A short Introduction to Dynamical Systems

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Abstract

In this manuscript the concept of dynamical systems is introduced. This definition is motivated and illustrated in detail by several examples. A classification of dynamical systems is given too.

Resumen

En este manuscrito se presenta el concepto de sistema dinámico. Esta definición es motivada e ilustrada en detalle a través de algunos ejemplos. También se presenta una clasificación de los sistemas dinámicos.

Keywords: Dynamical Systems, discrete-time systems, continuous-time systems, mathematical modeling.

Introduction

The notion of dynamical systems first appeared when Newton combined the concept of ordinary differential equations (ODEs) with mechanics. However, the modern approach of dynamical systems theory is due to Heinri Poincaré. In 1890 he analyzed the stability of the solar system and the three-body problem by means of ODEs. In order to simplify the analysis, Poincaré introduced a surface perpendicular to the orbits described by the planets, in such a way that instead of considering the whole trajectories, he studied their intersections with the transversal surface. In that way the study of discrete dynamical systems began. This approach gave later rise to the well-known Poincaré map.

Nowadays, dynamical systems play a central role in many branches of applied sciences, such as: physics, chemistry, biology, economy and even social sciences, among others. One

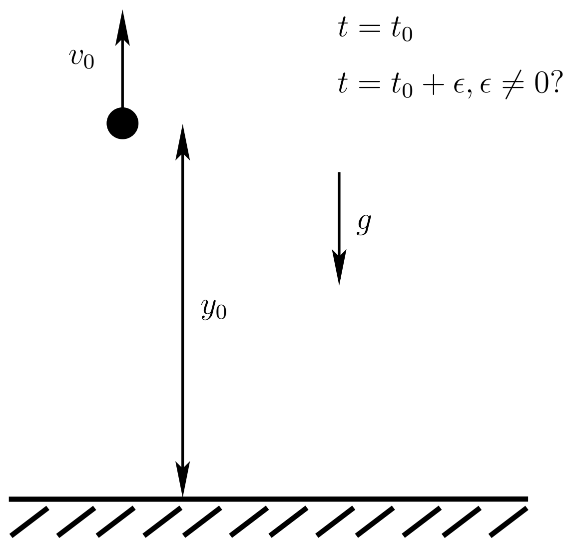

Fig. 1.1: Two-body system.

main reason for this is that a dynamical system is nothing else but the mathematical formalization of a deterministic process. This allows us to study very (apparently) different mathematical models under a compact setting, such that a broader overview and a deeper understanding of the process behind the models is achieved. For instance, by means of the dynamical systems theory we are able to understand that e.g. ODEs, difference equations, and some types of partial differential equations (PDEs) share abstract properties which will enable us to study these objects in the very same setting and in a compact manner.

Most of the material presented in this manuscript can be found in the lectures notes of Prof. Beyn, [3, 4]. Literature on this subject is numerous. The reader may find an interesting introduction, motivation and applications of dynamical systems theory in the monographs [1, 2, 5, 6, 8].

1 Basic Concepts

Let us first introduce the abstract definition of dynamical systems. For this purpose, we will motivate the reader by a very simple problem of physics that is known from high school, namely, the two-body problem. Consider an idealized piece of the earth's surface and a small body of mass negligible with respect to the mass of the earth, see Figure 1.1. In order to perform a mathematical description of this two-body system, we need first, as usual, to choose proper variables. In this physical setting we have in principle three variables at hand, namely, position, velocity and acceleration. It is clear that the position (or the velocity) of the body alone does not fully determine the state of the system. On the other hand, in this idealized problem, the acceleration is assumed to be constant and equal to the gravity. Therefore, the variables we must consider for the description of the system are position and velocity.

Of course we also need to have a notion of time, which in our case is a real number. Thus, the problem we are dealing with is how to predict the behavior of the system knowing its initial state at $t = t_0$, i.e.: What will the state of the system be at $t = t_0 + \epsilon$, $\epsilon \neq 0$? Note that we allow negative values of ϵ, which means that we want to know not only future states of the system but also past states of the system, provided we are given an initial state. These questions are answered by a set of equations which models the motion of the body. The underlying equations are given in terms of the following function $\varphi^{(\cdot)}(\cdot) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi^t \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} y_0 + v_0 t - \frac{1}{2} g t^2 \\ v_0 - g t \end{pmatrix},$$

where y_0, v_0 represent the initial position and velocity of the body, respectively, at $t_0 = 0$ (see Figure 1.1), and g is the gravity constant. In short, we have illustrated three ingredients of a dynamical system, which are:

- time,
- state set,
- function that describes the evolution of the system.

Let us now present the formal definition of dynamical systems:

Definition 1.1. *A dynamical system is a triple* $\{\mathbb{T}, X, \{\varphi^t\}_{t\in\mathbb{T}}\}$, *where* $\mathbb{T}$ *is a time set,* X *is a state space, and* $\varphi^t : X \to X$ *is a family of operators parametrized by* $t \in \mathbb{T}$, *such that:*

DS1 $\forall u \in X : \varphi^0(u) = u$, *i.e.,* $\varphi^0 = Id_X$,

DS2 $\forall u \in X, \forall s, t \in \mathbb{T} : \varphi^{t+s}(u) = \varphi^t(\varphi^s(u))$, *i.e.,* $\varphi^t \circ \varphi^s = \varphi^{t+s}$.

Some remarks about this definition are in order:

- Throughout this manuscript, X stands for a metric space[1].
- φ^t is referred to as evolution operator.
- The time set $\mathbb{T}$ has the following properties:
  - $\exists 0 \in \mathbb{T}, \forall t \in \mathbb{T} : t + 0 = t$,
  - $\forall t, s \in \mathbb{T} : s + t \in \mathbb{T}$,
  - $\forall t, s \in \mathbb{T} : s + t = t + s$.

  This means that $\mathbb{T}$ equipped with the operation + is a commutative semigroup.
- **DS1** reflects the fact that the system does not change its state spontaneously, i.e., "no time, no evolution".

[1] In general, it suffices that X is a topological space.

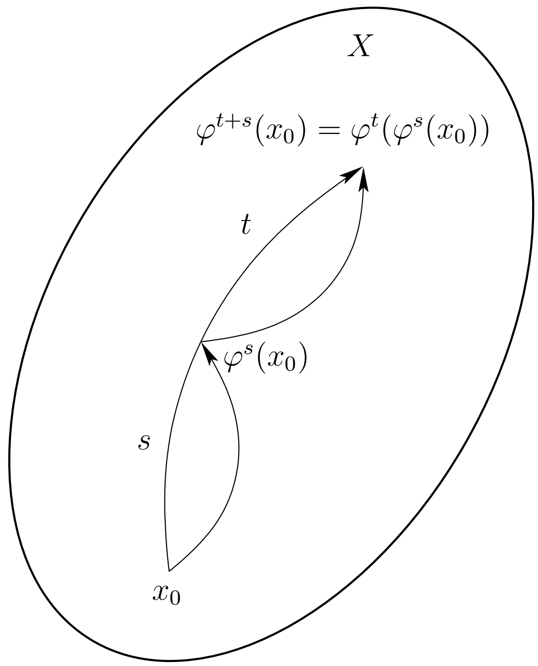

Fig. 1.2: Illustration of **DS2**.

- **DS2** means that the laws governing the system do not change in time, i.e., the process is autonomous and deterministic. This property is schematically depicted in Figure 1.2. In this picture we illustrate two ways of arriving at the same final state from an initial state. The first one consists in letting the system run from the starting point x_0 to the final state $\varphi^{t+s}(x_0)$, which takes $t + s$ units of time. The other one is first letting the system run from x_0 to the state $\varphi^s(x_0)$, and then using this point as initial state to arrive at the final point $\varphi^t(\varphi^s(x_0))$, after t units of time. **DS2** guarantees that this two procedures lead the system to the same final state.

Let us show some examples to illustrate the concept of dynamical systems:

Example 1.2. Consider the two-body problem introduced before (see Figure 1.1). We will verify that $\{\mathbb{R}, \mathbb{R}^2, \{\varphi^t\}_{t\in\mathbb{R}}\}$ is a dynamical system. Let us begin with **DS1**:

$$\forall \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} \in \mathbb{R}^2 : \varphi^0 \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} y_0 + v_0 t - \frac{1}{2} g t^2 \\ v_0 - gt \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}.$$

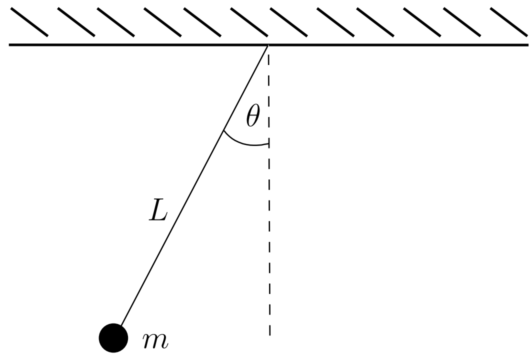

Fig. 1.3: Simple pendulum.

As for **DS2**, it holds that:

$$\forall \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} \in \mathbb{R}^2, \forall s,t \in \mathbb{R} : \varphi^t \left(\varphi^s \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} \right) = \varphi^t \begin{pmatrix} y_0 + v_0 s - \frac{1}{2} g s^2 \\ v_0 - gs \end{pmatrix},$$

$$= \begin{pmatrix} y_0 + v_0 s - \frac{1}{2} g s^2 + (v_0 - gs)t - \frac{1}{2} g t^2 \\ v_0 - gs - gt \end{pmatrix},$$

$$= \begin{pmatrix} y_0 + v_0 (t+s) - \frac{1}{2} g (t+s)^2 \\ v_0 - g(t+s) \end{pmatrix},$$

$$= \varphi^{t+s} \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}.$$

Therefore, the triple $\left\{ \mathbb{R}, \mathbb{R}^2, \{\varphi^t\}_{t\in\mathbb{R}} \right\}$ is a dynamical system.

Example 1.3. Consider a pendulum of mass m attached to a string of length L, which is displaced by an angle from the vertical rest position, see Figure 1.3. If we assume the amplitude of oscillation to be sufficiently small, then the dynamics of the pendulum is approximately described by the ODE

$$\theta'' + \frac{g}{L}\theta = 0,$$

whose general solution is given by

$$\theta(t) = \frac{\theta_0'}{\omega} \sin(\omega t) + \theta_0 \cos(\omega t), \quad \omega = \sqrt{\frac{g}{L}}, \quad t \in \mathbb{R}.$$

As in the two-body problem, it is clear that knowing the angle θ at certain value of time is not sufficient to uniquely define the state of the system. Therefore, the function that describes the evolution of the system should provide another physical quantity, e.g. angular velocity. Thus, such a function may be given by $\varphi^{(\cdot)}(\cdot) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi^t \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix} = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix} = \begin{pmatrix} \frac{\theta_0'}{\omega} \sin(\omega t) + \theta_0 \cos(\omega t) \\ \theta_0' \cos(\omega t) - \theta_0 \omega \sin(\omega t) \end{pmatrix},$$

where θ_0, θ_0' represent the initial angle and angular velocity of the pendulum, respectively, at $t=0$. Let us see whether $\left\{\mathbb{R}, \mathbb{R}^2, \{\varphi^t\}_{t\in\mathbb{R}}\right\}$ is a dynamical system. We begin with **DS1**:

$$\forall \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix} \in \mathbb{R}^2 : \varphi^0 \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix} = \left. \begin{pmatrix} \frac{\theta_0'}{\omega}\sin(\omega t) + \theta_0 \cos(\omega t) \\ \theta_0' \cos(\omega t) - \theta_0 \omega \sin(\omega t) \end{pmatrix} \right|_{t=0} = \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix}.$$

For **DS2** we have that:

$$\begin{aligned} &\forall \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix} \in \mathbb{R}^2, \forall s, t \in \mathbb{R} : \\ &\varphi^t \left(\varphi^s \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix} \right) = \varphi^t \begin{pmatrix} \frac{\theta_0'}{\omega}\sin(\omega s) + \theta_0 \cos(\omega s) \\ \theta_0' \cos(\omega s) - \theta_0 \omega \sin(\omega s) \end{pmatrix}, \\ &= \begin{pmatrix} \frac{\theta_0'}{\omega}\sin(\omega(t+s)) + \theta_0 \cos(\omega(t+s)) \\ \theta_0' \cos(\omega(t+s)) - \theta_0 \omega \sin(\omega(t+s)) \end{pmatrix}, \\ &= \varphi^{t+s} \begin{pmatrix} \theta_0 \\ \theta_0' \end{pmatrix}. \end{aligned}$$

Therefore, the triple $\left\{\mathbb{R}, \mathbb{R}^2, \{\varphi^t\}_{t\in\mathbb{R}}\right\}$ is a dynamical system.

2 Classification of Dynamical Systems

As mentioned in the Introduction, dynamical systems play an important role in many applied sciences, and consequently a large variety of dynamical systems has been developed. In order to present the theory in a systematic manner, it is very useful to introduce a classification of dynamical systems. Common criteria used for classification are: time, state space, and invertibility of the evolution operator. Let us briefly discuss them.

Classification with respect to the time

According to this criterion, dynamical systems can be divided into two groups: continuous- and discrete-time dynamical systems. The first group is characterized by one of the following conditions:

- $\mathbb{T} = \mathbb{R}$,
- $\mathbb{T} = \mathbb{R}^+ \cup \{0\}$,
- $\mathbb{T} = (a, b) \subset \mathbb{R}$, $a < 0 < b$.

For continuous-time dynamical systems the family $\{\varphi^t\}_{t\in\mathbb{T}}$ is referred to as flow (resp. semiflow, see below). It is readily seen that the systems presented in the Examples 1.2, 1.3 are continuous-time systems. As for the discrete-time systems, it holds that:

- $\mathbb{T} = \mathbb{N} \cup \{0\} =: \mathbb{N}_0$,
- $\mathbb{T} = \mathbb{Z}$.

Let us illustrate this type of systems by the following example:

Example 2.1. Let X be a metric space and $g : X \to X$. Define the operator $\varphi^{(\cdot)}(\cdot) : \mathbb{N}_0 \times X \to X$ as follows

$$\forall k \in \mathbb{N} : \varphi^{k+1} = \varphi^1 \circ \varphi^k, \quad \varphi^1 = g, \quad \varphi^0 = Id_X. \tag{2.1}$$

We will prove that $\left\{\mathbb{N}_0, X, \{\varphi^k\}_{k\in\mathbb{N}_0}\right\}$ is a dynamical system. It is clear that **DS1** holds. Let us work with **DS2**. We must show that

$$\forall u \in X, \forall m, n \in \mathbb{N}_0 : \varphi^{m+n}(u) = \varphi^m(\varphi^n(u)).$$

For this purpose we will use mathematical induction for the variable m. Let $n \in \mathbb{N}_0$ and $u \in X$ be arbitrary. **DS2** clearly holds for $m = 0$. For $m = 1$ we have that

$$\varphi^{1+n}(u) \overset{(2.1)}{=} \left(\varphi^1 \circ \varphi^n\right)(u) = \varphi^1(\varphi^n(u)).$$

Now assume that for some $m \in \mathbb{N}_0$ $\varphi^{m+n}(u) = \varphi^m(\varphi^n(u))$ holds. Then

$$\varphi^{m+1}(\varphi^n(u)) \overset{(2.1)}{=} \left(\varphi^1 \circ \varphi^m\right)(\varphi^n(u)) = \varphi^1(\varphi^m(\varphi^n(u))) = \varphi^1(\varphi^{m+n}(u)) \overset{(2.1)}{=} \varphi^{(m+1)+n}(u).$$

Hence $\left\{\mathbb{N}_0, X, \{\varphi^k\}_{k\in\mathbb{N}_0}\right\}$ is a discrete-time dynamical system. Note that we have also shown that any discrete-time system is completely defined by the function $\varphi^1 = g$. This function is referred to as the generator of the dynamical system.

Classification with respect to the state space

In this case we assume the state space X to be a vector space. Thus, according to the state space, dynamical systems are: finite-dimensional, if $\dim X < \infty$, and infinite-dimensional, if $\dim X = \infty$. It is easy to show that Examples 1.2, 1.3 correspond to systems of the first type. Let us present a system of the second type:

Example 2.2. Consider the following PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + x\dfrac{\partial u}{\partial x}(x,t) = 0 \\ \hfill u(x,0) = f(x) \end{cases}, \quad u \in C^1(\mathbb{R}^2, \mathbb{R}), \quad f \in C^1(\mathbb{R}). \tag{2.2}$$

This system has the general solution $u(x,t) = f(xe^{-t})$, $x, t \in \mathbb{R}$. Define the operator $\varphi^{(\cdot)}(\cdot) : \mathbb{R} \times C^1(\mathbb{R}) \to C^1(\mathbb{R})$, such that $(\varphi^t(f))(x) = f(xe^{-t})$. Is $\{\mathbb{R}, C^1(\mathbb{R}), \{\varphi^t\}_{t\in\mathbb{R}}\}$ a dynamical system? Note that

$$\forall f \in C^1(\mathbb{R}) : (\varphi^0(f))(x) = f(x) \Rightarrow \varphi^0(f) = f,$$

which means that **DS1** holds. Next, take any $f \in C^1(\mathbb{R})$ and $t, s \in \mathbb{R}$. Let $g := \varphi^s(f)$, i.e. $g(x) = f(xe^{-s})$, $x \in \mathbb{R}$. Then

$$(\varphi^t(\varphi^s(f)))(x) = (\varphi^t(g))(x) = g(xe^{-t}) = f(xe^{-(t+s)}) = (\varphi^{t+s}(f))(x),$$

that is, $\varphi^t(\varphi^s(f)) = \varphi^{t+s}(f)$. Hence, $\{\mathbb{R}, C^1(\mathbb{R}), \{\varphi^t\}_{t\in\mathbb{R}}\}$ is an infinite-dimensional, continuous-time dynamical system. Furthermore, this system is invertible too (see below).

Classification with respect to invertibility

For this type of classification we are interested in analyzing the invertibility of the evolution operator of the system, in the following sense. Let $\{\mathbb{T}, X, \{\varphi^t\}_{t\in\mathbb{T}}\}$ be a dynamical system with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. Observe that

$$\forall t \in \mathbb{T} : \varphi^t \circ \varphi^{-t} = \varphi^{t+(-t)} = \varphi^0 = Id_X,$$

which implies that for all fixed $t \in \mathbb{T}$, φ^t is invertible and further $(\varphi^t)^{-1} = \varphi^{-t}$. Hence, invertible dynamical systems are those that admit both positive and negative values of time, otherwise they are non-invertible. For example, the two-body problem, the simple pendulum, and the system defined by (2.2) are invertible. On the other hand, Example 2.1 provides us, by its construction, with a non-invertible system, however, if g^{-1} exists, then we could redefine the system in such a way that it admits negative values of time. Moreover, if we deal with continuous-time, invertible dynamical systems, then the family of evolution operators is called flow. Otherwise, the family is referred to as semiflow.

Another important type of dynamical systems are those that describe the so-called Symbolic Dynamics. Let $N \in \mathbb{N}$ and $[N] := \{0, 1, \ldots, N\}$. Consider the state space $X := [N]^{\mathbb{Z}}$, i.e.

$$X = \{u = (u_i)_{i\in\mathbb{Z}} : u_i \in [N]\}^2.$$

Consider the shift-operator $\varphi^{(\cdot)}(\cdot) : \mathbb{Z} \times X \to X$, defined as

$$(\varphi^k(u))_i = u_{i+k}, \quad i \in \mathbb{Z}\,.$$

We will show that $\left\{\mathbb{Z}, X, \{\varphi^k\}_{k\in\mathbb{Z}}\right\}$ defines a dynamical system. To begin with, we verify **DS1**:

$$\forall u \in X, \forall i \in \mathbb{Z} : (\varphi^0(u))_i = u_{i+0} = u_i \Rightarrow \varphi^0(u) = u.$$

Next, for **DS2** it holds that

$$\forall k, p \in \mathbb{Z}, \forall u \in X, \forall i \in \mathbb{Z} : (\varphi^{p+k}(u))_i = u_{i+k+p} = u_{(i+k)+p} = (\varphi^p(\varphi^k(u)))_i,$$

and thereby, $\left\{\mathbb{Z}, X, \{\varphi^k\}_{k\in\mathbb{Z}}\right\}$ is a discrete-time, invertible dynamical system.

Conclusions

In this manuscript we presented a brief motivation to the study of dynamical systems. Several examples were given in order to emphasize the advantage of having such a powerful concept at hand, which allows us to consider different types of mathematical models as a very same object. In a forthcoming manuscript ([7]) we will present some more robust connections between ODEs and dynamical systems, so that the reader will realize that the object here introduced did not arise artificially but it is rather a natural generalization of many already known dynamical processes.

[2] Note that X equipped with the metric $d : X \times X \to \mathbb{R}$, defined as $d(u, v) = \sum_{i=-\infty}^{\infty} |u_i - v_i| 2^{-|i|}$, becomes a complete metric space.

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