FOURIER ANALYSIS

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1. The best approximation onto trigonometric polynomials

Before we start the discussion of Fourier series we will review some basic results on inner–product spaces and orthogonal projections mostly presented in Section 4.6 of [1].

1.1. **Inner–product spaces.** Let V be an inner–product space. As usual we let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ denote the inner–product of $\boldsymbol{u}$ and $\boldsymbol{v}$. The corresponding norm is given by

$$\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}.$$

A basic relation between the inner–product and the norm in an inner–product space is the Cauchy–Scwarz inequality. It simply states that the absolute value of the inner–product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is bounded by the product of the corresponding norms, i.e.

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \le \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|. \tag{1.1}$$

An outline of a proof of this fundamental inequality, when $V = \mathbb{R}^n$ and $\| \cdot \|$ is the standard Eucledian norm, is given in Exercise 24 of Section 2.7 of [1]. We will give a proof in the general case at the end of this section.

Let W be an n dimensional subspace of V and let $P : V \mapsto W$ be the corresponding projection operator, i.e. if $\boldsymbol{v} \in V$ then $\boldsymbol{w}^* = P\boldsymbol{v} \in W$ is the element in W which is closest to $\boldsymbol{v}$. In other words,

$$\|\boldsymbol{v} - \boldsymbol{w}^*\| \le \|\boldsymbol{v} - \boldsymbol{w}\| \quad \text{for all } \boldsymbol{w} \in W.$$

It follows from Theorem 12 of Chapter 4 of [1] that $\boldsymbol{w}^*$ is characterized by the conditions

$$\langle \boldsymbol{v} - P\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v} - \boldsymbol{w}^*, \boldsymbol{w} \rangle = 0 \quad \text{for all } w \in W. \tag{1.2}$$

In other words, the error $\boldsymbol{v} - P\boldsymbol{v}$ is orthogonal to all elements in W.

It is a consequence of the characterization (1.2) and Cauchy–Schwarz inequality (1.1) that the norm of $P\boldsymbol{v}$ is bounded by the norm of $\boldsymbol{v}$, i.e.

$$\|P\boldsymbol{v}\| \le \|\boldsymbol{v}\| \quad \text{for all } \boldsymbol{v} \in V. \tag{1.3}$$

To see this simply take $\boldsymbol{w} = \boldsymbol{w}^*$ in (1.2) to obtain

$$\|\boldsymbol{w}^*\|^2 = \langle \boldsymbol{w}^*, \boldsymbol{w}^* \rangle = \langle \boldsymbol{v}, \boldsymbol{w}^* \rangle \le \|\boldsymbol{v}\| \, \|\boldsymbol{w}^*\|,$$

Notes written for for Mat 120B, Fall 2001, Preliminary version.

or

$$\|\boldsymbol{w}^*\| \leq \|\boldsymbol{v}\|.$$

Hence, since $P\boldsymbol{v} = \boldsymbol{w}^*$, we established the bound (1.3).

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be an orthogonal basis of the subspace W. Such an orthogonal basis can be used to give an explicit representation of the projection $P\boldsymbol{v}$ of $\boldsymbol{v}$. It follows from Theorem 13 of Chapter 4 of [1] that $P\boldsymbol{v}$ is given by

$$P\boldsymbol{v} = \sum_{j=1}^{n} c_j \boldsymbol{u}_j \quad \text{where the coefficients } c_j = \frac{\langle \boldsymbol{v}, \boldsymbol{u}_j \rangle}{\|\boldsymbol{u}_j\|^2}. \tag{1.4}$$

¿From the orthogonal basis we can also derive an expression for the norm of $P\boldsymbol{v}$. In fact, we have

$$\|P\boldsymbol{v}\|^2 = \sum_{j=1}^{n} c_j^2 \|\boldsymbol{u}_j\|^2. \tag{1.5}$$

This follows more or less directly from the orthogonality property of the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$. We have

$$\begin{aligned}
\|P\boldsymbol{v}\|^2 &= \langle P\boldsymbol{v}, P\boldsymbol{v} \rangle \\
&= \langle \sum_{j=1}^{n} c_j \boldsymbol{u}_j, \sum_{k=1}^{n} c_k \boldsymbol{u}_k \rangle \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \langle \boldsymbol{u}_j, \boldsymbol{u}_k \rangle \\
&= \sum_{j=1}^{n} c_j^2 \|\boldsymbol{u}_j\|^2.
\end{aligned}$$

The situation just described is very general. Some more concrete examples using orthogonal basises to compute projections are given Section 4.6 of [1]. Fourier analysis is another very important example which fits into the general framework described above, where V is a space of functions and W is a space of trigonometric polynomials. The Fourier series correspons to orthogonal projections of a given function onto the trigonometric polynomials, and the basic formulas of Fourier series can be derived as special examples of general discussion given above.

Proof of Cauchy–Schwarz inequality (1.1). If $\boldsymbol{v} = 0$ we have zero on both sides of (1.1). Hence, (1.1) holds in this case. Therefore, we can assume that $\boldsymbol{v} \neq 0$ in the rest of the proof.

For all $t \in \mathbb{R}$ we have

$$\|\boldsymbol{u} - t\boldsymbol{v}\|^2 \geq 0.$$

However,

$$\begin{aligned}\|\boldsymbol{u}-t\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}-t\boldsymbol{v}, \boldsymbol{u}-t\boldsymbol{v}\rangle \\ &= \langle \boldsymbol{u},\boldsymbol{u}\rangle - t\langle \boldsymbol{u},\boldsymbol{v}\rangle - t\langle \boldsymbol{v},\boldsymbol{u}\rangle + t^2\langle vv,\boldsymbol{v}\rangle \\ &= \|\boldsymbol{u}\|^2 - 2t\langle \boldsymbol{u},\boldsymbol{v}\rangle + t^2\|\boldsymbol{v}\|^2.\end{aligned}$$

Taking $t = \langle \boldsymbol{u},\boldsymbol{v}\rangle/\|\boldsymbol{v}\|^2$ we therefor obtain

$$0 \leq \|\boldsymbol{u}-t\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 - \frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle^2}{\|\boldsymbol{v}\|^2}$$

or

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle^2 \leq \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2.$$

By taking square roots we obtain (1.1). □

1.2. **Fourier series.** A trigonometric polynomial of order m is a function of t of the form

$$p(t) = a_0 + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt),$$

where the coefficients $a_0, a_1, \dots, a_m, b_1, \dots, b_m$ are real numbers. Hence, trigonometric polynomials of order zero are simply all constant functions, while first order trigonometric polynomials are functions of the form

$$p(t) = a_0 + a_1 \cos t + b_1 \sin t.$$

A function $f(t)$ is called *periodic* with period T if $f(t) = f(t+T)$ for all t. Such a function is uniquely determined by its values in the interval $[-T/2, T/2]$ or any other interval of length T. The trigonometric polynomials are periodic with period 2π. Hence we can regard them as elements of the space $C[-\pi,\pi]$.

The space of trigonometric polynomials of order m will be denoted by $\mathcal{T}_m$. More precisely,

$$\mathcal{T}_m = \{p \in C[-\pi,\pi] : p(t) = a_0 + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt), \quad a_k, b_k \in \mathbb{R}\}$$

$C[-\pi,\pi]$ is equipped with a natural inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$$

The norm is then given by

$$||f|| = (\int_{-\pi}^{\pi} f^2(t)dt)^{1/2}$$

We call this the L^2-norm of f on $[-\pi,\pi]$. Any periodic function can be regarded as a 2π-periodic function by a simple change of variable. Hence everything that follows can be applied to general periodic functions.

It is easy to see that the constant function 1, together with the functions $\sin(kt)$ and $\cos(kt)$, $1 \le k \le m$ constitute an orthogonal basis for $\mathcal{T}_m$. To prove this, it is sufficient to prove that for all integers j, k the following identities hold:

$$\int_{-\pi}^{\pi} \sin(jt)\sin(kt)\,dt = 0 \quad j \neq k,$$
$$\int_{-\pi}^{\pi} \cos(jt)\cos(kt)\,dt = 0 \quad j \neq k,$$
$$\int_{-\pi}^{\pi} \cos(jt)\sin(kt)\,dt = 0.$$

Notice that setting $j = 0$ the $\cos(jt)$ factor becomes the constant 1. To prove the first identity, we use the trigonometric formula

$$\sin(u)\sin(v) = \frac{1}{2}(\cos(u-v) - \cos(u+v)).$$

¿From this identity we obtain for $j \neq k$, using the fact that $\sin(l\pi) = 0$ for all integers l, that

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(jt)\sin(kt)\,dt &= \frac{1}{2}\int_{-\pi}^{\pi} (\cos((j-k)t) - \cos(j+k)t)\,dt \\ &= \frac{1}{2(j-k)}\sin((j-k)t) - \frac{1}{2(j+k)}\sin((j+k)t)\ |_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

The two other equalities follow in a similar fashion. Note that we can also compute the norm of these functions using the same equation. Clearly the norm of the constant function 1 is $(2\pi)^{1/2}$. Setting $j = k$ in the integrals above yields

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(kt)\,dt &= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos(2kt)\,dt \\ &= \frac{t}{2} - \frac{1}{4k}\sin(2kt)\ |_{-\pi}^{\pi} \\ &= \pi.\end{aligned}$$

(This also follows easily from the fact that $\sin^2 t + \cos^2 t = 1$, hence both of these functions have average value 1/2 over a whole period.) Hence the norm of $\sin(kt)$ and $\cos(kt)$ equals $\pi^{1/2}$.

The projection of a function $f \in C[-\pi, \pi]$ onto $\mathcal{T}_m$ is the best approximation in L^2-norm of f by a trigonometric polynomial of degree m and is denoted by $S_m(t)$. Notice that S_m depends on the function f, although this is suppressed in the notation. By (1.4) the coefficients are given by the formulae

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt$$

A central question in Fourier analysis is whether or not the approximations $S_m(t)$ converge to $f(t)$,i.e. if formula

$$f(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

holds. The series on the right is called the *Fourier series* of f, whether it converges to f or not.In Figure 1 we see the graphs of $f(t) = t$ and $S_m(t)$ for $m = 1, 3, 5, 10$.

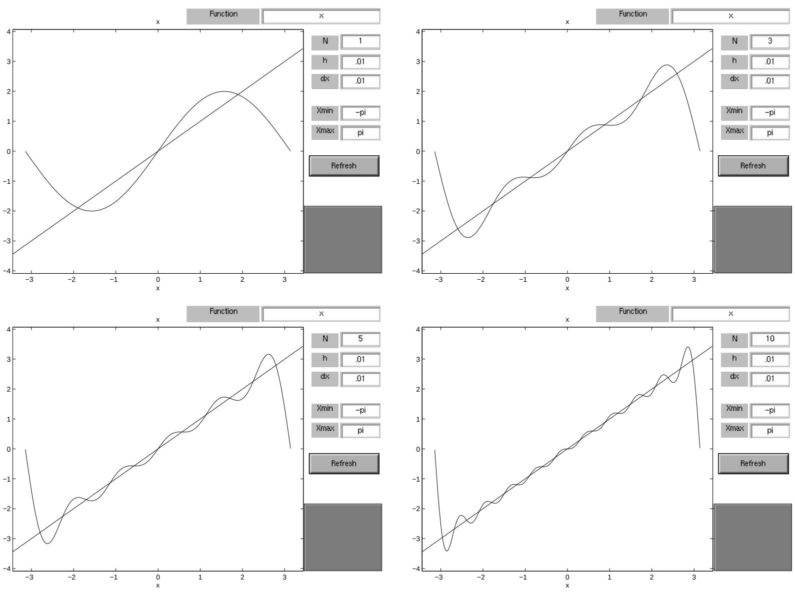

FIGURE 1. Fourier approximations to $f(t) = t$

There are three things to observe from these curves. First, the functions $S_m(t)$ do get closer to $f(t)$ as m incresases, at least in a somewhat smaller interval. Second, at the endpoints it is not possible for S_m to converge to f , since S_m is 2π-periodic and hence has the same values at these enpoints. In this case the value is 0. Finally, close to the endpoints there are blips which do not approach zero, although they do get closer to the endpoints. This is called the Gibbs phenomenon and again is related to the fact that f does not have the same values at the endpoints. We do not investigate this phenomenon further.

In Figure 2 we see the corresponding curves for $f(t) = t^2$. This function has the same values at the endpoints. Notice that the blips have diss-appeared and the convergence is faster, but it is still somewhat slower at the endpoints.

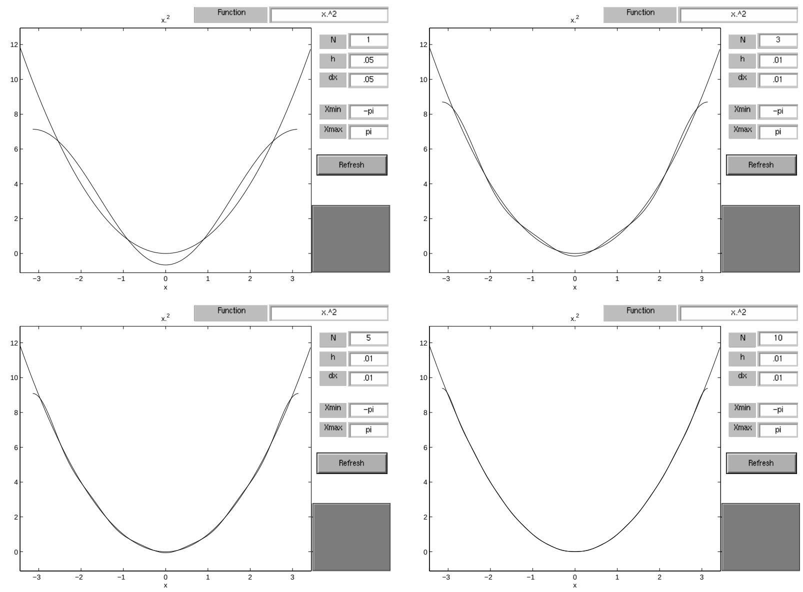

FIGURE 2. Fourier approximations to $f(t) = t^2$

1.3. **Exercises.** 1.Determine the Fourier coefficients of $f(t) = e^t$

2. a) A function f is called *even* if $f(-t) = f(t)$ for all t and *odd* if $f(-t) = -f(t)$. Which polynomials are even or odd ?
b)If $f \in C[-\pi, \pi]$ is even, prove that $b_k = 0$ for all $k > 0$. If f is odd, prove that $a_k = 0$ for all $k \geq 0$.
c)Prove that any f may be written as a sum of an even and an odd function. (Hint : Let $f_e(t) = (f(t) + f(-t))/2$ and $f_o(t) = (f(t) - f(-t))/2$.
d)Determine the Fourier coefficients of $sinht$ and $cosht$.

3.a) If p is an odd polynomial, prove that

$$\int_{-\pi}^{\pi} p(t)sin(kt)\,dt = (-1)^{k+1}\frac{2}{k}p(\pi) - \frac{1}{k^2}\int_{-\pi}^{\pi} p''(t)sin(kt)\,dt$$

b) If p is an even polynomial, prove that

$$\int_{-\pi}^{\pi} p(t)cos(kt)\,dt = (-1)^{k}\frac{2}{k^2}p'(\pi) - \frac{1}{k^2}\int_{-\pi}^{\pi} p''(t)cos(kt)\,dt$$

c) Find the Fourier coefficients of $f(t) = t^3 + t^2$.

4.The curves in the figures above were made by using the Matlab toolbox fourgraph. Download fourgraph from http://www.mathworks.com (for instance by writing fourgraph in the search box). Reproduce the curves above and try some other functions, for instance e^t or $t^3 + t^2$.

2. Trigonometric polynomials

2.1. **The complex exponential function.** The real exponential function e^x can be extended to complex values of the argument. Let $z = s + it$ be a complex number. If the complex exponential function satisfies the usual product rule, we must have

$$e^z = e^{s+it} = e^s e^{it}$$

Hence we must define the complex function $f(t) = e^{it}$. If f satisfies the usual chain rule for differentiation of the exponential function, we must have $f^{'}(t) = ie^{it} = if(t)$. Hence, if $g(t)$ and $h(t)$ denote the real and imaginary parts of $f(t)$, i.e. $f(t) = g(t) + ih(t)$, we get

$$g^{'}(t) + ih^{'}(t) = i(g(t) + ih(t))$$

and we must have

$$\begin{aligned} h^{'}(t) &= g(t) \\ g^{'}(t) &= -h(t) \end{aligned}$$

which gives

$$\begin{aligned} g^{''}(t) &= -h^{'}(t) = -g(t) \\ h^{''}(t) &= g^{'}(t) = -h(t) \end{aligned}$$

Since we also must have $f(0) = 1$, this gives $g(0) = 1$ and $h(0) = 0$. The solutions to these equations are $g(t) = \cos t$ and $h(t) = \sin t$.Hence we must have $e^{it} = \cos t + i \sin t$. We therefore define the complex exponential function e^z by

$$e^{s+it} = e^s(\cos t + i \sin t) \tag{2.1}$$

The complex number $e^{it} = \cos t + i \sin t$ is located on the unit circle in the complex plane at the angle t with the real axis. It follows that the complex number $w = e^{s+it}$ in the point in the complex plane whose length is e^s and angle is t. It follows from (2.1) that the complex exponential function satisfies the usual addition rule for the exponent:

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

From this we obtain the famous DeMoivre's formula:

$$\cos nt + i \sin nt = (\cos t + i \sin t)^n \tag{2.2}$$

since both sides equal $e^{int} = (e^{it})^n$. This formula contains the formulas for $\cos nt$ and $\sin nt$ as functions of $\cos t$ and $\sin t$ and is the most efficient way of expressing these formulas. It is also the easiest to remember.

If f is a real or complex function defined on the unit circle in the complex plane, then $f(e^{it})$, as a function of t, is 2π-periodic. Conversely, given any function f of t that is 2π-periodic, we may think of f as being defined on the unit circle. In this way we identify 2π-periodic functions with functions defined on the unit circle in the complex plane.

Also, if f is a function defined on $[-\pi, \pi)$, then f extends uniquely to a 2π-periodic function on the whole real line. This is actually true for any function defined on a (half-open) interval of length 2π.

2.2. **Complex representation of trigonometric polynomials.** In this section we shall write a trigonometric polynomial $p \in \mathcal{T}_m$ in the form

$$p(t) = \frac{1}{2}a_0 + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt).$$

Notice the factor $\frac{1}{2}$ in front of a_0. Hence if $p(t) = 4 + \cos t + \sin t$, then $a_0 = 8$, $a_1 = 1$ and $b_1 = 1$. This convention will make formulas simpler later.(Notice also that if p is the projection of some 2π periodic function f onto $\mathcal{T}_m$, then all the a_k-coefficients are given by the formula $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos kt\, dt$, also for $k = 0$.) By convention, we also set $b_0 = 0$. For any complex number $z = x + iy$, the real and imaginary parts are given by

$$\Re z = \frac{1}{2}(z + \bar{z})$$
$$\Im z = \frac{1}{2i}(z - \bar{z})$$

Applying this to $z = e^{ikt}$, we see that

$$\cos kt = \frac{1}{2}(e^{ikt} + e^{-ikt})\,, \sin kt = \frac{1}{2i}(e^{ikt} - e^{-ikt}) \tag{2.3}$$

so

$$\begin{aligned} p(t) &= \frac{1}{2}a_0 + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt) \\ &= \frac{1}{2}a_0 + \sum_{k=1}^{m} \frac{1}{2}a_k(e^{ikt} + e^{-ikt}) + \frac{1}{2i}b_k(e^{ikt} - e^{-ikt}) \\ &= \frac{1}{2}a_0 + \sum_{k=1}^{m} \frac{1}{2}(a_k - ib_k)e^{ikt} + \frac{1}{2}(a_k + ib_k)e^{-ikt} \end{aligned}$$

Hence if we define

$$c_k = \frac{1}{2}(a_k - ib_k), c_{-k} = \frac{1}{2}(a_k + ib_k) \tag{2.4}$$

we see that we can write this as

$$p(t) = \sum_{k=-m}^{m} c_k e^{ikt}. \tag{2.5}$$

The constants satisfy $c_{-k} = \bar{c_k}$. This will be called the complex representation of the trigonometric polynomial $p(t)$ Conversely, any function of the form (2.5) is a real trigonometric polynomial, provided the constants satisfy $c_{-k} = \bar{c_k}$. The coefficients are then given by

$$a_k = c_k + c_{-k}$$
$$b_k = i(c_k - c_{-k})$$

On the unit circle $z = e^{it}$, hence if we consider p as a function on the unit circle, we have

$$p(z) = \sum_{k=-m}^{m} c_k z^k \tag{2.6}$$

This shows the connection between real trigonometric polynomials and complex polynomials (if we admit negative powers in a polynomial).

By $\mathcal{P}_{2m+1}$ we denote the set of complex functions of the form $\sum_{k=-m}^{m} c_k z^k$. We have shown that $\mathcal{T}_m$ is naturally isomorphic to the real subspace of $\mathcal{P}_{2m+1}$ defined by the equations $c_{-k} = \bar{c_k}$ for $k = 0, 1, .., m$.If we had allowed complex values for the coefficients a_k, b_k in $\mathcal{T}_m$, i.e. considered the complex vector space $\mathbf{C}\mathcal{T}_m$ of complex trigonometric polynomials, the argument above shows that $\mathbf{C}\mathcal{T}_m$ and $\mathcal{P}_{2m+1}$ are isomorphic as complex vector spaces.

Finally, if $S_m(t)$ is the projection of a function f onto $\mathcal{T}_m$, then the coefficients of the complex representation of $S_m(t)$ are given by the one formula

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt \tag{2.7}$$

These coefficients are called the complex Fourier coefficients of f.

2.3. **Sampling of functions in $\mathcal{T}_m$.** The vector space $\mathcal{T}_m$ is a real vector space of dimension $2m + 1$. It is therefore reasonable to expect that a trigonometric polynomial $p \in \mathcal{T}_m$ is uniquely determined by its values at $2m + 1$ points. We shall see that this is the case if we divide the interval $[0, 2\pi]$ into $2m + 1$ equally long pieces, i.e. consider the points

$$t_j = j\frac{2\pi}{2m+1}, j = 0, 1, \cdots, 2m$$

Furthermore, we shall determine the precise relationship between the coefficients $a_0, a_1, \cdots, a_m, b_1, \cdots, b_m$ and the sampled values $v_j = p(t_j)$, $j = 0, 1, \cdots, 2m$.

To do this, we consider the evaluation map

$$L : \mathcal{T}_m \to \mathbf{R}^{2m+1}$$

defined by $L(p) = \boldsymbol{v}$ where the components of $\boldsymbol{v}$ are defined by evaluating p at t_j, i.e. $v_j = p(t_j)$. The colums of the matrix A of L with respect to the basis $\{1, \cos kt, \sin kt\}$ of $\mathcal{T}_m$ and the standard basis of $\mathbf{R}^{2m+1}$ are given by:

$$\boldsymbol{u}_0 = L(1) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\boldsymbol{u}_k = L(\cos kt) = \left[\cos kj\frac{2\pi}{2m+1}\right], j = 0, 1, \cdots, 2m$$

$$\boldsymbol{v}_k = L(\sin kt) = \left[\sin kj\frac{2\pi}{2m+1}\right], j = 0, 1, \cdots, 2m$$

for $k = 1, 2, \cdots, m$.

We shall prove that the vectors $\boldsymbol{u}_0, \cdots, \boldsymbol{u}_m, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_m$ are orthogonal, hence form an orthogonal basis for $\mathbf{R}^{2m+1}$. To prove this, we shall need the following formula:

$$\sum_{j=0}^{N} \cos j\alpha = \sum_{j=0}^{N} \Re(e^{ij\alpha}) = \Re\sum_{j=0}^{N}(e^{i\alpha})^j = \Re(\frac{1-e^{i(N+1)\alpha}}{1-e^{i\alpha}}) \tag{2.8}$$

where we have used the formula for the sum of a (finite) geometric series. Similarly, we also have

$$\sum_{j=0}^{N} \sin j\alpha = \Im(\frac{1-e^{i(N+1)\alpha}}{1-e^{i\alpha}}) \tag{2.9}$$

We shall prove that the vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m$ are orthogonal. The argument for the $\boldsymbol{u}$-vectors is similar and also that the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s are mutually orthogonal. Therefore let k and k' be two indices. The usual trig formula for the product of two sines then gives

$$\begin{aligned} \langle \boldsymbol{v}_k, \boldsymbol{v}_{k'} \rangle &= \sum_{j=0}^{2m} \sin kj\frac{2\pi}{2m+1} \sin k'j\frac{2\pi}{2m+1} \\ &= \frac{1}{2}\sum_{j=0}^{2m} \cos j\frac{2\pi(k-k')}{2m+1} - \frac{1}{2}\sum_{j=0}^{2m} \cos j\frac{2\pi(k+k')}{2m+1} \end{aligned}$$

Setting $N = 2m$ and $\alpha = \frac{2\pi(k+k')}{2m+1}$ we have

$$1 - e^{i(N+1)\alpha} = 1 - e^{i2\pi(k+k')} = 0$$

, hence the second sum is zero by (2.8). Also, the first sum is zero for $k \neq k'$ and equal to $\frac{2m+1}{2}$ for $k = k'$. This show that the vectors all are orthogonal and the norms are given by

$$\|\boldsymbol{u}_0\|^2 = 2m+1,$$
$$\|\boldsymbol{u}_k\|^2 = \|\boldsymbol{v}_k\|^2 = \frac{2m+1}{2}$$

Hence A is an ortogonal matrix whose inverse is given by

$$A^{-1} = \frac{2}{2m+1}\begin{bmatrix} \frac{1}{2}\boldsymbol{u}_0^T \\ \boldsymbol{u}_1^T \\ \vdots \\ \boldsymbol{u}_m^T \\ \boldsymbol{v}_1^T \\ \vdots \\ \boldsymbol{v}_m^T \end{bmatrix}$$

In other words, if p is a trigonometric polynomial whose values $v_j = p(t_j)$ are given by $\boldsymbol{v} = [v_j]$, then the coefficients of p are given by

$$a_0 = \frac{1}{2m+1}\langle \boldsymbol{u}_0, \boldsymbol{v}\rangle,$$
$$a_k = \frac{2}{2m+1}\langle \boldsymbol{u}_k, \boldsymbol{v}\rangle, 1 \le k \le m,$$
$$b_k = \frac{2}{2m+1}\langle \boldsymbol{v}_k, \boldsymbol{v}\rangle, 1 \le k \le m$$

2.4. **Signal processing.** The terms in a trigonometric polynomial have different *frequencies*. In figure 1 the top curve is $\sin t$ and the middle curve is $\sin 5t$. We see that the middle curve oscillates five times as fast as the top curve, i.e. the frequency is five times higher. So the constant k that appears in the terms $\sin kt$ and $\cos kt$ measures the frequency of the oscillation. If we draw the graph of a trigonometric polynomial $p(t)$, then we call the curve the time representation of p, or the *representation of p in the time domain.* The bottom curve is $p(t) = \sin t - 0.5\sin 2t + \sin 5t$. This curve therefore is the representation of p in the time domain.

A trigonometric polynomial, however, is defined by

$$p(t) = a_0 + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt).$$

We call this the *representation of p in the frequency domain* because this representation explicitly states how much (specified by the constants a_k and b_k) p contains of oscillations with frequency k. In the bottom curve, p contains the frequencies $1, 2$ and 5, or more explicitly, $b_1 = 1$, $b_2 = -0.5$ and $b_5 = 1$, while all the other a's and b's are 0.

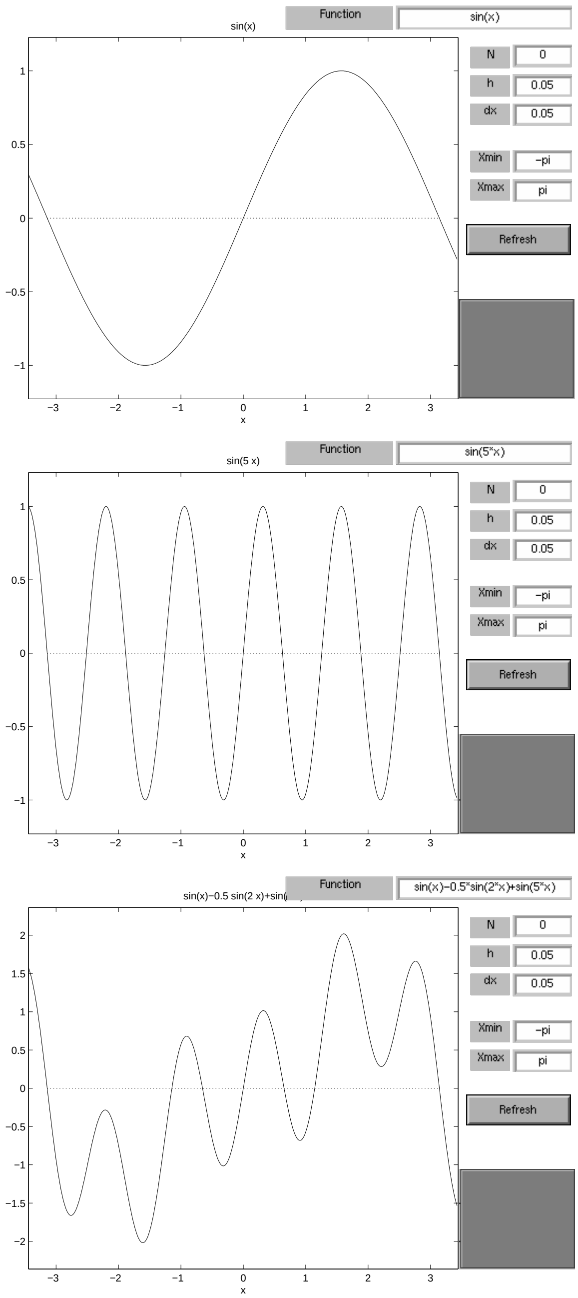

FIGURE 3. Time representations of trigonometric polynomials

We saw in the previous paragraph that p also was caracterized by its values at regularly spaced points. We may think of this as a discrete time representation of p. If the points are chosen very close (i.e. m is chosen big), then these points will trace out the graph. We also saw that one could pass from the frequency representation to the time representation and back. This is the central theme of Fourier analysis; that functions may be described either in the time domain (through its graph) or in the frequency domain, by breaking the function down into its frequency components (i.e. components of different frequencies).

It follows from the orthogonality of the basis $\{1, \cos kt, \sin kt\}$ that the projection of $\mathcal{T}_m$ onto $\mathcal{T}_n$ for some $n < m$ simply consists of omitting the highest frequencies, i.e.

$$P(a_0 + \sum_{k=1}^{m} a_k \cos kt + b_k \sin kt) = a_0 + \sum_{k=1}^{n} a_k \cos kt + b_k \sin kt$$

So, the best (L^2) approximation of a trigonometric polynomial of degree m by a trigonometric polynomial of degree n is simply the degree n part of the trigonometric polynomial.

2.5. **Convolution in $\mathcal{T}_m$.** If f and g are 2π-periodic functions, we define the convolution of f and g by

$$f * g(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) g(t-s)\, ds$$

It follows that we can actually perform the inegration over any interval of length 2π and that $f * g$ also is 2π-periodic and $f * g = g * f$. The functions can be real or complex. If $f, g \in \mathcal{T}_m$, the convolution is most easily described using the complex Fourier representation. Therefore, assume that $f(t) = e^{ikt}$ and $g(t) = e^{ik't}$. Then

$$\begin{aligned} f * g(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iks} e^{ik'(t-s)}\, ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik't} e^{i(k-k')s}\, ds \\ &= \frac{1}{2\pi} e^{ik't} \int_{-\pi}^{\pi} e^{i(k-k')s}\, ds = \begin{cases} 0 & \text{if } k \neq k' \\ e^{ikt} & \text{if } k = k' \end{cases} \end{aligned}$$

Hence, if $f = \sum_{k=-m}^{m} c_k e^{ikt}$ and $f = \sum_{k=-m}^{m} c'_k e^{ikt}$, then

$$f * g(t) = \sum_{k=-m}^{m} c_k c'_k e^{ikt}$$

Hence the complex Fourier coefficients of the convolution of f and g equal the product of the Fourier coefficients of f and g. In other words, convolution in the time domain corresponds to pointwise products in the frequency domain.

If we project a trigonometric polynomial $p \in \mathcal{T}_m$ to $\mathcal{T}_n$, then we simply omit the terms of degree greater than n, i.e. we multiply the coefficients c_k by 1 for $|k| \leq n$ and by 0 for $|k| > n$. By the discussion above, this means that we convolve p with the function

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt} \qquad (2.10)$$

which is called the *Dirichlet kernel* (of degree n). This is a finite geometric series, whose sum is

$$\begin{aligned} D_n(t) &= e^{-int}\frac{e^{i(2n+1)t}-1}{e^{it}-1} \\ &= \frac{e^{i(n+1)t}-e^{-int}}{e^{it}-1} \\ &= \frac{e^{i(n+\frac{1}{2})t}-e^{-i(n+\frac{1}{2})t}}{e^{i\frac{t}{2}}-e^{-i\frac{t}{2}}} \\ &= \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t} \end{aligned}$$

by (2.3).In Figure 2 we see the Dirichlet kernel for $n = 1, 3, 5, 10$.

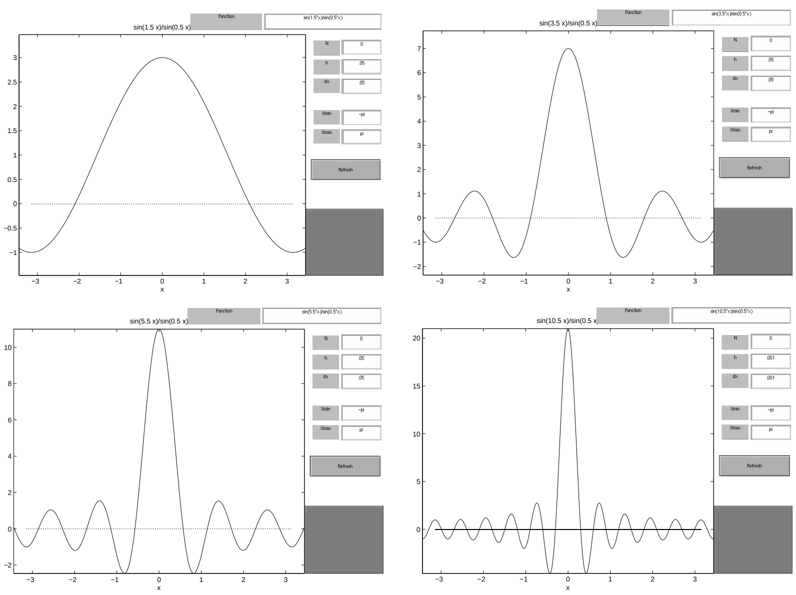

FIGURE 4. Dirichlet kernel for $n = 1, 3, 5, 10$

Notice that the peaks are higher and higher. We have $\int_{-\pi}^{\pi} D_n(t)dt = 2\pi$ for all n. From the figures above, we see that as n increases $D_n(t)$ more and more concentrates a mass (i.e. integral) of 2π at the origin.

If f is a 2π-periodic function, we have

$$\begin{aligned} S_m(t) &= \sum_{k=-m}^{m} c_k e^{ikt} \\ &= \sum_{k=-m}^{m} (\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-iks}\,ds)e^{ikt} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)(\sum_{k=-m}^{m} e^{ik(t-s)})\,ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)D_m(t-s)\,ds \\ &= f * D_m(t) \end{aligned}$$

We shall see later that if f is differentiable, this converges to f. This is easily understood from the fact that for large m, $\frac{1}{2\pi}D_m(t-s)$ concentrates a mass of 1 at s=t.

2.6. **Exercises.** 1.Show that the complex Fourier coefficients of an even/odd function are real/pure imaginary. Use exercise 3c) in Chapter 1 to find the complex Fourier coefficients of $f(t) = t^3 + t^2$

2 a)For $m = 1$, determine the 3x3 matrix A of the evaluation map $L : \mathcal{T}_1 \to \mathbf{R}^3$ described in this chapter.
b) Determine $p \in \mathcal{T}_1$ such that $p(0) = p(\frac{4\pi}{3}) = 1$ and $p(\frac{2\pi}{3}) = 4$.
c)Using MATLAB, compute the 5x5 matrix A for $L : \mathcal{T}_2 \to \mathbf{R}^5$. Use this to find a trigonometric polynomial $p \in \mathcal{T}_2$ such that $p(0) = p(\frac{6\pi}{5}) = 0$ and $p(\frac{2\pi}{5}) = p(\frac{4\pi}{5}) = p(\frac{8\pi}{5}) = 1$. (Plot your curve to check the result, for instance using fourgraph).

3. Let f be the 2π-periodic function whose values in $[-\pi, \pi)$ is given by $f(t) = t$.
a)Determine $f * f$ and draw the graph. (Requires some work!)
b)Determine the complex Fourier coefficients of f.
c)Determine $S_m * S_m$ for $m = 1$ and 2 and draw the graph.

4.(Parseval's identity. Move to Chapter 1 next year!) If $p(t) = a_0 + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt)$, show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |p(t)|^2\,dt = a_0^2 + \frac{1}{2}\sum_{k=1}^{m}(a_k^2 + b_k^2)$$

5.Using fourgraph, plot the projection of $f(t) = t^6 - 13t^4 + 36t^2 - 2$ onto $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_3, \mathcal{T}_5, \mathcal{T}_{10}$.

3. Orthogonal systems of functions

The basic properties of the Fourier expansion of a function f discussed above is a consequence of the orthogonality properties of the basis functions $\cos(kx)$ and $\sin(kx)$, i.e.

$$\int_{-\pi}^{\pi} \cos(jx)\cos(kx)\,dx = 0 \quad j \neq k,$$
$$\int_{-\pi}^{\pi} \sin(jx)\sin(kx)\,dx = 0 \quad j \neq k,$$
$$\int_{-\pi}^{\pi} \cos(jx)\sin(kx)\,dx = 0.$$

In fact, as soon as we have a set of basis functions satisfying similar orthogonality properties, the a corresponding "generalized Fourier expansion" can be derived.

A sequence of functions $\{\phi_k(x)\}_{k=1}^{\infty}$, defined on an interval $[a, b]$, is referred to as an orthogonal system of functions if

$$\int_{a}^{b} \phi_j(x)\phi_k(x)\,dx = 0 \quad j \neq k. \tag{3.1}$$

Of course, the standard trigonometric basis functions fits into this set up if we let the interval $[a, b]$ be $[-\pi, \pi]$, $\phi_{2k+1}(x) = \cos(kx)$ and $\phi_{2k}(x) = \sin(kx)$. However, there are many more examples of orthogonal systems.

Example 3.1 Let the interval $[a, b]$ be $[-\pi, \pi]$ and $\phi_k(x) = \sin(kx)$, $k \geq 1$. Then the system is orthogonal, since we already know that

$$\int_{-\pi}^{\pi} \sin(jx)\sin(kx)\,dx = 0 \quad j \neq k.$$

Example 3.2 Let the interval $[a, b]$ be $[0, \pi]$ and $\phi_k(x) = \sin(kx)$, $k \geq 1$. Note that we have changed the interval of definition. In order to show that the system is orthogonal we have to show that

$$\int_{0}^{\pi} \sin(jx)\sin(kx)\,dx = 0 \quad j \neq k.$$

To show this we use the identity

$$\sin(u)\sin(v) = \frac{1}{2}(\cos(u - v) - \cos(u + v)).$$

¿From this identity we obtain for $j \neq k$, using the fact that $\sin(k\pi) = 0$ for all integers k, that

$$\begin{aligned}\int_0^\pi \sin(jx)\sin(kx)\,dx &= \frac{1}{2}\int_0^\pi (\cos((j-k)x) - \cos(j+k)x)\,dx \\ &= \frac{1}{2(j-k)}\sin((j-k)\pi) - \frac{1}{2(j+k)}\sin((j+k)\pi) \\ &= 0.\end{aligned}$$

Hence, the system is orthogonal.

Example 3.3 Let the interval $[a,b]$ be $[0,\pi]$ and $\phi_k(x) = \sin((k+\frac{1}{2})x)$, $k \geq 1$. As above we have for $j \neq k$

$$\begin{aligned}\int_0^\pi \phi_j(x)\phi_k(x)\,dx &= \int_0^\pi \sin((j+\frac{1}{2})x)\sin((k+\frac{1}{2})x)\,dx \\ &= \frac{1}{2}\int_0^\pi (\cos((j-k)x) - \cos(j+k+1)x)\,dx \\ &= \frac{1}{2(j-k)}\sin((j-k)\pi) - \frac{1}{2(j+k+1)}\sin((j+k+1)\pi) \\ &= 0.\end{aligned}$$

Therefore, the system is orthogonal.

All the examples we have studied up to now are basesd on the trigonometric functions sin and cos. However, there also exists many other orthogonal systems, for example orthogonal polynomials.

Example 3.4 The Legendre polynomials are orthogonal functions with respect to the interval $[-1,1]$. For $k \geq 0$ these polynomials are of the form

$$L_k(x) = \alpha_k(\frac{d}{dx})^k(x^2-1)^k,$$

where α_k is a suitable constant and $(\frac{d}{dx})^k$ is the derivative of order k. For the discussion here we let $\alpha_k = 1$, even if other scalings, like $\alpha_k = 1/2^k k!$ is more standard.

Note that $(x^2-1)^k$ is a polynomial of degree $2k$. By differentiating this function k times we therefore end up with a polynomial of degree k. We have therefore established that $L_k(x)$ is a polynomial of degree k. In fact, from the definition above we can easily compute

$$\begin{aligned}L_0(x) &= 1, \\ L_1(x) &= 2x, \\ L_2(x) &= 12x^2 - 4.\end{aligned}$$

Next we like to establish that the polynomials L_k are orthogonal, i.e. we like to show that

$$\int_{-1}^1 L_j(x)L_k(x)\,dx = 0 \quad \text{for } j \neq k.$$

Consider first the polynomial $L_2(x)$. The integral of this polynomial over $[-1,1]$ is given by

$$\int_{-1}^{1} L_2(x)\,dx = (4x^3 - 4x)|_{-1}^{1} = 0.$$

Similarly, the first order moment of $L_2(x)$ is given by

$$\begin{aligned}\int_{-1}^{1} xL_2(x)\,dx &= \int_{-1}^{1} (12x^4 - 4x^2)\,dx \\ &= (3x^4 - 2x^2)|_{-1}^{1} = 1 - 1 = 0.\end{aligned}$$

By combining these two results it follows that for arbritary constants a_0 and a_1 we have

$$\int_{-1}^{1} (a_0 + a_1 x)L_2(x)\,dx = a_0 \int_{-1}^{1} L_2(x)\,dx + a_1 \int_{-1}^{1} xL_2(x)\,dx = 0.$$

In orther words, the quadratic polynomial $L_2(x)$ is orthogonal to all linear polynomials. In particular,

$$\int_{-1}^{1} L_2(x)L_1(x)\,dx = \int_{-1}^{1} L_2(x)L_0(x)\,dx = 0,$$

i.e. L_2 is orthogonal to L_1 and L_0.

A similar argument can be used to show that L_k is orthogonal to L_j for $j < k$. Since L_j is a polynomial of degree j it is enough to show that

$$\int_{-1}^{1} x^j L_k(x)\,dx = 0 \quad \text{for all } j < k.$$

Note that the function $(x^2-1)^k = (x-1)^k(x+1)^k$ has a root of order k at each endpoint ± 1. Therefore, if $i < k$ then

$$(\frac{d}{dx})^i (x^2 - 1)^k = 0 \quad \text{for } x = \pm 1.$$

Integration by parts therefore gives

$$\begin{aligned}\int_{-1}^{1} x^j L_k(x)\,dx &= \int_{-1}^{1} x^j (\frac{d}{dx})^k (x^2-1)^k\,dx \\ &= x^j (\frac{d}{dx})^{k-1} (x^2-1)^k)|_{-1}^{1} - \int_{-1}^{1} (\frac{d}{dx} x^j)(\frac{d}{dx})^{k-1}(x^2-1)^k\,dx \\ &= -\int_{-1}^{1} j x^{j-1} (\frac{d}{dx})^{k-1} (x^2-1)^k\,dx.\end{aligned}$$

By repeating this argument k times we get

$$\int_{-1}^{1} x^j L_k(x)\,dx = (-1)^k \int_{-1}^{1} ((\frac{d}{dx})^k x^j)(x^2-1)^k\,dx = 0,$$

where we have used that k^{th} derivative of x^j equals zero for $j < k$. Hence, we have shown that L_k is orthogonal to L_j for $j < k$, or, in other words, the sequence $\{L_k\}$ is an orthogonal system of polynomials.

Example 3.5 If $\{\psi_k(x)\}_{k=1}^{\infty}$ is any sequence of functions in $C[a, b]$, then we can produce an orthogonal sequence $\{\phi_k(x)\}_{k=1}^{\infty}$ by Gram-Schmidt orthogonalization :

$$\begin{aligned} \phi_1(x) &= \psi_1(x) \\ &\vdots \\ \phi_n(x) &= \psi_n(x) - \textstyle\sum_{k=1}^{n-1} \frac{\langle \psi_n, \phi_k \rangle}{\|\phi_k\|^2} \phi_k(x) \end{aligned}$$

Returning to the general situation, let $\{\phi_k(x)\}_{k=1}^{\infty}$ be an orthogonal system of continuous functions defined on an interval $[a, b]$. For each integer $n \geq 1$ we let W_n be the subspace of $C[a, b]$ spanned by $\{\phi_1, \phi_2, \ldots \phi_n\}$, i.e.

$$W_n = \{w \in C[a, b] \, : w(x) = \sum_{k=1}^{n} a_k \phi_k(x), \quad a_k \in \mathbb{R}\}.$$

It is clear that

$$W_1 \subset W_2 \subset \ldots \subset W_n \subset W_{n+1} \ldots$$

since an element in W_n corresponds to an element in W_{n+1} with $a_{n+1} = 0$.

Let $P_n : C[a, b] \mapsto W_n$ be the orthogonal projection. It follows from (1.4) that

$$P_n f = \sum_{k=1}^{n} a_k \phi_k, \tag{3.2}$$

where the coefficients a_k are given by

$$a_k = \frac{1}{\|\phi_k\|^2} \int_a^b f(x) \phi_k(x) \, dx \quad \text{and} \quad \|\phi_k\|^2 = \int_a^b (\phi_k(x))^2 \, dx. \tag{3.3}$$

The finite series (3.2) is a generalized finite Fourier expansion of f, and the function $P_n f$ is the best L^2-approximation of f by a function in W_n. We observe that, just as in the case of the ordinary Fourier expansion, the coefficients a_k are independent of n. Hence, in order to compute $P_{n+1} f$, when $P_n f$ is known, all we have to do is to add the extra term $a_{n+1} \phi_{n+1}$.

The coefficients a_k are called the *generalized Fourier coefficients* of f and the series

$$\sum_{k=1}^{\infty} a_k \phi_k(x)$$

is called the generalized Fourier series of f.

It also follows from (3.2) that

$$\|P_n f\|^2 = \sum_{k=1}^{n} a_k^2 \|\phi_k\|^2.$$

By combining this identity with (1.3) we obtain the inequality

$$\sum_{k=1}^{n} a_k^2 \|\phi_k\|^2 \le \|f\|^2 = \int_a^b (f(x))^2\,dx,$$

where the coefficients a_k are given by (3.3). We observe that the right hand side of this inequality is independent of n. Hence, the partial sums on the left hand side are bounded independent of n. Since all the terms in the series are positive it therefore follows that the inequality still holds with $n = \infty$, cf. Theorem 12.2.1 in [2]. In particular, the infinite series converges. This result is usually referred to as *Bessel's inequality*. We state the result precisely as a theorem.

Theorem 3.1 (Bessel's inequality)**.** *If $f \in C[a,b]$ and $\{\phi_k\}_{k=1}^{\infty}$ is an orthogonal system of continuous functions on $[a,b]$ then*

$$\sum_{k=1}^{\infty} a_k^2 \|\phi_k\|^2 \le \int_a^b (f(x))^2\,dx, \tag{3.4}$$

where the generalized Fourier coefficients a_k are given by (3.3)*.*

Since the infinite series (3.4) converges we must have, in particular, that the sequence $\{a_k\|\phi_k\|\}$ converges to zero. This result will be used later. We therefore state it as a corollary.

Corollary 3.1. *Let f, $\{\phi_k\}_{k=1}^{\infty}$ and $\{a_k\}$ be as in Theorem 3.1 above. Then*

$$a_k\|\phi_k\| \longrightarrow 0 \quad \text{as } k \to \infty.$$

In particular, if $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal family (or more generally, $\|\phi_k\| > c$ for all k for some $c > 0$), then $a_k \to 0$.

Example 3.6 If we consider the usual Fourier series

$$a_0 + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt$$

of a continuous 2π-periodic function $f(t)$ defined for all real t, then we have the following formula for the n-th partial sum :

$$\begin{aligned} S_n(t) &= a_0 + \sum_{k=1}^{n} a_k \cos kt + b_k \sin kt \\ &= f * D_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s) D_n(s)\,ds \end{aligned}$$

where D_n is the Dirichlet kernel

$$D_n(s) = \frac{\sin(n+\frac{1}{2})s}{\sin\frac{1}{2}s}$$

We have also shown that the integral of D_n is 2π, which gives

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(s)\,ds$$

Subtracting these two equations gives

$$\begin{aligned} f(t) - S_n(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t)-f(t-s))D_n(s)\,ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)-f(t-s)}{\sin\frac{1}{2}s}\sin(n+\frac{1}{2})s\,ds \end{aligned}$$

The function $g^t(s) = \frac{1}{2\pi}\frac{f(t)-f(t-s)}{\sin\frac{1}{2}s}$ is continuous for $s \neq 0$. It is easy to see, using L'Hoptal's rule, that it is also continuous for $s = 0$ if f is differentiable at t. Furthermore, the functions $\phi_n(s) = \sin(n+\frac{1}{2})s$ are orthogonal on $[-\pi,\pi]$, by example 3.3, and $\|\phi_n\| = \sqrt{\pi}$. Hence the integral above is just the generalized Fourier coefficients of g^t with respect to the orthogonal family ϕ_n, and therefore converges to zero, by Corollary 3.1.

Theorem 3.2 (Convergence of Fourier series)**.** *If f is a 2π-periodic continuous function which is differentiable at t, then*

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos kt + b_k\sin kt,$$

i.e. the Fourier series of f converges to f at t.

Example 3.7 If we use the orthononal family $\sin nt$ in $C[0,\pi]$ (see example 3.2), then the generalized Fourier series of $f \in C[0,\pi]$ is called the *sine series* of f and is given by

$$\sum_{k=1}^{\infty} b_k \sin kt \quad \text{and} \quad b_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin kt\,dt \tag{3.5}$$

It will converge to $f(t)$ for all $t \in (0,\pi)$ where f is differentiable. Since the series is always zero at $t = 0$ and $t = \pi$, it will converge there if and only if $f(0) = 0$ ($f(\pi) = 0$).

Example 3.8 It is also easy to see that the family $\{1, \cos t, \cos 2t, \cdots\}$ also is orthogonal in $C[0,\pi]$. In this case the generalized Fourier series of $f \in C[0,\pi]$ is called the *cosine series* of f and is given by

$$a_0 + \sum_{k=1}^{\infty} a_k\cos kt \quad \text{and} \quad \begin{aligned} a_0 &= \tfrac{1}{\pi}\textstyle\int_0^{\pi} f(t)\,dt \\ a_k &= \tfrac{2}{\pi}\textstyle\int_0^{\pi} f(t)\cos kt\,dt\,(k>0) \end{aligned} \tag{3.6}$$

It will converge to $f(t)$ for all $t \in (0, \pi)$ where f is differentiable and at the endpoints if f has one-sided derivatives at both endpoints.

3.1. **Exercises.** 1. Let $\{\phi_k(x)\}_{k=1}^{\infty} = \{\sin(kx)\}_{k=1}^{\infty}$ be the orthogonal system studied in Example 3.1 above.
a) Compute $P_n f$ for $f(x) = x$.
b) Compute $P_n g$ for $g(x) = 1$.

2. Let $\{\phi_k(x)\}_{k=1}^{\infty} = \{\sin(kx)\}_{k=1}^{\infty}$ be the orthogonal system studied in Example 3.2 above, i.e. the functions are considered on the interval $[0, \pi]$.
a) Compute $P_n f$ for $f(x) = x$.
b) Compute $P_n g$ for $g(x) = 1$.

3. a) Let $f(x)$ be the sign–function, i.e. $f(x) = 1$ for $x > 0$, $f(x) = -1$ for $x < 0$ and $f(0) = 0$. Compute the ordinary Fourier expansion of f and compare the result with the result of Exercise 2b above.
b) Let f be a function defined on $[0, \pi]$. Show that the generalized Fourier expansion of f, with respect to the orthogonal system $\{\sin(kx)\}_{k=1}^{\infty}$, is exactly the ordinary Fourier expansion of the odd extension of f.

4. Let $f(x)$ be the sign–function. Use Bessel's inequality to show that

$$\sum_{k=1}^{\infty} (\frac{1}{2k-1})^2 \leq \frac{\pi^2}{8}.$$

5. Let $\{L_k(x)\}_{k=1}^{\infty}$ be the Legendre polynomials studied in Example 3.4.
a) It can be shown that the polynomials $L_k(x)$ satisfies the recurrence relation

$$L_{k+1}(x) = 2(2k+1)xL_k(x) - 4k^2 L_{k-1}(x) \tag{3.7}$$

for $k \geq 1$. Use this relation, and the fact that $L_0(x) = 1$ and $L_1(x) = 2x$, to compute the polynomials $L_2(x)$, $L_3(x)$ and $L_4(x)$.
b) Establish the relation (3.7).

References

[1] L.W. Johnson, R.D. Riess and J.T. Arnold *Linear algebra,* 4. edition, Addison Wesley 1998.
[2] Tom Lindstrøm, *Kalkulus,* Universitetsforlaget 1995.