Starting Homoclinic Tangencies near 1:1 Resonances

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Overview

- Description of the problem
- 1:1 Resonances
- Starting Procedure
- Numerical Examples

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Description of the problem

Consider

$$x \mapsto f(x, \alpha) \qquad \text{(DS)}$$

- $f \in C^k(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^N)$, $k \geq 1$ sufficiently large
- $f(\cdot, \alpha)$ is a diffeomorphism for all $\alpha \in \mathbb{R}^2$
- $\xi \in \mathbb{R}^N$ is a saddle fixed point of (DS) at some $\alpha = \alpha_0$

An orbit $x_\mathbb{Z} \in \left(\mathbb{R}^N\right)^\mathbb{Z}$ of (DS) is called homoclinic if

$$\lim_{n \to \pm\infty} x_n = \xi$$

Further we call the homoclinic orbit $x_\mathbb{Z}$ tangential if the stable and unstable manifolds of ξ intersect tangentially along the connecting orbit $x_\mathbb{Z}$

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Numerical Approximation (Transversal)
(Beyn, Kleinkauf, SIAM J. on Num. Anal., 1997)

Let $N_+, N_- \in \mathbb{Z}$, $N_- < 0 < N_+$. Define the discrete intervals

$$J = [N_-, N_+] \cap \mathbb{Z} \text{ and } \hat{J} = [N_-, N_+ - 1] \cap \mathbb{Z}$$

and let $S_J^N \subset \left(\mathbb{R}^N\right)^J$ be the space of bounded sequences in $\mathbb{R}^N$. A homoclinic orbit can be approximated by zeroes of the operator

$$\Gamma : \begin{array}{ccc} S_J^N \times \mathbb{R}^2 & \to & S_J^N \\ (x_J, \alpha) & \mapsto & \left((x_{n+1} - f(x_n, \alpha))_{n \in \hat{J}}, b(x_{N_-}, x_{N_+}, \alpha) \right) \end{array}$$

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A tangential homoclinic orbit can be approximated by turning points of the operator Γ defined previously, for example

$$\Upsilon : \begin{array}{ccc} S_J^N \times S_J^N \times \mathbb{R}^2 & \to & S_J^N \times S_J^N \times \mathbb{R} \\ (x_J, u_J, \alpha) & \mapsto & \begin{pmatrix} \Gamma(x_J, \alpha) \\ \Gamma_{x_J}(x_J, \alpha) u_J \\ \sum_{i=N_-}^{N_+} \|u_i\|^2 - 1 \end{pmatrix} \end{array}$$

In this setting, we have a free parameter (α_1 or α_2) for the continuation of a curve of tangential homoclinic orbits.

Main Question: How to find a first solution of $\Upsilon(x_J, u_J, \alpha) = 0$?

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Some Methods

- Set a first approximating orbit to

$$(\xi, \dots, \xi, x_1, \dots, x_r, \xi, \dots, \xi)$$

  where $x_i \in \mathbb{R}^N$ are, basically, randomly chosen vectors
  - Trial and error
  - Brute force, "luck"
  - Spurious solutions are easily obtained
- Compute the stable and unstable manifolds of ξ and use the intersections as an approximating orbit
  - Computation of the manifolds can be expensive
  - Applicable for planar systems
  - We should know a priori the parameter values at which the homoclinic connection occur

Thus we will propose a theory-based starting method for constructing an "educated" initial guess of tangential homoclinic orbits near 1:1 resonances (for arbitrary dimension $N \geq 2$)

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1:1 Resonances (R1)

- $f(x_0, \alpha_0) = x_0$
- Jordan block of $f_x(x_0, \alpha_0)$: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- $ab \neq 0$ (nondegeneracy condition)

Where a, b are coefficients of the R1 Normal Form:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto G(u, \delta) := \begin{pmatrix} u_1 + u_2 \\ u_2 + \delta_1 + \delta_2 u_2 + a u_1^2 + b u_1 u_2 \end{pmatrix}$$

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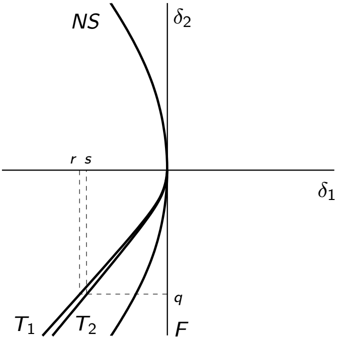
Bifurcation Diagram ($a = b = 1$)



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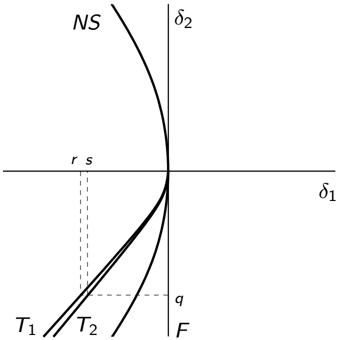
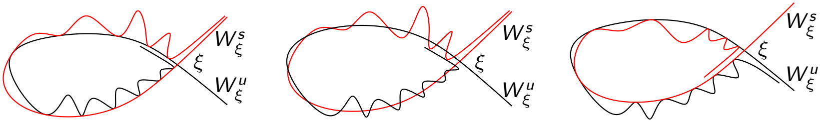
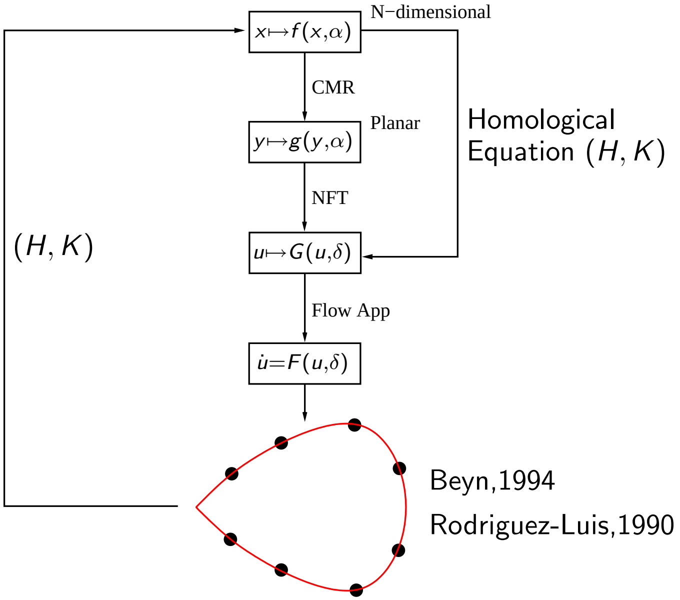




$\delta_1 = r$ $\qquad$ $r < \delta_1 < s$ $\qquad$ $\delta_1 = s$

Starting Procedure



Center Manifold Reduction

Homological Equation (Meijer, PhD Thesis, Utrecht, 2006)

- Assume that (DS) has a R1 point at the origin
- Local representation of a parameter-dependent center manifold:

$$W_\delta^c = \left\{ x \in \mathbb{R}^N : x = H(u,\delta), (u,\delta) \in \mathbb{R}^2 \times \mathbb{R}^2 \right\}$$

- By the invariance of W_δ^c, there exists a smooth parameter transformation $K : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f\left(H(u,\delta), K(\delta)\right) = H\left(G(u,\delta),\delta\right) \qquad \text{(HE)}$$

- A rigorous discussion about the local validity of (HE) can be found in Páez, PhD Thesis, Bielefeld, 2009

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At a R1 point (Páez, Int. J. of Bif. and Chaos, 2010)

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$$G(u,\delta) = \begin{pmatrix} u_1 + u_2 \\ u_2 + \delta_1 + \delta_2 u_2 + a u_1^2 + b u_1 u_2 + \mathcal{O}((u_1^2 + |u_1 u_2|)||\delta||) \end{pmatrix} + \mathcal{O}(||u||^3)$$

$$K(\delta) = K_1 \delta + \mathcal{O}(||\delta||^2)$$

$$H(u,\delta) = \begin{pmatrix} v_0 & v_1 \end{pmatrix} u + D\delta + \mathcal{O}\left(||u||^2 + ||u||||\delta|| + ||\delta||^2\right)$$

where a, b are the normal form coefficients, v_0, v_1 denote critical eigenvectors of $f_x^0 := f_x(0,0)$, and $K_1 \in \mathbb{R}^{2,2}$, $D \in \mathbb{R}^{N,2}$ are constants to be computed. Consider also the Taylor expansion of f

$$f(x,\alpha) = f_x^0 x + f_\alpha^0 \alpha + \frac{1}{2} B(x,x) + \mathcal{O}\left(||x||^3 + ||\alpha||^2 + ||x||||\alpha||\right)$$

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Inserting f, G, H, K into the Homological Equation yields the following relations:

$$a = \frac{1}{2} p_0^T B(v_0, v_0), \quad b = p_1^T B(v_0, v_0) + p_0^T B(v_0, v_1) \text{ (Meijer, 2006)}$$

where p_0, p_1 denote critical left eigenvectors of f_x^0,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} K_1$$

where $0 \neq \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} := p_0^T f_\alpha^0$, and

$$(f_x^0 - I_N) D = \begin{pmatrix} v_1 & 0 \end{pmatrix} - f_\alpha^0 K_1$$

Remark:These relations do not define K_1 and D uniquely

Center Manifold Reduction

At a R1 point (Páez, Int. J. of Bif. and Chaos, 2010)

Inserting f, G, H, K into the Homological Equation yields the following relations:

$$a = \frac{1}{2} p_0^T B(v_0, v_0), \quad b = p_1^T B(v_0, v_0) + p_0^T B(v_0, v_1) \text{ (Meijer, 2006)}$$

where p_0, p_1 denote critical left eigenvectors of f_x^0,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} K_1$$

where $0 \neq \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} := p_0^T f_\alpha^0$, and

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Flow Approximation
(Kuznetsov, Elements of Applied Bifurcation Theory, 2004)

Let $\phi^t(\cdot, \delta)$ be the t-flow of

$$\dot{u} = F_{app}(u, \delta) := F_0(\delta) + F_1(u, \delta) + F_2(u)$$

where:

$$F_0(\delta) = \begin{pmatrix} -\frac{1}{2}\delta_1 \\ \delta_1 \end{pmatrix}$$

$$F_1(u, \delta) = \begin{pmatrix} u_2 + \left(\frac{1}{3}b - \frac{1}{2}a\right)\delta_1 u_1 + \left(\left(\frac{1}{5}a - \frac{5}{12}b\right)\delta_1 - \frac{1}{2}\delta_2\right) u_2 \\ \left(\frac{2}{3}a - \frac{1}{2}b\right)\delta_1 u_1 + \left(\left(\frac{1}{2}b - \frac{1}{6}a\right)\delta_1 + \delta_2\right) u_2 \end{pmatrix}$$

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Then the dynamics of the normal form

$$u \mapsto G(u, \delta)$$

are approximated quantitatively by the system

$$u \mapsto \phi^1(u, \delta)$$

Therefore a homoclinic $\phi^1(\cdot, \delta)$-orbit can be used as an approximation of a tangential homoclinic orbit of the normal form. But we also need an approximation of the solution of

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Assume $u_{\mathbb{Z}} \in \left(\mathbb{R}^2\right)^{\mathbb{Z}}$ to be a homoclinic $\phi^1(\cdot, \delta)$-orbit. We claim that the sequence

$$U_i := F_{app}(u_i, \delta), \quad i \in \mathbb{Z}$$

is a solution of the variational equation along $u_{\mathbb{Z}}$ because:

$$\begin{aligned} \text{Take any } n \in \mathbb{Z}: \ U_{n+1} &= F_{app}(u_{n+1}, \delta) \\ &= F_{app}(\phi^{n+1}(u_0, \delta), \delta) \\ &= \frac{d}{dt}\left(\phi^{t+1}(u_0, \delta)\right)_{t=n} \\ &= \frac{d}{dt}\left(\phi^1(\phi^t(u_0, \delta), \delta)\right)_{t=n} \\ &= \phi^1_u(u_n, \delta)\frac{d}{dt}\left(\phi^t(u_0, \delta)\right)_{t=n} \\ &= \phi^1_u(u_n, \delta)F_{app}(\phi^n(u_0, \delta), \delta) \\ &= \phi^1_u(u_n, \delta)U_n \end{aligned}$$

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Numerical Examples

Normal form of the 1:1 resonance ($a = b = 1$)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + \alpha_1 + \alpha_2 x_2 + x_1^2 + x_1 x_2 \end{pmatrix}$$

Applying the starting method we obtain the linear transformations:

$$\alpha = \widetilde{K}(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta$$

$$x = \widetilde{H}(u, \delta) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} u + \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \delta$$

and an ϵ-dependent flow approximation (Beyn's method):

$$\begin{aligned} \delta_1 &= -\frac{1}{4}\epsilon^4 \\ \delta_2 &= -0.35714285714052\epsilon^2 \\ u_1(t) &= \frac{\epsilon^2}{2}\left(1 - 3\,\mathrm{sech}^2\left(\frac{\epsilon}{2}t\right)\right) \\ u_2(t) &= \frac{3\epsilon^3}{2}\,\mathrm{sech}^2\left(\frac{\epsilon}{2}t\right)\tanh\left(\frac{\epsilon}{2}t\right) + \frac{1}{2}\delta_1 \end{aligned}$$

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Numerical Examples

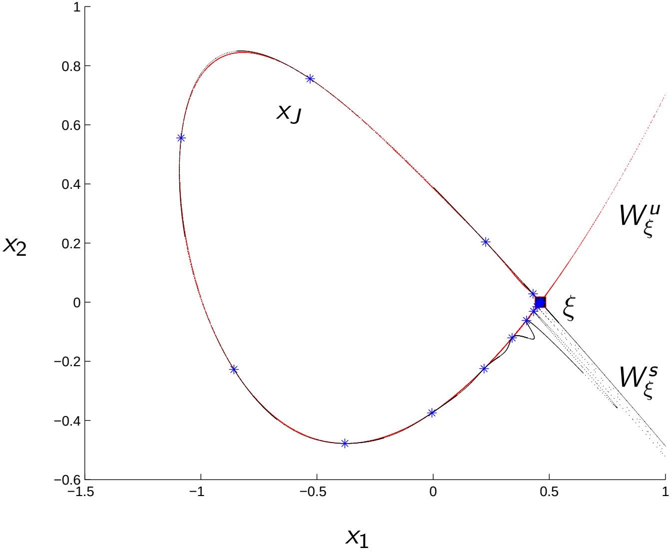
Normal form of the 1:1 resonance ($a = b = 1$)

Choose $\epsilon = 0.9$, $N_- = -40$, $N_+ = 40$, $J = [N_-, N_+] \cap \mathbb{Z}$. After some Newton iterations we find a homoclinic tangency x_J, X_J at

$$(\alpha_1, \alpha_2) = (-0.213581806538199, -0.289285714285714)$$

Numerical Examples

Normal form of the 1:1 resonance ($a = b = 1$)



Numerical Examples

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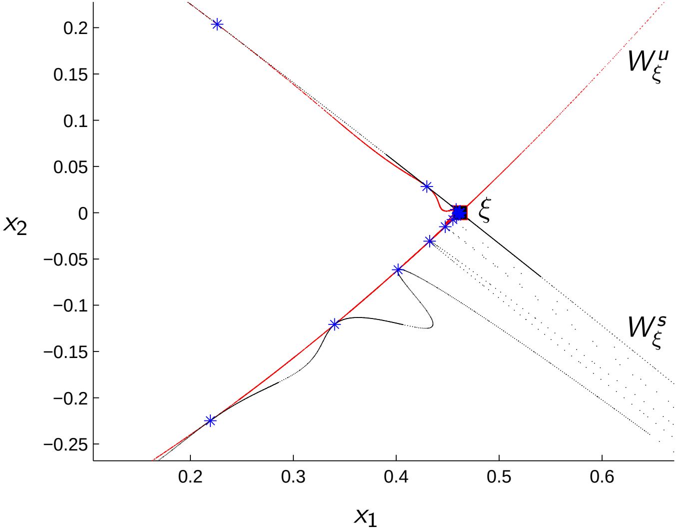

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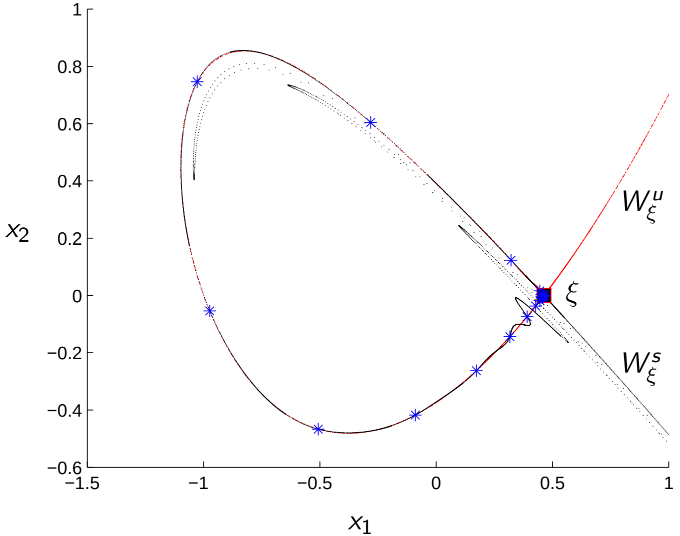
$$(\alpha_1, \alpha_2) = (-0.21528248893748, -0.28928571428382)$$



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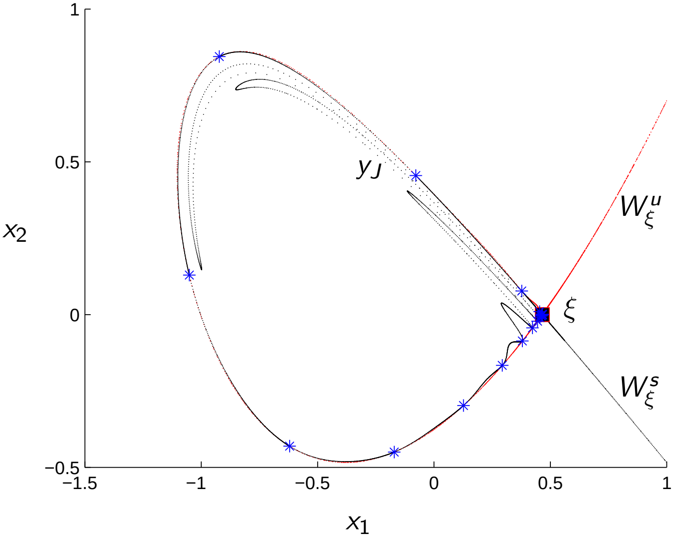
$$(\alpha_1, \alpha_2) = (-0.2167231774074, -0.28928571428382)$$



Numerical Examples

Discretization of the Normal form of the Bogdanov-Takens Bifurcation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \alpha_1 + \alpha_2 x_2 + x_1^2 + x_1 x_2 \end{cases}$$

The discretization via Euler's method has a R1 point at the origin (Lóczi, Páez, Int. J. Qual. Theory Differ. Equ. Appl., 2009)
By applying the starting method with $h = 0.3$, $\epsilon = 0.15$, $N_- = -70$, $N_+ = 70$, $J = [N_-, N_+] \cap \mathbb{Z}$ we find a homoclinic tangency x_J, X_J at

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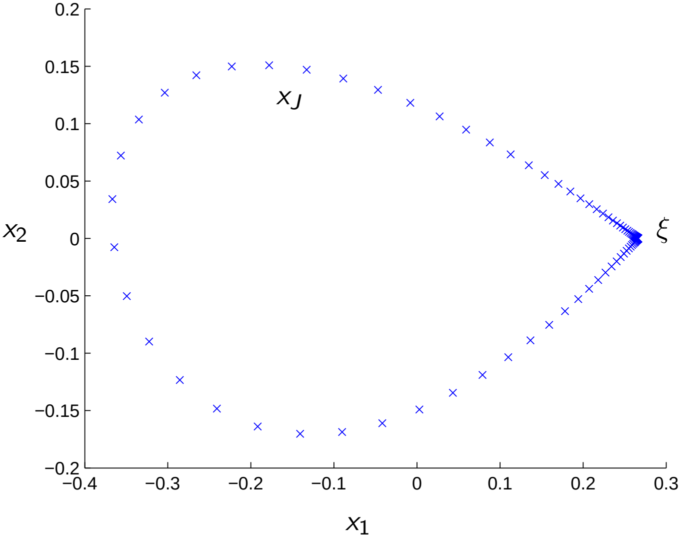

Numerical Examples
Hénon 3D map

Consider the following three-dimensional version of the Hénon map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \alpha_2 + \alpha_1 z - x^2 \\ x \\ y \end{pmatrix}$$

This system undergoes an R1 bifurcation at $(x, y, z) = (-0.75, -0.75, -0.75)$, $(\alpha_1, \alpha_2) = (-0.5, -0.5625)$. By applying the starting procedure with $\epsilon = 0.8$, $N_- = -50$, $N_+ = 50$, $J = [N_-, N_+] \cap \mathbb{Z}$ we find a homoclinic tangency x_J, X_J at

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This system undergoes an R1 bifurcation at $(x, y, z) = (-0.75, -0.75, -0.75)$, $(\alpha_1, \alpha_2) = (-0.5, -0.5625)$. By applying the starting procedure with $\epsilon = 0.8$, $N_- = -50$, $N_+ = 50$, $J = [N_-, N_+] \cap \mathbb{Z}$ we find a homoclinic tangency x_J, X_J at

$$(\alpha_1, \alpha_2) = (-0.992975759928172, -0.478387567729272)$$

Numerical Examples
Hénon 3D map

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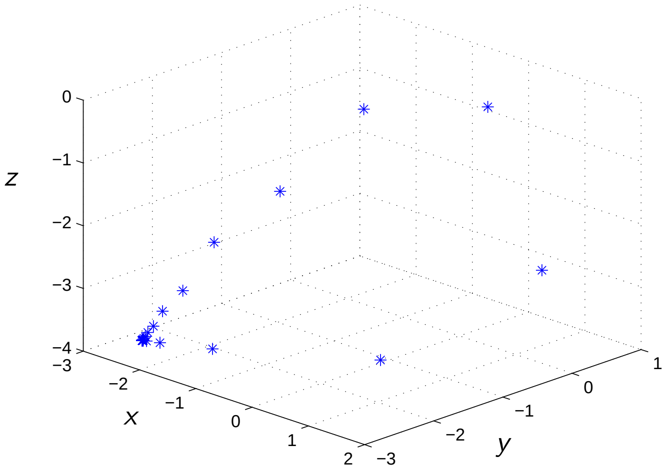
Numerical Examples

Hénon 3D map



Numerical Examples

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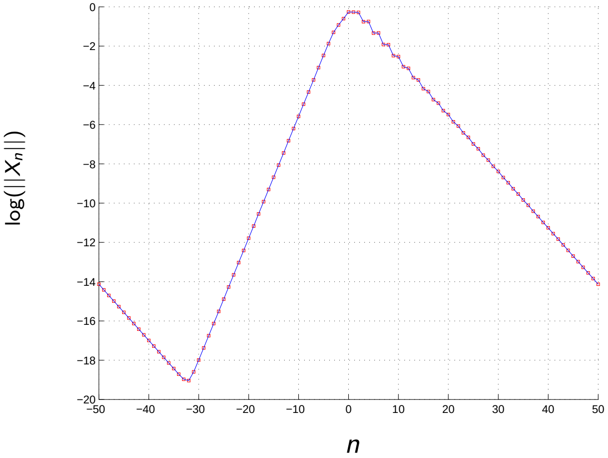

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