Dynamical Systems and Differential Equations

Joseph Páez Chávez

Instituto de Ciencias Matemáticas,
Escuela Superior Politécnica del Litoral,
Km. 30.5 Vía Perimetral, P.O. Box 09-01-5863
Guayaquil, Ecuador
jpaez@espol.edu.ec

August 26, 2010

Abstract

In this manuscript we introduce some important concepts concerning dynamical systems theory. We devote special attention to studying differential equations from a dynamical systems viewpoint. The introduced concepts are illustrated by examples.

Resumen

En este manuscrito presentamos algunos conceptos importantes concernientes a la teoría de los sistemas dinámicos. Se presta especial atención al estudio de ecuaciones diferenciales desde el punto de vista de sistemas dinámicos. Los conceptos presentados son ilustrados mediante ejemplos.

Keywords: Dynamical Systems, discrete-time systems, continuous-time systems, differential equations, vector fields.

Introduction

Nowadays dynamical systems phenomena appear in almost every area of science, from the oscillating Belousov-Zhabotinsky reaction in chemistry to the chaotic Lorenz system in meteorology, from complicated behavior in celestial mechanics to the bifurcations arising in ecological models. It turns out that many of the phenomena mentioned above can be described by means of differential equations. For this reason, it is an important task to understand the connections between differential equations and dynamical systems. By doing this, we obtain a powerful tool which allows us to study the qualitative behavior of differential equations without having to solve them analytically. This is specially useful when a general solution is not available or the numerical simulations are too expensive.

In this article, we continue the study started in [7]. We recall some basic concepts introduced in that manuscript and bring some new ones. We also present a theorem concerning existence and uniqueness of the solution of initial value problems and show the connection between these problems and dynamical systems.

Most of the material presented in this manuscript can be found in the lectures notes of Prof. Beyn, [4, 5]. There is nevertheless plenty of literature on this subject, e.g., see [1, 2, 3, 6, 8].

1 Basic Concepts and Theorems

To begin with, we recall the definition of dynamical system (cf. [7]):

Definition 1.1. *A dynamical system is a triple* $\{\mathbb{T}, X, \{\varphi^t\}_{t\in\mathbb{T}}\}$, *where* $\mathbb{T}$ *is a time set,* X *is a state space and* $\varphi^t : X \to X$ *is a family of operators parametrized by* $t \in \mathbb{T}$, *such that:*

DS1 $\forall u \in X : \varphi^0(u) = u$, *i.e.,* $\varphi^0 = Id_X$,

DS2 $\forall u \in X, \forall s, t \in \mathbb{T} : \varphi^{t+s}(u) = \varphi^t(\varphi^s(u))$, *i.e.,* $\varphi^t \circ \varphi^s = \varphi^{t+s}$.

Here, the set X stands for a metric space. The function φ^t is known as evolution operator. This operator can be thought of as a "law" that governs the behavior of the system. Furthermore, the time set $\mathbb{T}$ has the following properties:

- $\exists 0 \in \mathbb{T}, \forall t \in \mathbb{T} : t + 0 = t$,
- $\forall t, s \in \mathbb{T} : s + t \in \mathbb{T}$,
- $\forall t, s \in \mathbb{T} : s + t = t + s$.

This means that $\mathbb{T}$ equipped with the operation $+$ is a commutative semigroup.

In [7, Example 2.1], the author explains the fact that a discrete-time system is completely defined by knowing the function $g := \varphi^1$. Hence the evolution operator can be constructed as follows

$$\varphi^0 = Id_X, \text{ and } \varphi^k = \underbrace{g \circ g \circ \ldots \circ g}_{k \text{ times}}, \tag{1.1}$$

$k \in \mathbb{N}$. Further, if g is invertible, the system admits negative values of k. For this reason, the function g is said to be the generator of the dynamical system. Now the natural question that arises from this fact is whether continuous-time systems also have, in some sense, generators. Note that in this case we deal with values of time on the real line, so the function φ^1 does not allow us to construct the evolution operator, at least not on the whole real line. These considerations lead us to the following definition:

Definition 1.2. *Let* $\{\mathbb{T}, X, \{\varphi^t\}_{t\in\mathbb{T}}\}$, *where* $\mathbb{T} = \mathbb{R}$ *or* $\mathbb{R}^+ \cup \{0\}$, *be a dynamical system such that* $\varphi^{\cdot}(x)$ *is differentiable for all* $x \in X$. *Then the function* $f : X \to X$ *given by*

$$f(x) := \frac{d}{dt}\left(\varphi^t(x)\right)\Big|_{t=0} = \lim_{\substack{h\to 0\\ h\in\mathbb{T}}} \frac{\varphi^h(x) - \varphi^0(x)}{h}$$

is referred to as infinitesimal generator of the dynamical system[1].

Of course, the spirit of a generator is that we can in some way construct the evolution operator from the generator. In (1.1) we explained how the evolution operator of a discrete-time system can be obtained from its generator. Now we will show that this is possible for continuous-time systems, too:

Theorem 1.3. *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be the infinitesimal generator of a dynamical system $\left\{\mathbb{T}, \mathbb{R}^N, \{\varphi^t\}_{t\in\mathbb{T}}\right\}$. Then the function $u \in C^1(\mathbb{T}, \mathbb{R}^N)$ defined as $u(t) = \varphi^t(u_0)$ is a solution of the initial value problem*

$$\dot{y}(t) = f(y(t)), \quad y(0) = u_0.$$

Proof. By **DS1** we have that $u(0) = \varphi^0(u_0) = u_0$, so the initial condition is satisfied. According to the definition of infinitesimal generator we have that

$$\begin{aligned}
\forall t \in \mathbb{T} : f(u(t)) &= \lim_{\substack{h\to 0\\ h\in\mathbb{T}}} \frac{\varphi^h(u(t)) - u(t)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{T}}} \frac{\varphi^h\left(\varphi^t(u_0)\right) - u(t)}{h} \\
&\overset{\textbf{DS2}}{=} \lim_{\substack{h\to 0\\ h\in\mathbb{T}}} \frac{\varphi^{h+t}(u_0) - u(t)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{T}}} \frac{u(t+h) - u(t)}{h} \\
&= \dot{u}(t).
\end{aligned}$$

□

This theorem asserts that every dynamical system (of the type introduced in Definition 1.2) is completely defined by its infinitesimal generator, or, more precisely, by the initial value problem presented above. Now the natural question is whether every initial value problem represents a dynamical system. For this issue to be dealt with, we first present a standard result about the existence and uniqueness of the solution of initial value problems:

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N$ be open and $f \in C^1(\Omega, \mathbb{R}^N)$. Then the initial value problem*

$$\dot{y}(t) = f(y(t)), \quad y(0) = u_0 \tag{1.2}$$

has for each $u_0 \in \Omega$ exactly one nonextendible solution $u(t, u_0) \in \Omega$, where $t \in J(u_0) = (t_-(u_0), t_+(u_0)) \ni 0$. The domain of the function u

$$D = \{(t, u_0) \in \mathbb{R} \times \Omega : t \in J(u_0)\}$$

is open and $u \in C^1(D, \mathbb{R}^N)$. Furthermore, if $f \in C^k(\Omega, \mathbb{R}^N)$, then $u \in C^k(D, \mathbb{R}^N)$, $k \geq 1$.

[1]Also called vector field of the dynamical system.

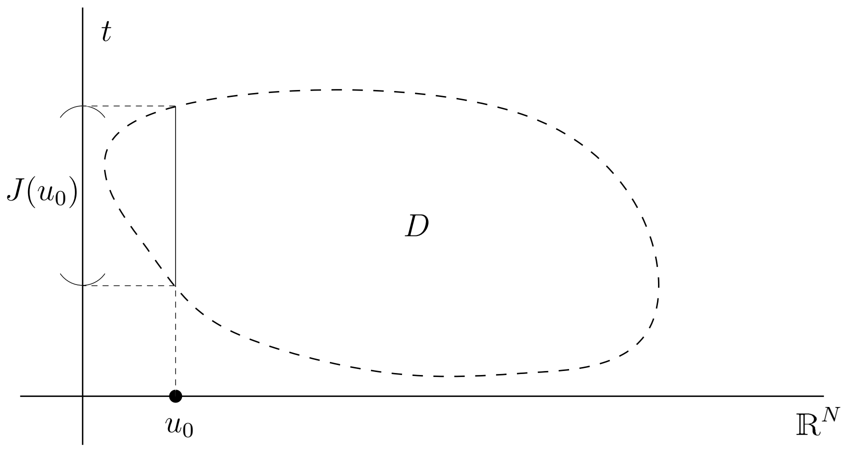

Fig. 1.1: Domain of definition of the solution of (1.2).

Proof. This is a classical result and its proof can be found in any book on Differential Equations, e.g., see [1, Chapter II]. □

Besides guaranteeing existence and uniqueness, this theorem also gives valuable information about the domain D of definition of the solution. This domain turns out to be open and furthermore the solution of the initial value problem (1.2) does not exist outside D. For this reason, the interval $J(u_0)$ is referred to as the maximal interval of existence. Here, it is important to point out that this interval varies with the initial value u_0, see Figure 1.1. With this few remarks we can turn back to the question we outlined before, that is, whether an initial value problem defines a dynamical system. We will see that this is true, but in a local sense:

Theorem 1.5. *Let the assumptions of Theorem 1.4 hold. Then the operator $\varphi^{\cdot}(\cdot) : D \to \mathbb{R}^N$, given by $\varphi^t(u_0) = u(t, u_0)$, defines a local dynamical system.*

Proof. Let us first show **DS1**. By the initial value condition in (1.2), we have that

$$\forall u_0 \in \Omega : \varphi^0(u_0) = u(0, u_0) = u_0.$$

Now let us work with **DS2**. Let $u_0 \in \Omega$ and $s \in J(u_0)$ be arbitrary, but fixed. Then the function $v(t) := \varphi^t\left(\varphi^s(u_0)\right)$, $t \in J\left(\varphi^s(u_0)\right)$, is a solution of

$$\dot{y}(t) = f(y(t)), \quad y(0) = \varphi^s(u_0). \tag{1.3}$$

Now consider the function $w(t) := \varphi^{t+s}(u_0)$. If follows that

$$\dot{w}(t) = \frac{d}{dt}\left(u(t+s, u_0)\right) \overset{(1.2)}{=} f(u(t+s, u_0)) = f(w(t)) \text{ and } w(0) = \varphi^{0+s}(u_0) = \varphi^s(u_0).$$

Therefore, w is another solution of (1.3) and by uniqueness (cf. Theorem 1.4), $v = w$, i.e. $\varphi^t\left(\varphi^s(u_0)\right) = \varphi^{t+s}(u_0)$. □

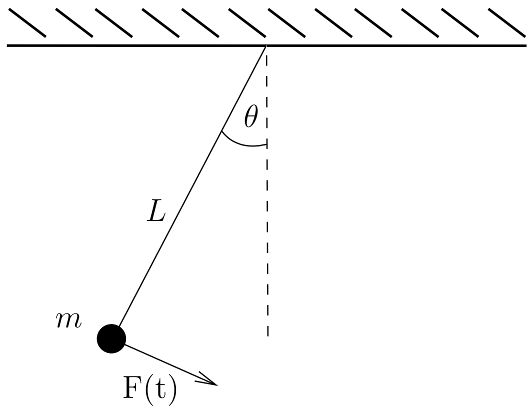

Fig. 1.2: Forced pendulum.

According to this theorem, the initial value problem (1.2) always represents an autonomous dynamical system. Thus, by combining this result with Theorem 1.3, we can realize that the concept of dynamical system is closely related to initial value problems, and, more generally, to differential equations (see [7, Example 2.2]).

In many cases, physical phenomena includes the action of an external time-dependent "force", which leads us to modeling the underlying phenomena by means of non-autonomous differential equations of the form

$$\dot{y}(t) = f(t, y(t)), \quad y(t_0) = u_0, \quad t \in [t_0, t_E], \tag{1.4}$$

where $f \in C^1(\mathbb{R} \times \Omega, \mathbb{R}^N)$. Studying in detail non-autonomous dynamical systems is beyond the scope of this article, however, we do want to point out that such systems can be written in an autonomous way as the following example shows.

Example 1.6. Consider a pendulum of mass m attached to a string of length L, which is displaced by an angle from the vertical rest position, see Figure 1.2. Suppose that there exists an external sinusoidal force $F(t)$ acting on the system. The dynamics of the pendulum can then be described by the ODE

$$\ddot{\theta}(t) + \frac{g}{L}\sin(\theta(t)) = A\sin(\beta t) + B\cos(\beta t), \quad t \in \mathbb{R}, \tag{1.5}$$

where $A, B, \beta \in \mathbb{R}$, $\beta > 0$ are fixed and g stands for the gravity constant. The case where there is no external force (i.e. $A = B = 0$) was studied in [7, Example 1.3]. There, it is proved that the pendulum can be seen as an autonomous dynamical system. Now we will show how to deal with the external force in order to preserve the autonomous character of the system. One way to achieve this is adding a nonlinear oscillator to the system. An example of such an oscillator is given by

$$\begin{cases} \dot{x}(t) = x(t) + \beta y(t) - x(t)(x(t)^2 + y(t)^2), \\ \dot{y}(t) = y(t) - \beta x(t) - y(t)(x(t)^2 + y(t)^2), \end{cases}$$

which has the solution $x(t) = \sin(\beta t)$, $y(t) = \cos(\beta t)$. Now consider the functions $u(t) = \theta(t)$, $v(t) = \dot{\theta}(t)$, $t \in \mathbb{R}$, and define

$$G(u, v, x, y) := \begin{pmatrix} v \\ -\frac{g}{L}\sin(u) + Ax + By \\ x + \beta y - x(x^2 + y^2) \\ y - \beta x - y(x^2 + y^2) \end{pmatrix},$$

where $z := (u, v, x, y) \in \mathbb{R}^4$. Thus, it is easy to see that the non-autonomous system (1.5) can be written as

$$\dot{z}(t) = G(z(t)),$$

which is an autonomous differential equation of the type of (1.2). This discussion provides us with a way of applying the theory developed for autonomous dynamical systems to the present non-autonomous case.

In many cases it may happen that the external force is not periodic, or difficult to model by an autonomous oscillator. If this is so, we can resort to writing system (1.4) in autonomous form as follows:

$$\begin{cases} \dot{y}(t) = f(h(t), y(t)), \quad t \in [t_0, t_E], \\ \dot{h}(t) = 1, \\ y(t_0) = u_0, \\ h(t_0) = t_0. \end{cases}$$

This is an autonomous system of $N + 1$ ODEs with $N + 1$ initial conditions.

2 Equilibria, Orbits, and Phase Diagrams

A common approach for starting the study of dynamical systems consists in introducing geometrical objects that allow us to visualize dynamical properties, thereby making their analysis easier. To achieve this, we will begin with the concept of orbit. Then we will realize that a dynamical system can be qualitatively described by drawing some "typical" orbits. This process leads us to the so-called phase diagram.

To begin with, let us first introduce the notion of equilibrium, which is the simplest object of study in dynamical systems. To this end, we consider the Example 1.6 without forcing, see Figure 1.2. In [7, Example 2.1], the (approximate) evolution operator $\varphi^{(\cdot)}(\cdot) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is found to be

$$\varphi^t \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix} = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix} = \begin{pmatrix} \frac{\theta_0'}{\omega}\sin(\omega t) + \theta_0 \cos(\omega t) \\ \theta_0' \cos(\omega t) - \theta_0 \omega \sin(\omega t) \end{pmatrix}, \quad \omega = \sqrt{\frac{g}{L}}, \tag{2.1}$$

where θ_0 and θ_0' represent the angle and angular velocity of the pendulum, respectively, at $t = 0$. Now suppose that we let the pendulum run with the initial conditions $\theta_0 = \theta_0' = 0$. How does the system evolve in time? From the formula for the evolution operator presented above, it is easy to see that $\varphi^t(\theta_0, \theta_0') = 0$, for all t. We can also arrive at this conclusion from a physical point of view. Initializing the system with $\theta_0 = \theta_0' = 0$ amounts to placing the pendulum at the vertical position with initial angular velocity equal to zero. It is then clear that the pendulum will remain at the vertical position $\theta = 0$ forever. This illustrates the simplest behavior that a dynamical system may present. However, this simple behavior can be seen in more complicated/abstract systems, too. Consider for example the PDE

$$\left\{ \begin{array}{r} \dfrac{\partial u}{\partial t}(x,t) + x\dfrac{\partial u}{\partial x}(x,t) = 0 \\ u(x,0) = f(x) \end{array} \right. , \quad u \in C^1(\mathbb{R}^2, \mathbb{R}), \quad f \in C^1(\mathbb{R}),$$

with evolution operator $\varphi^{(\cdot)}(\cdot) : \mathbb{R} \times C^1(\mathbb{R}) \to C^1(\mathbb{R})$ given by $(\varphi^t(f))(x) = f(xe^{-t})$ (cf. [7, Example 2.2]). Choose the initial condition $f(x) = K$ for all $x \in \mathbb{R}$, K being a real constant. Then it follows that

$$\forall t \in \mathbb{R} : (\varphi^t(f))(x) = f(xe^{-t}) = K = f(x) \Rightarrow \varphi^t(f) = f.$$

This means that if we initialize the system at a constant function, the system will remain at that function forever. These two examples illustrate the concept of equilibrium of a dynamical system, which is formally defined as follows:

Definition 2.1. *Let $\{\mathbb{T}, X, \{\varphi^t\}_{t\in\mathbb{T}}\}$ be a dynamical system. A point $x_0 \in X$ is referred to as equilibrium point if*

$$\forall t \in \mathbb{T} : \varphi^t(x_0) = x_0.$$

In other words, we can say that if a dynamical system is placed at an equilibrium point, it will remain there forever. This fact was already seen in the examples above.

In the literature, equilibrium points are also called “steady states”, “equilibrium solutions”, “stationary points”, “rest points”, and “fixed points”, among others. Some authors reserve the name “equilibrium” for continuous-time systems, while the term “fixed point” is used when dealing with discrete-time systems. However, the reader should have in mind that both terms stand for the same dynamical object.

Now that we have introduced our first (and the simplest) dynamical object of study, our next task will be to investigate how to detect such objects, provided the (infinitesimal) generator of the system is known. This task is accomplished in the following:

Theorem 2.2. *Let $f, g \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. Consider the systems*

$$\dot{x}(t) = f(x(t)), \quad x(0) = \xi, \tag{2.2}$$

$$x_n = g(x_{n-1}), \quad x_0 = \widetilde{\xi}, \quad n \in \mathbb{N}\,. \tag{2.3}$$

Then $z_0 \in \mathbb{R}^N$ is an equilibrium (resp. fixed point) of (2.2) *(resp.* (2.3)*), if and only if $f(z_0) = 0$ (resp. $g(z_0) = z_0$).*

Proof. Let us first work with system (2.2). Assume that z_0 is an equilibrium of (2.2). This means that $\varphi^t(z_0) = z_0$ for all t. Thus, according to Definition 1.2, we have that

$$f(z_0) = \frac{d}{dt}\left(\varphi^t(z_0)\right)\bigg|_{t=0} = \frac{d}{dt}(z_0)\bigg|_{t=0} = 0.$$

Now suppose that $f(z_0) = 0$. Define the constant function $x(t) = z_0$, $t \in \mathbb{R}$. It is then easy to check that x is a solution of (2.2), and by uniqueness (cf. Theorem 1.4), we can conclude that $\varphi^t(z_0) = z_0$ for all $t \in \mathbb{R}$. Hence z_0 is an equilibrium of (2.2). Now let us turn to the discrete-time case. Assume z_0 to be a fixed point of (2.3), i.e., $\varphi^n(z_0) = z_0$ for all $n \in \mathbb{N} \cup \{0\}$. By (1.1), it is readily seen that $g(z_0) = \varphi^1(z_0) = z_0$. Now suppose that $g(z_0) = z_0$, and that $\varphi^n(z_0) = z_0$ for some fixed $n \in \mathbb{N}$ holds. It follows by induction that

$$\varphi^{n+1}(z_0) \overset{\textbf{DS2}}{=} \varphi^n(\underbrace{\varphi^1(z_0)}_{=z_0}) = \varphi^n(z_0) = z_0.$$

Hence $\varphi^n(z_0) = z_0$ for all $n \in \mathbb{N} \cup \{0\}$, i.e., z_0 is a fixed point of (2.3). □

The principal significance of the theorem is that it provides us with a way of finding and characterizing equilibrium points of dynamical systems. In other words, if we are interested in equilibrium points (resp. fixed points) of system (2.2) (resp. (2.3)), we should look for the solutions of the equation $f(x) = 0$ (resp. $g(x) = x$).

Now that we have understood the meaning of equilibrium point, we can proceed with the concept of a somewhat more elaborated dynamical object, the so-called orbit.

Definition 2.3. *Let $\{\mathbb{T}, X, \{\varphi^t\}_{t\in\mathbb{T}}\}$ be a dynamical system and $x_0 \in X$. The set*

$$O_r(x_0) = \{x \in X : x = \varphi^t(x_0), t \in \mathbb{T}\}$$

is referred to as orbit of x_0.

Before showing some examples, it is worth presenting a few remarks:

- $\forall x_0 \in X : O_r(x_0) \subset X$,
- if x_0 is an equilibrium point, then $O_r(x_0) = \{x_0\}$, i.e., an equilibrium point is the simplest orbit,
- in continuous-time dynamical systems, the orbits are curves parametrized by the time t,
- in discrete-time dynamical systems, the orbits are sequences in X, i.e., $O_r(x_0) \in X^{\mathbb{T}}$ for all $x_0 \in X$.

Let us illustrate the concept of orbit by some examples.

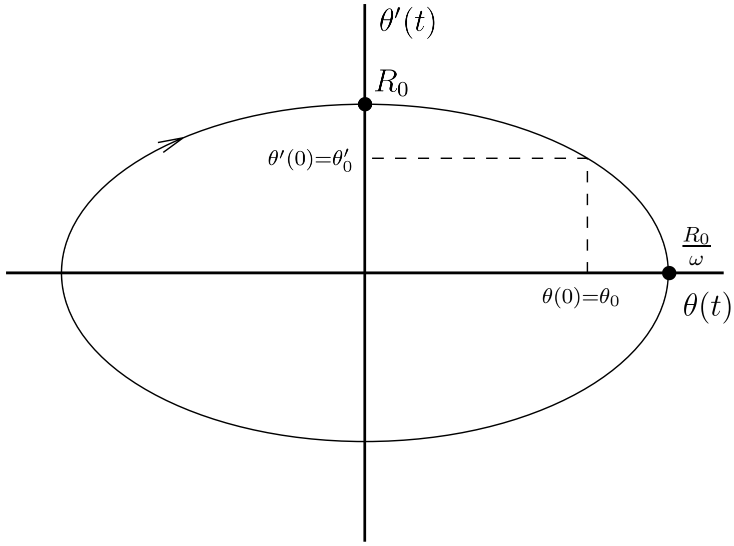

Fig. 2.1: Orbit of the pendulum system.

Example 2.4. Consider again the pendulum system of Example 1.6, without external forcing. The evolution operator can be written as (cf. (2.1))

$$\varphi^t \begin{pmatrix} R_0 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} \dfrac{R_0}{\omega} \sin(\omega t + \phi_0) \\ R_0 \cos(\omega t + \phi_0) \end{pmatrix}, \text{ where } R_0 = \sqrt{(\theta'_0)^2 + (\theta_0 \omega)^2} \text{ and } \sin(\phi_0) = \frac{\omega \theta_0}{R_0}.$$

Here, we must choose $\dfrac{R_0}{\omega}$ small (why?). Thus, an orbit of the pendulum system is described by the parametric curve

$$O_r(\theta_0, \theta'_0) = \left\{ \begin{pmatrix} \dfrac{R_0}{\omega} \sin(\omega t + \phi_0) \\ R_0 \cos(\omega t + \phi_0) \end{pmatrix} : t \in \mathbb{R} \right\}.$$

A typical example of an orbit of this system is depicted in Figure 2.1. Note that, in this case, the orbits are always closed curves, which reveals the periodic nature of the system (in the absence of friction!). In the figure, the arrow stands for the direction of the evolution as the time increases. How does the orbit $O_r(0,0)$ look like?

Example 2.5. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Consider the system

$$x_n = g(x_{n-1}), \quad n \in \mathbb{N}\,.$$

Choose $x_0 = 2$ as initial point. Then we have that $O_r(2) = \{x \in \mathbb{R} : x = \varphi^n(2), n = 0, 1, \ldots\} = \{2, 4, 16, \ldots\} = \left(2^{2^n}\right)_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N} \cup \{0\}}$. Clearly, $x = 1$ and $x = 0$ are fixed points of the system, and so it follows that $O_r(1) = \{1\}$ and $O_r(0) = \{0\}$.

In the examples above we have illustrated the concept of orbit for both continuous- and discrete-time dynamical systems. As we pointed out before, an orbit is a subset of the state space, and so we can ask ourselves whether the state space could be, in some sense, decomposed into a collection of orbits of a dynamical system. To this end, the following theorem gives us important information:

Theorem 2.6. *Let $\{\mathbb{T}, X, \{\varphi^t\}_{t\in\mathbb{T}}\}$ be an invertible (cf. [7, Section 2]) dynamical system. Let $u_0, v_0 \in X$. Then $O_r(u_0) \cap O_r(v_0) = \emptyset$ or $O_r(u_0) = O_r(v_0)$.*

Proof. Suppose that $O_r(u_0) \cap O_r(v_0) \neq \emptyset$. This means that

$$\exists y \in X : y \in O_r(u_0) \wedge y \in O_r(v_0) \Longleftrightarrow \exists s, t \in \mathbb{T} : y = \varphi^t(u_0) = \varphi^s(v_0).$$

Therefore, we have that

$$\forall \tau \in \mathbb{T} : \varphi^\tau(u_0) \overset{\mathbf{DS2}}{=} \varphi^{\tau-t}(\underbrace{\varphi^t(u_0)}_{=\varphi^s(v_0)}) = \varphi^{\tau-t}(\varphi^s(v_0)) \overset{\mathbf{DS2}}{=} \varphi^{\tau-t+s}(v_0) \in O_r(v_0).$$

This implies that $O_r(u_0) \subseteq O_r(v_0)$. We can prove analogously that $O_r(v_0) \subseteq O_r(u_0)$ and hence $O_r(u_0) = O_r(v_0)$. $\square$

From this Theorem, we can conclude the following:

- Two orbits satisfying the conditions of the theorem above are either disjoint or identical,
- through every point in the state space passes only one orbit. Consequently, it follows that
$$X = \dot{\bigcup_{x_0\in X}} O_r(x_0).$$

This means that the state space can be represented as the disjoint union of orbits of the underlying dynamical system. This motivates the definition of phase diagram (see below).

Definition 2.7. *Let $\{\mathbb{T}, X, \{\varphi^t\}_{t\in\mathbb{T}}\}$ be a dynamical system. The partitioning of the state space into orbits is referred to as phase diagram[2] of the dynamical system.*

Let us restrict our attention to initial value problems (cf. (1.2)). In this case, the phase diagram consists of a family of solution curves of the system (1.2) obtained by varying the initial condition u_0. It is clear that if x is any point in $O_r(u_0)$, then $f(x)$ represents a tangent vector of the solution curve at x. For this reason, the ODE (1.2) is also referred to as vector field. Let us consider again the pendulum system of Example 2.4. How does its phase diagram look like? We have seen that any orbit of that system is given by

$$O_r(\theta_0, \theta_0') = \left\{ \begin{pmatrix} \frac{R_0}{\omega}\sin(\omega t + \phi_0) \\ R_0\cos(\omega t + \phi_0) \end{pmatrix} : t \in \mathbb{R} \right\}.$$

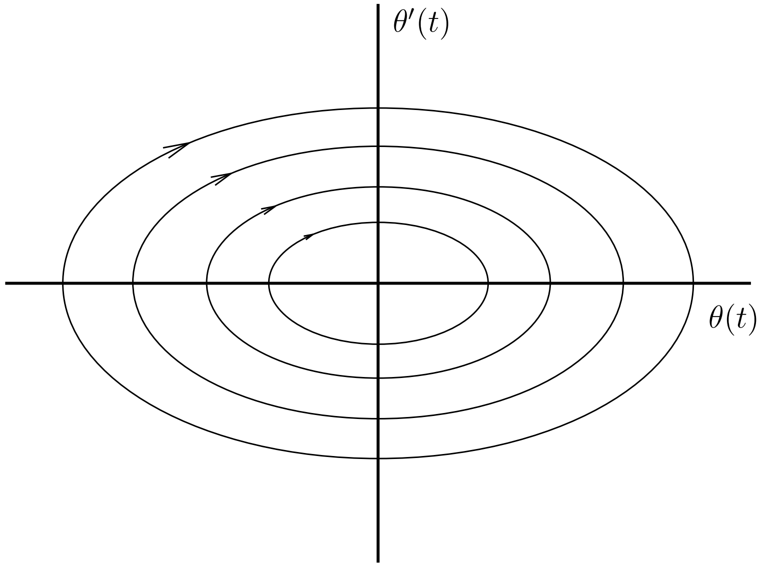

Fig. 2.2: Phase diagram of the pendulum system.

If we vary the initial conditions θ_0, θ'_0, we will then obtain several closed curves centered at the origin, see Figure 2.2. The phase diagram provides us with an easy way of visualizing the behavior of a dynamical system. Useful information can be obtained from the phase plot, even if we do not know the evolution operator. For instance, suppose that a pendulum system presents the phase plot as shown in Figure 2.3. From this picture, we can see that the orbits are no longer periodic, but they spiral into the origin, which is an equilibrium point. Hence this point is called a spiral sink. What does this mean from a physical point of view? Note that the phase diagram tells us that any initial point will, after a long time, end at the equilibrium point. This fact reveals the presence of friction, which prevents the system from oscillating forever as it happens in the pendulum system described in Figure 2.2.

Conclusions

In this manuscript we have seen that dynamical systems and autonomous differential equations are intimately related objects. In particular, Theorems 1.3 and 1.5 reflect this fact. We have also learned how to deal with non-autonomous differential equations, which appear when an external forcing is present. More importantly, we have introduced several geometrical objects such as equilibrium points, orbits and phase diagrams. With these concepts, our goal was to provide the reader with tools for facilitating the study of the qualitative behavior of dynamical systems. In forthcoming articles we will explain in more detail how the objects mentioned above can be used for the analysis of dynamical systems.

[2] Also called phase portrait and phase plot.

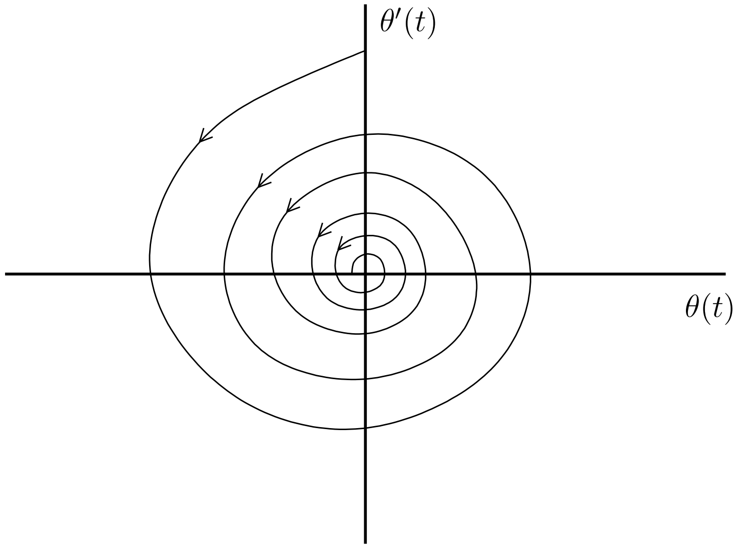

Fig. 2.3: Phase diagram of a damped pendulum system.

References

[1] Amann, H. *Gewöhnliche Differentialgleichungen*, second ed. Walter de Gruyter, Berlin, 1995.

[2] Arrowsmith, D., and Place, C. M. *An Introduction to Dynamical Systems*. Cambridge University Press, 1990.

[3] Arrowsmith, D., and Place, C. M. *Dynamical Systems: Differential Equations, Maps, and Chaotic Behaviour*. Chapman & Hall, 1992.

[4] Beyn, W.-J. Dynamische Systeme. Vorlesungsskriptum, Bielefeld University, 1991.

[5] Beyn, W.-J. Numerik dynamischer Systeme. Vorlesungsskriptum, Bielefeld University, 2009.

[6] Guckenheimer, J., and Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, vol. 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York. Fourth printing, 1993.

[7] Páez Chávez, J. A short Introduction to Dynamical Systems. *Matemática: Una Publicación del ICM - ESPOL 8*, 1 (2010), 28–33.

[8] Wiggins, S. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, second ed., vol. 2 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 2003.