ESCUELA SUPERIOR POLITÉCNICA DEL LITORAL
INSTITUTO DE CIENCIAS MATEMÁTICAS
SEGUNDA EVALUACIÓN DE ALGEBRA LINEAL

Nombre: ....

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1. (20 ptos) Califique como verdaderas o falsas a las siguientes proposiciones. **Justifique sus respuestas.**

a. Una transformación lineal cuyo núcleo es $\{0_V\}$ es invertible. (FALSO).

Contraejemplo: $T: P_2 \rightarrow M_{2x2}$

$T(ax^2+bx+c) = \begin{pmatrix} a & b \\ c & c \end{pmatrix}$

$\Rightarrow Nu(T) = \{0x^2+0x+0\} \wedge Im(T) = \mathcal{L}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \neq M_{2x2}$

$\Rightarrow$ T es inyectiva $\wedge$ T no es sobreyectiva

$\Rightarrow$ T no es invertible

b. $\forall r, t \in \mathbb{R}$: $A = \begin{pmatrix} r\,\text{sen}(t) & \cos(t) \\ \cos(t) & -r\,\text{sen}(t) \end{pmatrix}$ es ortogonal. (FALSO)

A es Ortogonal $\equiv A^{-1} = A^T$ (Definición).

Por Teorema: A_{nxn} es Ortogonal $\equiv$ Las Columnas de A, son una Base Ortonormal de IR^n

$* \left\langle \begin{pmatrix} r\,\text{Sen}\,t \\ \text{Cos}\,t \end{pmatrix} / \begin{pmatrix} \text{Cos}\,t \\ -r\,\text{Sen}\,t \end{pmatrix} \right\rangle = r\,\text{Sen}\,t\,\text{Cos}\,t - r\,\text{Sen}\,t\,\text{Cos}\,t = 0$ ✓

$* \left\langle \begin{pmatrix} r\,\text{Sen}\,t \\ \text{Cos}\,t \end{pmatrix} / \begin{pmatrix} r\,\text{Sen}\,t \\ \text{Cos}\,t \end{pmatrix} \right\rangle = r^2\,\text{Sen}^2 t + \text{Cos}^2 t \neq 1$ (Ej: $t = \frac{\pi}{2}$, $r = 2$) *

$\Rightarrow$ A no es ortogonal.

Contraejemplo: $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ no es Ortogonal.

c. Sea V un espacio vectorial real con producto interno, sean $u,v\in V$ dos vectores ortonormales. Si los vectores $\alpha u+\beta v$ y $\alpha u-\beta v$ son ortogonales, entonces $|\alpha|=|\beta|$. (VERDADERO)

$u, v\in V$ dos vectores ortonormales $\Rightarrow \langle u/v\rangle=0 \wedge \|u\|=1 \wedge \|v\|=1$

$* \langle \alpha u+\beta v/\alpha u-\beta v\rangle = 0 \Rightarrow \langle \alpha u/\alpha u\rangle - \langle \alpha u/\beta v\rangle + \langle \beta v/\alpha u\rangle - \langle \beta v/\beta v\rangle = 0$

$\Rightarrow \alpha^2 \underbrace{\langle u/u\rangle}_{1} - \alpha\beta \cancel{\langle u/v\rangle} + \alpha\beta \cancel{\langle u/v\rangle}^{0} - \beta^2 \underbrace{\langle v/v\rangle}_{1} = 0$

$\Rightarrow \alpha^2 = \beta^2 \Rightarrow \sqrt{\alpha^2} = \sqrt{\beta^2}$

$\Rightarrow |\alpha| = |\beta|$

d. Si λ es un valor propio de $A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, entonces $\left(A+A^{-1}\right)^{\lambda} = 2^{\lambda} A$. (VERDADERO)

$\begin{vmatrix}1-\lambda&0\\0&-1-\lambda\end{vmatrix}=0 \Rightarrow \lambda=1 \vee \lambda=-1$; A es simétrica ; $A=A^{-1}$

$*$ Para $\lambda=1$: $(A+A)^{\lambda} = 2^{\lambda} A \Rightarrow (2A)^{1} = 2^{1} A \Rightarrow \underline{\underline{2A=2A}}$ ✓

$*$ Para $\lambda=-1$: $(A+A)^{\lambda} = 2^{\lambda} A \Rightarrow (2A)^{-1} = 2^{-1} A$

$$2A=\begin{pmatrix}2&0\\0&-2\end{pmatrix} \Rightarrow \underbrace{\left(\begin{array}{cc|cc}2&0&1&0\\0&-2&0&1\end{array}\right)}_{2A} \sim \underbrace{\left(\begin{array}{cc|cc}1&0&1/2&0\\0&1&0&-1/2\end{array}\right)}_{(2A)^{-1}}$$

$$\Rightarrow (2A)^{-1} = \begin{pmatrix}1/2&0\\0&-1/2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{1}{2}A = 2^{-1}A$$ ✓

2. (15 puntos) Sea $L: M_{2\times 2} \to \mathbb{R}^2$ tal que: $L\begin{pmatrix}0&1\\1&0\end{pmatrix} = \underset{\checkmark}{L\begin{pmatrix}1&0\\0&1\end{pmatrix}} = \underset{\checkmark}{L\begin{pmatrix}1&0\\1&0\end{pmatrix}} = \begin{pmatrix}1\\1\end{pmatrix}$, y $L\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. Determine:

a. Nu(L), Im(L).

b. La matriz asociada a L respecto a las bases canónicas.

$$* \boxed{L\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}} ; * L\begin{pmatrix}1&0\\1&0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \overset{\begin{pmatrix}0\\0\end{pmatrix}}{\cancel{L\begin{pmatrix}1&0\\0&0\end{pmatrix}}} + L\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \boxed{L\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}}$$

$$* L\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow L\begin{pmatrix}0&1\\0&0\end{pmatrix} + \boxed{L\begin{pmatrix}0&0\\1&0\end{pmatrix}} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow L\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \boxed{L\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}}$$

$$* L\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \boxed{L\begin{pmatrix}1&0\\0&0\end{pmatrix}} + L\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \boxed{L\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}}$$

ⓑ $$A_{L_{\beta_C}} = \left(\left[L\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]_{\beta_{C_{\mathbb{R}^2}}} \left[L\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]_{\beta_{C_{\mathbb{R}^2}}} \left[L\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]_{\beta_{C_{\mathbb{R}^2}}} \left[L\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]_{\beta_{C_{\mathbb{R}^2}}} \right)$$

$$\Rightarrow \boxed{A_{L_{\beta_C}} = \begin{pmatrix}0&0&1&1\\0&0&1&1\end{pmatrix}}$$

$$L\begin{pmatrix}a&b\\c&d\end{pmatrix} = a \overset{\begin{pmatrix}0\\0\end{pmatrix}}{\cancel{L\begin{pmatrix}1&0\\0&0\end{pmatrix}}} + b \overset{\begin{pmatrix}0\\0\end{pmatrix}}{\cancel{L\begin{pmatrix}0&1\\0&0\end{pmatrix}}} + c \overset{\begin{pmatrix}1\\1\end{pmatrix}}{\cancel{L\begin{pmatrix}0&0\\1&0\end{pmatrix}}} + d \overset{\begin{pmatrix}1\\1\end{pmatrix}}{\cancel{L\begin{pmatrix}0&0\\0&1\end{pmatrix}}}$$

$$\Rightarrow \boxed{L\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}c+d\\c+d\end{pmatrix}} \Rightarrow Nu(L) = \left\{ \begin{pmatrix}a&b\\c&d\end{pmatrix} \Big/ c+d=0 \right\}$$

$$\Rightarrow \boxed{Nu(L) = \left\{ \begin{pmatrix}a&b\\c&-c\end{pmatrix} \right\}} \Rightarrow \beta_{Nu(L)} = \left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&-1\end{pmatrix} \right\}$$

$$* Im(T) = \left\{ \begin{pmatrix}c+d\\c+d\end{pmatrix} \right\} \Rightarrow Im(T) = \left\{ c\begin{pmatrix}1\\1\end{pmatrix} + \cancel{d\begin{pmatrix}1\\1\end{pmatrix}} \right\} \Rightarrow \boxed{Im(T) = \left\{ \begin{pmatrix}c\\c\end{pmatrix}\right.}$$

3. (15 ptos) Sea $A = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix}$. Determine:

a. Los valores propios de A.

b. Una base para cada espacio propio de A.

$$\begin{vmatrix} \overset{+}{\alpha-\lambda} & \overset{-}{1} & \overset{+}{1} \\ 1 & \alpha-\lambda & 1 \\ 1 & 1 & \alpha-\lambda \end{vmatrix} = 0 \Rightarrow (\alpha-\lambda)\left[(\lambda-\alpha)^2-1\right] - \overbrace{\left[(\alpha-\lambda)-1\right]}^{\lambda-\alpha+1} + \overbrace{\left[1-(\alpha-\lambda)\right]}^{\lambda-\alpha+1} = 0$$

$$\Rightarrow (\alpha-\lambda)\left[\underbrace{(\lambda-\alpha)^2-1}_{\boxed{[(\lambda-\alpha)-1][(\lambda-\alpha)+1]}}\right] + 2(\lambda-\alpha+1) = 0 \Rightarrow \underline{(\lambda-\alpha)}(\lambda-\alpha-1)\overline{(\lambda-\alpha+1)} - 2\overline{(\lambda-\alpha+1)} = 0$$

$$\Rightarrow (\lambda-\alpha+1)\left[\underbrace{(\lambda-\alpha)(\lambda-\alpha-1)-2}\right] = 0$$

$$\boxed{\lambda = -1+\alpha}$$

$$\to (\lambda-\alpha)\left[(\lambda-\alpha)-1\right]-2$$

$$\to (\lambda-\alpha)^2-(\lambda-\alpha)-2$$

$$\to (\lambda-\alpha) = \frac{(\lambda-\alpha) \pm \sqrt{(\lambda-\alpha)^2 - 4(\lambda-\alpha)^2(-2)}}{2(\lambda-\alpha)}$$

$$\to (\lambda-\alpha) = \frac{\cancel{(\lambda-\alpha)} \pm \cancel{(\lambda-\alpha)}\sqrt{1+8}}{2\cancel{(\lambda-\alpha)}}$$

$$\to (\lambda-\alpha) = \frac{1 \pm 3}{2} \Rightarrow \lambda-\alpha = 2 \vee \lambda-\alpha = -1$$

$$\to \boxed{\lambda = 2+\alpha} \vee \boxed{\lambda = -1+\alpha}$$

$$\underline{\underline{E_{\lambda=-1+\alpha}}} = \beta$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} a & b & c & \\ 1 & 1 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$$

$$\Rightarrow E_{\lambda=-1+\alpha} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle/ a=-b-c \right\} = \left\{ \begin{pmatrix} -b-c \\ b \\ c \end{pmatrix} \right\}$$

$$\Rightarrow \boxed{\beta_{E_{\lambda=-1+\alpha}} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

$$\underline{\underline{E_{\lambda=2+\alpha}}} = \beta$$

$$\left(\begin{array}{ccc|c} a & b & c & \\ \cancel{-2} & \cancel{1} & \cancel{1} & \cancel{0} \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} a & b & c & \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow E_{\lambda=2+\alpha} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle/ \begin{matrix} a=c \\ b=c \end{matrix} \right\}$$

$$\Rightarrow \boxed{\beta_{E_{\lambda=2+\alpha}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}}$$

4. (5ptos) Si $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, determine si A es diagonalizable.

$$\begin{vmatrix} \overset{\oplus}{1-\lambda} & \overset{\ominus}{0} & 1 \\ 0 & 1-\lambda^{\oplus} & 1 \\ 1 & 0^{\ominus} & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[\overbrace{(\lambda-1)^2-1}^{\lambda^2-2\lambda+1-1}\right] = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-2\lambda)=0 \Rightarrow \lambda(\lambda-1)(\lambda-2)=0$$

$\Rightarrow$ A es diagonalizable puesto que todos sus valores propios son diferentes $\Rightarrow ma(\lambda_i) = mg(\lambda_i) \Rightarrow$ A tiene 3 vectores propios lin. ind. en $\mathbb{R}^3 \Rightarrow$ A es diagonalizable.

$$z = -\frac{1}{2}x + \frac{1}{3}y$$

5. (15 ptos) Sea $V = \mathbb{R}^3$, y $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 / 3x - 2y + 6z = 0 \right\}$

a. Determine $W^{\perp}$. Utilice el producto interno estándar.

b. Si $v = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$, determine la proyección de v en W.

ⓐ $W^{\perp} = ?$

1.) $\beta_W = \left\{ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$

$$W = \left\{ \begin{pmatrix} x \\ y \\ -\frac{1}{2}x + \frac{1}{3}y \end{pmatrix} \right\} \Rightarrow W = \mathcal{L}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1/3 \end{pmatrix} \right\}$$

2.) Condiciones

$$\left\langle \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} / \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = 0 \Rightarrow 2a - c = 0 \Rightarrow a = \frac{1}{2}c$$

$$\left\langle \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} / \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = 0 \Rightarrow 3b + c = 0 \Rightarrow b = -\frac{1}{3}c$$

3.) $$W^{\perp} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} / \begin{matrix} a = 1/2\, c \\ \wedge\, b = -1/3\, c \end{matrix} \right\} \Rightarrow W^{\perp} = \left\{ \begin{pmatrix} 1/2\, c \\ -1/3\, c \\ c \end{pmatrix} \right\} = \mathcal{L}\left\{ \begin{pmatrix} 1/2 \\ -1/3 \\ 1 \end{pmatrix} \right\}$$

$$\Rightarrow \beta_{W^{\perp}} = \left\{ \underbrace{\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}}_{u} \right\} \Rightarrow \|u\| = \sqrt{9+4+36} \Rightarrow \|u\| = 7$$

$$\Rightarrow \beta_{\substack{O.N \\ W^{\perp}}} = \left\{ \begin{pmatrix} 3/7 \\ -2/7 \\ 6/7 \end{pmatrix} \right\}$$

ⓑ $$\operatorname{proy}_{W^{\perp}} \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \underbrace{\left\langle \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} / \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} \right\rangle}_{(-9)+(-2)+(24)} \left(\frac{1}{49}\right) \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$$

$$\Rightarrow \operatorname{proy}_{W^{\perp}} \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \frac{13}{49} \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}; \quad \operatorname{proy}_{W} \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -39/49 \\ 26/49 \\ -78/49 \end{pmatrix}$$

$$\Rightarrow \operatorname{proy}_{W} \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -186/49 \\ 75/49 \\ 118/49 \end{pmatrix}$$