Numerical Study of a Jeffcott Rotor Model with a Snubber Ring

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Joint work with Prof. Marian Wiercigroch

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Modern Practice in Stress and Vibration Analysis (MPSVA 2012)
Overview

- Description of the rotor system.
- Equations of motion.
- Bifurcation analysis with TC-HAT.
- Analytical grazing curve.
- Future work.
Description of the rotor system

Physical model

![Diagram of a rotor system with components labeled: k1, c, k2, m, ρ, γ, M, snubber ring, rotor, and Snubber Ring.]}
Description of the rotor system

Physical model

Assumptions:

- Massless snubber ring.
- No dry friction between rotor and snubber ring.
- Snubber ring elastically supported (no viscous damping).
- No gyroscopic forces.
- Gravity loads are neglected.

Imbalance caused by:

- Thermal deformation.
- Blade/tooth breakage.

⇒ Serious malfunctions.
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Description of the rotor system

Geometrical representation

$\mathbf{O}_r$: Rotor static position.

$\mathbf{O}_{s0}$: Snubber ring resting position.

$(\varepsilon_x, \varepsilon_y)$: Eccentricity of the rotor.

$F_N$: Normal force produced by contact.

$\omega$: Frequency of rotation.

$m\rho\omega^2$: Centrifugal force.
**Description of the rotor system**

**Geometrical representation**

\[ O_{r0} : \text{Rotor static position.} \]

\[ O_{s0} : \text{Snubber ring resting position.} \]

\((x,y)\): Coordinates of the center of the rotor \(O_r\).

\((\varepsilon_x,\varepsilon_y)\): Eccentricity of the rotor.

\(F_N\): Normal force produced by contact.

\(\omega\): Frequency of rotation.

\(m \rho \omega^2\): Centrifugal force.
Equations of motion

No contact ($R < \gamma$):

\[ Mx'' + cx' + k_1 x = m\rho\omega^2 \cos(\omega t + \varphi_0), \]
\[ My'' + cy' + k_1 y = m\rho\omega^2 \sin(\omega t + \varphi_0). \]
Equations of motion

No contact \((R < \gamma)\):

\[
M x'' + cx' + k_1 x = m \rho \omega^2 \cos(\omega t + \varphi_0),
\]
\[
M y'' + cy' + k_1 y = m \rho \omega^2 \sin(\omega t + \varphi_0).
\]
Equations of motion

No contact \( (R < \gamma) \):

\[ Mx'' + cx' + k_1x = m\rho\omega^2 \cos(\omega t + \varphi_0), \]
\[ My'' + cy' + k_1y = m\rho\omega^2 \sin(\omega t + \varphi_0). \]

Contact \( (R \geq \gamma) \):

\[ Mx'' + cx' + k_1x + k_2(R - \gamma) \cos(\psi) = m\rho\omega^2 \cos(\omega t + \varphi_0), \]
\[ My'' + cy' + k_1y + k_2(R - \gamma) \sin(\psi) = m\rho\omega^2 \sin(\omega t + \varphi_0). \]
Equations of motion
Nondimensionalization

Dimensionless variables and parameters:

\[ \omega_n = \sqrt{\frac{k_1}{M}} \]
\[ \eta = \frac{\omega}{\omega_n} \]
\[ t \leftarrow \omega_n t \]
\[ x \leftarrow \frac{x}{\gamma} \]
\[ y \leftarrow \frac{y}{\gamma} \]
\[ z = x' \]
\[ w = y' \]
\[ s = \eta t \in [0, 2\pi) \]
\[ \nu = (x, y, z, w, s) \]
\[ \epsilon_x \leftarrow \frac{\epsilon_x}{\gamma} \]
\[ \epsilon_y \leftarrow \frac{\epsilon_y}{\gamma} \]
\[ \eta_m = \frac{m}{M} \]
\[ \nu = \frac{c}{2\sqrt{k_1 M}} \]
\[ K = \frac{k_2}{k_1} \]
\[ \rho \leftarrow \frac{\rho}{\gamma} \]
\[ \alpha = (\eta_m, \rho, \nu, K, \varphi_0, \epsilon_x, \epsilon_y) \]
Equations of motion
First order ODE

\[ v' = \begin{cases} \ f_{\text{contact}}(v, \alpha, \eta), & R \geq \gamma, \\ f_{\text{no contact}}(v, \alpha, \eta), & R < \gamma, \end{cases} \]

where

\[
\begin{align*}
 f_{\text{no contact}}(v, \alpha, \eta) &= \begin{pmatrix} z \\ w \\ \eta \end{pmatrix}, \\
 f_{\text{contact}}(v, \alpha, \eta) &= \begin{pmatrix} z \\ w \\ \eta \end{pmatrix},
\end{align*}
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Equations of motion
First order ODE

\[ v' = \begin{cases} 
  f_{\text{contact}}(v, \alpha, \eta), & R \geq \gamma, \\
  f_{\text{no contact}}(v, \alpha, \eta), & R < \gamma,
\end{cases} \]

where

\[
f_{\text{no contact}}(v, \alpha, \eta) = \begin{pmatrix}
  z \\
  w \\
  \eta m \rho \eta^2 \cos(s + \varphi_0) - 2\nu z - x \\
  \eta m \rho \eta^2 \sin(s + \varphi_0) - 2\nu w - y \\
  \eta
\end{pmatrix},
\]

\[
f_{\text{contact}}(v, \alpha, \eta) = \begin{pmatrix}
  z \\
  w \\
  \eta m \rho \eta^2 \cos(s + \varphi_0) - 2\nu z - x - K(x - \varepsilon_x) \left(1 - \frac{\gamma}{R}\right) \\
  \eta m \rho \eta^2 \sin(s + \varphi_0) - 2\nu w - y - K(y - \varepsilon_y) \left(1 - \frac{\gamma}{R}\right) \\
  \eta
\end{pmatrix}.
\]
Bifurcation analysis with TC-HAT

Assume $\varepsilon_y = 0, \varepsilon_x \neq 0$. 

![Graph showing impact boundary and contact points](image)

- Impact boundary: $(x - \varepsilon_x)^2 + (y - \varepsilon_y)^2 = 1$
- Contact point at $s = 0$
- No contact points at $s \in [0, 2\pi)$
Bifurcation analysis with TC-HAT

impact boundary \((x - \varepsilon_x)^2 + (y - \varepsilon_y)^2 = 1\)

near grazing segment

near grazing point

no contact \(2\pi\)
Bifurcation analysis with TC-HAT
Segments, event and jump functions

<table>
<thead>
<tr>
<th>Index</th>
<th>Segment</th>
<th>Vector Field</th>
<th>Event</th>
<th>Jump</th>
</tr>
</thead>
<tbody>
<tr>
<td>I₁</td>
<td>no contact</td>
<td>$f_{\text{no contact}}$</td>
<td>$h_{\text{contact}}$</td>
<td>$g_{\text{id}}$</td>
</tr>
<tr>
<td>I₂</td>
<td>contact</td>
<td>$f_{\text{contact}}$</td>
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</tr>
<tr>
<td>I₃</td>
<td>no contact $2\pi$</td>
<td>$f_{\text{no contact}}$</td>
<td>$h_{2\pi}$</td>
<td>$g_{2\pi}$</td>
</tr>
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</tr>
</tbody>
</table>

Where:

\[ h_{\text{contact}}(v, \alpha) := (x - \varepsilon_x)^2 + (y - \varepsilon_y)^2 - 1 = 0, \]
\[ h_{2\pi}(v, \alpha) := s - 2\pi = 0, \]
\[ h_{\text{near grazing}}(v, \alpha) := z = x' = 0, \]
\[ g_{\text{id}}(v) := v, \]
\[ g_{2\pi}(v) := (x, y, z, w, s - 2\pi). \]
Bifurcation analysis with TC-HAT

Grazing orbit found during continuation at $\eta \approx 0.85281$ for $\eta_m = 0.0017$, $\rho = 70$, $\nu = 0.125$, $K = 30$, $\varepsilon_x = 0.75$ and $\varepsilon_y = 0$ fixed.
Bifurcation analysis with TC-HAT

Two-parameter continuation of the grazing curve in the $\eta$-$\eta_m$ plane

Mass imbalance vs Frequency

Contact

No contact

Initial grazing point

Minimum $\eta_m = 0.000886567$
Bifurcation analysis with TC-HAT
Two-parameter continuation of the grazing curve in the $\eta-\eta_m$ plane

Mass imbalance vs Frequency

Contact

initial grazing point

minimum $\eta_m = 0.000886567$

No contact
Bifurcation analysis with TC-HAT

Grazing curve

One-parameter continuation w.r.t. $\eta$

[Graphs showing grazing curve and one-parameter continuation]
Bifurcation analysis with TC-HAT

Two-parameter continuation in the $\eta-\eta_m$ plane (grazing curve in red)

Fold curve

Period-doubling curve

The values on the $\eta_m$-axis are multiplied by the factor $10^{-2}$. 
Bifurcation analysis with TC-HAT

Chaotic Regime

For $\eta_m = 0.0017$, $\rho = 70$, $\nu = 0.11$, $K = 30$, $\varepsilon_x = 1$ and $\varepsilon_y = 0$, cf. Karpenko et al., Chaos, Solitons & Fractals, 2002.
Bifurcation analysis with TC-HAT

Initial **period-1** orbit (stable) for $\eta = 2.4$

The continuation task reveals a torus bifurcation at $\eta \approx 2.45989$ and a period-doubling at $\eta \approx 2.55$. 
Bifurcation analysis with TC-HAT

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The continuation task reveals a torus bifurcation at $\eta \approx 2.45989$ and a period-doubling at $\eta \approx 2.55$. 
Bifurcation analysis with TC-HAT

**Period-1** orbit (unstable) and **quasi-periodic** motion (stable) after the torus bifurcation, for $\eta = 2.47$.
Bifurcation analysis with TC-HAT

**Period-1** (unstable) and **period-2** orbits (unstable) after the PD bifurcation, for $\eta = 2.55128$. 
Bifurcation analysis with TC-HAT

Coexisting **period-1**, **period-2** (both unstable) and **chaotic motion** (stable) for $\eta = 2.6$. 
Analytical grazing curve
Relation between frequency ($\eta$) and mass imbalance ($\eta_m$) producing grazing contact

Where $Amp(\eta_m, \eta, \rho, \nu) = \eta_m \sqrt{A(\eta, \rho, \nu)^2 + B(\eta, \rho, \nu)^2}$ and $A(\eta, \rho, \nu) = \frac{\rho \eta^2 (1 - \eta^2)}{4 \nu^2 \eta^2 + (1 - \eta^2)^2}$, $B(\eta, \rho, \nu) = \frac{2 \nu \rho \eta^3}{4 \nu^2 \eta^2 + (1 - \eta^2)^2}$. According to the geometry of the system, grazing contact occurs when

$$Amp(\eta_m, \eta, \rho, \nu) = 1 - \sqrt{\varepsilon_x^2 + \varepsilon_y^2}$$
Analytical grazing curve
Relation between frequency ($\eta$) and mass imbalance ($\eta_m$) producing grazing contact

Where $\text{Amp}(\eta_m, \eta, \rho, \nu) = \eta_m \sqrt{A(\eta, \rho, \nu)^2 + B(\eta, \rho, \nu)^2}$ and $A(\eta, \rho, \nu) = \frac{\rho \eta^2 (1 - \eta^2)}{4 \nu^2 \eta^2 + (1 - \eta^2)^2}$, $B(\eta, \rho, \nu) = \frac{2 \nu \rho \eta^3}{4 \nu^2 \eta^2 + (1 - \eta^2)^2}$. According to the geometry of the system, grazing contact occurs when

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$\text{Amp}(\eta_m, \eta, \rho, \nu) = 1 - \sqrt{\varepsilon_x^2 + \varepsilon_y^2}$
Analytical grazing curve
Relation between frequency ($\eta$) and mass imbalance ($\eta_m$) producing grazing contact

From the last condition we obtain the grazing curve:

$$\eta_m = G(\eta, \rho, \nu, \varepsilon_x, \varepsilon_y) = \frac{1 - \sqrt{\varepsilon_x^2 + \varepsilon_y^2}}{\sqrt{A(\eta, \rho, \nu)^2 + B(\eta, \rho, \nu)^2}}$$

$$\eta_m^\infty = \lim_{\eta \to \infty} G(\eta, \rho, \nu, \varepsilon_x, \varepsilon_y) = \frac{1 - \sqrt{\varepsilon_x^2 + \varepsilon_y^2}}{\rho}$$

$$\eta_m^{\min} = \frac{1}{\sqrt{1 - 2\nu^2}}, \text{ for } 0 < \nu < \frac{1}{\sqrt{2}}$$

$$\eta_m^{\min} = \frac{2\nu \left(1 - \sqrt{\varepsilon_x^2 + \varepsilon_y^2}\right) \sqrt{1 - \nu^2}}{\rho}$$
Analytical grazing curve
Relation between frequency ($\eta$) and mass imbalance ($\eta_m$) producing grazing contact

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$$\eta_m^{\min} = \frac{2\nu \left(1 - \sqrt{\epsilon_x^2 + \epsilon_y^2}\right) \sqrt{1 - \nu^2}}{\rho}$$
Analytical grazing curve
For different values of viscous damping

\[ \eta_m = \begin{cases} 
\eta_{\nu=0.005} = 0.005, \\
\eta_{\nu=0.2} = 0.2, \\
\eta_{\nu=0.5} = 0.5, \\
\eta_{\nu=1} = 1, \\
\eta_{\nu=1.5} = 1.5 
\end{cases} \]
Future work
Experimental verification of the bifurcation scenarios

Rotor in action!
(Thanks to Vahid Vaziri, University of Aberdeen)
Acknowledgements

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Thanks a lot for your attention!