Numerical Bifurcation Analysis of an Impact Oscillator with Drift

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Joint work with Prof. M. Wiercigroch and Dr. E. Pavlovskaia

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Nonlinear Dynamics in Engineering: Modelling, Analysis and Applications

Overview

- Physical Description.
- Equations of motion.
- Mathematical modeling with TC-HAT.
- Numerical results.
- Closing remarks.

Physical Description

Pavlovskaia et al., Physical Review E, 2001

m: Impactor.

$f(t)$: External forcing.

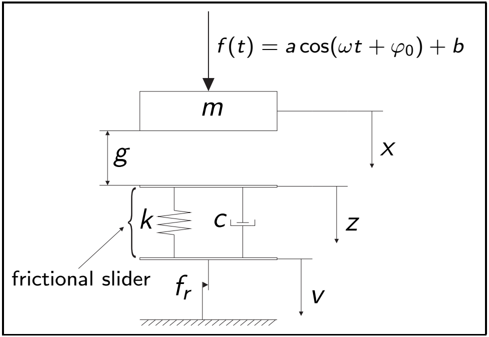
a,ω,b: Control parameters.

c,k: Viscoelastic properties of the slider.

f_r: Dry friction threshold.

g: Gap between impactor and slider.

x,z,v: System coordinates.



In our study, we will consider the linear coordinate transformation

$$\mathbf{p} = \mathbf{x} - \mathbf{v}, \quad \mathbf{q} = \mathbf{z} - \mathbf{v}$$

and the dimensionless parameters $m = 1$, $f_r = 1$, $k = 1$ and $c = 2\xi$.

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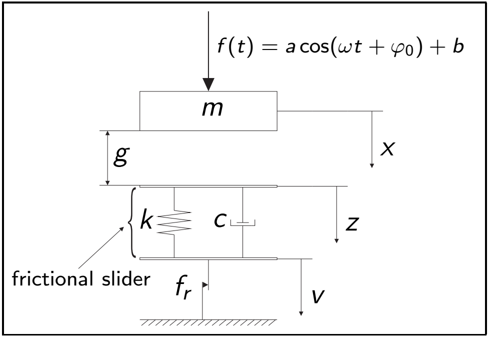

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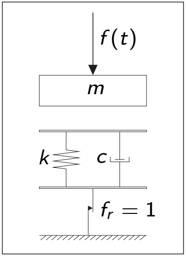
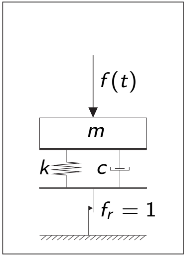
Equations of motion



No contact:

$$
\begin{aligned}
p' &= y,\\
y' &= a\cos(s+\varphi_0)+b,\\
q' &= -\frac{1}{2\xi}q,\\
v' &= 0,\\
s' &= \omega.
\end{aligned}
$$



Contact without progression:

$$(0 < 2\xi y + q < 1)$$

$$
\begin{aligned}
p' &= y,\\
y' &= a\cos(s+\varphi_0)+b-(2\xi y+q),\\
q' &= y,\\
v' &= 0,\\
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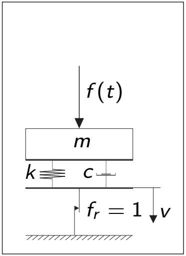

Contact with progression:

$$(2\xi y + q \geq 1)$$

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p' &= -\frac{1}{2\xi}(q-1),\\
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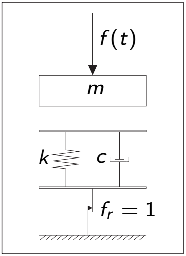

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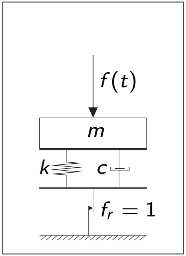

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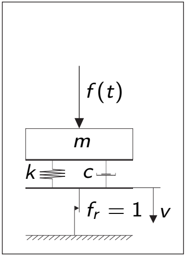

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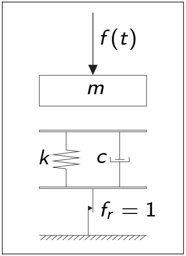

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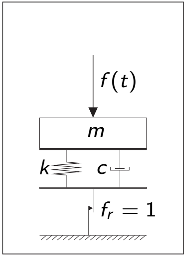

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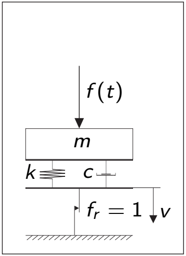

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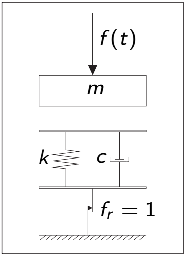

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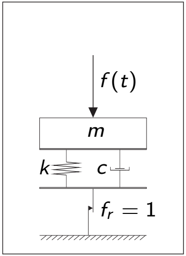

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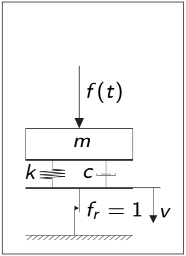

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Equations of motion

Piecewise-smooth ODE

$$u' = \begin{cases} f_{\mathrm{NC}}(u,\alpha), & p < q+g \text{ or } 2\xi y + q \le 0, \\ f_{\mathrm{CwoP}}(u,\alpha), & p = q+g \text{ and } 0 < 2\xi y + q < 1, \\ f_{\mathrm{CwP}}(u,\alpha), & p = q+g \text{ and } 2\xi y + q \ge 1, \end{cases}$$

where $u = (p, y, q, s)$, $\alpha = (b, \omega, a, \xi, g, \varphi_0)$, and:

$$f_{\mathrm{NC}}(u,\alpha) := \begin{pmatrix} y \\ a\cos(s+\varphi_0) + b \\ -\frac{1}{2\xi} q \\ \omega \end{pmatrix},$$

$$f_{\mathrm{CwoP}}(u,\alpha) := \begin{pmatrix} y \\ a\cos(s+\varphi_0) + b - (2\xi y + q) \\ y \\ \omega \end{pmatrix},$$

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Equations of motion

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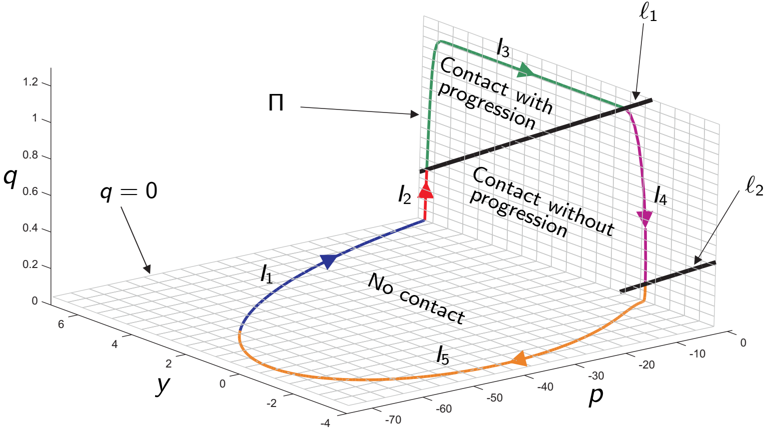
Mathematical modeling with TC-HAT

Auto 97 toolbox for the continuation of periodic orbits of non-smooth systems (Thota & Dankowicz, SIAM J. Appl. Dyn. Sys., 2008)



$$\Pi := \{(p, y, q) \in \mathbb{R}^3 : p - q - g = 0\},$$
$$\ell_1 := \{(p, y, q) \in \mathbb{R}^3 : 2\xi y + q - 1 = 0, \quad p - q - g = 0\},$$
$$\ell_2 := \{(p, y, q) \in \mathbb{R}^3 : 2\xi y + q = 0, \quad p - q - g = 0\}.$$

Mathematical modeling with TC-HAT

Segments, event and jump functions

Index	Segment	Vector field	Event function	Jump function
I_1	No contact	f_{NC}	h_{C}	g_{id}
I_2	C. without progression 1	f_{CwoP}	$h_{\text{CwoP-1}}$	g_{id}
I_3	C. with progression	f_{CwP}	$h_{\text{CwoP-1}}$	g_{id}
I_4	C. without progression 2	f_{CwoP}	$h_{\text{CwoP-2}}$	g_{id}
I_5	No contact 2π	f_{NC}	$h_{2\pi}$	$g_{2\pi}$
I_6	Near grazing	f_{CwoP}	h_{NG}	g_{id}

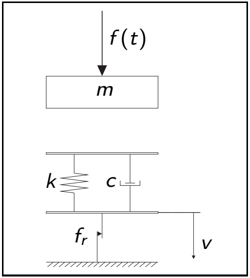
Where:

$$\begin{aligned}
h_{\text{C}}(u,\alpha) &:= p - q - g = 0,\\
h_{2\pi}(u,\alpha) &:= s - 2\pi = 0,\\
h_{\text{CwoP-1}}(u,\alpha) &:= 2\xi y + q - 1 = 0,\\
h_{\text{CwoP-2}}(u,\alpha) &:= 2\xi y + q = 0,\\
h_{\text{NG}}(u,\alpha) &:= y(1-4\xi^2) + 2\xi(a\cos(s+\varphi_0) + b - q) = 0,\\
g_{\text{id}}(u) &:= u,\\
g_{2\pi}(u) &:= (p, y, q, s - 2\pi).
\end{aligned}$$

Mathematical modeling with TC-HAT

Solution measures

Consider a periodic solution $(p(t), y(t), q(t))$ with period $T > 0$.



Rate of progression (average rate of progression per period)

$$\mathrm{ROP} := \frac{1}{T}\int_0^T v'(t)\,dt.$$

Damping dissipation (average power dissipated by the damper c)

$$P_c := \frac{1}{T}\int_0^T w'(t)\,dt,$$

where

$$w'(t) = \begin{cases} \frac{1}{2\xi}q(t)^2, & p(t) < q(t) + g \text{ or } 2\xi y(t) + q(t) \le 0, \\ 2\xi y(t)^2, & p(t) = q(t) + g \text{ and } 0 < 2\xi y(t) + q(t) < 1, \\ \frac{1}{2\xi}(q(t)-1)^2, & p(t) = q(t) + g \text{ and } 2\xi y(t) + q(t) \ge 1. \end{cases}$$

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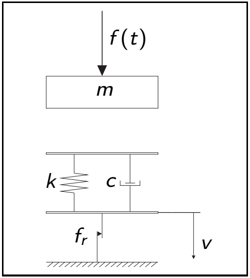

Mathematical modeling with TC-HAT

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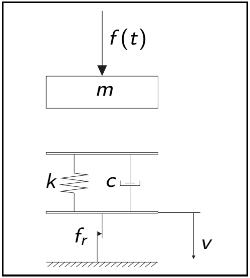

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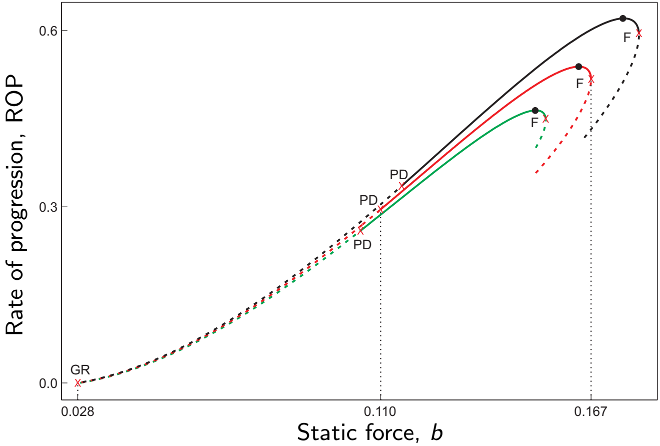
Numerical results

1-parameter analysis w.r.t. static force b

Continuation computed for $\omega = 0.1$, $\xi = 0.05$, $g = 0.02$ and $a = 0.28$, $a = 0.3$, $a = 0.32$.



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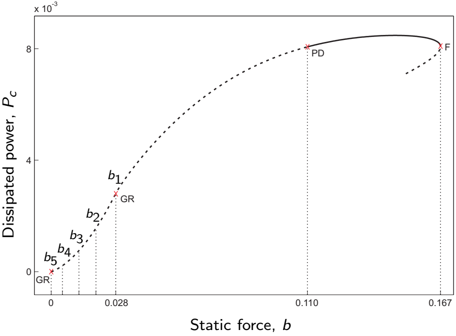

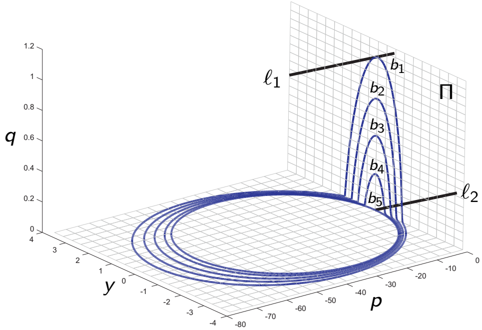

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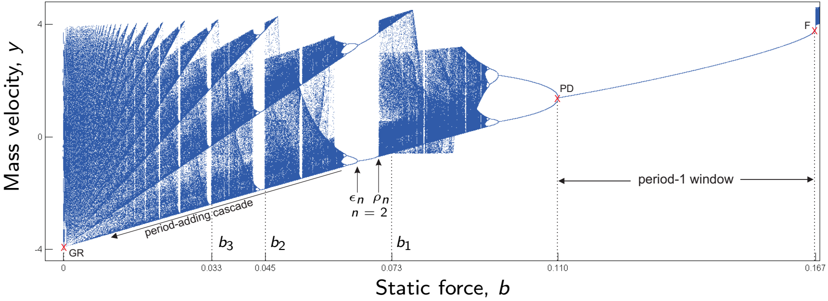

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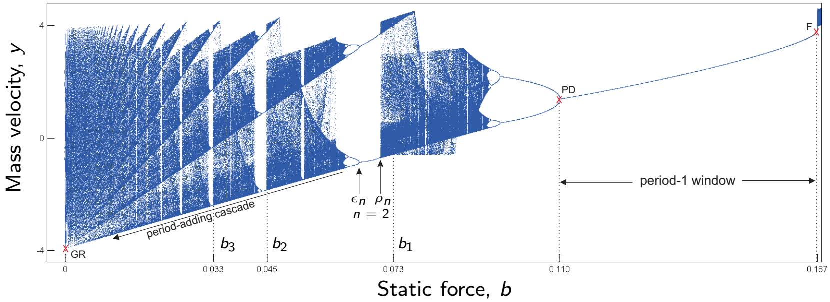
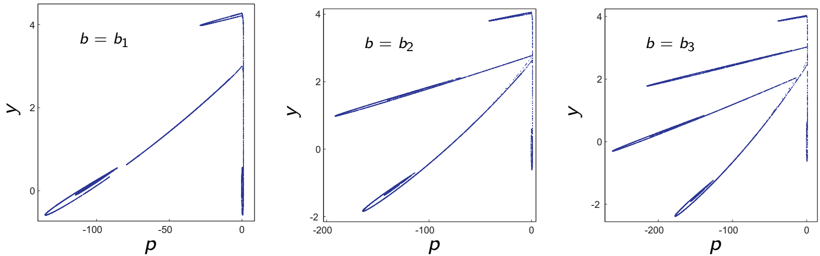
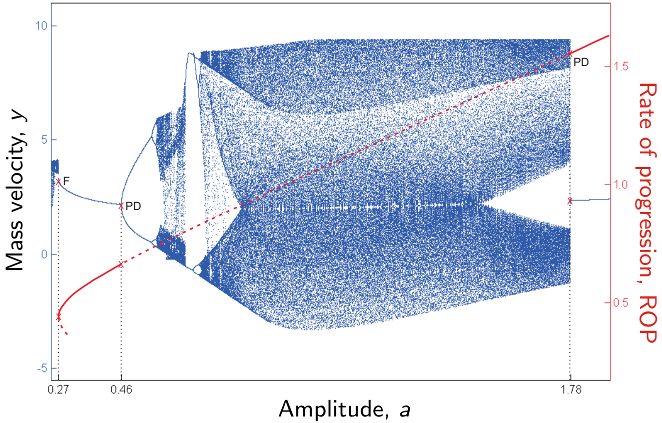




Numerical results

1-parameter analysis w.r.t. amplitude a

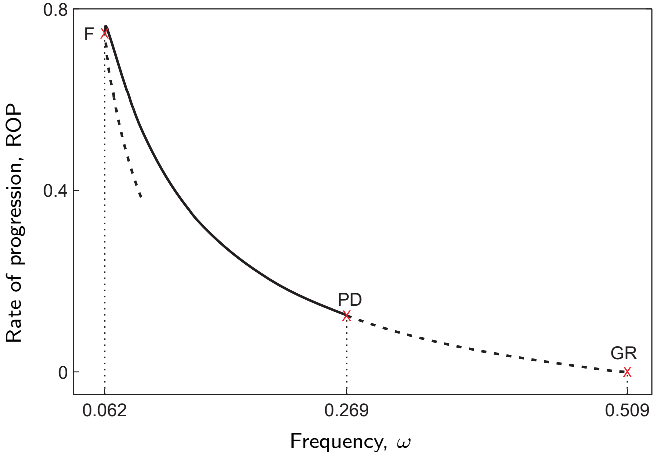
Bifurcation diagram for $\omega = 0.1$, $b = 0.15$, $\xi = 0.05$ and $g = 0.02$.



Numerical results

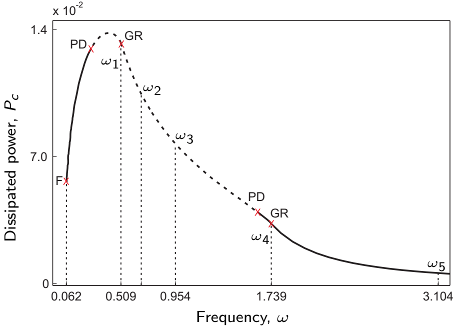
1-parameter analysis w.r.t. frequency ω

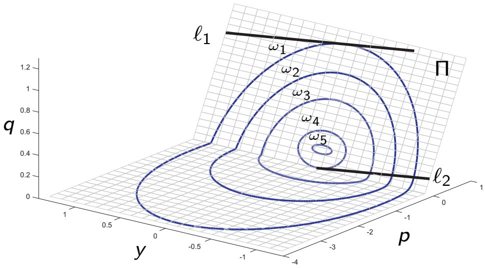
Behaviour of the ROP for $b = 0.15$, $a = 0.3$, $\xi = 0.05$ and $g = 0.02$.



Numerical results

1-parameter analysis w.r.t. frequency ω

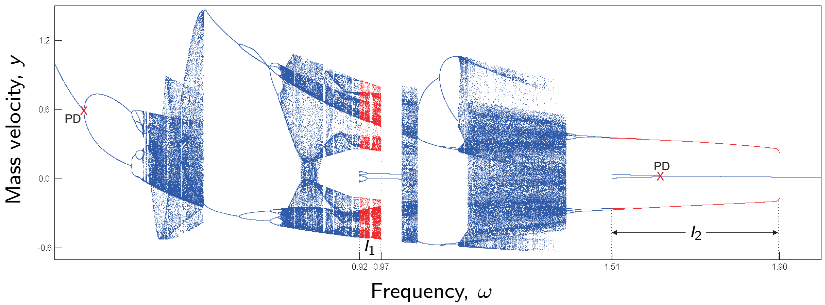




Numerical results

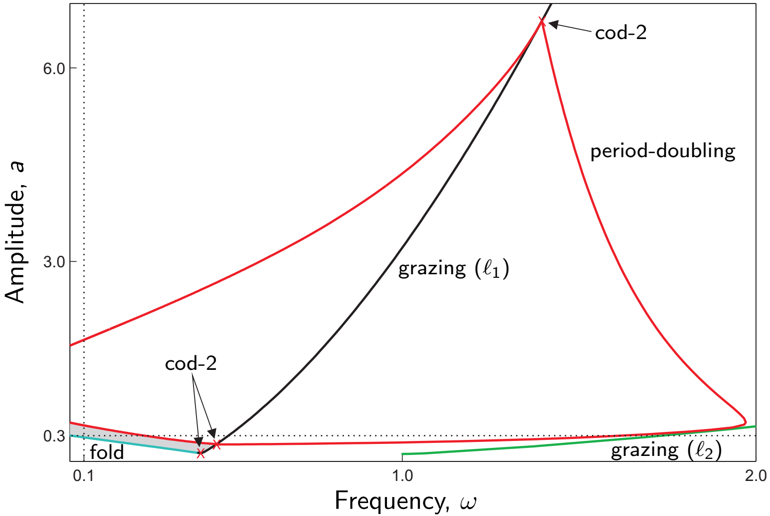
1-parameter analysis w.r.t. frequency ω

Bifurcation diagram



Numerical results

2-parameter continuation of codimension-1 bifurcations



Numerical results

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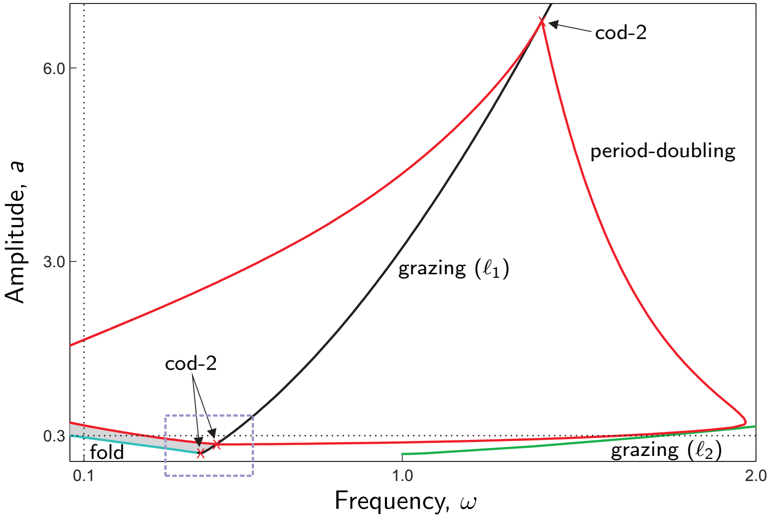
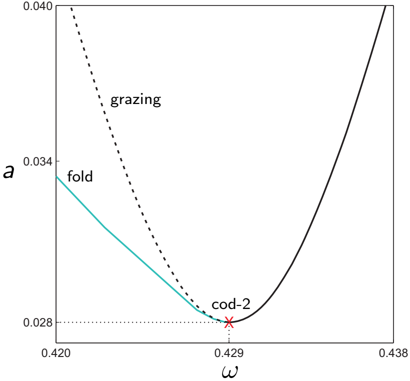


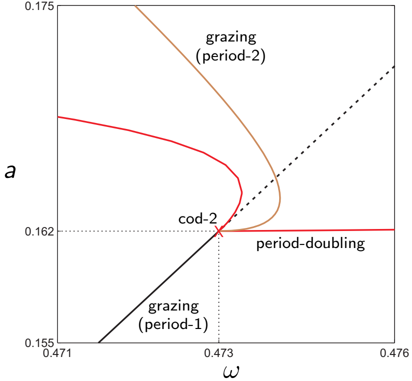
Numerical results

2-parameter continuation of codimension-1 bifurcations (blow-up of boxed region)

Fold-grazing bifurcation

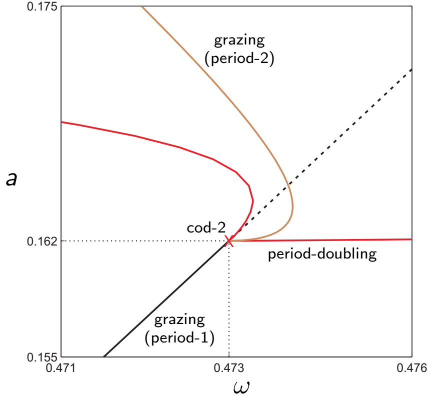


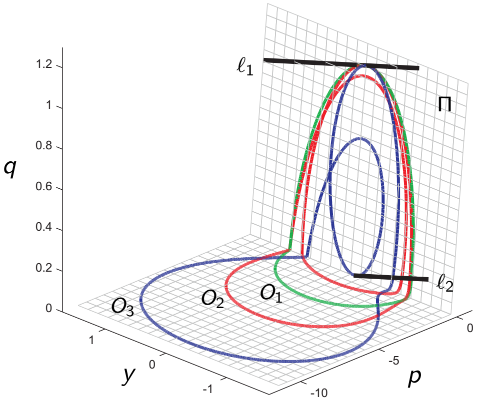
Flip-grazing bifurcation



Numerical results

Period-2 orbits along the grazing curve





Closing remarks

- Rich dynamical behaviour.
- Codimension-1 bifurcations of limit cycles: fold, period-doubling, grazing.
- Codimension-2 bifurcations of limit cycles: flip-grazing, fold-grazing, double grazing.
- Further phenomena: hysteresis loops, period-adding cascades, fingered strange attractors.
- Effect of the control parameters on the rate of progression (ROP) studied using path-following methods (TC-HAT).

Acknowledgements

The speaker wishes to thank:

- Centre for Applied Dynamics Research, University of Aberdeen, Scotland,
- Organizers of the 10th Anniversary CADR International Conference, and
- The Audience!

Thanks a lot for your attention!