CHAPTER 15 Differential Equations

15.1 Concepts Review

1. $r^2 + a_1 r + a_2 = 0$; complex conjugate roots

2. $C_1 e^{-x} + C_2 e^x$

3. $(C_1 + C_2 x)e^x$

4. $C_1 \cos x + C_2 \sin x$

Problem Set 15.1

1. Roots are 2 and 3. General solution is $y = C_1 e^{2x} + C_2 e^{3x}$.

2. Roots are –6 and 1. General solution is $y = C_1 e^{-6x} + C_2 e^x$.

3. Auxiliary equation: $r^2 + 6r - 7 = 0$, $(r + 7)(r - 1) = 0$ has roots –7, 1.
General solution: $y = C_1 e^{-7x} + C_2 e^x$
$y' = -7C_1 e^{-7x} + C_2 e^x$
If $x = 0$, $y = 0$, $y' = 4$, then $0 = C_1 + C_2$ and $4 = -7C_1 + C_2$, so $C_1 = -\frac{1}{2}$ and $C_2 = \frac{1}{2}$.
Therefore, $y = \frac{e^x - e^{-7x}}{2}$.

4. Roots are –2 and 5. General solution is $y = C_1 e^{-2x} + C_2 e^{5x}$. Particular solution is $y = \left(\frac{12}{7}\right)e^{5x} - \left(\frac{5}{7}\right)e^{-2x}$.

5. Repeated root 2. General solution is $y = (C_1 + C_2 x)e^{2x}$.

6. Auxiliary equation: $r^2 + 10r + 25 = 0, (r + 5)^2 = 0$ has one repeated root -5.
General solution: $y = C_1 e^{-5x} + C_2 x e^{-5x}$ or $y = (C_1 + C_2 x)e^{-5x}$

7. Roots are $2 \pm \sqrt{3}$. General solution is $y = e^{2x}(C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x})$.

8. Roots are $-3 \pm \sqrt{11}$. General solution is $y = e^{-3x}\left(C_1 e^{\sqrt{11}x} + C_2 e^{-\sqrt{11}x}\right)$.

9. Auxiliary equation: $r^2 + 4 = 0$ has roots $\pm 2i$.
General solution: $y = C_1 \cos 2x + C_2 \sin 2x$
If $x = 0$ and $y = 2$, then $2 = C_1$; if $x = \frac{\pi}{4}$ and $y = 3$, then $3 = C_2$.
Therefore, $y = 2\cos 2x + 3\sin 2x$.

10. Roots are $\pm 3i$. General solution is $y = (C_1 \cos 3x + C_2 \sin 3x)$. Particular solution is $y = -\sin 3x - 3\cos 3x$.

11. Roots are $-1 \pm i$. General solution is $y = e^{-x}(C_1 \cos x + C_2 \sin x)$.

12. Auxiliary equation: $r^2 + r + 1 = 0$ has roots $\frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$.
General solution:
$$y = C_1 e^{(-1/2)x} \cos\left(\frac{\sqrt{3}}{2}\right)x + C_2 e^{(-1/2)x} \sin\left(\frac{\sqrt{3}}{2}\right)x$$
$$y = e^{-x/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}\right)x + C_2 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$$

13. Roots are 0, 0, –4, 1.
General solution is $y = C_1 + C_2 x + C_3 e^{-4x} + C_4 e^x$.

14. Roots are –1, 1, $\pm i$. General solution is $y = C_1 e^{-x} + C_2 e^x + C_3 \cos x + C_4 \sin x$.

15. Auxiliary equation: $r^4 + 3r^2 - 4 = 0$, $(r + 1)(r - 1)(r^2 + 4) = 0$ has roots –1, 1, $\pm 2i$.
General solution:
$y = C_1 e^{-x} + C_2 e^x + C_3 \cos 2x + C_4 \sin 2x$

16. Roots are –2, 3, $\pm i$. General solution is $y = C_1e^{-2x} + C_2e^{3x} + C_3\cos x + C_4\sin x$.

17. Roots are –2, 2. General solution is $y = C_1e^{-2x} + C_2e^{2x}$.

$y = C_1(\cosh 2x - \sinh 2x) + C_2(\sinh 2x + \cosh 2x) = (-C_1 + C_2)\sinh 2x + (C_1 + C_2)\cosh 2x$

$= D_1\sinh 2x + D_2\cosh 2x$

18. $e^u = \cosh u + \sinh u$ and $e^{-u} = \cosh u - \sinh u$.

Auxiliary equation: $r^2 - 2br - c^2 = 0$

Roots of auxiliary equation: $\dfrac{2b \pm \sqrt{4b^2 + 4c^2}}{2} = b \pm \sqrt{b^2 + c^2}$

General solution: $y = C_1e^{(b+\sqrt{b^2+c^2})x} + C_2e^{(b-\sqrt{b^2+c^2})x}$

$$= e^{bx}\left[C_1\left(\cosh\left(\sqrt{b^2+c^2}\,x\right) + \sinh\left(\sqrt{b^2+c^2}\,x\right)\right) + C_2\left(\cosh\left(\sqrt{b^2+c^2}\,x\right) - \sinh\left(\sqrt{b^2+c^2}\,x\right)\right)\right]$$

$$= e^{bx}\left[(C_1 + C_2)\cosh\left(\sqrt{b^2+c^2}\,x\right) + (C_1 + C_2)\sin\left(\sqrt{b^2+c^2}\,x\right)\right] = e^{bx}\left[D_1\cosh\left(\sqrt{b^2+c^2}\,x\right) + D_2\sinh\left(\sqrt{b^2+c^2}\,x\right)\right]$$

19. Repeated roots $\left(-\dfrac{1}{2}\right) \pm \left(\dfrac{\sqrt{3}}{2}\right)i$.

General solution is $y = e^{-x/2}\left[(C_1 + C_2x)\cos\left(\dfrac{\sqrt{3}}{2}\right)x + (C_3 + C_4x)\sin\left(\dfrac{\sqrt{3}}{2}\right)x\right]$.

20. Roots $1 \pm i$. General solution is

$y = e^x(C_1\cos x + C_2\sin x)$

$= e^x(c\sin\gamma\cos x + c\cos\gamma\sin x) = ce^x\sin(x + \gamma)$.

21. $(*) x^2y'' + 5xy' + 4y = 0$

Let $x = e^z$. Then $z = \ln x$;

$y' = \dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx} = \dfrac{dy}{dz}\dfrac{1}{x}$;

$y'' = \dfrac{dy'}{dx} = \dfrac{d}{dx}\left(\dfrac{dy}{dz}\dfrac{1}{x}\right) = \dfrac{dy}{dz}\dfrac{-1}{x^2} + \dfrac{1}{x}\dfrac{d^2y}{dz^2}\dfrac{dz}{dx}$

$= \dfrac{dy}{dz}\dfrac{-1}{x^2} + \dfrac{1}{x}\dfrac{d^2y}{dz^2}\dfrac{1}{x}$

$\left(-\dfrac{dy}{dz} + \dfrac{d^2y}{dz^2}\right) + \left(5\dfrac{dy}{dz}\right) + 4y = 0$

(Substituting y' and y'' into (*))

$\dfrac{d^2y}{dz^2} + 4\dfrac{dy}{dz} + 4y = 0$

Auxiliary equation: $r^2 + 4r + 4 = 0$, $(r+2)^2 = 0$

has roots –2, –2.

General solution: $y = (C_1 + C_2z)e^{-2z}$,

$y = (C_1 + C_2\ln x)e^{-2\ln x}$

$y = (C_1 + C_2\ln x)x^{-2}$

22. As done in Problem 21,

$\left[-a\left(\dfrac{dy}{dz}\right) + a\left(\dfrac{d^2y}{dx^2}\right)\right] + b\left(\dfrac{dy}{dz}\right) + cy = 0.$

Therefore, $a\left(\dfrac{d^2y}{dz^2}\right) + (b-a)\left(\dfrac{dy}{dz}\right) + cy = 0.$

23. We need to show that $y'' + a_1y' + a_2y = 0$ if r_1 and r_2 are distinct real roots of the auxiliary equation.

We have,

$y' = C_1r_1e^{r_1x} + C_2r_2e^{r_2x}$

$y'' = C_1r_1^2e^{r_1x} + C_2r_2^2e^{r_2x}$.

When put into the differential equation, we obtain

$$y'' + a_1y' + a_2y = C_1r_1^2e^{r_1x} + C_2r_2^2e^{r_2x} + a_1\left(C_1r_1e^{r_1x} + C_2r_2e^{r_2x}\right) + a_2\left(C_1e^{r_1x} + C_2e^{r_2x}\right) \quad (*)$$

The solutions to the auxiliary equation are given by

$r_1 = \dfrac{1}{2}\left(-a_1 - \sqrt{a_1^2 - 4a_2}\right)$ and

$r_2 = \dfrac{1}{2}\left(-a_1 + \sqrt{a_1^2 - 4a_2}\right)$.

Putting these values into (*) and simplifying yields the desired result: $y'' + a_1y' + a_2y = 0$.

24. We need to show that $y''+a_1y'+a_2y=0$ if $\alpha \pm \beta i$ are complex conjugate roots of the auxiliary equation. We have,

$$y'=e^{\alpha x}\left(\left(\alpha C_1+\beta C_2\right)\cos\left(\beta x\right)+\left(\alpha C_2-\beta C_1\right)\sin\left(\beta x\right)\right)$$

$$y''=e^{\alpha x}\left(\left(\alpha^2C_1-\beta^2C_1+2\alpha\beta C_2\right)\cos\left(\beta x\right)+\left(\alpha^2C_2-\beta^2C_2-2\alpha\beta C_1\right)\sin\left(\beta x\right)\right).$$

When put into the differential equation, we obtain

$$y''+a_1y'+a_2y=e^{\alpha x}\left(\left(\alpha^2C_1-\beta^2C_1+2\alpha\beta C_2\right)\cos\left(\beta x\right)+\left(\alpha^2C_2-\beta^2C_2-2\alpha\beta C_1\right)\sin\left(\beta x\right)\right)$$

$$+a_1e^{\alpha x}\left(\left(\alpha C_1+\beta C_2\right)\cos\left(\beta x\right)+\left(\alpha C_2-\beta C_1\right)\sin\left(\beta x\right)\right)+a_2\left(C_1e^{\alpha x}\cos\left(\beta x\right)+C_2e^{\alpha x}\sin\left(\beta x\right)\right)\quad (*)$$

From the solutions to the auxiliary equation, we find that

$\alpha=\frac{-a_1}{2}$ and $\beta=-\frac{1}{2}i\sqrt{a_1{}^2-4a_2}$.

Putting these values into (*) and simplifying yields the desired result: $y''+a_1y'+a_2y=0$.

25. a. $$e^{bi}=1+(bi)+\frac{(bi)^2}{2!}+\frac{(bi)^3}{3!}+\frac{(bi)^4}{4!}+\frac{(bi)^5}{5!}+\ldots=\left(1-\frac{b^2}{2!}+\frac{b^4}{4!}-\frac{b^6}{6!}+\right)\ldots+i\left(b-\frac{b^3}{3!}+\frac{b^5}{5!}-\frac{b^7}{7!}+\ldots\right)$$

$= \cos(b) + i\sin(b)$

b. $e^{a+bi}=e^ae^{bi}=e^a[\cos(b)+i\sin(b)]$

c. $$D_x\left[e^{(\alpha+\beta i)x}\right]=D_x[e^{\alpha x}(\cos\beta x+i\sin\beta x)]=\alpha e^{\alpha x}(\cos\beta x+i\sin\beta x)+e^{\alpha x}(-i\beta\sin\beta x+i\beta\cos\beta x)$$

$$=e^{\alpha x}[(\alpha+\beta i)\cos\beta x+(\alpha i-\beta)\sin\beta x]$$

$$(\alpha+\beta)e^{(\alpha+\beta i)x}=(\alpha+\beta i)[e^{\alpha x}(\cos\beta x+i\sin\beta x)]=e^{\alpha x}[(\alpha+\beta i)\cos\beta x+(\alpha i-\beta)\sin\beta x]$$

Therefore, $D_x[e^{(\alpha+\beta i)x}]=(\alpha+i\beta)e^{(\alpha+\beta i)x}$

26. $c_1e^{(\alpha+\beta i)x}+c_2e^{(\alpha+\beta i)x}$ [c_1 and c_2 are complex constants.]

$$=c_1e^{\alpha x}[\cos\beta x+i\sin\beta x]+c_2e^{\alpha x}[\cos(-\beta x)+i\sin(-\beta x)]=e^{\alpha x}[(c_1+c_2)\cos\beta x+(c_1-c_2)i\sin\beta x]$$

$=e^{\alpha x}[C_1\cos\beta x+C_2\sin\beta x]$, where $C_1=c_1+c_2$, and $C_2=c_1-c_2$.

Note: If c_1 and c_2 are complex conjugates, then C_1 and C_2 are real.

27. $y=0.5e^{5.16228x}+0.5e^{-1.162278x}$

28. $y=3.5xe^{-2.5x}+2e^{-2.5x}$

29. $y=1.29099e^{-0.25x}\sin(0.968246x)$

30. $y=e^{0.333333x}[2.5\cos(0.471405x)-4.94975\sin(0.471405x)]$

15.2 Concepts Review

1. particular solution to the nonhomogeneous equation; homogeneous equation

2. $-6+C_1e^{-2x}+C_2e^{3x}$

3. $y=Ax^2+Bx+C$

4. $y=Bxe^{\frac{1}{3}x}$

Problem Set 15.2

1. $y_h=C_1e^{-3x}+C_2e^{3x}$

$y_p=\left(-\frac{1}{9}\right)x+0$

$y=\left(-\frac{1}{9}\right)x+C_1e^{-3x}+C_2e^{3x}$

2. $y_h=C_1e^{-3x}+C_2e^{2x}$

$y_p=\left(-\frac{1}{3}\right)x^2+\left(-\frac{1}{9}\right)x+\left(-\frac{7}{54}\right)$

$y=\left(-\frac{1}{3}\right)x^2-\left(\frac{1}{9}\right)x-\left(\frac{7}{54}\right)+C_1e^{-3x}+C_2e^{2x}$

3. Auxiliary equation: $r^2-2r+1=0$ has roots 1, 1.

$y_h=(C_1+C_2x)e^x$

Let $y_p=Ax^2+Bx+C;\ y_p'=2Ax+B;$

$y_p''=2A$.

Then $(2A)-2(2Ax+B)+(Ax^2+Bx+C)=x^2+x.$

$Ax^2+(-4A+B)x+(2A-2B+C)=x^2+x$

Thus, $A = 1$, $-4A + B = 1$, $2A - 2B + C = 0$, so
$A = 1$, $B = 5$, $C = 8$.

General solution: $y=x^2+5x+8+(C_1+C_2x)e^x$

4. $y_h=C_1e^{-x}+C_2\cdot y_p=2x^2+(-4)x$

$y=2x^2-4x+C_1e^{-x}+C_2$

5. $y_h=C_1e^{2x}+C_2e^{3x}\cdot y_p=\left(\frac{1}{2}\right)e^x\cdot y$

$=\left(\frac{1}{2}\right)e^x+C_1e^{2x}+C_2e^{3x}$

6. Auxiliary equation: $r^2 + 6r + 9 = 0, (r+3)^2 = 0$ has roots –3, –3.

$y_h = (C_1 + C_2 x)e^{-3x}$

Let $y_p = Be^{-x}; y_p' = -Be^{-x}; y_p'' = Be^{-x}$.

Then $(Be^{-x}) + 6(-Be^{-x}) + 9(Be^{-x}) = 2e^{-x};\ \ 4Be^{-x} = 2e^{-x}; B = \frac{1}{2}$

General solution: $y = \left(\frac{1}{2}\right)e^{-x} + (C_1 + C_2 x)e^{-3x}$

7. $y_h = C_1 e^{-3x} + C_2 e^{-x}$

$y_p = \left(-\frac{1}{2}\right)xe^{-3x}$

$y = \left(-\frac{1}{2}\right)xe^{-3x} + C_1 e^{-3x} + C_2 e^{-x}$

8. $y_h = e^{-x}(C_1 \cos x + C_2 \sin x)$

$y_p = \left(\frac{3}{2}\right)e^{-2x}$

$y = \left(\frac{3}{2}\right)e^{-2x} + e^{-x}(C_1 \cos x + C_2 \sin x)$

9. Auxiliary equation: $r^2 - r - 2 = 0$,
$(r + 1)(r - 2) = 0$ has roots –1, 2.

$y_h = C_1 e^{-x} + C_2 e^{2x}$

Let $y_p = B\cos x + C\sin x;\ \ y_p' = -B\sin x + C\cos x;\ \ y_p'' = -B\cos x - C\sin x.$

Then $(-B\cos x - C\sin x) - (-B\sin x + C\cos x)$
$-2(B\cos x + C\sin x) = 2\sin x.$

$(-3B - C)\cos x + (B - 3C)\sin x = 2\sin x$, so $-3B - C = 0$ so $-3B - C = 0$ and $B - 3C = 2$; $B = \frac{1}{5}; C = \frac{-3}{5}.$

General solution: $\left(\frac{1}{5}\right)\cos x - \left(\frac{3}{5}\right)\sin x + C_1 e^{2x} + C_2 e^{-x}$

10. $y_h = C_1 e^{-4x} + C_2$

$y_p = \left(-\frac{1}{17}\right)\cos x + \left(\frac{4}{17}\right)\sin x$

$y = \left(-\frac{1}{17}\right)\cos x + \left(\frac{4}{17}\right)\sin x + C_1 e^{-4x} + C_2$

11. $y_h = C_1 \cos 2x + C_2 \sin 2x$

$y_p = (0)x\cos 2x + \left(\frac{1}{2}\right)x\sin 2x$

$y = \left(\frac{1}{2}\right)x\sin 2x + C_1 \cos 2x + C_2 \sin 2x$

12. Auxiliary equation: $r^2 + 9 = 0$ has roots $\pm 3i$, so $y_h = C_1 \cos 3x + C_2 \sin 3x$.

Let $y_p = Bx\cos 3x + Cx\sin 3x$; $y_p' = (-3bx + C)\sin 3x + (B + 3Cx)\cos 3x$;

$y_p'' = (-9Bx + 6C)\cos 3x + (-9Cx - 6B)\sin 3x$.

Then substituting into the original equation and simplifying, obtain $6C\cos 3x - 6B\sin 3x = \sin 3x$, so $C = 0$ and $B = -\frac{1}{6}$.

General solution: $y = \left(-\frac{1}{6}\right)x\cos 3x + C_1 \cos 3x + C_2 \sin 3x$

13. $y_h = C_1 \cos 3x + C_2 \sin 3x$

$y_p = (0)\cos x + \left(\frac{1}{8}\right)\sin x + \left(\frac{1}{13}\right)e^{2x}$

$y = \left(\frac{1}{8}\right)\sin x + \left(\frac{1}{13}\right)e^{2x} + C_1 \cos 3x + C_2 \sin 3x$

14. $y_h = C_1 e^{-x} + C_2$

$y_p = \left(\frac{1}{2}\right)e^x + \left(\frac{3}{2}\right)x^2 + (-3)x$

$y = \left(\frac{1}{2}\right)e^x + \left(\frac{3}{2}\right)x^2 - 3x + C_1 e^{-x} + C_2$

15. Auxiliary equation: $r^2 - 5r + 6 = 0$ has roots 2 and 3, so $y_h = C_1 e^{2x} + C_2 e^{3x}$.

Let $y_p = Be^x$; $y_p' = Be^x$; $y_p'' = Be^x$.

Then $(Be^x) - 5(Be^x) + 6(Be^x) = 2e^x$; $2Be^x = 2e^x$; $B = 1$.

General solution: $y = e^x + C_1 e^{2x} + C_2 e^{3x}$

$y' = e^x + 2C_1 e^{2x} + 3C_2 e^{3x}$

If $x = 0$, $y = 1$, $y' = 0$, then $1 = 1 + C_1 + C_2$ and $0 = 1 + 2C_1 + 3C_2$; $C_1 = 1, C_2 = -1$.

Therefore, $y = e^x + e^{2x} - e^{3x}$.

16. $y_h = C_1 e^{-2x} + C_2 e^{2x}$

$y_p = (0)\cos x + \left(-\frac{4}{5}\right)\sin x$

$y = \left(-\frac{4}{5}\right)\sin x + C_1 e^{-2x} + C_2 e^{2x}$

$y = \left(-\frac{4}{5}\right)\sin x + \left(\frac{9}{5}\right)e^{-2x} + \left(\frac{11}{5}\right)e^{2x}$ satisfies the conditions.

17. $y_h = C_1 e^x + C_2 e^{2x}$

$y_p = \left(\frac{1}{4}\right)(10x + 19)$

$y = \left(\frac{1}{4}\right)(10x + 19) + C_1 e^x + C_2 e^{2x}$

18. Auxiliary equation: $r^2 - 4 = 0$ has roots 2, –2, so $y_h = C_1e^{2x} + C_2e^{-2x}$.

Let $y_p = v_1e^{2x} + v_2e^{-2x}$, subject to $v_1'e^{2x} + v_2'e^{-2x} = 0$, and $v_1'(2e^{2x}) + v_2'(-2e^{-2x}) = e^{2x}$.

Then $v_1'(4e^{2x}) = e^{2x}$ and $v_2'(-4e^{-2x}) = e^{2x}$; $v_1' = \frac{1}{4}$ and $v_2' = -e^{4x/4}$; $v_1 = \frac{x}{4}$ and $v_2 = -\frac{e^{4x}}{16}$.

General solution: $y = \frac{xe^{2x}}{4} - \frac{e^{2x}}{16} + C_1e^{2x} + C_2e^{-2x}$

19. $y_h = C_1 \cos x + C_2 \sin x$

$y_p = -\cos \ln|\sin x| - \cos x - x \sin x$

$y = -\cos x \ln|\sin x| - x \sin x + C_3 \cos x + C_2 \sin x$ (combined $\cos x$ terms)

20. $y_h = C_1 \cos x + C_2 \sin x$

$y_p = -\sin x \ln|\csc x + \cot x|$

$y = -\sin x \ln|\csc x + \cot x| + C_1 \cos x + C_2 \sin x$

21. Auxiliary equation: $r^2 - 3r + 2 = 0$ has roots 1, 2, so $y_h = C_1e^x + C_2e^{2x}$.

Let $y_p = v_1e^x + v_2e^{2x}$ subject to $v_1'e^x + v_2'e^{2x} = 0$, and $v_1'(e^x) + v_2'(2e^{2x}) = e^x(ex+1)^{-1}$.

Then $v_1' = \frac{-e^x}{e^x(e^x+1)}$ so $v_1 = \int \frac{-e^x}{e^x(e^x+1)}dx = \int \frac{-1}{u(u+1)}du$

$= \int\left(\frac{-1}{u} + \frac{1}{u+1}\right)du = -\ln u + \ln(u+1) = \ln\left(\frac{u+1}{u}\right) = \ln\frac{e^x+1}{e^x} = \ln(1+e^{-x})$

$v_2' = \frac{e^x}{e^{2x}(e^x+1)}$ so $v_2 = -e^{-x} + \ln(1+e^{-x})$

(similar to finding v_1)

General solution: $y = e^x \ln(1+e^{-x}) - e^x + e^{2x}\ln(1+e^{-x}) + C_1e^x + C_2e^{2x}$

$y = (e^x + e^{2x})\ln(1+e^{-x}) + D_1e^x + D_2e^{2x}$

22. $y_h = C_1e^{2x} + C_2e^{3x}$; $y_p = e^x$

$y = e^x + C_1e^{2x} + C_2e^{3x}$

23. $L(y_p) = (v_1u_1 + v_2u_2)'' + b(v_1u_1 + v_2u_2)' + c(v_1u_1 + v_2u_2)$

$= (v_1'u_1 + v_1u_1' + v_2'u_2 + v_2u_2') + b(v_1'u_1 + v_1u_1' + v_2'u_2 + v_2u_2') + c(v_1u_1 + v_2u_2)$

$= (v_1''u_1 + v_1'u_1' + v_1'u_1' + v_1u_1'' + v_2''u_2 + v_2'u_2' + v_2'u_2' + v_2u_2'') + b(v_1'u_1 + v_1u_1' + v_2'u_2 + v_2u_2') + c(v_1u_1 + v_2u_2)$

$= v_1(u_1'' + bu_1' + cu_1) + v_2(u_2'' + bu_2' + cu_2) + b(v_1'u_1 + v_2'u_2) + (v_1''u_1 + v_1'u_1' + v_2''u_2 + v_2'u_2) + (v_1'u_1' + v_2'u_2')$

$= v_1(u_1'' + bu_1' + cu_1) + v_2(u_2'' + bu_2' + cu_2) + b(v_1'u_1 + v_2'u_2) + (v_1'u_1 + v_2'u_2)' + (v_1'u_1' + v_2'u_2')$

$= v_1(0) + v_2(0) + b(0) + (0) + k(x) = k(x)$

24. Auxiliary equation: $r^2 + 4 = 0$ has roots $\pm 2i$.

$y_h = C_1 \cos 2x + C_2 \sin 2x$

Now write $\sin^3 x$ in a form involving sin βx's or cos βx's.

$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

(C.R.C. Standard Mathematical Tables, or derive it using half-angle and product identities.)

Let $y_p = A\sin x + B\cos x + C\sin 3x + D\cos 3x$;

$y_p' = A\cos x - B\sin x + 3C\cos 3x - 3D\sin 3x$;

$y_p'' = -A\sin x - B\cos x - 9C\sin 3x - 9D\cos 3x$.

Then

$y_p'' + 4y_p = 3A\sin x + 3B\cos x - 5C\sin 3x - 5D\cos 3x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$, so

$A = \frac{1}{4},\ B = 0,\ C = \frac{1}{20},\ D = 0.$

General solution: $y = \frac{1}{4}\sin x + \frac{1}{20}\sin 3x + C_1\cos 2x + C_2 \sin 2x$

15.3 Concepts Review

1. 3; π

2. π; decreases

3. 0

4. electric circuit

Problem Set 15.3

1. $k = 250, m = 10, B^2 = k/m = 250/10 = 25, B = 5$

(the problem gives the mass as $m = 10$ kg)

Thus, $y'' = -25y$. The general solution is $y = C_1\cos 5t + C_2 \sin 5t$. Apply the initial condition to get $y = 0.1\cos 5t$.

The period is $\frac{2\pi}{5}$ seconds.

2. k = 100 lb/ft, w = 1 lb, g = 32 ft/s^2, $y_0 = \frac{1}{12}$ ft,

$B = 40\sqrt{2}$. Then $y = \left(\frac{1}{12}\right)\cos(40\sqrt{2})t$.

Amplitude is $\frac{1}{12}$ ft $= 1$ in.

Period is $\frac{2\pi}{40\sqrt{2}} \approx 0.1111$ s.

3. $y = 0.1\cos 5t = 0$ whenever $5t = \frac{\pi}{2} + \pi k$ or $t = \frac{\pi}{10} + \frac{\pi}{5}k$.

$\left| y'\left(\frac{\pi}{10} + \frac{\pi}{5}k\right)\right| = 0.5\left|\sin 5\left(\frac{\pi}{10} + \frac{\pi}{5}k\right)\right| = 0.5\left|\sin\left(\frac{\pi}{2} + \pi k\right)\right| = 0.5$ meters per second

4. $|10| = k\left(\frac{1}{3}\right)$, so $k = 30$ lb/ft, $w = 20$ lb,

$g = 32$ ft/s^2, $y_0 = -1$ ft, $v_0 = 2$ ft/s, $B = 4\sqrt{3}$

Then $y = C_1 \cos(4\sqrt{3}t) + C_2 \sin(4\sqrt{3}t)$.

$y = \cos(4\sqrt{3}t) + \left(\frac{\sqrt{3}t}{6}\right)\sin(4\sqrt{3}t)$ satisfies the initial conditions.

5. $k = 20$ lb/ft; $w = 10$ lb; $y_0 = 1$ ft; $q = \frac{1}{10}$ s-lb/ft, $B = 8$, $E = 0.32$

$E^2 - 4B^2 < 0$, so there is damped motion. Roots of auxiliary equation are approximately $-0.16 \pm 8i$.

General solution is $y \approx e^{-0.16t}(C_1 \cos 8t + C_2 \sin 8t)$. $y \approx e^{-0.16t}(\cos 8t + 0.02 \sin 8t)$ satisfies the initial conditions.

6. $k = 20$ lb/ft; $w = 10$ lb; $y_0 = 1$ ft; $q = 4$ s-lb/ft

$B = \sqrt{\frac{(20)(32)}{10}} = 8$; $E = \frac{(4)(32)}{10} = 12.8$; $E^2 - 4B^2 < 0$, so damped motion.

Roots of auxiliary equation are $\frac{-E \pm \sqrt{E^2 - 4B^2}}{2} = -6.4 \pm 4.8i$.

General solution is $y = e^{-6.4t}(C_1 \cos 4.8t + C_2 \sin 4.8t)$.

$y' = e^{-6.4t}(-4.8C_1 \sin 4.8t + 4.8C_2 \cos 4.8t) - 6.4e^{-6.4t}(C_1 \cos 4.8t + C_2 \sin 4.8t)$

If $t = 0$, $y = 1$, $y' = 0$, then $1 = C_1$ and $0 = 4.8C_2 - 6.4C_1$, so $C_1 = 1$ and $C_2 = \frac{4}{3}$.

Therefore, $y = e^{-6.4t}\left[\cos 4.8t + \left(\frac{4}{3}\right)\sin 4.8t\right]$.

7. Original amplitude is 1 ft. Considering the contribution of the sine term to be negligible due to the 0.02 coefficient, the amplitude is approximately $e^{-0.16t}$.

$e^{-0.16t} \approx 0.1$ if $t \approx 14.39$, so amplitude will be about one-tenth of original in about 14.4 s.

8. $C_1 = 1$ and $C_2 = -0.105$, so $y = e^{-0.16t}(\cos 8t + 0.105 \sin 8t)$.

9. $LQ'' + RQ' + \frac{Q}{C} = E(t)$; $10^6 Q' + 10^6 Q = 1$; $Q' + Q = 10^{-6}$

Integrating factor: e^t

$D[Qe^t] = 10^{-6}e^t$; $Qe^t = 10^{-6}e^t + C$;

$Q = 10^{-6} + Ce^{-t}$

If $t = 0$, $Q = 0$, then $C = -10^{-6}$.

Therefore, $Q(t) = 10^{-6} - 10^{-6}e^{-t} = 10^{-6}(1 - e^{-t})$.

10. Same as Problem 9, except $C = 4 - 10^{-6}$, so $Q(t) = 10^{-6} + (4 - 10^{-6})e^{-t}$.

Then $I(t) = Q'(t) = -(4 - 10^{-6})e^{-t}$.

11. $\frac{Q}{[2(10^{-6})]} = 120 \sin 377t$

a. $Q(t) = 0.00024 \sin 377t$

b. $I(t) = Q'(t) = 0.09048 \cos 377t$

12. $LQ'' + RQ' + \frac{Q}{C} = E;\ 10^{-2}Q'' + \frac{Q}{10^{-7}} = 20;\ Q'' + 10^9 Q = 2000$

The auxiliary equation, $r^2 + 10^9 = 0$, has roots $\pm 10^{9/2} i$.

$Q_h = C_1 \cos 10^{9/2} t + C_2 \sin 10^{9/2} t$

$Q_p = 2000(10^{-9}) = 2(10^{-6})$ is a particular solution (by inspection).

General solution: $Q(t) = 2(10^{-6}) + C_1 \cos 10^{9/2} t + C_2 \sin 10^{9/2} t$

Then $I(t) = Q'(t) = -10^{9/2} C_1 \sin 10^{9/2} t + 10^{9/2} C_2 \cos 10^{9/2} t$.

If $t = 0$, $Q = 0$, $I = 0$, then $0 = 2(10^{-6}) + C_1$ and $0 = C_2$.

Therefore, $I(t) = -10^{9/2}(-2[10^{-6}]) \sin 10^{9/2} t = 2(10^{-3/2}) \sin 10^{9/2} t$.

13. $3.5Q'' + 1000Q + \frac{Q}{[2(10^{-6})]} = 120 \sin 377t$

(Values are approximated to 6 significant figures for the remainder of the problem.)

$Q'' + 285.714Q' + 142857Q = 34.2857 \sin 377t$

Roots of the auxiliary equation are
$-142.857 \pm 349.927i$.

$Q_h = e^{-142.857t}(C_1 \cos 349.927t + C_2 \sin 349.927t)$

$Q_p = -3.18288(10^{-4}) \cos 377t + 2.15119(10^{-6}) \sin 377t$

Then, $Q = -3.18288(10^{-4}) \cos 377t + 2.15119(10^{-6}) \sin 377t + Q_h$.

$I = Q' = 0.119995 \sin 377t + 0.000810998 \cos 377t + Q_h'$

0.000888 cos 377t is small and $Q_h' \to 0$ as $t \to \infty$, so the steady-state current is $I \approx 0.12 \sin 377t$.

14. a. Roots of the auxiliary equation are $\pm Bi$.

$y_h = C_1 \cos Bt + C_2 \sin Bt$.

$y_p = \left[\frac{c}{(B^2 - A^2)}\right] \sin At$

The desired result follows.

b. $y_p = \left(-\frac{c}{2B}\right) t \cos Bt$, so

$y = C_1 \cos Bt + C_2 \sin Bt - \left(\frac{c}{2B}\right) t \cos Bt$.

c. Due to the t factor in the last term, it increases without bound.

15. $A \sin(\beta t + \gamma) = A(\sin \beta t \cos \gamma + \cos \beta t \sin \gamma)$

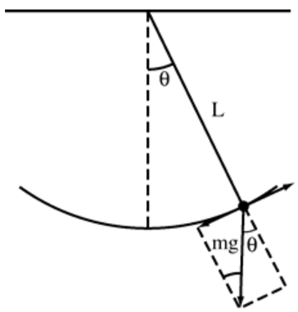
$= (A \cos \gamma) \sin \beta t + (A \sin \gamma) \cos \beta t$

$= C_1 \sin \beta t + C_2 \cos \beta t$, where $C_1 = A \cos \gamma$ and $C_2 = A \sin \gamma$.

[Note that
$C_1^2 + C_2^2 = A^2 \cos^2 \gamma + A^2 \sin^2 \gamma = A^2$.)

16. The first two terms have period $\frac{2\pi}{B}$ and the last has period $\frac{2\pi}{A}$. Then the sum of the three terms is periodic if $m\left(\frac{2\pi}{B}\right) = n\left(\frac{2\pi}{B}\right)$ for some integers m, n; equivalently, if $\frac{B}{A} = \frac{m}{n}$, a rational number.

17. The magnitudes of the tangential components of the forces acting on the pendulum bob must be equal.



Therefore, $-m\frac{d^2s}{dt^2} = mg\sin\theta.$

$s = L\theta$, so $\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2}.$

Therefore, $-mL\frac{d^2\theta}{dt^2} = mg\sin\theta.$

Hence, $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta.$

18. a. Since the roots of the auxiliary equation are $\pm\sqrt{\frac{g}{L}}i$, the solution of $\theta''(t) + \left(\frac{g}{L}\right)\theta = 0$ is $\theta = C_1\cos\sqrt{\frac{g}{L}}t + C_2\sin\sqrt{\frac{g}{L}}t,$ which can be written as $\theta = C\left(\sqrt{\frac{g}{L}}t + \gamma\right)$ (by Problem 15).

The period of this function is

$$\frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi\frac{L}{\sqrt{G}} = 2\pi\sqrt{\frac{LR^2}{GM}} = 2\pi R\sqrt{\frac{L}{GM}}.$$

Therefore, $\frac{p_1}{p_2} = \frac{2\pi R_1\sqrt{\frac{L_1}{GM}}}{2\pi R_2\sqrt{\frac{L_2}{GM}}} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}.$

b. To keep perfect time at both places, require $p_1 = p_2$. Then $1 = \frac{R_2\sqrt{80.85}}{3960\sqrt{81}}$, so

$R_2 \approx 3963.67.$

The height of the mountain is about $3963.67 - 3960 = 3.67$ mi (about 19,387 ft).

15.4 Chapter Review

Concepts Test

1. False: y^2 is not linear in y.

2. True: y and y'' are linear in y and y'', respectively.

3. True: $y' = \sec^2 x + \sec x\tan x$

$2y' - y^2 = (2\sec^2 x + 2\sec x\tan x)$

$-(\tan^2 x + 2\sec x\tan x + \sec^2 x)$

$= \sec^2 x - \tan^2 x = 1$

4. False: It should involve 6.

5. True: D^2 adheres to the conditions for linear operators.

$D^2(kf) = kD^2(f)$

$D^2(f+g) = D^2 f + D^2 g$

6. False: Replacing y by $C_1u_1(x) + C_2u_2(x)$ would yield, on the left side, $C_1 f(x) + C_2 f(x) = (C_1 + C_2) f(x)$ which is $f(x)$ only if $C_1 + C_2 = 1$ or $f(x) = 0$.

7. True: –1 is a repeated root, with multiplicity 3, of the auxiliary equation.

8. True: $L(u_1 - u_2) = L(u_1) - L(u_2)$

$= f(x) - f(x) = 0$

9. False: That is the form of y_h. y_p should have the form $Bx\cos 3x + Cx\sin 3x$.

10. True: See Problem 15, Section 15.3.

Sample Test Problems

1. $u' + 3u = e^x$. Integrating factor is e^{3x}.

$D[ue^{3x}] = e^{4x}$

$y = \left(\frac{1}{4}\right)e^x + C_1e^{-3x}$

$y' = \left(\frac{1}{4}\right)e^x + C_1e^{-3x}$

$y = \left(\frac{1}{4}\right)e^x + C_3e^{-3x} + C_2$

2. Roots are –1, 1. $y = C_1e^{-x} + C_2e^x$

3. (Second order homogeneous)
The auxiliary equation, $r^2 - 3r + 2 = 0$, has roots 1, 2. The general solution is $y = C_1e^x + C_2e^{2x}$.
$y' = C_1e^x + 2C_2e^{2x}$
If $x = 0$, $y = 0$, $y' = 3$, then $0 = C_1 + C_2$ and $3 = C_1 + 2C_2$, so $C_1 = -3, C_2 = 3$.
Therefore, $y = -3e^x + 3e^{2x}$.

4. Repeated root $-\frac{3}{2}$. $y = (C_1 + C_2x)e^{(-3/2)x}$

5. $y_h = C_1e^{-x} + C_2e^x$ (Problem 2)
$y_p = -1 + C_1e^{-x} + C_2e^x$

6. (Second-order nonhomogeneous) The auxiliary equation, $r^2 + 4r + 4 = 0$, has roots –2, –2.
$y_h = C_1e^{-2x} + C_2xe^{-2x} = (C_1 + C_2x)e^{-2x}$
Let $y_p = Be^x; y_p' = Be^x; y_p'' = Be^x$.
$(Be^x) + 4(Be^x) + 4(Be^x) = 3e^x$, so $B = \frac{1}{3}$.
General solution: $y = \frac{e^x}{3} + (C_1 + C_2x)e^{-2x}$

7. $y_h = (C_1 + C_2x)e^{-2x}$ (Problem 12)
$y_p = \left(\frac{1}{2}\right)x^2e^{-2x}$
$y = \left[\left(\frac{1}{2}\right)x^2 + C_1 + C_2x\right]e^{-2x}$

8. Roots are ±2*i*.
$y = C_1 \cos 2x + C_2 \sin 2x$
$y = \sin 2x$ satisfies the conditions.

9. (Second-order homogeneous)
The auxiliary equation, $r^2 + 6r + 25 = 0$, has roots –3 ± 4*i*. General solution:
$y = e^{-3x}(C_1 \cos 4x + C_2 \sin 4x)$

10. Roots are ±*i*. $y_h = C_1 \cos x + C_2 \sin x$
$y_p = x \cos x - \sin x + \sin x \ln|\cos x|$
$y = x \cos x - \sin x \ln|\cos x| + C_1 \cos x + C_3 \sin x$
(combining the sine terms)

11. Roots are –4, 0, 2. $y = C_1e^{-4x} + C_2 + C_3e^{2x}$

12. (Fourth-order homogeneous)
The auxiliary equation, $r^4 - 3r^2 - 10 = 0$ or $(r^2 - 5)(r^2 + 2) = 0$, has roots $-\sqrt{5}, \sqrt{5}, \pm\sqrt{2}i$.
General solution:
$y = C_1e^{\sqrt{5}x} + C_2e^{-\sqrt{5}x} + C_3 \cos \sqrt{2}x + C_4 \sin \sqrt{2}x$

13. Repeated roots $\pm\sqrt{2}$
$y = (C_1 + C_2x)e^{-\sqrt{2}x} + (C_3 + C_4x)e^{\sqrt{2}x}$

14. **a.** $Q'(t) = 3 - 0.02Q$

b. $Q'(t) + 0.02Q = 3$
Integrating factor is $e^{0.02t}$
$D[Qe^{0.02t}] = 3e^{0.02t}$
$Q(t) = 150 + Ce^{-0.02t}$
$Q(t) = 150 - 30e^{-0.02t}$ goes through (0, 120).

c. $Q \to 150$ g, as $t \to \infty$.

15. (Simple harmonic motion)
$k = 5$; $w = 10$; $y_0 = -1$
$B = \sqrt{\frac{(5)(32)}{10}} = 4$
Then the equation of motion is $y = -\cos 4t$.
The amplitude is $|-1| = 1$; the period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

16. It is at equilibrium when $y = 0$ or $-\cos 4t = 0$, or $t = \frac{\pi}{8}, \frac{3\pi}{8}, \ldots$.
$y'(t) = 4 \sin 4t$, so at equilibrium $|y'| = |\pm 4| = 4$.

17. $Q'' + 2Q' + 2Q = 1$
Roots are $-1 \pm i$.
$Q_h = e^{-t}(C_1 \cos t + C_2 \sin t)$ and $Q_p = \frac{1}{2}$;
$Q = e^{-t}(C_1 \cos t + C_2 \sin t) + \frac{1}{2}$
$I(t) = Q'(t) = -e^{-t}[(C_1 - C_2)\cos t + (C_1 + C_2)\sin t]$
$I(t) = e^{-t} \sin t$ satisfies the initial conditions.