Discretizing Dynamical Systems with a Codimension two Singularity

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Overview

- Description of the problem.
- Discretizing systems having a BT_2 (Bogdanov-Takens) point via Runge-Kutta methods.
- Discretizing systems having a FH (fold-Hopf) via general one-step discretization methods.
- Analysis of the emanating discretized path of Hopf points.
- Numerical example.

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Description of the problem

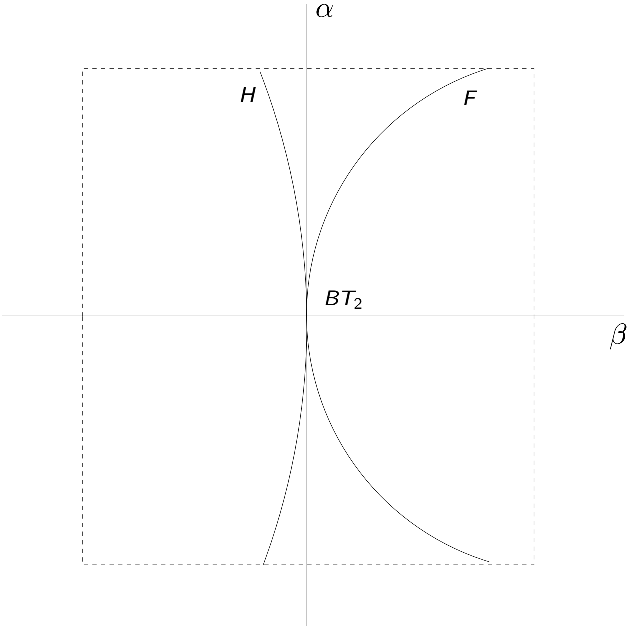
Let

$$\dot{x}(t) = f(x(t), \beta, \alpha), \tag{CTDS}$$

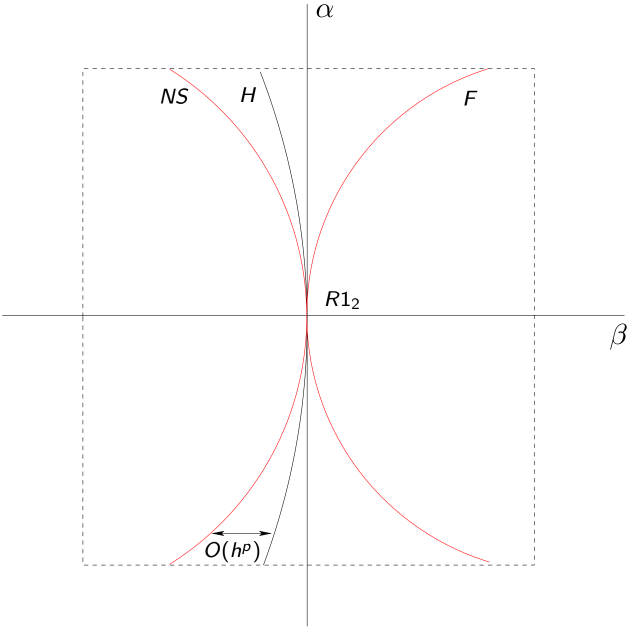
$$x \mapsto g(x, \beta, \alpha), \tag{DTDS}$$

- $f, g \in C^k(\Omega \times \Lambda, \mathbb{R}^n)$ with $0 \in \Omega \subset \mathbb{R}^N$, $0 \in \Lambda \subset \mathbb{R}^2$, Ω, Λ open sets, $k \geq 1$ sufficiently large,
- (CTDS) undergoes a BT_2 or FH bifurcation at $(x_0, \beta_0, \alpha_0) = (0, 0, 0)$.

BT_2 - local bifurcation diagram



BT_2 - discretized local bifurcation diagram



Discretized local bifurcation diagram

Already known:

- Fold points persist under s-stage implicit Runge-Kutta methods (cf. L. Lóczi. PhD Thesis, Budapest, 2006).
- Hopf points are $O(h^p)$-shifted under general one-step methods (cf. X. Wang, E. Blum, Q. Li. J. Difference Eq. Appl. 4, 29-57, 1998).

We will show that:

- BT_2 points persist under s-stage implicit Runge-Kutta methods (arguments follow closely Lóczi's).
- Fold-Hopf points are $O(h^p)$-shifted under general one-step methods.
- Emanating discretized curve of Hopf points is $O(h^p)$-shifted in both cases by general one-step methods.

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Discretizing systems having a BT_2 point via Runge-Kutta methods

s-stage implicit Runge-Kutta method for $\dot{x}(t) = f(x(t), \beta, \alpha)$ (CTDS)

Let

$$x \mapsto \psi^h(x, \beta, \alpha) := x + h\Phi(h, x, \beta, \alpha), \qquad \text{(RKmap)}$$

$h > 0$ step-size, with

$$\Phi(h, x, \beta, \alpha) := \sum_{i=1}^{s} \gamma_i k_i(h, x, \beta, \alpha)$$

and $(k_i)_{i=1,\dots,s}$ solve

$$k_i(h, x, \beta, \alpha) = f(W_i(h, x, \beta, \alpha), \beta, \alpha), \;\; i = 1, \dots, s,$$

where

$$W_i(h, x, \beta, \alpha) := x + h \sum_{j=1}^{s} \tau_{ij} k_j(h, x, \beta, \alpha), \;\; i = 1, \dots, s.$$

Main result

Theorem

Let system (CTDS) have a BT_2 point at the origin $(x, \beta, \alpha) = (0, 0, 0)$. Consider a general s-stage implicit Runge-Kutta method with step-size $h > 0$ given by (RKmap). Then there exist a constant $\xi > 0$, such that (RKmap) has a $R1_2$ point at the origin for all $h \in (0, \xi)$.

Ideas of proof

Recall definition of a BT_2 and $R1_2$ point

BT_2:

1c $f(x_0, \beta_0, \alpha_0) = 0$,

2c Jordan block of $f_x(x_0, \beta_0, \alpha_0)$: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

3c $ab \neq 0$ (nondegeneracy condition).

$R1_2$:

1d $g(x_0, \beta_0, \alpha_0) - x_0 = 0$,

2d Jordan block of $g_x(x_0, \beta_0, \alpha_0)$: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

3d $\tilde{a}\tilde{b} \neq 0$ (nondegeneracy condition).

Also recall the BT_2 Normal Form:

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= \delta_1 + \delta_2 u_1 + a u_1^2 + b u_1 u_2. \end{aligned}$$

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Equilibrium condition

- f has a local Lipschitz constant.
- So system

$$k_i(h, x, \beta, \alpha) = f(W_i(h, x, \beta, \alpha), \beta, \alpha), \;\; i = 1, ..., s,$$

is solvable and has at the BT_2 point a unique solution $k_i^0(h) = 0, \; i = 1, ..., s.$
- Then

$$\psi^h(x_0, \beta_0, \alpha_0) - x_0 = \psi^0(h) - 0 = h \sum_{i=1}^{s} k_i^0(h) = 0,$$

for all sufficiently small h.

Equilibrium condition

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1d $g(x_0, \beta_0, \alpha_0) - x_0 = 0$, ✓

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Jordan block condition

- Show that null(f_x^0) =null($\psi_x^0(h) - I_N$) (which implies dim null($\psi_x^0(h) - I_N$) $= 1$) for all sufficiently small h.
- Show that there exist only one generalized eigenvector.

It holds that

-
$$\psi_x^0(h) - I_N = h f_x^0 A(h), \;\; A(h) := I_N + h \sum_{i=1}^{s} \sum_{j=1}^{s} \gamma_i \tau_{ij} k_{jx}^0(h).$$

- We find that for all sufficiently small h, $A(h)$ is invertible and $A(h) v_0 = v_0$, $0 \neq v_0 \in$ null(f_x^0). Thus the Jordan structure of $\psi_x^0(h) - I_N$ and f_x^0 is related.

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Nondegeneracy condition

Let $c(h) := v_1^T (A^{-1}(h))^T p_1 = O(h)$. The following relations hold

$$\begin{array}{ll} \tilde{v}_0(h) = v_0, & \tilde{v}_1(h) = \frac{1}{h} A^{-1}(h) v_1, \\ \tilde{p}_0(h) = h p_0, & \tilde{p}_1(h) = p_1 - c(h) p_0, \end{array},$$

- $v_0, v_1,\ p_0, p_1$ are right and left generalized eigenvectors of f_x^0 and
- $\tilde{v}_0, \tilde{v}_1,\ \tilde{p}_0, \tilde{p}_1$ are right and left generalized eigenvectors of $\psi_x^0(h) - I_N$.

Hence

$$\tilde{a}(h) = h^2 a,\ \tilde{b}(h) = 2h(h\omega - c(h))a + hb.$$

So nondegeneracy, for all sufficiently small h, is guaranteed.

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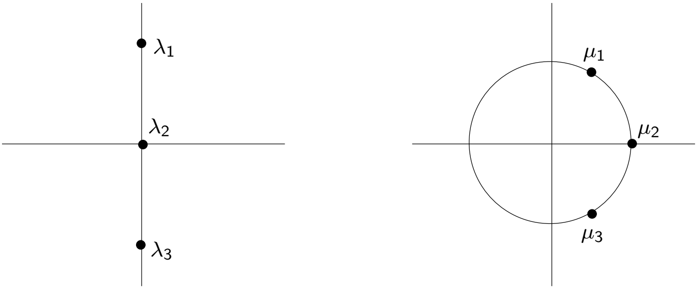
3d $\tilde{a}\tilde{b} \neq 0$ (nondegeneracy condition). ✓

Discretizing systems having a *FH* point via general one-step discretization methods

Recall definition of a *FH* and *FN* point

A *FH* (resp. *FN*) point is an equilibrium of (CTDS) (resp. (DTDS)) with the eigenvalue condition



FH *FN*

Basic Setup for the section

Consider a general one-step method of order $p \geq 1$ applied to (CTDS), given by

$$x \mapsto \psi^h(x, \beta, \alpha) := x + h\Phi(h, x, \beta, \alpha), \qquad \text{(OSM)}$$

with $\psi, \Phi : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}^N$ smooth.

(OSM) is called standard if there exists smooth functions $\Upsilon, \Xi : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}^N$ such that:

$$\begin{aligned}
\psi^h(x, \beta, \alpha) &= \varphi^h(x, \beta, \alpha) + \Upsilon(h, x, \beta, \alpha)h^{p+1}, \\
\psi^h_w(x, \beta, \alpha) &= \varphi^h_w(x, \beta, \alpha) + \Upsilon_w(h, x, \beta, \alpha)h^{p+1}, \\
\Phi(h, x, \beta, \alpha) &= f(x, \beta, \alpha) + \Xi(h, x, \beta, \alpha)h, \\
\Phi_w(h, x, \beta, \alpha) &= f_w(x, \beta, \alpha) + \Xi_w(h, x, \beta, \alpha)h,
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hold locally. w stands for any of the variables of $f(\cdot, \cdot, \cdot)$ and $\varphi^{\cdot}(\cdot, \cdot, \cdot)$ for the flow of (CTDS).

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hold locally. w stands for any of the variables of $f(\cdot, \cdot, \cdot)$ and $\varphi^{\cdot}(\cdot, \cdot, \cdot)$ for the flow of (CTDS).

Main result

Theorem

Consider a general one-step discretization method of order $p \geq 1$ *applied to (CTDS) and given by* (OSM). *Assume:*

1. (OSM) *is standard,*
2. *System (CTDS) has a generic FH point* $(x_{FH}, \beta_{FH}, \alpha_{FH})$ *at the origin,*

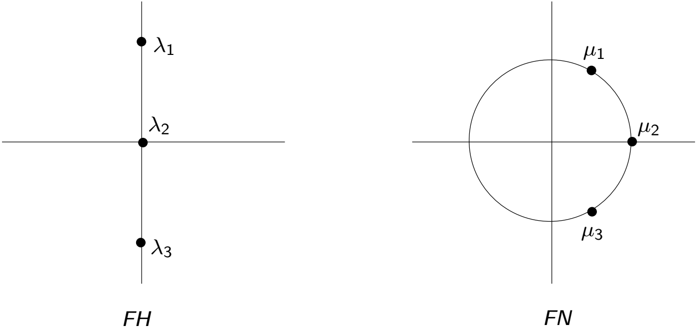
then (OSM) *possesses a FN point* $(x_{FN}(h), \beta_{FN}(h), \alpha_{FN}(h))$ *that depends smoothly on* h*, for all sufficiently small* h*. Furthermore, the following estimate holds*

$$\|(x_{FN}(h), \beta_{FN}(h), \alpha_{FN}(h)) - (x_{FH}, \beta_{FH}, \alpha_{FH})\| \leq C|h|^p,$$

$C > 0$*,* h *sufficiently small.*

Ideas of proof

A *FH* (resp. *FN*) point is an equilibrium of (CTDS) (resp. (DTDS)) with the eigenvalue condition



So a *FH* (resp. *FN*) point is a solution of

$$\tilde{F}_{FH}(x,\beta,\alpha) := \begin{pmatrix} f(x,\beta,\alpha) \\ \det(2f_x(x,\beta,\alpha) \odot I_N) \\ \det(f_x(x,\beta,\alpha)) \end{pmatrix} = 0, \qquad \text{(FHsys)}$$

resp.,

$$\tilde{G}_{FN}(h,x,\beta,\alpha) := \begin{pmatrix} \psi^h(x,\beta,\alpha) - x \\ \det(\psi_x^h(x,\beta,\alpha) \odot \psi_x^h(x,\beta,\alpha) - I_m) \\ \det(\psi_x^h(x,\beta,\alpha) - I_N) \end{pmatrix} = 0, \qquad \text{(FNsys)}$$

where $\odot$ stands for the bialternate product of matrices and $m = \frac{N}{2}(N-1)$ (cf. J. Guckenheimer, M. Myers, B. Sturmfels. SIAM Journal on Numerical Analysis 34, 1-21, 1997). By assumption, *FH* point is generic, so (FHsys) is regular at the origin.

System (FHsys) can be written in terms of the flow φ as

$$\tilde{F}_1(h,x,\beta,\alpha) := \begin{pmatrix} \varphi^h(x,\beta,\alpha) - x \\ \det(\varphi^h_x(x,\beta,\alpha) \odot \varphi^h_x(x,\beta,\alpha) - I_m) \\ \det(\varphi^h_x(x,\beta,\alpha) - I_N) \end{pmatrix} = 0.$$

But as $h \to 0$, system becomes trivial, so we transform the system further:

$$F_{FH}(h,x,\beta,\alpha) := \begin{pmatrix} \frac{1}{h}\left(\varphi^h(x,\beta,\alpha) - x\right) \\ \det\left(\frac{1}{h}\left(\varphi^h_x(x,\beta,\alpha) \odot \varphi^h_x(x,\beta,\alpha) - I_m\right)\right) \\ \det\left(\frac{1}{h}\left(\varphi^h_x(x,\beta,\alpha) - I_N\right)\right) \end{pmatrix} = 0.$$

Similarly, we can transform (FNsys) into

$$G_{FN}(h,x,\beta,\alpha) := \begin{pmatrix} \frac{1}{h}\left(\psi^h(x,\beta,\alpha) - x\right) \\ \det\left(\frac{1}{h}\left(\psi^h_x(x,\beta,\alpha) \odot \psi^h_x(x,\beta,\alpha) - I_m\right)\right) \\ \det\left(\frac{1}{h}\left(\psi^h_x(x,\beta,\alpha) - I_N\right)\right) \end{pmatrix} = 0.$$

The following relations hold

$$\begin{aligned} F_{FH}(h,x,\beta,\alpha) &:= \tilde{F}_{FH}(x,\beta,\alpha) + O(h), \\ G_{FN}(h,x,\beta,\alpha) &:= \tilde{F}_{FH}(x,\beta,\alpha) + O(h), \\ F_{FH}(h,x,\beta,\alpha) &:= G_{FN}(h,x,\beta,\alpha) + O(h^p). \end{aligned}$$

- By applying Vainikko's Lemma (Inverse Lipschitz Mapping Theorem) to $G_{FN}(h,\cdot,\cdot,\cdot)$, we obtain the existence (and uniqueness) of the *FN* point and the closeness result.
- By applying the Implicit Function Theorem to G_{FN}, we obtain the smooth dependence of the *FN* point on h.

Analysis of the emanating discretized path of Hopf points

Theorem

Consider a general one-step discretization method of order $p \geq 1$ *applied to (CTDS) and given by* (OSM). *Assume:*

1. (OSM) *is standard,*
2. *System (CTDS) has a generic* BT_2 *or FH point at the origin,*

then there exist curves of Hopf and Neimark-Sacker points

$$\begin{aligned} d_H(\alpha) &:= (x_H(\alpha), \beta_H(\alpha)), \\ d_{NS}(h,\alpha) &:= (x_{NS}(h,\alpha), \beta_{NS}(h,\alpha)), \end{aligned}$$

with $x_H(\alpha), x_{NS}(h,\alpha) \in \mathbb{R}^N$, $\beta_H(\alpha), \beta_{NS}(h,\alpha) \in \mathbb{R}$ *locally defined and the following estimate holds*

$$\|d_{NS}(h,\alpha) - d_H(\alpha)\| \leq C|h|^p, \;\; C > 0,$$

uniformly in α *and for all sufficiently small* h.

Ideas of proof

Similar as before. Genericity of the BT_2 (resp. FH) point implies that

$$\begin{cases} f(x,\beta,\alpha) = 0, \\ \det(2f_x(x,\beta,\alpha) \odot I_N) = 0, \\ \det(f_x(x,\beta,\alpha)) = 0, \end{cases}$$

is regular at the origin, hence

$$\tilde{F}_H(x,\beta,\alpha) := \begin{pmatrix} f(x,\beta,\alpha) \\ \det(2f_x(x,\beta,\alpha) \odot I_N) \end{pmatrix} = 0, \qquad \text{(Hopfsys)}$$

is regular at the origin. So the Implicit Function Theorem applied to $\tilde{F}_H$ guarantees the existence of a curve of Hopf points.

As before, we can write (Hopfsys) in terms of the flow φ as

$$F_H(h,x,\beta,\alpha) := \begin{pmatrix} \frac{1}{h}\left(\varphi^h(x,\beta,\alpha) - x\right) \\ \det\left(\frac{1}{h}\left(\varphi_x^h(x,\beta,\alpha) \odot \varphi_x^h(x,\beta,\alpha) - I_m\right)\right) \end{pmatrix} = 0.$$

Similarly a curve of Neimark-Sacker points is a solution to

$$G_{NS}(h,x,\beta,\alpha) := \begin{pmatrix} \frac{1}{h}\left(\psi^h(x,\beta,\alpha) - x\right) \\ \det\left(\frac{1}{h}\left(\psi_x^h(x,\beta,\alpha) \odot \psi_x^h(x,\beta,\alpha) - I_m\right)\right) \end{pmatrix} = 0.$$

The following relations hold

$$\begin{aligned} F_H(h,x,\beta,\alpha) &:= \tilde{F}_H(x,\beta,\alpha) + O(h), \\ G_{NS}(h,x,\beta,\alpha) &:= \tilde{F}_H(x,\beta,\alpha) + O(h), \\ F_H(h,x,\beta,\alpha) &:= G_{NS}(h,x,\beta,\alpha) + O(h^p). \end{aligned}$$

- By applying the Implicit Function Theorem to G_{NS}, we obtain the existence of a path of NS points.
- By applying Vainikko's Lemma (Inverse Lipschitz Mapping Theorem) to $F_H(h,\cdot,\cdot,\alpha)$, we obtain the closeness result.

Numerical example

Consider the following system (cf. A. Algaba, et al. Nonlinear Dynamics 16, 369-404, 1998):

$$\begin{aligned}
\dot{x} &= -\frac{\beta+\alpha}{R}x + \frac{\alpha}{R}y - \frac{C}{R}x^3 + \frac{D}{R}(y-x)^3 - \frac{E}{R}x^5 + \frac{F}{R}(y-x)^5, \\
\dot{y} &= \alpha x - (\alpha+G)y - z - D(y-x)^3 - Hy^3 - F(y-x)^5 - Iy^5, \\
\dot{z} &= y,
\end{aligned}$$

with parameters $\beta, \alpha, C, D, E, F, G, H, I \in \mathbb{R}$ and $0 < R \in \mathbb{R}$, which describes the dynamics of a modified van der Pol-Duffing oscillator. We choose β, α as our bifurcation parameters and we set $C = 1$, $D = -5$, $E = 1$, $F = 1$, $G = -1.5$, $H = 1$, $I = 1$, $R = 3$. For our computations we use the continuation software CONTENT.

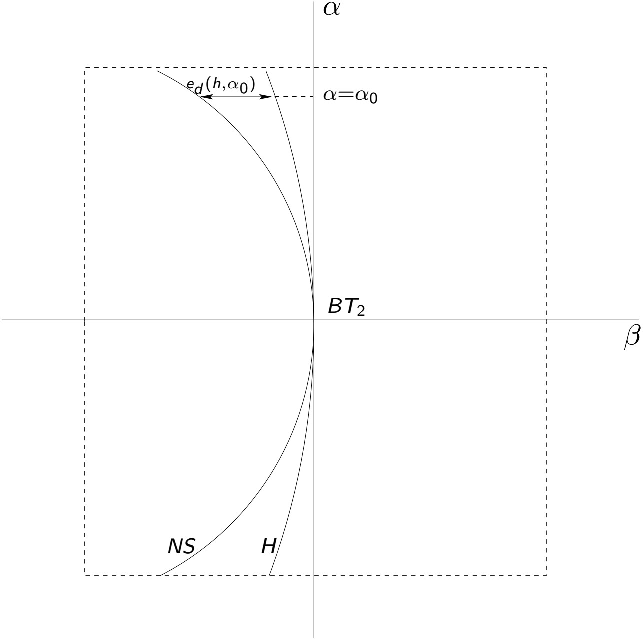
- We find a BT_2 point at (up to 9 digits) $(x_{BT}, y_{BT}, z_{BT}) = (1, 0, 4.267949192)$, $(\beta_{BT}, \alpha_{BT}) = (-6.267949192, 8.267949192)$.
- Then we discretize the system via Runge's method (order 3), step-size $h_0 = 0.13$.
- A $R1_2$ point is found at $(x_{R1}, y_{R1}, z_{R1}) = (1, 0, 4.267949192)$, $(\beta_{R1}, \alpha_{R1}) = (-6.267949192, 8.267949192)$.

For estimation of the order of approximation between paths of Hopf and Neimark-Sacker points, we define

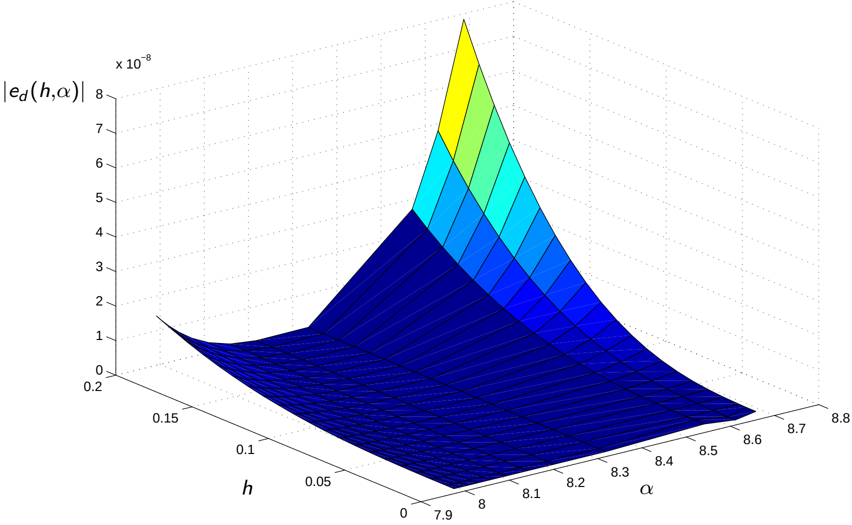
$$\beta_{NS}(h, \alpha) - \beta_H(\alpha) =: e_d(h, \alpha).$$

Thus we will plot $e_d(h, \alpha)$ in a small neighborhood of the singularity.

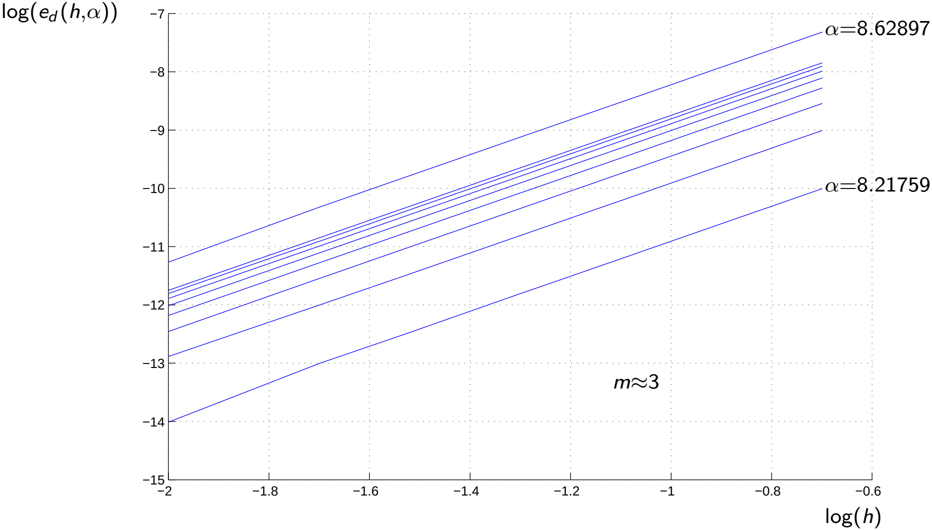
Estimation of the order of approximation



Behavior of $e_d(h, \alpha)$ w.r.t. (h, α)



Behavior of $e_d(h, \alpha)$ w.r.t. h, with α fixed



According to this figure have

$$p \approx 3,$$

this means

$$e_d(h, \alpha) \approx O(h^3) \checkmark$$

As predicted.

Thanks a lot for your attention!