2}\int_0^\pi (1 - \cos 2t)\, dt \\ &= \frac{1}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^\pi \\ &= \frac{1}{2}\left[\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(2\cdot 0)\right)\right] \\ &= \frac{\pi}{2} \end{aligned}$$

18.
$$\begin{aligned} \int_0^\pi \sin t \cos t\, dt &= \int_0^\pi \tfrac{1}{2}\sin(2t)\, dt \\ &= \left[-\tfrac{1}{4}\cos(2t)\right]_0^\pi \\ &= -\frac{1}{4}\left(\cos(2\pi) - \cos(0)\right) \\ &= -\frac{1}{4}(1 - 1) = 0 \end{aligned}$$

19. $\int_0^1 \int_1^2 xy\ dy\,dx = \int_0^1 \left[\int_1^2 xy\,dy \right] dx = \int_0^1 \left[\frac{xy^2}{2} \right]_{y=1}^{y=2} dx$

$= \int_0^1 \left(\frac{3x}{2} \right) dx = \left[\frac{3x^2}{4} \right]_0^1 = \frac{3}{4}$

20. $\int_{-1}^1 \int_1^4 (x^2+2y)\ dy\,dx = \int_{-1}^1 \left[\int_1^4 (x^2+2y)\,dy \right] dx =$

$\int_{-1}^1 \left[x^2 y + y^2 \right]_{y=1}^{y=4} dx = \int_{-1}^1 \left(3x^2 + 15 \right) dx =$

$\left[x^3 + 15x \right]_{-1}^1 = 16 + 16 = 32$

21. $\int_0^{2\pi} \int_1^2 r^2\,dr\,d\theta = \int_0^{2\pi} \left[\int_1^2 r^2\,dr \right] d\theta = \int_0^{2\pi} \left[\frac{r^3}{3} \right]_{r=1}^{r=2} d\theta =$

$\int_0^{2\pi} \frac{7}{3}\,d\theta = \frac{7}{3}\theta \Big|_0^{2\pi} = \frac{14\pi}{3} \approx 14.66$

22. $\int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta =$

$\int_0^{2\pi} \int_0^{\pi} \left[\int_1^2 \rho^2 \sin\phi\,d\rho \right] d\phi\,d\theta =$

$\int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3} \sin\phi \right]_{\rho=1}^{\rho=2} d\phi\,d\theta =$

$\int_0^{2\pi} \left[\int_0^{\pi} \frac{7}{3} \sin\phi\,d\phi \right] d\theta = \int_0^{2\pi} \left[-\frac{7}{3}\cos\phi \right]_{\phi=0}^{\phi=\pi} d\theta =$

$\int_0^{2\pi} \frac{14}{3}\,d\theta = \frac{28\pi}{3} \approx 29.32$

23. Note that

$\int_0^{2\pi} \int_0^{\pi} \int_r^R \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta =$

$\int_0^{2\pi} \int_0^{\pi} \left[\int_r^R \rho^2 \sin\phi\,d\rho \right] d\phi\,d\theta =$

$\int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3} \sin\phi \right]_{\rho=r}^{\rho=R} d\phi\,d\theta =$

$\int_0^{2\pi} \left[\int_0^{\pi} \frac{1}{3}(R^3 - r^3) \sin\phi\,d\phi \right] d\theta =$

$\int_0^{2\pi} \left[-\frac{1}{3}(R^3 - r^3)\cos\phi \right]_{\phi=0}^{\phi=\pi} d\theta =$

$\int_0^{2\pi} \left[\frac{2}{3}(R^3 - r^3) \right] d\theta = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$

which we recognize as the difference between the volume of a sphere with radius R and the volume of a sphere with radius r. Thus the volume in problem 22 is that of a spherical shell with center at (0,0,0) and, in this case, outer radius = 2 and inner radius = 1 .

24. Let $f(x,y) = z = 144 - x^2 - y^2$; then:

$f_x(x,y) = -2x \qquad f_y(x,y) = -2y$

We note that $z = 36$ when $x^2 + y^2 = 108$; thus the surface area we seek projects onto the circular region inside $S = \left\{ (x,y) \mid x^2 + y^2 = 108 \right\}$

Hence $A = \iint_S \sqrt{4x^2 + 4y^2 + 1}\ dA$; or, converting to polar coordinates $(r^2 = x^2 + y^2$ and $r = 6\sqrt{3}$ when $r^2 = 108)$,

$A = \int_0^{2\pi} \int_0^{6\sqrt{3}} (\sqrt{4r^2 + 1})\ r\,dr\,d\theta =$

$\int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2} \right]_{r=0}^{r=6\sqrt{3}} d\theta =$

$\frac{1}{12} \int_0^{2\pi} [433^{3/2} - 1]\,d\theta \approx 1501.7\pi \approx 4717.7$

25. This will be the unit vector, at the point $(3,4,12)$, in the direction of ∇F, where $F(x,y,z) = x^2 + y^2 + z^2$.

Now $\nabla F(x,y,z) = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ so that $\nabla F(3,4,12) = \langle 6,8,24\rangle$ and the unit vector in the same direction is

$$\frac{\nabla F}{\|\nabla F\|} = \left\langle \frac{6}{26}, \frac{8}{26}, \frac{24}{26} \right\rangle = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle.$$

This agrees with our geometric intuition, since $x^2 + y^2 + z^2 = 169$ is the surface of a sphere with center at $O = (0,0,0)$ and radius = 13. Now the plane tangent to a sphere (center at (0,0,0) and radius r) at any point $P = (a,b,c)$ is perpendicular to the radius at that point; so it would follow that the vector $\overrightarrow{OP} = \langle a,b,c\rangle$ is perpendicular to the tangent plane and hence normal to the surface. The unit normal in the direction of $\overrightarrow{OP}$ is simply $\frac{1}{r}\langle a,b,c\rangle$, or in our case $\frac{1}{13}\langle 3,4,12\rangle$.