

STARTING HOMOCLINIC TANGENCIES NEAR 1:1 RESONANCES

Joseph Páez Chávez

Instituto de Ciencias Matemáticas
 Escuela Superior Politécnica del Litoral
 Guayaquil, Ecuador
 Email: jpaez@espol.edu.ec

ABSTRACT

In this presentation we construct a theory-based numerical method for starting the continuation of homoclinic tangencies near 1 : 1 resonances, for systems with arbitrary dimension ≥ 2 . The method is based on numerical center manifold reduction and flow approximation. The effectiveness of the method is illustrated by numerical examples.

INTRODUCTION

A typical problem in the numerical analysis of homoclinic orbits is the choice of an appropriate initial guess that could lead us, via e.g. Newton iterations, to the homoclinic connection we want to analyze. In our case we have a well-posed problem given in terms of an operator, whose solutions correspond to the numerical approximation of the homoclinic tangencies we are interested in (see e.g. [1]). Thus we will construct a theory-based starting procedure, by means of which we can obtain an “educated” initial guess of the solution of the underlying well-posed problem.

What is commonly done is to set a first approximating orbit to

$$(\dots, \xi, \dots, \xi, x_1, \dots, x_r, \xi, \dots, \xi, \dots),$$

where ξ represents an equilibrium point and $x_i, i = 1, \dots, r, r \in \mathbb{N}$ are, basically, randomly chosen vectors on the state space. This method is successfully applied, e.g., in [1]. This is of course a purely trial-and-error-based method, where the user relies entirely on “luck” or brute force. Therefore, the user has essen-

tially no control over the outcome of the Newton iterations, and it can easily happens that a spurious solution is obtained, which is a significant disadvantage. Thus, our main concern throughout this presentation will be the construction of a theory-based method that allows us to start the continuation of tangential homoclinic orbits near 1 : 1 resonances, with no restriction of the dimension of the system. This method will enable us to quantitatively explore the homoclinic structure in various interesting examples, since homoclinic orbits are one of the most fascinating objects of study in the theory of dynamical systems, because their presence leads to nontrivial dynamics.

BASIC SETUP

Let $f(\cdot, \alpha), f \in C^k(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^N), k \geq 1$, be a diffeomorphism for all $\alpha \in \mathbb{R}^2$. Throughout this presentation we consider the discrete-time system

$$x \mapsto f(x, \alpha). \tag{1}$$

Suppose that $\xi \in \mathbb{R}^N$ is a hyperbolic equilibrium of (1) at $\alpha = \alpha_0$. An orbit $x_{\mathbb{Z}} \in (\mathbb{R}^N)^{\mathbb{Z}}$ of (1) is referred to as homoclinic with respect to ξ (in short homoclinic) if

$$\lim_{n \rightarrow \pm\infty} x_n = \xi.$$

Further, a homoclinic orbit $x_{\mathbb{Z}}$ of (1) is referred to as tangential, if the stable and unstable manifolds of ξ intersect tangentially along the connecting orbit $x_{\mathbb{Z}}$.

We will restrict our study to tangential homoclinic orbits near 1 : 1 resonances. We say that the point $(x_R, \alpha_R) \in \mathbb{R}^N \times \mathbb{R}^2$ is a 1 : 1 resonance of (1), if $f(x_R, \alpha_R) = x_R$ and the matrix $f_x(x_R, \alpha_R)$ has a double, defective eigenvalue $\mu_{1,2} = 1$ and no other eigenvalues on the unit circle.

The normal form of the 1 : 1 resonance is known to be (cf. [2, Lemma 9.7])

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto N_v(\xi) := \begin{pmatrix} \xi_1 + \xi_2 \\ \xi_2 + v_1 + v_2 \xi_2 + A(v) \xi_1^2 + B(v) \xi_1 \xi_2 \end{pmatrix} + O(\|\xi\|^3),$$

where $A(\cdot), B(\cdot)$ depend smoothly on $v := (v_1, v_2) \in \mathbb{R}^2$, and $A(0)B(0) \neq 0, \xi_{1,2} \in \mathbb{R}$. This normal form can be approximated by the flow of a vector field for all sufficiently small $\|v\|$, that is

$$N_v(\xi) = \phi_v^1(\xi) + O(\|v\|^2) + O(\|\xi\|^2 \|v\|) + O(\|\xi\|^3),$$

where ϕ_v^1 is the flow of a smooth planar system, whose dynamics are described by the Bogdanov-Takens bifurcation theory. Therefore, a tangential homoclinic orbit of the normal form can be approximated by a homoclinic orbit of the vector field, for which several starting procedures are available, see e.g. [3]. Once we have constructed an approximating homoclinic orbit for the normal form of the 1 : 1 resonance, we can transform this orbit to an approximating orbit for a general system (1), via numerical center manifold reduction. For this purpose, the homological equation (cf. [2]) plays a central role. This equation has the form

$$f(H(\xi, v), K(v)) = H(N_v(\xi), v), \quad (2)$$

where $H : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^N$ and $K : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are locally defined, smooth functions that represent a parametrization of a center manifold of (1) and a parameter transformation, respectively. By (2), we can obtain linear approximations of the functions H, K , and hence a homoclinic orbit of the normal form can be transformed into an approximating homoclinic orbit of the general system (1). The resulting starting procedure can be found in [4].

NUMERICAL EXAMPLES

Consider the following three-dimensional version of the Hénon map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \alpha_2 + \alpha_1 z - x^2 \\ x \\ y \end{pmatrix}.$$

This system undergoes a 1 : 1 resonance at $(x, y, z) = (-0.75, -0.75, -0.75), (\alpha_1, \alpha_2) = (-0.5, -0.5625)$. With the procedure described in the previous section, we find a homoclinic tangency at $(\alpha_1, \alpha_2) \approx (-1.144, -0.233)$. In the following picture we show the phase plot of the computed orbit.

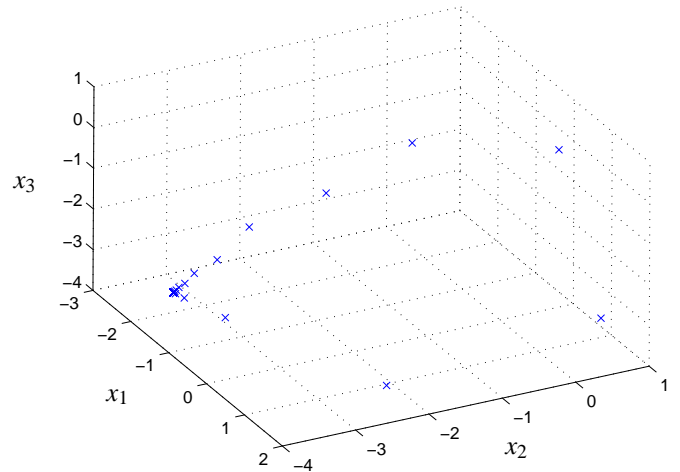


Figure 1. TANGENTIAL HOMOCLINIC ORBIT OF THE HÉNON MAP.

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