Discretizing the Generalized Hopf Bifurcation

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Overview

- Description of the Problem
- Discretizing a GH Point
- Discretizing the Bifurcation Diagram
- Discretizing the First Lyapunov Coefficients
- Numerical Experiments

Description of the Problem

Consider

$$\dot{x}(t) = f(x(t), \alpha), \tag{ODE}$$

where $f \in C^k(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^N)$, $k \geq 1$ sufficiently large, $N \geq 2$.

A Generalized Hopf Bifurcation (GH) is an equilibrium $(x_0, \alpha_0) \in \mathbb{R}^N \times \mathbb{R}^2$ of (ODE) such that:

- $f_x(x_0, \alpha_0)$ has eigenvalues $\pm i\omega_0$, $0 < \omega_0 \in \mathbb{R}$, and
- the first Lyapunov Coefficient vanishes at (x_0, α_0).

The GH point is generic if:

- The critical eigenvalues cross the imaginary axis with non-zero velocity, and
- the first Lyapunov coefficient vanishes with non-zero velocity.

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Description of the Problem

One-Step Discretization

We consider general one-step methods of order $p \geq 1$ applied to (ODE), given by

$$x \mapsto \psi^h(x,\alpha) := x + h\Phi(h,x,\alpha), \qquad \text{(OSM)}$$

where $\Phi(\cdot,\cdot,\cdot)$ is defined over compact sets, h is the step-size. Order $p \geq 1$: There exists $C_0 > 0$ such that

$$||\varphi^h(x,\alpha) - \psi^h(x,\alpha)|| \leq C_0|h|^{p+1},$$

where $\varphi^t(\cdot,\alpha)$ stands for the t-flow of (ODE). In this setting, there exist smooth functions Υ, Ξ such that

$$\begin{aligned} \psi^h(x,\alpha) &= \varphi^h(x,\alpha) + \Upsilon(h,x,\alpha)h^{p+1} \\ \Phi(h,x,\alpha) &= f(x,\alpha) + \Xi(h,x,\alpha)h \end{aligned} \qquad \text{(TAYLOR)}$$

hold locally.

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Description of the Problem
First Lyapunov Coefficient

We use the following operators:

$$A_\theta := \theta_x(x,\alpha),$$

$$B_\theta(v,w) := \theta_{xx}(x,\alpha)[v,w] := \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2\theta(x,\alpha)}{\partial x_j \partial x_i} v_i w_j,$$

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where $v,w,z \in \mathbb{R}^N$, and $\theta : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}^N$ is sufficiently smooth.

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Description of the Problem

First Lyapunov Coefficient

The First Lyapunov Coefficient is computed via (Kuznetsov, Elements of Applied Bifurcation Theory, 2004)

$$L_H := \frac{1}{2}\operatorname{Re}\langle p, C_f(q,q,\overline{q}) - 2B_f(q, A_f^{-1}B_f(q,\overline{q})) + B_f(\overline{q}, (2i\omega_0 I_N - A_f)^{-1}B_f(q,q))\rangle,$$

where $q, p \in \mathbb{C}^N$ satisfy

$$A_f q = i\omega_0 q, \qquad p^T A_f = -i\omega_0 p^T$$

and $\langle p, q\rangle := \overline{p}^T q = 1$. In the discrete-time case (Generalized Neimark–Sacker bifurcation, in short GN) we have

$$L_{NS} := \frac{1}{2}\operatorname{Re} e^{-i\theta_0} d,$$

where

$$d := \langle \widetilde{p}, C_g(\widetilde{q},\widetilde{q},\overline{\widetilde{q}}) - 2B_g(\widetilde{q}, (A_g - I_N)^{-1}B_g(\widetilde{q},\overline{\widetilde{q}})) + B_g(\overline{\widetilde{q}}, (e^{2i\theta_0} I_N - A_g)^{-1}B_g(\widetilde{q},\widetilde{q}))\rangle.$$

The vectors $\widetilde{q}, \widetilde{p} \in \mathbb{C}^N$ satisfy analogous relations (see above) for a discrete-time system $x \mapsto g(x, \alpha)$.

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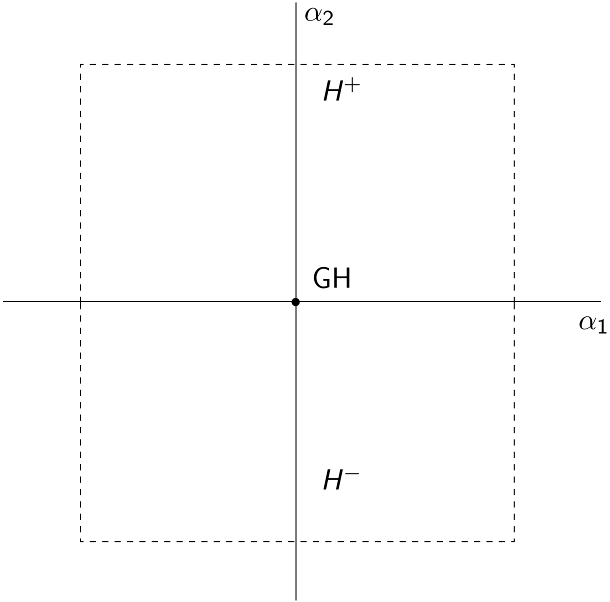
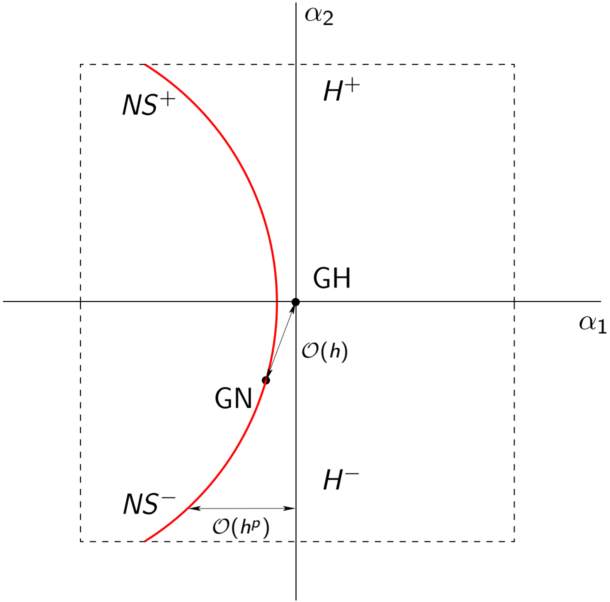
Description of the Problem

GH - Bifurcation Diagram (without the fold curve of cycles)



Description of the Problem
GH - Discretized Bifurcation Diagram



Discretizing a GH Point

Theorem

Let (ODE) have a generic GH point at $(x_{GH}, \alpha_{GH}) \in \mathbb{R}^N \times \mathbb{R}^2$*. Apply a one-step method (OSM) of order* $p \geq 1$*. Then (OSM) has a GN point* $(x_{GN}(h), \alpha_{GN}(h))$ *that satisfies*

$$||(x_{GN}(h), \alpha_{GN}(h)) - (x_{GH}, \alpha_{GH})|| \leq C|h|,$$

for all $h \in (-\rho, \rho)$*, where* C *and* ρ *are positive constants.*

Remark. For the fold-Hopf case we have that (J. Páez Chávez, Internat. J. of Bif. and Chaos 20, No. 5, 2010)

$$||(x_{FN}(h), \alpha_{FN}(h)) - (x_{FH}, \alpha_{FH})|| \leq C|h|^p.$$

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Discretizing a GH Point

Ideas of the Proof

A GH point is a solution of the system (Govaerts et al., SIAM J. Numer. Anal. 38, No. 1, 2000)

$$
\begin{cases}
f(x,\alpha) = 0, \\
f_x(x,\alpha)q - i\omega q = 0, \\
f_x^T(x,\alpha)p + \lambda p = 0, \\
\langle q, q_0 \rangle - 1 = 0, \\
\langle p, q \rangle - 1 = 0, \\
A_f v - B_f(q,\overline{q}) = 0, \\
(2i\omega I_N - A_f)w - B_f(q,q) = 0, \\
\mathrm{Re}\langle p, C_f(q,q,\overline{q}) - 2B_f(q,v) + B_f(\overline{q},w)\rangle = 0.
\end{cases}
$$

Write the real form of this system as $F(x,\alpha,z) = 0$, where $z = (q_1, q_2, p_1, p_2, v, w_1, w_2, \omega, \lambda_1, \lambda_2)$, $q_1 = \mathrm{Re}\, q$, etc.Then $F'(x_{GH}, \alpha_{GH}, z_{GH})$ is invertible.

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Similarly, a GN point of the one-step map (OSM) is a solution of

$$\begin{cases} \frac{1}{h}\left(\psi^h(x,\alpha) - x\right) = 0, \\ \frac{1}{h}\left(\psi^h_x(x,\alpha)q - e^{ih\omega}q\right) = 0, \\ \frac{1}{h}\left((\psi^h_x(x,\alpha))^T p - e^{-h\lambda}p\right) = 0, \\ \langle q, q_0\rangle - 1 = 0, \\ \langle p, q\rangle - 1 = 0, \\ \frac{1}{h}\left((A_{\psi^h} - I_N)v - B_{\psi^h}(q,\overline{q})\right) = 0, \\ \frac{1}{h}\left((e^{2ih\omega} I_N - A_{\psi^h})w - B_{\psi^h}(q,q)\right) = 0, \\ \frac{1}{h}\operatorname{Re} e^{-ih\omega}\langle p, C_{\psi^h}(q,q,\overline{q}) - 2B_{\psi^h}(q,v) + B_{\psi^h}(\overline{q},w)\rangle = 0. \end{cases}$$

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We need this system to be nontrivial at $h = 0$. Write the real form of this system as $G(h, x, \alpha, z) = 0$.

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$$G(h, x, \alpha, z) = F(x, \alpha, z) + \mathcal{O}(h),$$

because

$$\frac{1}{h}\left(\psi^h(x, \alpha) - x\right) = \Phi(h, x, \alpha) = f(x, \alpha) + \mathcal{O}(h)$$

and so on. Thus $G(0, x_{GH}, \alpha_{GH}, z_{GH}) = 0$ and $G_{(x,\alpha,z)}(0, x_{GH}, \alpha_{GH}, z_{GH}) = F'(x_{GH}, \alpha_{GH}, z_{GH})$ is invertible.Therefore, the Implicit Function Theorem gives functions $(x_{GN}(h), \alpha_{GN}(h), z_{GN}(h))$ with

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for all h near zero. And also we have

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$$||(x_{GN}(h), \alpha_{GN}(h)) - (x_{GH}, \alpha_{GH})|| \leq C|h|.$$

Discretizing a GH Point

Ideas of the Proof

Due to (TAYLOR) we have that

$$G(h, x, \alpha, z) = F(x, \alpha, z) + \mathcal{O}(h),$$

because

$$\frac{1}{h}\left(\psi^h(x, \alpha) - x\right) = \Phi(h, x, \alpha) = f(x, \alpha) + \mathcal{O}(h)$$

and so on. Thus $G(0, x_{GH}, \alpha_{GH}, z_{GH}) = 0$ and $G_{(x,\alpha,z)}(0, x_{GH}, \alpha_{GH}, z_{GH}) = F'(x_{GH}, \alpha_{GH}, z_{GH})$ is invertible.Therefore, the Implicit Function Theorem gives functions $(x_{GN}(h), \alpha_{GN}(h), z_{GN}(h))$ with

$$(x_{GN}(0), \alpha_{GN}(0), z_{GN}(0)) = (x_{GH}, \alpha_{GH}, z_{GH}) \text{ and}$$
$$G(h, x_{GN}(h), \alpha_{GN}(h), z_{GN}(h)) = 0$$

for all h near zero. And also we have

$$||(x_{GN}(h), \alpha_{GN}(h)) - (x_{GH}, \alpha_{GH})|| \leq C|h|.$$

Discretizing the Bifurcation Diagram

Theorem

Let (ODE) *have a generic GH point at the origin* $(x_{GH}, \alpha_{GH}) = (0, 0)$*. Apply a one-step method* (OSM) *of order* $p \geq 1$*. Then* (OSM) *has for every* h *sufficiently small a curve of Neimark–Sacker points which lies* $O(h^p)$*-close to the Hopf curve of* (ODE) *emanating from the GH point.*

Discretizing the Bifurcation Diagram

Ideas of the Proof

A curve of Hopf points of (ODE) is a solution of (Griewank et al., IMA J. Numer. Anal. 3, No. 3, 1983)

$$\begin{cases} f(x,\alpha) = 0, \\ f_x(x,\alpha)q - i\omega q = 0, \\ \langle q, q_0 \rangle - 1 = 0, \end{cases}$$

Write the real form of this system as $\widetilde{F}(x,\alpha,z) = 0$, $z := (\mathrm{Re}\, q, \mathrm{Im}\, q, \omega)$. This system can be also be written in terms of the flow of (ODE) $\left(\varphi_x^h(x_{eq},\alpha_{eq}) = e^{hf_x(x_{eq},\alpha_{eq})}\right)$

$$\begin{cases} \frac{1}{h}\left(\varphi^h(x,\alpha) - x\right) = 0, \\ \frac{1}{h}\left(\varphi_x^h(x,\alpha)q - e^{ih\omega}q\right) = 0, \\ \langle q, q_0 \rangle - 1 = 0. \end{cases}$$

We write its real form as $F(h,x,\alpha,z) = 0$.

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We write its real form as $F(h,x,\alpha,z) = 0$.

Similarly, a curve of Neimark–Sacker points of the one-step map (OSM) is a solution of

$$\begin{cases} \frac{1}{h}\left(\psi^h(x,\alpha) - x\right) = 0, \\ \frac{1}{h}\left(\psi_x^h(x,\alpha)q - e^{ih\omega}q\right) = 0, \\ \langle q, q_0\rangle - 1 = 0. \end{cases}$$

We write its real form as $G(h, x, \alpha, z) = 0$. Again, due to (TAYLOR) we have that

$$G(h, x, \alpha, z) = F(h, x, \alpha, z) + \mathcal{O}(h^p),$$

because

$$\frac{1}{h}\left(\psi^h(x,\alpha) - x\right) = \frac{1}{h}\left(\varphi^h(x,\alpha) - x\right) + \mathcal{O}(h^p)$$

and so on. It also follows that

$$G(h, x, \alpha, z) = \widetilde{F}(x, \alpha, z) + \mathcal{O}(h)$$

$$\Rightarrow G_{(x,\alpha,z)}(0, x_{GH}, \alpha_{GH}, z_{GH}) = \widetilde{F}'(x_{GH}, \alpha_{GH}, z_{GH}) \text{ has full rank.}$$

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Discretizing the Bifurcation Diagram

Similarly, a curve of Neimark–Sacker points of the one-step map (OSM) is a solution of

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Discretizing the Bifurcation Diagram
Ideas of the Proof

Summarizing, we have that

- $G(h, x, \alpha, z) = \widetilde{F}(x, \alpha, z) + \mathcal{O}(h)$
- $G(h, x, \alpha, z) = F(h, x, \alpha, z) + \mathcal{O}(h^p)$

Thus

- By applying the Implicit Function Theorem to G we prove the existence of a Neimark–Sacker curve of (OSM).
- By applying the Inverse Lipschitz Mapping Theorem (perturbation form) to $G(h, \cdot, \cdot, \cdot)$ we show the $\mathcal{O}(h^p)$-closeness.

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Discretizing the First Lyapunov Coefficients

Recall the GH-defining-system $F(x, \alpha, z) = 0$:

$$\begin{cases} f(x,\alpha)=0, \\ f_x(x,\alpha)q-i\omega q=0, \\ f_x^T(x,\alpha)p+\lambda p=0, \\ \langle q,q_0\rangle-1=0, \\ \langle p,q\rangle-1=0, \\ A_f v-B_f(q,\overline{q})=0, \\ (2i\omega I_N-A_f)w-B_f(q,q)=0, \\ \mathrm{Re}\langle p,C_f(q,q,\overline{q})-2B_f(q,v)+B_f(\overline{q},w)\rangle=0. \end{cases}$$

and the GN-defining-system $G(h, x, \alpha, z) = 0$:

$$\begin{cases} \frac{1}{h}\left(\psi^h(x,\alpha)-x\right)=0, \\ \frac{1}{h}\left(\psi_x^h(x,\alpha)q-e^{ih\omega}q\right)=0, \\ \frac{1}{h}\left((\psi_x^h(x,\alpha))^T p-e^{-h\lambda}p\right)=0, \\ \langle q,q_0\rangle-1=0, \\ \langle p,q\rangle-1=0, \\ \frac{1}{h}\left((A_{\psi^h}-I_N)v-B_{\psi^h}(q,\overline{q})\right)=0, \\ \frac{1}{h}\left((e^{2ih\omega}I_N-A_{\psi^h})w-B_{\psi^h}(q,q)\right)=0, \\ \frac{1}{h}\,\mathrm{Re}\,e^{-ih\omega}\langle p,C_{\psi^h}(q,q,\overline{q})-2B_{\psi^h}(q,v)+B_{\psi^h}(\overline{q},w)\rangle=0. \end{cases}$$

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Discretizing the First Lyapunov Coefficients

We have seen before that

$$G(h, x, \alpha, z) = F(x, \alpha, z) + \mathcal{O}(h).$$

From the last component of this relation it can be shown that

$$L_{NS}(h, s) = h\left(L_H(s) + \mathcal{O}(h)\right)$$

holds along the emanating Hopf curve (s being a real parameter).

- Hence we can prove that L_H and $L_{NS}(h, \cdot)$ change their sign in the same direction, as they pass through the critical points (GH and GN, respectively), for every positive, sufficiently small h.
- In particular, $L_{NS}(h, \cdot)$ changes sign with non-zero velocity (genericity is also preserved!).

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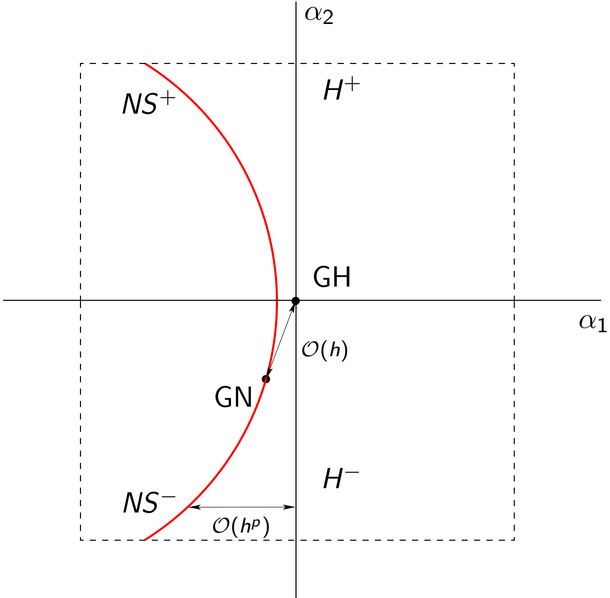
Discretizing the First Lyapunov Coefficients

For the discretized bifurcation diagram this means:



Numerical Experiments

Consider the following system

$$\dot{x} = \mu^{+}\left(\frac{\alpha}{\beta + x} - k\right) - \mu^{-}\gamma xyz,$$
$$\dot{y} = \epsilon x - \gamma z - \rho,$$
$$\dot{z} = \sigma\gamma y - \delta,$$

which is a modified version of a system that describes the dynamics of corruption in democratic societies (Rinaldi et al., Complexity 3, No. 5, 1998). We choose (δ, ϵ) as our bifurcation parameters and let $\mu^{+} = k = \gamma = 1$, $\mu^{-} = 10$, $\alpha = 1.5$, $\beta = 0.5$, $\rho = 0.1$ and $\sigma = 2$ fixed. For our computations we use the continuation software MATCONT.

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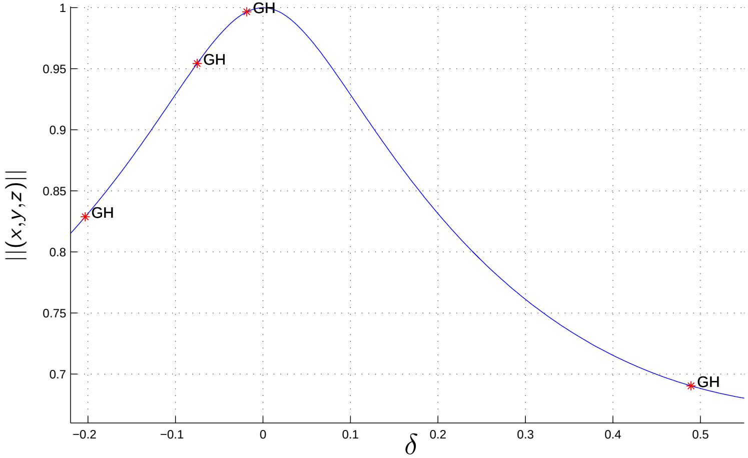
Numerical Experiments
Bifurcation Diagram of the System



Numerical Experiments

- We find a GH point at $(x, y, z) \approx (0.591512, 0.244627, 0.258631)$, $(\delta, \epsilon) \approx (0.489254, 0.606295)$ and $\omega_0 \approx 1.711003$.
- After Applying the classical Runge–Kutta method of order 4 (h=0.3) we find a GN point at $(x, y, z) \approx (0.592544, 0.243995, 0.257952)$, $(\delta, \epsilon) \approx (0.48799, 0.604093)$ and $\frac{\theta_0}{h} \approx 1.709335$.

Now define the distance function

$$Dist_{GH}(h) := ||(x_{GN}(h), y_{GN}(h), z_{GN}(h), \delta_{GN}(h), \epsilon_{GN}(h)) - (x_{GH}, y_{GH}, z_{GH}, \delta_{GH}, \epsilon_{GH})||,$$

for $h > 0$ small, where $|| \cdot ||$ represents the Euclidean norm and $(x_{GN}(h), y_{GN}(h), z_{GN}(h), \delta_{GN}(h), \epsilon_{GN}(h))$ stand for a GN point of the Runge–Kutta map.

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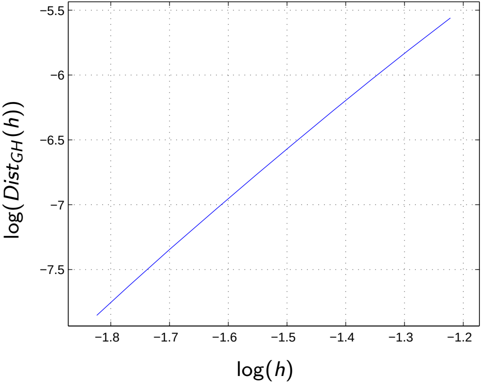
Numerical Experiments

Behavior of $Dist_{GH}$ for the classical Runge-Kutta method

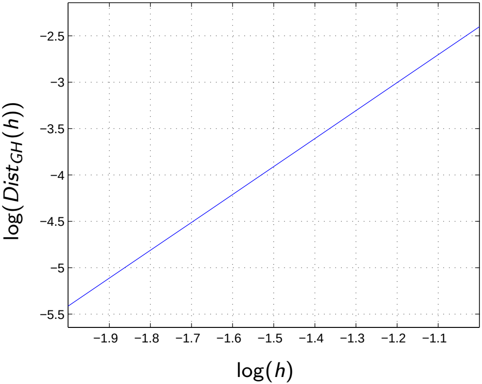


According to this figure we have $Dist_{GH}(h) = \mathcal{O}(h^r)$, where

$$r = \text{slope} \approx 4$$

Numerical Experiments

Behavior of $Dist_{GH}$ for the method of Runge (order $p = 3$)



According to this figure we have $Dist_{GH}(h) = \mathcal{O}(h^r)$, where

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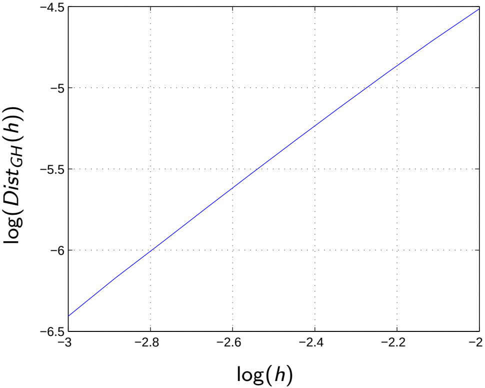
Numerical Experiments

Behavior of $Dist_{GH}$ for the method of Heun (order $p = 2$)



According to this figure we have $Dist_{GH}(h) = \mathcal{O}(h^r)$, where

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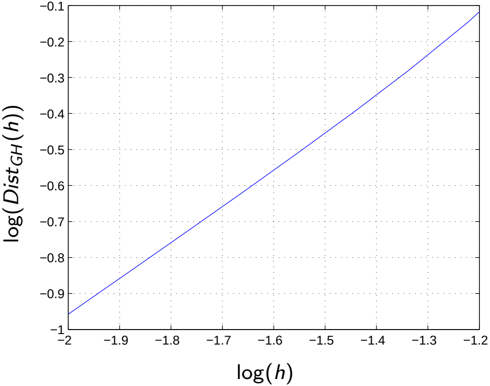
Numerical Experiments

Behavior of $Dist_{GH}$ for the method of Euler (order $p = 1$)



According to this figure we have $Dist_{GH}(h) = \mathcal{O}(h^r)$, where

$$r = \text{slope} \approx 1$$

Numerical Experiments

Remarks

- Our analysis predicts (for all methods) that $Dist_{GH}(h) = \mathcal{O}(h)$.
- Question: What is the optimal order?
- In this experiment: $Dist_{GH}(h) = \mathcal{O}(h^p)$, p being the order of the method.

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