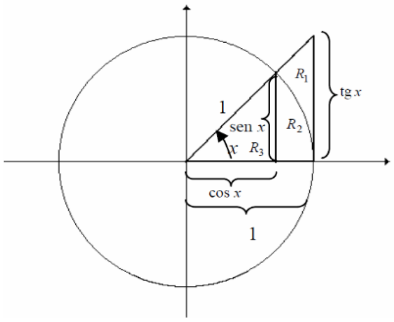
- $\lim\limits_{x\to 0}\dfrac{\operatorname{sen}(x)}{x}\equiv\dfrac{\operatorname{sen}((0))}{(0)}=\dfrac{0}{0}$, por tanto simplificamos la expresión:



$$Area_{R1}=\frac{(\tan(x))(1)}{2}\qquad Area_{R2}=\frac{(x)(1)^2}{2}\qquad Area_{R3}=\frac{(\operatorname{sen}(x))(\cos(x))}{2}$$

$$Area_{R1}\geq Area_{R2}\geq Area_{R3}$$

$$\frac{\tan(x)}{2}\geq\frac{x}{2}\geq\frac{\operatorname{sen}(x)\cos(x)}{2}\quad\Rightarrow\quad\frac{\operatorname{sen}(x)\cos(x)}{2}\leq\frac{x}{2}\leq\frac{(\tan(x))(1)}{2}$$

$$\frac{\operatorname{sen}(x)\cos(x)}{2}\left[\frac{2}{\operatorname{sen}(x)}\right]\leq\frac{x}{2}\left[\frac{2}{\operatorname{sen}(x)}\right]\leq\frac{\operatorname{sen}(x)}{2\cos(x)}\left[\frac{2}{\operatorname{sen}(x)}\right]$$

$$\cos(x)\leq\frac{x}{\operatorname{sen}(x)}\leq\frac{1}{\cos(x)}\quad\Rightarrow\quad\frac{1}{\cos(x)}\geq\frac{\operatorname{sen}(x)}{x}\geq\cos(x)$$

Por el teorema del Emparedado

$$\cos(x)\leq\frac{\operatorname{sen}(x)}{x}\leq\frac{1}{\cos(x)}\quad\Rightarrow\quad\lim_{x\to 0}\cos(x)\leq\lim_{x\to 0}\frac{\operatorname{sen}(x)}{x}\leq\lim_{x\to 0}\frac{1}{\cos(x)}$$

$$\cos(0)\leq\lim_{x\to 0}\frac{\operatorname{sen}(x)}{x}\leq\frac{1}{\cos(0)}\quad\Rightarrow\quad 1\leq\lim_{x\to 0}\frac{\operatorname{sen}(x)}{x}\leq 1$$

$$\therefore\ \lim_{x\to 0}\frac{sen(x)}{x}=1$$

Con este análisis se puede llegar a los siguientes limites notables:

$$\therefore\ \lim_{x\to 0}\frac{sen(kx)}{kx}=1\qquad\therefore\ \lim_{x\to 0}\frac{sen(kx)}{x}=k\qquad\therefore\ \lim_{x\to 0}\frac{sen^n(kx)}{x^n}=k^n$$

- $\lim\limits_{x\to 0}\dfrac{1-\cos(kx)}{x}\equiv\dfrac{1-\cos(k(0))}{(0)}=\dfrac{0}{0}$, por tanto simplificamos la expresión:

$$\equiv\lim_{x\to 0}\frac{1-cos(kx)}{x}=\lim_{x\to 0}\frac{1-cos(kx)}{x}\left[\frac{1+cos(kx)}{1+cos(kx)}\right]=\lim_{x\to 0}\frac{1-cos^2(kx)}{x[1+cos(kx)]}$$

$$\equiv\lim_{x\to 0}\frac{sen^2(kx)}{x[1+cos(kx)]}=\left[\lim_{x\to p0}\frac{sen(kx)}{x}\right]\left[\lim_{x\to 0}\frac{sen(kx)}{[1+cos(kx)]}\right]$$

$$\equiv\left[\lim_{x\to 0}\frac{sen(kx)}{x}\right]\left[\frac{sen(k(0))}{[1+cos(k(0))]}\right]=[k][0]=0$$

$$\therefore\ \lim_{x\to 0}\frac{1-cos(kx)}{x}=0$$

- $\lim\limits_{x\to 0}\dfrac{1-\cos(kx)}{x^2}\equiv\dfrac{1-\cos(k(0))}{(0)^2}=\dfrac{0}{0}$, por tanto simplificamos la expresión:

$$\equiv\lim_{x\to 0}\frac{1-cos(kx)}{x^2}=\lim_{x\to 0}\frac{1-cos(kx)}{x^2}\left[\frac{1+cos(kx)}{1+cos(kx)}\right]=\lim_{x\to 0}\frac{1-cos^2(kx)}{x^2[1+cos(kx)]}$$

$$\equiv\lim_{x\to 0}\frac{sen^2(kx)}{x^2[1+cos(kx)]}=\left[\lim_{x\to 0}\frac{sen(kx)}{x}\right]\left[\lim_{x\to 0}\frac{sen(kx)}{x}\right]\left[\lim_{x\to 0}\frac{1}{[1+cos(kx)]}\right]$$

$$\equiv\left[\lim_{x\to 0}\frac{sen(kx)}{x}\right]\left[\lim_{x\to 0}\frac{sen(kx)}{x}\right]\left[\frac{1}{[1+cos(k(0))]}\right]=[k][k]\left[\frac{1}{2}\right]=\frac{k^2}{2}$$

$$\therefore\ \lim_{x\to 0}\frac{1-cos(kx)}{x^2}=\frac{k^2}{2}$$

➢ $\lim\limits_{x\to 0}\frac{\sqrt[n]{1+kx}-1}{x}\equiv\frac{\sqrt[n]{1+k(0)}-1}{(0)}=\frac{0}{0}$, por tanto simplificamos la expresión:

$$\equiv\lim_{x\to 0}\frac{\sqrt[n]{1+kx}-1}{x}=\lim_{x\to 0}\frac{\sqrt[n]{1+kx}-1}{x}\left[\frac{\left(\sqrt[n]{1+kx}\right)^{n-1}+\left(\sqrt[n]{1+kx}\right)^{n-2}+\cdots+1}{\left(\sqrt[n]{1+kx}\right)^{n-1}+\left(\sqrt[n]{1+kx}\right)^{n-2}+\cdots+1}\right]$$

$$\equiv\lim_{x\to 0}\frac{\left(\sqrt[n]{1+kx}\right)^{n}-(1)^{n}}{x\left[\left(\sqrt[n]{1+kx}\right)^{n-1}+\left(\sqrt[n]{1+kx}\right)^{n-2}+\cdots+1\right]}=\lim_{x\to 0}\frac{(1+kx)-(1)}{x\left[\left(\sqrt[n]{1+kx}\right)^{n-1}+\left(\sqrt[n]{1+kx}\right)^{n-2}+\cdots+1\right]}$$

$$\equiv\lim_{x\to 0}\frac{kx}{x\left[\left(\sqrt[n]{1+kx}\right)^{n-1}+\left(\sqrt[n]{1+kx}\right)^{n-2}+\cdots+1\right]}=\lim_{x\to 0}\frac{k}{\left[\left(\sqrt[n]{1+kx}\right)^{n-1}+\left(\sqrt[n]{1+kx}\right)^{n-2}+\cdots+1\right]}$$

$$\equiv\frac{k}{\left[\left(\sqrt[n]{1+k(0)}\right)^{n-1}+\left(\sqrt[n]{1+k(0)}\right)^{n-2}+\cdots+1\right]}=\frac{k}{[1+1+\cdots+1]}=\frac{k}{n}$$

$$\therefore\ \lim_{x\to 0}\frac{\sqrt[n]{1+kx}-1}{x}=\frac{k}{n}$$

➢ $\lim\limits_{x\to 0}\frac{a^{kx}-1}{x}\equiv\frac{a^{k(0)}-1}{(0)}=\frac{0}{0}$, por tanto simplificamos la expresión:

Utilizando el cambio de variable de:

$$u=a^{kx}-1$$
$$x=\frac{\ln(u+1)}{k\ln(a)}$$
$$u\to 0$$

$$\equiv\lim_{u\to 0}\frac{u}{\frac{\ln(u+1)}{k\ln(a)}}=k\ln(a)\lim_{u\to 0}\frac{1}{\left(\frac{1}{u}\right)\cdot\ln(u+1)}=k\ln(a)\lim_{u\to 0}\frac{1}{(\ln(u+1))^{\frac{1}{u}}}$$

$$\equiv k\ln(a)\frac{1}{\ln\left[\lim\limits_{u\to 0}\left(1+\underbrace{u}_{f(u)}\right)^{\underbrace{\frac{1}{u}}_{f(u)}}\right]}=k\ln(a)\frac{1}{\ln\left[\underbrace{\lim\limits_{u\to 0}(1+u)^{\frac{1}{u}}}_{e}\right]}=k\ln(a)\frac{1}{\ln(e)}=k\ln(a)$$

$$\therefore\ \lim_{x\to 0}\frac{a^{kx}-1}{x}=k\ln(a)$$

➢ $\lim\limits_{x\to 0}\frac{\tan(kx)}{x}\equiv\frac{\tan(k(0))}{(0)}=\frac{0}{0}$, por tanto simplificamos la expresión:

$$\equiv\lim_{x\to 0}\frac{\tan(kx)}{x}=\lim_{x\to 0}\frac{\frac{sen(kx)}{cos(kx)}}{x}=\lim_{x\to 0}\frac{sen(kx)}{x\,cos(kx)}=\left[\lim_{x\to 0}\frac{sen(kx)}{x}\right]\left[\lim_{x\to 0}\frac{1}{[cos(kx)]}\right]$$

$$\equiv\left[\lim_{x\to 0}\frac{sen(kx)}{x}\right]\left[\frac{1}{[cos(k(0))]}\right]=[k][1]=k$$

$$\therefore\ \lim_{x\to 0}\frac{tan(kx)}{x}=k$$

➢ $\lim\limits_{x\to 0}\frac{arcsen(kx)}{x}\equiv\frac{arcsen(k(0))}{(0)}=\frac{0}{0}$, por tanto simplificamos la expresión:

Utilizando el cambio de variable de:

$$u=arcsen(kx)\ \Rightarrow\ x=\frac{\operatorname{sen}(u)}{k}$$
$$u\to 0$$

$$\equiv\lim_{x\to 0}\frac{arcsen(x)}{x}=\lim_{u\to 0}\frac{u}{\frac{\operatorname{sen}(u)}{k}}=\lim_{u\to 0}\frac{k}{\frac{\operatorname{sen}(u)}{u}}$$

$$\equiv\frac{k}{\lim\limits_{u\to 0}\frac{\operatorname{sen}(u)}{u}}=\frac{k}{\lim\limits_{u\to 0}\frac{\operatorname{sen}(u)}{u}}=k$$

$$\therefore\ \lim_{x\to 0}\frac{arcsen(kx)}{x}=k$$

- $\lim\limits_{x \to 0} \dfrac{arctan(kx)}{x} \equiv \dfrac{arctan(k(0))}{(0)} = \dfrac{0}{0}$, por tanto simplificamos la expresión:

Utilizando el cambio de variable de:
$u = arctan(kx) \Rightarrow x = \dfrac{\tan(u)}{k}$
$u \longrightarrow 0$

$$\equiv \lim_{x \to 0} \frac{arctan(kx)}{x} = \lim_{u \to 0} \frac{u}{\frac{\tan(u)}{k}} = \lim_{u \to 0} \frac{k}{\frac{\tan(u)}{u}}$$

$$\equiv \frac{k}{\lim\limits_{u \to 0} \frac{\tan(u)}{u}} = k$$

$$\therefore \lim_{x \to 0} \frac{arctan(kx)}{x} = k$$

$$\therefore \lim_{x \to 0} \left(1 + f(x)\right)^{\frac{1}{f(x)}} = e \qquad \therefore \lim_{x \to \infty} \left(1 + \frac{1}{f(x)}\right)^{f(x)} = e$$

$$\lim_{x \to \infty} \frac{ax^m + \cdots + c}{bx^n + \cdots + d} = \begin{cases} m < n; & 0 \\ m = n; & \dfrac{a}{b} \\ m > n; & \infty \end{cases}$$